Math 312, Spring 2013 Jerry L. Kazdan

# Problem Set 4

DUE: In class Thurs. Feb. 7 [*Late papers will be accepted until 1:00 on Friday*].

> **Reminder:** Exam 1 is on Tuesday, Feb. 12, 9:00–10:20. No books or calculators but you may always use one 3" × 5" card with handwritten notes on both sides.

1. a). Use Theorems from Section 3.3 (or from class) to explain the following carefully.
   a) If $V$ and $W$ are subspaces with $V$ contained inside of $W$, why is $\dim V \le \dim W$?
   b) If $\dim V = \dim W$, explain why $V = W$.

2. Let $A$ be a square matrix. If $A^2$ is invertible, show that $A$ is invertible.

3. Find a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is exactly the plane

$$\{(x_1,\, x_2,\, x_3) \subset \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\}.$$

4. In class we considered the interpolation problem of finding a polynomial of degree $n$ passing through $n+1$ specified distinct points in the plane. To be definite, take $n = 3$, and say our points are $(a_1, b_1)$, $(a_2, b_2)$, $(a_3, b_3)$, and $(a_4, b_4)$. This problem involves $\mathcal{P}_3$, and so we could work in the usual basis $\{1, x, x^2, x^3\}$. However, it is easier to use the *Lagrange basis*. The point of this problem is to see vividly why choosing a basis adapted to the problem may involve much less work.
   a) Setup the linear equations you would need to solve to find the polynomial of degree 3 passing through the points $(0, -3)$, $(1, -1)$, $(2, 11)$, and $(-1, -7)$ if you use the usual basis $\{1, x, x^2, x^3\}$. But don't take time to solve these.
   b) Solve the same problem explicitly using the Lagrange basis.

5. [BRETSCHER, SEC. 2.4 #35] An $n \times n$ matrix $A$ is called *upper triangular* if all the elements below the *main diagonal*, $a_{11}$ $a_{22}$, $\dots a_{nn}$ are zero, that is, if $i > j$ then $a_{ij} = 0$.
   a) Let $A$ be the upper triangular matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

   For which values of $a$, $b$, $c$, $d$, $e$, $f$ is $A$ invertible? HINT: Write out the equations $AX = Y$ explicitly.

   b) If $A$ is invertible, is its inverse also upper triangular?

c) Show that the product of two $n \times n$ upper triangular matrices is also upper triangular.

d) Show that an upper triangular $n \times n$ matrix is invertible if none of the elements on the main diagonal are zero.

e) Conversely, if an upper triangular $n \times n$ matrix is invertible show that none of the elements on the main diagonal can be zero.

6. [See Bretscher, Sec. 3.2 #6] Let $U$ and $V$ both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of $W$.

7. [See Bretscher, Sec. 3.2 #50] Let $U$ and $V$ both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

   a) Show that $W$ is a linear space.

   b) Find all possible values for the dimension of $W$.

8. Say you have $k$ linear algebraic equations in $n$ variables; in matrix form we write $A\vec{x} = \vec{y}$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

   b) If $n > k$ you can *always* solve $A\vec{x} = \vec{y}$.

   c) If $n > k$ the nullspace (= kernel) of $A$ has dimension greater than zero.

   d) If $n < k$ then for *some* $\vec{y}$ there is *no* solution of $A\vec{x} = \vec{y}$.

   e) If $n < k$ the *only* solution of $A\vec{x} = 0$ is $\vec{x} = 0$.

9. [Bretscher, Sec. 3.3 #30] Find a basis for the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

10. Let $V$ the vector space of $n \times n$ matrices $A$ with real entries. Define a transformation $L : V \to V$ where $L(A) = \frac{1}{2}(A + A^T)$. (Here, $A^T$ is the matrix transpose of $A$.)

    a) Verify that $L$ is linear. You may use familiar facts about transpose.

    b) Describe the image of $L$ , and find its dimension.

    c) Describe the kernel of $L$ , and find its dimension.

    d) Verify the rank and nullity add up what you would expect. (Final note: $L$ is called the *symmetrization operator*.)

11. Let $\mathcal{P}_2$ be the linear space of polynomials of degree at most 2 and $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the transformation

$$(T(p))(t) = \frac{1}{t} \int_0^t p(s)\, ds.$$

For instance, if $p(t) = 2 + 3t^2$, then $T(p) = 2 + t^2$.

a) Prove that $T$ is a linear transformation.

b) Find the kernel of $T$, and find its dimension.

c) Find the range (=image) of $T$, and compute its dimension.

d) Verify the dimension of the kernel and the dimension of the range add up to what you would expect.

e) Using the standard basis $\{1, t, t^2\}$ for $\mathcal{P}_2$, represent the linear transformation $T$ as a matrix $A$.

f) Using your matrix represention from (e), find $T(p)$ where $p(t) = t - 2$.

## Bonus Problem

[Please give this directly to Professor Kazdan]

1-B Let $L\colon V \to V$ be a linear map on a linear space $V$.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^k \subset \ker L^{k+1}$ for all $k \geq 1$.

b) If $\ker L^j = \ker L^{j+1}$ for some integer $j$, show that $\ker L^k = \ker L^{k+1}$ for all $k \geq j$.

c) Let $A$ be an $n \times n$ matrix. If $A^j = 0$ for some integer $j$ (perhaps $j > n$), show that $A^n = 0$.

[Last revised: May 5, 2013]