WHAT IS LINEAR PROGRAMMING?

We content ourselves with

A PRACTICAL EXAMPLE: You are a traveling snake-oil salesman, selling brand A at \$5 per gallon and brand B at \$3 per gallon. You ply your trade in a territory that craves snake oil, and so you can be sure of selling all you carry at the prices mentioned. Things are not quite so simple, however. Snake oil requires the magical ingredient M. More precisely, brand A requires 4 oz of ingredient M per 100 gal of oil, and brand B requires 3 oz per 100 gal. And you have only 12 oz of ingredient M. See Fig. 3.17.

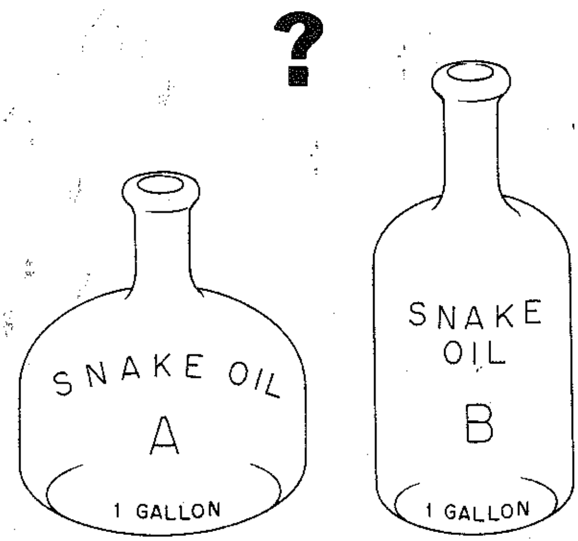

FIGURE 3.17

If the scarcity of ingredient M were the only constraint, then it is easy to see that you would maximize gross income by producing and selling only brand A. There is one more fly in the ointment. Brand A weighs 15 lb per gal; brand B weighs one-third this much. And you are capable of transporting only 3,000 lb of snake oil. How much should be brand A and how much brand B?

THE APPROACH: We proceed as follows. Let $x =$ amount of brand A in hundreds of gallons, $y =$ amount of brand B in hundreds of gallons.

We wish to compute x and y so that the gross income is maximized. This leads to a linear functional

$$\text{Gross income} = L(x, y) = 500x + 300y$$

since, for example, brand A sells for \$500 per 100 gal.

Now we may not use more than 12 oz of ingredient M; that is

$$4x + 3y \leq 12.$$

Also brands A and B weigh 1,500 and 500 lb per 100 gal, respectively, so that we must satisfy

$$1{,}500x + 500y \leq 3{,}000,$$

which is the same as

$$3x + y \leq 6.$$

Hence the problem may be stated in mathematical terms thus: maximize $L(x, y) = 500x + 300y$ subject to the constraints

$$\begin{array}{ll} x \geq 0, \quad y \geq 0 & \text{amounts nonnegative,} \\ 4x + 3y \leq 12 & \text{ingredient } M \text{ limited,} \\ 3x + y \leq 6 & \text{weight limited.} \end{array}$$

A Graphical Solution: It is not hard to see that the set $\mathscr{K}$ of points (x, y) satisfying all four inequalities above looks as shown in Fig. 3.18. This set is **convex** in that if (x_1, y_1) and (x_2, y_2) are two points of $\mathscr{K}$, then the straight-line segment connecting them is also in $\mathscr{K}$.

Now for certain numbers c, the straight line $L(x, y) = 500x + 300y = c$ actually intersects the set $\mathscr{K}$—for example, $c = 0$, $(x, y) = (0, 0)$. What we seek is the *largest* number c having this property and the point (or points) (x, y) of intersection. The lines $500x + 300y = c$ all have the same slope; they are parallel. Examination of the picture should convince you that the maximum value of $L(x, y)$ for $(x, y) \in \mathscr{K}$ occurs at the point $X^* = (x^*, y^*) = (\frac{6}{5}, \frac{12}{5})$, one of the "vertices" (corners) of the convex set $\mathscr{K}$. This means that you should produce 120 gal of brand A and 240 gal of brand B in order to maximize your total income, which will be \$1,320.

For the general linear-programming problem, we have a linear functional $L: \mathbb{R}^n \to \mathbb{R}$, like $L(x, y) = 500x + 300y$ above, and some convex set $\mathscr{K}$, not a subspace, of $\mathbb{R}^n$. This constraint set $\mathscr{K}$ of admissible values is usually defined by many inequalities (we had three in the example). The problem is to find X^* in $\mathscr{K}$ that maximizes (or minimizes) L, so that $LX^* \geq LX$ for all X in the set $\mathscr{K}$ of admissible values. An important

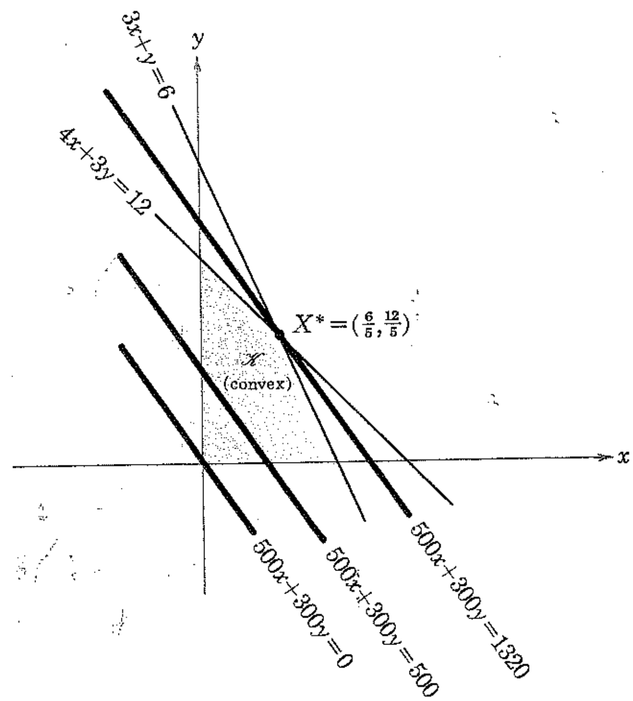

FIGURE 3.18

theorem asserts that the desired "best point" X^* is always one of the vertices (corners) of $\mathscr{K}$. Computers are essential in solving these problems.

REMARK: Note that we are maximizing a *linear* functional L here. Problems of maxima and minima for *nonlinear* functions are discussed in Chap. 6.