Formula for $1^2 + 2^2 + \cdots + n^2$

Let

$$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2. \tag{1}$$

We would like to FIND a formula for this, using ideas of induction. Say we know S_n. Then

$$S_{n+1} - S_n = (n+1)^2, \qquad \text{with} \qquad S_0 = 0. \tag{2}$$

This is a *first order linear difference equation*. Motivated by the formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

we seek a formula for S_n as a cubic polynomial:

$$S_n = An^3 + Bn^2 + Cn + D.$$

The problem is to find the coefficient A, B, C, and D. IDEA: plug into (2) and match coefficients:

$$\begin{aligned}
S_{n+1} - S_n =& [A(n+1)^3 + B(n+1)^2 + c(n+1) + D] \\
&- [An^3 + Bn^2 + Cn + D] \\
=& [A(n^3 + 3n^2 + 3n + 1) + B(n^2 + 2n + 1) + C(n+1) + D] \\
&- [An^3 + Bn^2 + Cn + D] \\
=& 3An^2 + (3A + 2B)n + (A + B + C)
\end{aligned}$$

We want this to be $(n+1)^2 = n^2 + 2n + 1$. Match coefficients of the powers of n to obtain the equations

$$3A = 1 \tag{3}$$

$$(3A + 2B) = 2 \tag{4}$$

$$A + B + C = 1. \tag{5}$$

Thus

$$A = 1/3, \; B = 1/2, \; \text{and} \; C = 1/6.$$

Notice that we still do not know D. Our formula is

$$\begin{aligned} S_n =& \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + D \\ =& \frac{n(2n^2 + 3n + 1)}{6} + D \ . \\ =& \frac{n(2n+1)(n+1)}{6} + D \end{aligned}$$

To find D we use the initial condition $S_0 = 0$. This gives $D = 0$. Consequently

$$S_n = \frac{n(2n+1)(n+1)}{6}.$$

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