Math 202 | Exam 1 | Jerry L. Kazdan
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Exam 1

DIRECTIONS: Part A has 8 shorter problems (5 points each) while Part B has 3 traditional problems (10 points each). [70 points total].
To receive full credit your solution should be clear and correct. Neatness counts. You have 1 hour 20 minutes. Closed book, no calculators, but you may use one 3×5 with notes on both sides.

PART A: Eight shorter Problems, 5 points each [40 points]

A-1. Show that $\sqrt{5}$ is not a rational number.

A-2. If a and b are *rational* numbers, consider the set S of real numbers of the form $a + b\sqrt{5}$. Show that the non-zero elements in S have multiplicative inverses in S. [This is the key step in showing that S is a field.]

A-3. Determine if the set $S = \{x \in \mathbb{R} : \ 2x^2 > x^3 - 3x\}$ is bounded above and/or below, and if so, find $\inf(S)$ and $\sup(S)$ — if they exist.

A-4. Give an example of a sequence of real numbers that is not monotone but that does converge to some limit.

A-5. If x_1 is a given real number and $x_{n+1} = \sqrt{1 + x_n^2}$ for $n = 1, 2, \ldots$, show that the sequence x_n diverges.

A-6. Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be bounded functions such that $f(x) \le g(x)$ for all x. Let F denote the image of f and G the image of g. Give an example (a picture) of pairs of such functions with $\sup(F) > \inf(G)$.

A-7. Compute $\lim_{n\to\infty} \dfrac{1 + 2n - 5n^2}{4 + 3n^2}$. Carefully note any standard theorems you use.

A-8. Give an example of a sequence x_n of real numbers with at least two subsequences that converge to different limits.

PART B: Three traditional problems, 10 points each[30 points[

B-1. a) For which real numbers $c > 0$ does $\lim_{n\to\infty} n^2 c^n = 0$? Why?

b) Repeat this if c is a complex number.

B-2. Let the real sequence $b_n > 0$ converge to a limit $B > 0$. Show with your bare hands (an ϵ argument) that $1/b_n \to 1/B$.

B-3. A sequence $x_n \in \mathbb{R}$ is called *contracting* if for some constant $0 < c < 1$ (such as $c = \frac{1}{2}$) it has the property that for all $n = 1, 2, 3, \ldots$

$$|x_{n+1} - x_n| \le c|x_n - x_{n-1}|.$$

The point of this problem is to show that a contracting sequence converges.

a) Show that $|x_{n+1} - x_n| \le c^n |x_1 - x_0|$ for all n.

b) Use $x_{n+1} - x_0 = (x_{n+1} - x_n) + (x_n - x_{n-1}) + \cdots + (x_1 - x_0)$ to show that

$$|x_{n+1} - x_0| \le \left(c^n + c^{n-1} + \cdots + c + 1\right) |x_1 - x_0|$$

c) More generally, if $n > k$ show that

$$\begin{aligned} |x_{n+1} - x_k| &\le \left(c^n + c^{n-1} + \cdots + c^k\right) |x_1 - x_0| \\ &= c^k \left(\frac{1 - c^{n-k+1}}{1 - c}\right) |x_1 - x_0| < c^k \frac{|x_1 - x_0|}{1 - c}. \end{aligned}$$

REMARK: Since $0 < c < 1$, this shows that the x_n are a Cauchy sequence and hence converge.