Multiple Integral: Change of Variable

Say we have a multiple integral

$$K := \iint_{\mathbb{R}^2} \frac{1}{[1+(x+2y-1)^2+(3x+y+2)^2]^2}\, dx\, dy \qquad (1)$$

and would like to make the change of variable

$$u = x+2y-1, \qquad v = 3x+y++2 \qquad (2)$$

since that would clean-up the integrand. How is this done?

Here is the general rule for

$$J := \iint_{\mathcal{D}} h(v_1,v_2)\, dv_1 dv_2$$

under the change of variable $\vec{v} = F(\vec{u})$ where $F(\vec{u}) = \begin{pmatrix} f_1(\vec{u}) \\ f_2(\vec{u}) \end{pmatrix}$ is given by

$$v_1 = f_1(u_1, u_2) \qquad v_2 = f_2(u_1, u_2).$$

Note that here we have defined the old variables, (v_1, v_2) in terms of the new variables, (u_1, u_2), while in equations (1)-(2) we defined the new variables, (u, v) in terms of the old ones, (x, y). In practice, one uses whichever is more convenient.

To begin, compute the first derivative (or *Jacobian*) matrix:

$$F'(\vec{u}) := \begin{pmatrix} \frac{\partial f_1(u_1,u_2)}{\partial u_1} & \frac{\partial f_1(u_1,u_2)}{\partial u_2} \\ \frac{\partial f_2(u_1,u_2)}{\partial u_1} & \frac{\partial f_2(u_1,u_2)}{\partial u_1} \end{pmatrix}. \qquad (3)$$

Then the rule is

$$dv_1\, dv_2 = |\det F'(\vec{u})|\, du_1\, du_2$$

so in the new variables

$$J = \iint_{\mathcal{D}'} h(f_1(u_1,u_2),\ f_2(u_1,u_2))\, |\det(F'(\vec{u})|\, du_1 du_2,$$

where $\mathcal{D}'$ is the region in the u_1u_2 plane corresponding to $\mathcal{D}$.

Example 1 Compute $\displaystyle\iint_{R^2} \frac{1}{(1+x^2+y^2)^2}\,dx\,dy$.

We change to polar coordinates $\begin{pmatrix} x \\ y \end{pmatrix} = F(r,\theta)$ with the usual formulas

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Then, as in equation (3), the first derivative matrix is

$$F'(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Since $\det F'(r,\theta) = r$ we have $dx\,dy = r\,dr\,d\theta$ so

$$\begin{aligned} \iint_{R^2} \frac{1}{(1+x^2+y^2)^2}\,dx\,dy &= \int_0^{2\pi} \left(\int_0^{\infty} \frac{1}{(1+r^2)^2}\,r\,dr \right) d\theta \\ &= 2\pi \int_0^{\infty} \frac{1}{(1+r^2)^2}\,r\,dr = \pi \end{aligned} \tag{4}$$

Example 2 For the integral in equation (1)-(2) if we write $\begin{pmatrix} u \\ v \end{pmatrix} = G(x,y)$ then the first derivative matrix is

$$G'(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \qquad \text{so} \qquad du\,dv = 5\,dx\,dy.$$

Therefore, using polar coordinates, from equation (4)

$$\begin{aligned} K &= \iint_{\mathbb{R}^2} \frac{1}{[1+(x+2y-1)^2+(3x+y+2)^2]^2}\,dx\,dy \\ &= \iint_{\mathbb{R}^2} \frac{1}{(1+u^2+v^2)^2}\,\frac{du\,dv}{5} = \frac{\pi}{5} \end{aligned} \tag{5}$$

The identical procedure works in in higher dimensions. In $\mathbb{R}^n$ say we have a multiple integral

$$J := \int \cdots \int_{\mathcal{D}} h(v_1,\ldots,v_n)\,dv_1 \cdots dv_n$$

and want to make the change of variable $\vec{v} = F(\vec{u})$. As above, compute the first derivative matrix

$$F'(\vec{u}) = \begin{pmatrix} \frac{\partial f_1(\vec{u})}{\partial u_1} & \cdots & \frac{\partial f_1(\vec{u})}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\vec{u})}{\partial u_1} & \cdots & \frac{\partial f_n(\vec{u})}{\partial u_n} \end{pmatrix}.$$

Then the element of "volume" becomes

$$dv_1 \cdots dv_n = |\det F'(\vec{u})| du_1 \cdots du_n.$$

This is particularly simple if one makes a *linear* change of variable, $\vec{v} = A\vec{u}$ where A is an invertible matrix whose elements are constants, so $F(\vec{u}) = A\vec{u}$. Then $F'(\vec{u}) = A$ and we obtain

$$dv_1 \cdots dv_n = |\det A| du_1 \cdots du_n \tag{6}$$

and the change of variable formula is simply

$$J := \int \cdots \int_{\mathcal{D}} h(\vec{v})\, dv_1 \cdots dv_n = \int \cdots \int_{\mathcal{D}'} h(A\vec{u})\, |\det A| du_1 \cdots du_n.$$

Example 3 Compute $J = \iint_{R^2} \dfrac{1}{(1 + 2x_1^2 + 6x_1x_2 + 9x_2^2)^2}\, dx_1\, dx_2$.

SOLUTION Write $2x_1^2 + 6x_1x_2 + 9x_2^2 = \langle \mathbf{x}, A\mathbf{x} \rangle$, where $A = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 9 \end{smallmatrix}\right)$. Idea: If A were the identity matrix, this would be straightforward, just use polar coordinates as in equation (4). Diagonalizing A is thus the essential step. Since A is symmetric, it is orthogonally similar to a diagonal matrix, $A = RDR^*$, where $D = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ has the eigenvalues of A on its diagonal and R is an orthogonal matrix.

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \langle \mathbf{x}, RDR^*\mathbf{x} \rangle = \langle R^*\mathbf{x}, DR^*\mathbf{x} \rangle.$$

Make the change of variable $\mathbf{y} = R^*\mathbf{x}$. In the integral, since $|\det R| = 1$, then, by (6),

$$dy_1\, dy_2 = |\det R^*| dx_1\, dx_2 = dx_1\, dx_2$$

we find

$$J = \iint_{R^2} \frac{1}{(1+\lambda_1 y_1^2 + \lambda_2 y_2^2)^2}\, dy_1\, dy_2.$$

Because A is positive definite (there is a simple test for 2×2 matrices), its eigenvalues are positive so we make the further change of variable $z_j = \sqrt{\lambda_j} y_j$. This gives

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = z_1^2 + z_2^2.$$

and

$$dz_1\, dz_2 = \sqrt{\lambda_1 \lambda_2}\, dy_1\, dy_2 = \sqrt{\det A}\, dy_1\, dy_2 = 3\, dy_1\, dy_2.$$

Thus, as in equation (4),

$$J = \frac{1}{3} \iint_{R^2} \frac{1}{(1+z_1^2+z_2^2)^2}\, dz_1\, dz_2 = \frac{\pi}{3}.$$

It is interesting that although we used the theory that we could orthogonally diagonalize A, we *never* needed to compute explicitly its eigenvalues or eigenvectors.

ALTERNATE For this and other examples where $\langle \mathbf{x}, A\mathbf{x} \rangle$ with A positive definite arise, it is often faster (and clearer) to use that A has a positive definite square root, that is, there is a positive dedinite (symmetric) matrix B with $A = B^2$. Then

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \langle \mathbf{x}, B^2\mathbf{x} \rangle = \langle B\mathbf{x}, B\mathbf{x} \rangle = \|B\mathbf{x}\|^2,$$

which suggests making the change of variables $\mathbf{y} = B\mathbf{x}$ to find

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \|\mathbf{y}\|^2.$$

If we use this approach in the above integral, then

$$dy_1 dy_2 = |\det B|\, dx_1\, dx_2 = \sqrt{|\det A|}\, dx_1\, dx_2$$

so

$$J = \frac{1}{\sqrt{|\det A|}} \iint_{R^2} \frac{1}{(1+\|\mathbf{y}\|^2)^2}\, dy_1\, dy_2.$$

As before, we now use polar coordinates (equation (4)) to conclude

$$J = \frac{1}{3}\int_0^{2\pi}\left(\int_0^{\infty}\frac{1}{(1+r^2)^2}\,r\,dr\right)d\theta = \frac{\pi}{3}.$$

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