ODE-Coupled

As a mapping, the matrix $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an orthogonal reflection across the line $x_1 = x_2$. The eigenvectors V have the property that $A\vec{v} = \lambda\vec{v}$ for some constant λ. On geometric grounds, under this reflection the points on this line $x_1 = x_2$ are fixed whilethe points on the line $x_2 = -x_1$ are reflected. In particular

$$A : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad A : \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

If we let $\vec{v}_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $A\vec{v}_1 = \vec{v}_1$ and $A\vec{v}_2 = \vec{v}_2$, so $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. These vectors form a basis of $\mathbb{R}^2$ that is particularly useful to use for problems involving this matrix A.

To illustrate, we solve the differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_1 \end{aligned} \qquad \text{that is,} \qquad \frac{d\vec{x}}{dt} = A\vec{x}, \tag{1}$$

with *initial conditions* $x_1(0) = 4$ and $x_2(0) = 0$. In the above, $\vec{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. These equations are *coupled* since they both involve $x_1(t)$ and $x_2(t)$.

METHOD 1 We use the eigenvectors of A as our new basis

$$\vec{x}(t) = u_1(t)\vec{v}_1 + u_2(t)\vec{v}_2, \tag{2}$$

where the coefficients $u_1(t)$ and $u_2(t)$ are to be found. Substitute this into both sinde of equation (1). Since neither $\vec{v}_1$ nor $\vec{v}_2$ depend on t we find:

$$\frac{d\vec{x}(t)}{dt} = \frac{du_1(t)}{dt}\vec{v}_1 + \frac{du_2(t)}{dt}\vec{v}_2.$$

Also, since the $\vec{v}_j$ are eigenvectors of A:

$$A\vec{x} = u_1(t)A\vec{v}_1 + u_2(t)A\vec{v}_2 = u_1(t)\vec{v}_1 - u_2(t)\vec{v}_2.$$

Thus, from equation (1)

$$0 = \frac{d\vec{x}(t)}{dt} - A\vec{x}(t) = \left[\frac{du_1(t)}{dt} - u_1(t)\right]\vec{v}_1 + \left[\frac{du_2(t)}{dt} + u_2(t)\right]\vec{v}_2.$$

Because $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, their coefficients must both be zero:

$$\begin{aligned} \frac{du_1(t)}{dt} &= \quad u_1(t) \\ \frac{du_2(t)}{dt} &= -u_2(t). \end{aligned} \tag{3}$$

Note these equations are *uncoupled* – and are easy to solve:

$$u_1(t) = c_1e^t \qquad \text{and} \qquad u_2(t) = c_2e^{-t},$$

where c_1 and c_2 are any constants. Shortly they will be determined by the initial conditions.

Substituting this into equation (2), we find that

$$\vec{x}(t) == c_1e^t\begin{pmatrix}1\\1\end{pmatrix} + c_2e^{-t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}c_1e^t + c_2e^{-t}\\c_1e^t - c_2e^{-t}\end{pmatrix}.$$

Now we use the initial condition to find the constants c_1 ans c_2:

$$\begin{pmatrix}4\\0\end{pmatrix} = \vec{x}(0) = \begin{pmatrix}c_1 + c_2\\c_1 - c_2\end{pmatrix}.$$

Therefore $c_1 = c_2 = 2$ so the desired solution is

$$\vec{x}(t) = \begin{pmatrix}2e^t + 2e^{-t}\\2e^t - 2e^{-t}\end{pmatrix},$$

that is,

$$x_1(t) = 2e^t + 2e^{-t}, \qquad x_2(t) = 2e^t - 2e^{-t}.$$

METHOD 2 This is essentially identical, but here we explicitly introduce the change of coordinates S from the standard basis to the new basis consisting of the eigenvectors of A. We want $S^{-1}AS = D$ where D is the diagonal matrix consisting of the eigenvalues of A, so

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

COMPUTATIONAL NOTE If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then

$$SD = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 a & \lambda_2 b \\ \lambda_1 c & \lambda_2 d \end{pmatrix}$$

so the *columns* of S are multiplied by the λ_j's (DS multiplies the *rows* of S by the λ_j's).

By general theory, the *columns* S are the corresponding eigenvectors of A

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since $A = SDS^{-1}$, we substitute this into equation (1)

$$\frac{d\vec{x}}{dt} = A\vec{x} = SDS^{-1}\vec{x}, \qquad \text{that is,} \qquad \frac{d(S^{-1}\vec{x})}{dt} = DS^{-1}\vec{x}$$

and are let to make the change of variable $\vec{u} = S^{-1}\vec{x}$ to find

$$\frac{d\vec{u}}{dt} = D\vec{u}, \qquad \text{that is,} \qquad \begin{aligned} \frac{du_1}{dt} &= u_1 \\ \frac{du_2}{dt} &= -u_2 \end{aligned}.$$

These are exactly the equations (3) we found above. Thus

$$\vec{u}(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{-t} \end{pmatrix},$$

so, just as before,

$$\vec{x}(t) = S\vec{u}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^{-t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{pmatrix}.$$

Again, we can use the initial condition to find the constants c_1 and c_2.

EXERCISE: Say you have a sequence of vectors $\vec{x}_1$, $\vec{x}_2, \ldots$ with the property that $\vec{x}_{k+1} = A\vec{x}_k$, where A is the above 2×2 matrix, and say the initial vector $X_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Compute $\vec{x}_k$ by using a basis consisting of the eigenvectors of A: $x_k = a_k \vec{v}_1 + b_k \vec{v}_2$.

Since our map A is just an orthogonal reflection, without any computation (or mention of eigenvectors) the answer is obviously that if k is even, then $\vec{x}_k = \vec{x}_0$ while if k is odd, then $\vec{x}_k = \vec{x}_1$ is the reflected vector. The point of this problem is that the identical computation works in the general case where A is any $n \times n$ matrix that can be diagonalized.