Kernel of $Lu := u'' + 4u$

Let $Lu := u'' + 4u$. This differential equation describes the motion of a mass on a spring. These notes complete the proof in class concerning the kernel of L. The (standard) procedure we will use is outlined in our text, Section 4.1 #58.

Theorem *The most general solution of $u'' + 4u = 0$ is $u(t) = a\cos 2t + b\sin 2t$ for any constants a and b. Thus the dimension of the kernel of L is two.*

Note that is is simple to verify that $u(t) = a\cos 2t + b\sin 2t$ is in the kernel of L. We want to show that everything in the kernel of L has this form.
The key step is the

Lemma *If $w'' + 4w = 0$ with both $w(0) = 0$ and $w'(0) = 0$, then $w(t) = 0$ for all t.*

PROOF: Introduce the function $E(t) := \frac{1}{2}[w'^2 + 4w^2]$ (this function $E(t)$ is motivated by the sum of kinetic and potential energy of the vibrating spring). Then differentiating $E(t)$ and using the differential equation

$$\frac{dE}{dt} = w'w'' + 4ww' = w'(w'' + 4w) = 0.$$

Thus the derivative of E is zero so $E(t) = \text{constant}$. But from the initial conditions $w(0) = 0$ and $w'(0) = 0$ we see that $E(0) = 0$. Thus $E(t) = 0$ for all t. Because $E(t)$ is a sum of squares, this implies that $w(t) = 0$ for all t, just as claimed.

PROOF OF THE THEOREM: Let $v(t)$ be any solution of the differential equation. We want to find constants a and b so that $v(t) = a\cos 2t + b\sin 2t$. Letting $t = 0$ in this we see that $a = v(0)$. Then taking the derivative and setting $t = 0$ we find $b = \frac{1}{2}v'(0)$.
Using this let $w(t) = v(t) - [v(0)\cos 2t + \frac{1}{2}v'(0)\sin 2t]$. Then $w'' + 4w = 0$ and also $w(0) = 0$ and $w'(0) = 0$. Therefore, by the Lemma, $w(t) = 0$ for all t, that is, this solution $v(t)$ does have the form $v(t) = a\cos 2t + b\sin 2t$.

APPLICATION: Similarily one can show that the only solutions of $u'' + u = 0$ are $u(x) = a\cos x + b\sin x$ for any constants a and b. We will apply this to prove the standard trigonometric identity

$$\cos(x + c) = \cos x\cos c - \sin x\sin c$$

Let $v(x) := \cos(x + c)$. Then v satisfies $v'' + v = 0$ so by the above result $v(x) = a\cos x + b\sin x$ where $a = v(0) = \cos c$ and $b = v'(0) = -\sin c$. Thus,

$$\cos(x + c) = \cos c\cos x - \sin c\sin x,$$

as claimed. Similarly $\sin(x + c) = \sin x\cos c + \cos x\sin c$.

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