Rook Theory and Matchings

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ABSTRACT

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In this paper we study an analogue of classical rook theory for new types of boards where rook placements represent matchings from graph theory. We survey recent results in this area, including the presentation of a statistic for combinatorially determining the q-hit numbers of such boards. In addition, we provide some results and conjectures on the zeros of rook polynomials associated with matchings of weighted graphs.

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Chapter 1

Motivation: Why Rook Theory?

Classical rook theory was developed in the 1940's by Riordan and Kaplansky as a framework for studying permutations with restricted position. In the game of chess, rooks are permitted to move horizontally or vertically across the board to attack an opposing piece. Thus, one can envision a permutation as a placement of n rooks on an n x n chessboard such that no two rooks have the ability to attack one another (i.e. no two rooks are in the same row or column). The notions of rook numbers and hit numbers were formulated, with the k-th hit number counting the number of ways that n non-attacking rooks can be placed on an n x n chessboard such that k of those rooks lie on a certain predefined subset of the squares on that board. A fundamental result of rook theory shows that these hit numbers can be calculated in terms of the more easily calculated rook numbers, where the k-th rook number is defined to be the number of ways of placing k non-attacking rooks on some predefined subset of squares on a chessboard. One can likely already see the potential applications of this theory to permutations with restricted position. For instance, if we take the predefined subset to be the squares along the diagonal from bottom left to top right on an n x n chessboard, the 0-th hit number corresponds to the number of permutations in S_n with no fixed points (i.e. the number of derangements in S_n). Similarly, any problem of permutations with restricted position can be formulated in terms of rook numbers and hit numbers.

Over the second half of the twentieth century, especially in the 1990's, mathematicians became increasingly interested in developing the theory of rooks, both for its own intrinsic appeal and for using it as a tool to prove theorems from other areas of enumerative combinatorics. Garsia, Remmel, Haglund, Dworkin, and many others used rook theory in connection with graphs, hypergeometric series (see [3]), Stirling numbers and permutations, Bessel polynomials, Abel polynomials and forests, among other combinatorial objects. In this paper, we focus on connections between rook theory and graph theory, particularly with regard to matchings of graphs. In Chapter 2, we provide a rapid introduction to the basics of classical rook theory before delving into a rook theory for perfect matchings of graphs in Chapter 3. These concepts will be examined from both a combinatoric and an algebraic perspective. Chapter 4 examines some of the recent results and conjectures pertaining to the zeros of polynomials that arise in the study of rook theory and matchings of graphs.

Chapter 2

Classical Rook Theory

As a warm-up for the rook theory for perfect matchings in Chapter 3, we give a quick review of the basics of classical rook theory. Here we provide definitions of rook numbers and hit numbers, their q-analogues for Ferrers boards, examples of how to calculate these numbers, and an application of classical rook theory to the problème des ménages.

2.1 Rook and Hit Numbers for Boards in A_n

In classical rook theory, our setting is an n x n array, which we will denote by A_n. By a square $(i, j) \in A_n$, we mean the square lying in the i-th column from the left and the j-th row from the bottom. A *board* B is defined to be a subset of the squares of A_n, which we will denote pictorially as shaded squares. On such a board we can realize a partial permutation as a rook placement, defined as follows.

Definition 2.1.1. Let $B \subseteq A_n$ be a board and k a natural number. A k-element *rook placement* p on B consists of a k-element subset of B such that no two squares of p lie in the same row or column. The collection of all placements of k rooks on B will be denoted $R_k(B)$, and the *k-th rook number (relative to B)*, denoted $r_k(B)$, is defined to be $|R_k(B)|$.

A permutation $\sigma \in S_n$ corresponds to a rook placement $p_\sigma \in R_n(B)$ in the obvious way: $(i,j) \in p_\sigma \iff \sigma(i) = j$. This leads us to the definition of a hit number.

Definition 2.1.2. Let $B \subseteq A_n$. Then $T_{k,n}(B)$ denotes the set of all permutations σ such that p_σ intersects B in k squares and the *k-th hit number (relative to B)*, denoted $t_{k,n}(B)$, is defined to be $|T_{k,n}(B)|$.

It turns out that the rook numbers and hit numbers are inherently linked, as evidenced by the following equivalence, first proved in [7].

Theorem 2.1.3. *Let $B \subseteq A_n$. Then $\sum_{k=0}^{n} t_{k,n}(B) z^k = \sum_{k=0}^{n} r_k(B)(n-k)!(z-1)^k$.*

Proof: We will count in two ways the number of pairs (σ, C), where $\sigma \in S_n$ and C is a k-subset of $B \cap p_\sigma$. First, for each $0 \leq j \leq n$ we can choose σ in $t_{j,n}(B)$ ways such that $|B \cap p_\sigma| = j$ and then select C in $\binom{j}{k}$ ways. Alternatively, we could first choose C in $r_k(B)$ ways and extend this partial permutation to a full permutation $\sigma \in S_n$ in $(n-k)!$ ways. Hence, $\sum_{j=0}^{n} \binom{j}{k} t_{j,n}(B) = r_k(B)(n-k)!$. Summing over k, we get $\sum_{k=0}^{n} r_k(B)(n-k)! y^k = \sum_{k=0}^{n} t_{k,n}(B)(y+1)^k$, and the result follows by substituting $z-1$ for y.

Example 2.1.4. Consider the board $B \subseteq A_3$ consisting in the shaded squares in the figure below, along with a sample rook placement $p \in R_2(B)$:

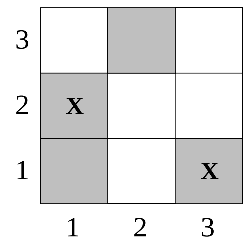

Figure 2.1: $p \in R_2(B)$

We leave it to the reader to verify the following rook numbers and hit numbers: $r_0(B) = 1$ (the empty placement), $r_1(B) = 4$, $r_2(B) = 4$, $r_3(B) = 1$, $t_{0,3}(B) = 1$, $t_{1,3}(B) = 3$, $t_{2,3}(B) = 1$, and $t_{3,3}(B) = 1$. The reader can then check that $\sum_{k=0}^{3} t_{k,3}(B)z^k$ and $\sum_{k=0}^{3} r_k(B)(3-k)!(z-1)^k$ both sum to $1 + 3z + z^2 + z^3$.

Example 2.1.5. (Problème des ménages): This classic problem ponders the question of how many ways to seat n married couples around a circular table, alternating male-female, such that no couple sits side-by-side. Equivalently, we seek the number of permutations $\sigma \in S_n$ such that $\sigma(i) \not\equiv i, i+1 \pmod{n}$ for all $i \in \{0, 1, ...n\}$. Putting this into rook theoretic terminology, we want to find $t_{0,n}(B)$ for the board $B = \{(1,1), (2,2), ..., (n,n), (1,2), (2,3), ...(n-1,n), (n,1)\}$ shown at the top of the following page.

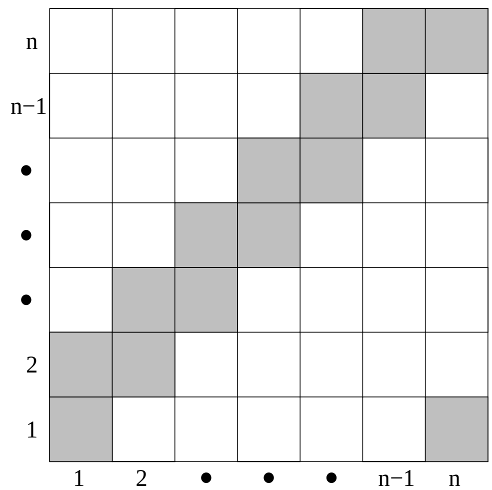

Figure 2.2: $B = \{(1,1),(2,2),...,(n,n),(1,2),(2,3)...(n-1,n),(n,1)\}$

Here, the kth rook number relative to B corresponds to the number of ways of choosing k pairwise non-adjacent points from a collection of $2n$ points arranged in a circle. In [9], Stanley proves that the number of ways is $\frac{2n}{2n-k}\binom{2n-k}{k}$. Therefore, by letting $z = 0$ in Theorem 2.1.3, we see that there are precisely $\sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$ ways to seat the n couples so that no husband and wife sit side-by-side.

2.2 q-Rook and q-Hit Numbers for Ferrers Boards

In the mid-80's, Garsia and Remmel in [1] posited a q-analogue of the rook numbers and hit numbers for a special collection of boards in A_n called Ferrers boards. This description of the q-hit numbers was given algebraically in terms of the q-rook numbers. A decade later, Dworkin and Haglund independently came up with statistics that gave the q-hit numbers a combinatorial interpretation. Shortly, we will present

Haglund's statistic, but first we need some definitions.

Definition 2.2.1. A *skyline board* $A(a_1, a_2, ..., a_n) \subseteq A_n$ is the board with column heights $a_1, a_2, ..., a_n$ from left to right containing the squares $\{(i,j) : j \le a_i\}$. If, in addition, $a_i \le a_j$ for $1 \le i < j \le n$, we call such a board a *Ferrers board.*

Given a rook placement $p \in R_k(F)$ on a Ferrers board $F \subseteq A_n$, by letting each rook in p cancel all squares to its right and below, we get a statistic $u_F(p)$ denoting the number of squares of F that are neither in p nor canceled by a rook in p. Throughout this paper, rooks will be shown as x's and the squares they cancel as dots •. This statistic allows us to define the k-th q-rook number $r_k(F,q)$ to be $\sum_{p \in R_k(F)} q^{u_F(p)}$.

Example 2.2.2. The Ferrers board $A(1,2,4,5,5,5) \subseteq A_6$ in the figure has $u_F(p) = 10$.

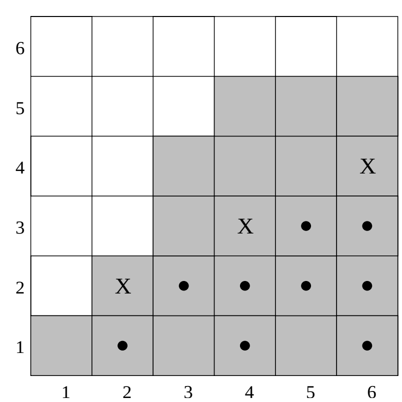

Figure 2.3: $F = A(1,2,4,5,5,5) \subseteq A_6$

Garsia and Remmel then defined in [1] the k-th q-hit number for Ferrers boards in terms of the q-rook numbers by the formula

$$\sum_{k=0}^{n} t_{k,n}(F,q)z^k = \sum_{k=0}^{n} r_k(F,q)[n-k]! \prod_{i=n-k+1}^{n} (z-q^i),$$

where $[n]! = [n][n-1]\cdots[2][1]$ and $[n] = 1+q+q^2+...+q^{n-1}$.

In the late 90's, Dworkin and Haglund each gave descriptions of statistics for calculating the q-hit numbers of Ferrers boards. We will only describe the Haglund statistic here; the Dworkin statistic is very similar. It was later proved that they both give rise to the same q-hit numbers that Garsia and Remmel postulated. Haglund's statistic $s_{F,h}(p)$, as presented in [5], is calculated by the following cancellation method. For a rook placement $p \in T_{k,n}(F)$, begin by letting each rook cancel all squares to its right, both shaded and unshaded. Then, let each rook $\mathbf{r}$ in F cancel all squares above it, regardless of whether or not they are in F, while letting each rook $\mathbf{r}$ in A_n/F cancel the squares below it that are not in F. The statistic $s_{F,h}(p)$ is then defined to be the number of uncanceled squares remaining in all of A_n, and the k-th q-hit number can then by described by the formula $t_{k,n}(F,q) = \sum_{p \in T_{k,n}(F)} q^{s_{F,h}(p)}$. The figure at the top of the following page illustrates an example of Haglund cancellation for a placement p on the board $F = A(1,3,3,4,6,6,6,8) \subseteq A_8$. The reader can verify that for this placement, $s_{F,h}(p) = 21$.

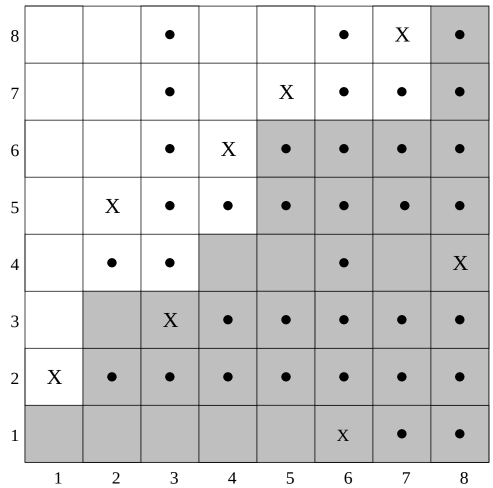

Figure 2.4: $F = A(1, 3, 3, 4, 6, 6, 6, 8) \subseteq A_8$

The next chapter develops notions of rook numbers, hit numbers, and their q-analogues in relation to matchings from graph theory. Like the exposition on q-hit numbers for Ferrers boards above, we give both algebraic and combinatoric definitions of these q-hit numbers as formulated by Haglund and Remmel in [5]. In the final section of Chapter 3, we go on to show that these two interpretations do in fact give rise to the same q-analogue.

Chapter 3

Rook Theory & Matchings

In this chapter, we illustrate Haglund and Remmel's theory of rooks for matchings. Most importantly, we present a statistic they developed that allows one, for a special kind of board, to combinatorially determine the same q-hit numbers that they previously defined algebraically. The last section of this chapter is devoted to demonstrating that for such boards the algebraic and combinatoric interpretations coincide.

3.1 Basic Results on Rook/Hit Numbers in B_{2n}

Recall from graph theory that a *matching* of a graph $G = (V, E)$, where V is a set of vertices and E is a set of edges, is a subset $E' \subseteq E$ of pairwise vertex disjoint edges. A *perfect matching* is a matching such that every vertex in G is the endpoint of some edge in the matching.

To study matchings in the framework of rook theory, we need to consider different types of boards from the ones in Chapter 2. The boards we will now work with consist in subsets of squares of the board B_{2n} shown below. To avoid confusion, in this chapter when we speak of square (i, j), we mean the square in row i, column j.

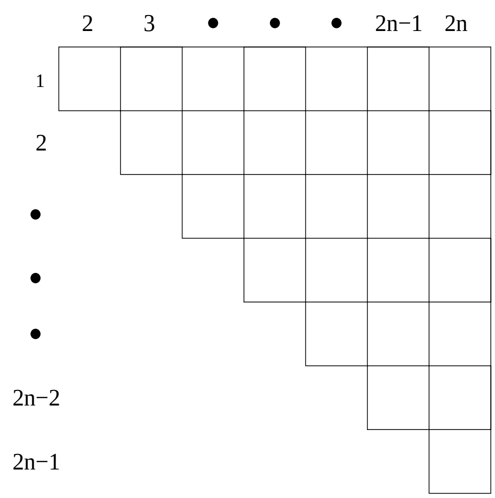

Figure 3.1: B_{2n}

Two classes of boards are of particular interest, nearly Ferrers boards and a subset of these boards called shifted Ferrers boards.

Definition 3.1.1. A board $B \subseteq B_{2n}$ is a *nearly Ferrers board* if $(i, j) \in B \implies \{(s, j) : s < i\} \cup \{(t, i) : t < i\} \subseteq B$. If, in addition, $(i, j) \in B \implies (i, r) \in B$ for all $r < j$, then we call such a board a *shifted Ferrers board.*

Thus, a shifted Ferrers board is a special type of shifted skyline board, where we now specify the row lengths instead of the column heights when we write $F =$

$B(a_1, a_2, ..., a_n)$. A shifted Ferrers board $F = B(a_1, a_2, ..., a_n)$ has the property that $2n-1 \geq a_1 \geq a_2 \geq ... \geq a_{2n-1} \geq 0$ with the nonzero row lengths strictly decreasing. The shifted Ferrers board $F = B(6, 5, 3, 2, 0, 0, 0) \subseteq B_8$ is pictured below.

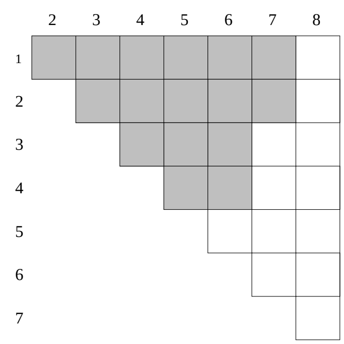

Figure 3.2: $F = B(6, 5, 3, 2, 0, 0, 0) \subseteq B_8$

Whereas partial permutations are modeled by rook placements in classical rook theory, rook placements will now correspond to matchings of the complete graph K_{2n} on vertices $1, 2, ..., 2n$. We associate a matching m with the rook placement p_m, using the rule $(i, j) \in p_m \iff i < j$ and $\{i, j\} \in m$. The rook placement p_m in the figure below corresponds to the (perfect) matching $m = \{\{1, 2\}, \{3, 8\}, \{4, 6\}, \{5, 7\}\}$.

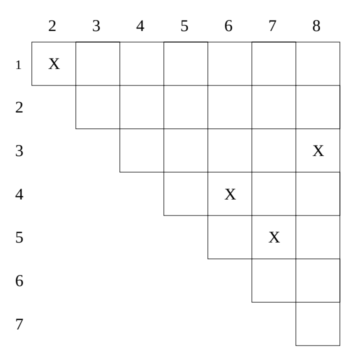

Figure 3.3: p_m for $m = \{\{1, 2\}, \{3, 8\}, \{4, 6\}, \{5, 7\}\}$

Now we are at a point where we can formally define rook placements, rook numbers, and hit numbers for boards in B_{2n}. For notational purposes, let $PM(B_{2n}) = \{p_m : m$ is a perfect matching of $K_{2n}\}$.

Definition 3.1.2. Let $p \subseteq B \subseteq B_{2n}$. Then p is a *rook placement of* B if $p \subseteq B \cap p_m$ for some $p_m \in PM(B_{2n})$. For any nonnegative integer k, $M_k(B)$ will denote the set of all k-element rook placements of B, and we subsequently define the *k-th rook number (relative to B)* to be $|M_k(B)|$ and denote this by $m_k(B)$.

Definition 3.1.3. For any nonnegative integer k, we define $F_{k,2n}(B)$ to be the set $\{p_m \in PM(B_{2n}) : |p_m \cap B| = k\}$. Then the *$k$-th hit number (relative to B)* is defined to be $|F_{k,2n}(B)|$ and will be denoted as $f_{k,2n}(B)$.

Having defined rook and hit numbers for boards in B_{2n}, we can now give some basic results involving these numbers, immediately following a few notational conventions that we will use throughout the rest of the paper, repeating a couple from Chapter 2 for convenience. We define $[n] = 1+q+q^2+...+q^{n-1}$, $[n]\downarrow_k = [n][n-1]\cdots[n-k+1]$, $[n]! = [n][n-1]\cdots[2][1]$, $n!! = \prod_{i=1}^{n}(2i-1)$, $[n]!! = \prod_{i=1}^{n}[2i-1]$, $(x)\downarrow\downarrow_k = x(x-2)(x-4)\cdots(x-2k+2)$, $[x]\downarrow\downarrow_k = [x][x-2][x-4]\cdots[x-2k+2]$, and $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$. We now present an analogue of Theorem 2.1.3 that demonstrates the close relationship between rook numbers and hit numbers for boards in B_{2n}.

Theorem 3.1.4. *Let* $B \subseteq B_{2n}$. *Then* $\sum_{k=0}^{n} f_{k,2n}(B)z^k = \sum_{k=0}^{n} m_k(B)(n-k)!!(z-1)^k$.

Next, we introduce a method from [5] for obtaining two new boards from a given board. First, if $B \subseteq B_{2n}$ and $(i,j) \in B$ with $i < j$, we obtain the board $B/(i,j) \subseteq B_{2n}$ simply by removing the square (i,j) from B. Second, we obtain the board $B/\overline{(i,j)} \subseteq B_{2n-2}$ by getting rid of any square that have either i or j as a coordinate. In doing so, we will delete two entire rows and columns, giving us our new board $B/\overline{(i,j)}$ lying on a board isomorphic to B_{2n-2}. We can map this to an actual board in B_{2n-2} by an explicit isomorphism that can be found in [5]. For our purposes, it suffices to show an example, which will elucidate the process of obtaining these two boards. In the figure at the top of the next page, we denote with a $\bullet$ the squares that share a common coordinate with the rook on square $(2,6)$. These squares, along with the square $(2,6)$ itself, are the squares that are deleted to form $B/\overline{(2,6)} \subseteq B_6$ from $B = B(6,5,3,2,0,0,0) \subseteq B_8$. This construction allows us to now present two recursions involving the rook and hit numbers for boards in B_{2n}, whose proofs can be found in [5].

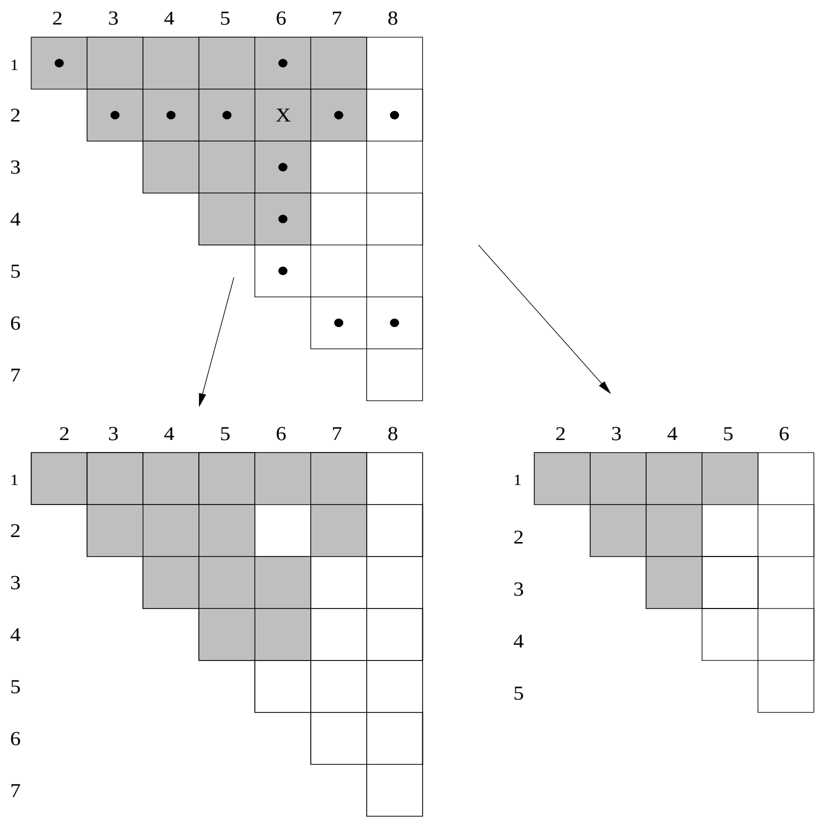

Figure 3.4: $B/(2,6)$ (bottom left) and $B/\overline{(2,6)}$ (right)

Theorem 3.1.5. *Let $B \subseteq B_{2n}$ and $(i,j) \in B$. Then the following recursions hold:*

(1) $m_k(B) = m_k(B/(i,j)) + m_{k-1}(B/\overline{(i,j)})$.

(2) $f_{k,2n}(B) = f_{k,2n}(B/(i,j)) + f_{k-1,2n-2}(B/\overline{(i,j)}) - f_{k,2n-2}(B/\overline{(i,j)})$.

The significance of these recursions is that they provide recursive algorithms for calculating the rook and hit numbers of a board in terms of the smaller boards $B/(i,j)$ and $B/\overline{(i,j)}$. Before moving on to an important theorem that will yield as a corollary the number of perfect matchings of a graph, we present another important recursion from [5], whose q-analogue will be presented in the next section.

Theorem 3.1.6. *Let $B \subseteq B_{2n}$ such that B has no squares in the last column of B_{2n}. Then*

$$f_{k,2n}(B) = \sum_{i=1}^{2n-1} f_{k,2n-2}(B/\overline{(i,2n)}).$$

The next theorem we will present is a type of theorem that arises frequently in rook theory, usually referred to as a "factorization" theorem. The following factorization of shifted Ferrers boards was first proved recursively by Reiner and White, and a direct bijection is given in [5] by Haglund and Remmel, which provides a more general factorization applicable to all nearly Ferrers boards.

Theorem 3.1.7. ***(Factorization Theorem for B_{2n}):*** *Let $B \subseteq B_{2n}$ be a nearly Ferrers board, and let a_i be the number of squares of B in row i for $i = 1, \ldots, 2n-1$. Then*

$$\prod_{i=1}^{2n-1}(x + a_{2n-i} - 2i + 2) = \sum_{k=0}^{n} m_k(B)(x) \downarrow\downarrow_{2n-1-k}.$$

From this theorem we derive a corollary about perfect matchings in graphs.

Corollary 3.1.8. *Let $B \subseteq B_{2n}$ be a nearly Ferrers board, and let a_i be the number of squares of B in row i for $i = 1, 2, \ldots, 2n-1$. Then the number of perfect matchings in the graph $G_B = (\{1, 2, \ldots, 2n\}, \{\{i,j\} : (i,j) \in B\})$ is*

$$\prod_{i=1}^{2n-1}(a_{2n-i} + 2(n-i))/2^{n-1}(n-1)!.$$

Proof. Set $x = 2n - 2$ in the conclusion of the previous theorem. Then all terms in the right-hand sum will drop out except $m_n(B)(2n-2)\downarrow\downarrow_{n-1}$. Since $(2n-2)\downarrow\downarrow_{n-1}= 2^{n-1}(n-1)!$, we solve for $m_n(B)$ and get the desired result. $\square$

In the next section, we will develop q-analogues of rook and hit numbers for matchings, including the presentation of Haglund and Remmel's statistic for combinatorially determining the q-hit numbers.

3.2 q-Rook and q-Hit numbers in B_{2n}

As was the case in Chapter 2 for boards in A_n, we need a type of cancellation in B_{2n} in order to define q-rook numbers.

Definition 3.2.1. Let $B \subseteq B_{2n}$ and $\mathbf{r}$ a rook lying on square $(i, j) \in B$. Then we say $\mathbf{r}$ *rook-cancels* the squares $\{(r, i) : r < i\} \cup \{(i, s) : i+1 \leq s < j\} \cup \{(t, j) : t < i\}$. We also define a statistic $u_B(p)$ for a rook placement p, which denotes the number of squares in $B - p$ that are not rook-canceled by a rook in p.

We now define q-rook numbers in terms of the statistic $u_B(p)$.

Definition 3.2.2. Let $B \subseteq B_{2n}$ and k a nonnegative integer. Then we define the *k-th q-rook number relative to B*, denoted $m_k(B, q)$, by the following formula:

$$m_k(B, q) = \sum_{p \in M_k(B)} q^{u_B(p)}.$$

We also define $m_0(B, q) = q^{|B|}$.

Example 3.2.3. The figure below depicts the four rook placements in $M_1(B)$ for the board $B = \{(1,3),(2,3),(2,4),(3,4)\}$.

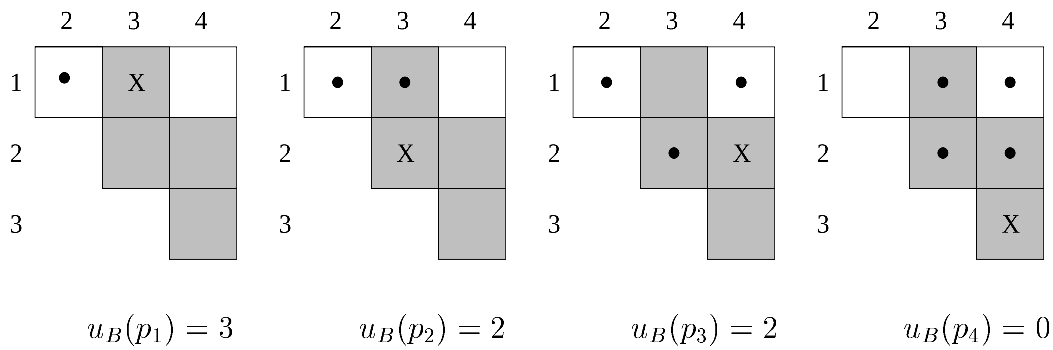

Figure 3.5: $B = \{(1,3),(2,3),(2,4),(3,4)\}$

Thus, $m_1(B,q) = 1 + 2q^2 + q^3$. The reader can check that $m_0(B,q) = q^4$ and $m_2(B,q) = q$. We next provide a definition related to the statistic $u_B(p)$, which we will need in the next section.

Definition 3.2.4. We say that a square (i,j) on a board $B \subseteq B_{2n}$ is a *corner square* of B if there does not exist a square $(s,t) \in B_{2n}$ that could rook-cancel (i,j) according to the $u_B(p)$ statistic.

In the special case where B is a shifted Ferrers board, the quality of a square (i,j) being a corner square reduces to (i,j) having no squares in B to the southeast in B_{2n}.

Our next task is to present Haglund and Remmel's algebraic and combinatoric definitions of the q-hit numbers. The entire section to follow will be devoted to showing that for shifted Ferrers boards these interpretations yield the same q-hit numbers. We first give the algebraic formulation.

Definition 3.2.5. Let $B \subseteq B_{2n}$ and k a nonnegative integer. Then we define the *k-th q-hit number (relative to B)*, denoted $f^a_{k,2n}(B,q)$ via the formula

$$\sum_{k=0}^{n} f^a_{k,2n}(B,q) z^k = \sum_{k=0}^{n} m_k(B,q)[n-k]!! \prod_{i=n-k+1}^{n} (z - q^{2i-1}).$$

The $f^a_{k,2n}(B,q)$ are not necessarily polynomials in q with nonnegative coefficients, but in Section 3.3 we will show they are as such in the case where B is a shifted Ferrers board.

We now define a new type of cancellation and a statistic based on that cancellation similar to the Haglund statistic $s_{F,h}(p)$ presented in Section 2.2, which will allow us to give a combinatorial interpretation of the q-hit numbers.

Definition 3.2.6. Let $B \subseteq B_{2n}$ and $p \in F_{k,2n}(B)$. For a rook $\mathbf{r}$ on cell $(i,j) \in p \cap B$, we say $\mathbf{r}$ *p_m-cancels* all squares $\{(r,i) : r < i\} \cup \{(i,s) : i+1 \le s < j\} \cup \{(t,j) : t < i\} \cup \{(u,j) : u > j \text{ and } (u,j) \notin B\}$. For a rook $\mathbf{r}$ on $(i,j) \notin B$, we say $\mathbf{r}$ *p_m-cancels* all squares in $\{(r,i) : r < i\} \cup \{(i,s) : i+1 \le s < j\} \cup \{(t,j) : t < i \text{ and } (t,j) \notin B\}$. For a rook placement $p \in F_{k,2n}(B)$, we define the statistic $t_B(p)$ to be the number of squares in $B_{2n} - p$ that are not p_m-canceled.

In other words, a rook $\mathbf{r}$ on the board p_m-cancels all squares that it would rook-cancel, plus all squares below $\mathbf{r}$ in its column that are not in B, while a rook $\mathbf{r}$ off the board p_m-cancels all squares to the left of $\mathbf{r}$ that it would rook-cancel plus all squares above $\mathbf{r}$ in its column that are not in B. The statistic $t_B(p)$ then counts the number of uncanceled squares in $B_{2n} - p$. The following example illustrates this process.

Example 3.2.7. For the rook placement p shown on the board $F = B(7, 5, 4, 2, 0, 0, 0)$, we see that $t_F(p) = 7$.

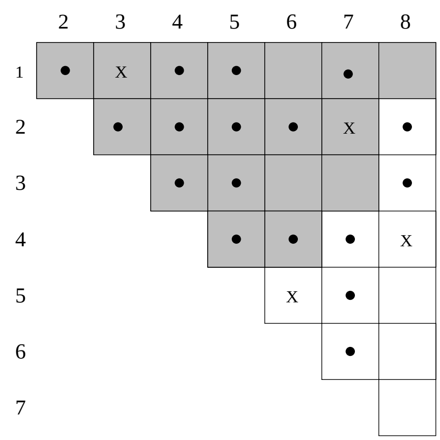

Figure 3.6: $F = B(7, 5, 4, 2, 0, 0, 0)$

We are now in a position to give the combinatorial interpretation of the q-hit numbers as defined in [5].

Definition 3.2.8. Let $B \subseteq B_{2n}$ and k a positive integer. Then we define the combinatorial object $f^c_{k,2n}(B, q)$ according to the formula $f^c_{k,2n}(B, q) = \sum_{p \in F_{k,2n}(B)} q^{t_F(p)}$.

The purpose of the following section is to show that for shifted Ferrers boards, the combinatorial interpretation of q-hit numbers coincides with the algebraic definition (i.e. $f^a_{k,2n}(B,q) = f^c_{k,2n}(B,q)$). The following analogue of the second recursion in Theorem 3.1.5, with which we conclude this section, will be utilized in that endeavor.

Theorem 3.2.9. *Let $B \subseteq B_{2n}$ such that $B \cap \{(j,2n) : 1 \leq j \leq 2n-1\} = \{(j,2n) : j \leq i\}$, where $i \geq 1$ (i.e. the last column of B contains exactly the squares $(1,2n),(2,2n),\ldots,(i,2n)$). Then*

$$f^c_{k,2n}(B,q) = qf^c_{k,2n}(B/(i,2n),q) + f^c_{k-1,2n-2}(B/\overline{(i,2n)},q) - q^{2n-1}f^c_{k,2n-2}(B/\overline{(i,2n)},q).$$

Proof. See [5]. □

3.3 $f^a_{k,2n}(B,q) = f^c_{k,2n}(B,q)$

Our goal in this section is to show that the q-hit numbers of shifted Ferrers boards are polynomials in q with nonnegative coefficients. We do this by showing the formal algebraic definition of the q-hit numbers, $f^a_{k,2n}(B,q)$, is equivalent to the combinatorial object $f^c_{k,2n}(B,q)$, which is clearly a polynomial in q with nonnegative coefficients.

Theorem 3.3.1. *Let $B \subseteq B_{2n}$ be a shifted Ferrers board and k a nonnegative integer. Then $f^a_{k,2n}(B,q) = f^c_{k,2n}(B,q)$.*

Sketch of Proof: The proof of the theorem is broken up into cases for different types of shifted Ferrers boards B:

(1) B is the empty board

(2) B contains at least one square in the last column of B_{2n}

(3) B contains no squares in the last column of B_{2n}

Case (1) is easy. If B is the empty board, then $f^c_{k,2n}(B,q) = \delta_{k,0}[n]!!$. Since $m_k(B,q) = \delta_{k,0}$, then by the recursion defining q-hit numbers we see that $f^a_{k,2n}(B,q) = \delta_{k,0}[n]!! = f^c_{k,2n}(B,q)$.

To prove Case (2), we will show that $f^a_{k,2n}(B,q)$ and $f^c_{k,2n}(B,q)$ satisfy the same recursion with the same initial conditions for a board B with at least one square in the last column of B_{2n}. We start by giving an analogue of the first recursion in Theorem 3.1.5.

Theorem 3.3.2. *Let $B \subseteq B_{2n}$, (i,j) a corner square of B, and k a positive integer. Then*

$$m_k(B,q) = qm_k(B/(i,j),q) + m_{k-1}(B/\overline{(i,j)},q).$$

Proof. We can partition the rook placements in $M_k(B)$ according to whether or not they contain (i,j). Let $M_k^{(i,j)}(B)$ denote the set of rook placements in $M_k(B)$ which contain a rook on (i,j). For a rook placement $p \in M_k^{(i,j)}(B)$, the rook $\mathbf{r}$ on (i,j), by virtue of being a corner square, will rook-cancel any square in B with which it shares a common coordinate (i.e. there will be no uncancelled squares sharing a coordinate with $\mathbf{r}$). This tells us that there exists a 1:1 weight-preserving correspondence between $M_k^{(i,j)}(B)$ and $M_{k-1}(B/\overline{(i,j)})$, and consequently

$$\sum_{p \in M_k^{(i,j)}(B)} q^{u_B(p)} = m_{k-1}(B/\overline{(i,j)}, q).$$

Next, for a rook placement $p \in M_k(B) \setminus M_k^{(i,j)}(B)$, the cell (i,j) is not rook-canceled by any rook in p by definition of what it is to be a corner square. Hence, $u_B(p) = 1 + u_{B/(i,j)}(p)$, and thus

$$\sum_{p \in M_k(B) \setminus M_k^{(i,j)}(B)} q^{u_B(p)} = \sum_{p \in M_k(B/(i,j))} q^{1+u_{B/(i,j)}(p)} = q m_k(B/(i,j), q),$$

and the theorem follows. □

By using Theorem 3.3.2 to manipulate the algebraic formula defining the q-hit numbers, the following corollary can be reached.

Corollary 3.3.3. *Let $B \subseteq B_{2n}$, (i,j) a corner square of B, and $k > 0$. Then*

$$f^a_{k,2n}(B,q) = q f^a_{k,2n}(B/(i,j), q) + f^a_{k-1,2n-2}(B/\overline{(i,j)}, q) - q^{2n-1} f^a_{k,2n-2}(B/\overline{(i,j)}, q).$$

Corollary 3.3.3 clearly can be applied to a shifted Ferrers board B that has at least one square in the last column, because the southernmost square in the last column that is in B is a corner square (since it can't possibly have any squares in B to the southeast). Thus, we see from Theorem 3.2.9 and Corollary 3.3.3 that $f^a_{k,2n}(B,q)$ and $f^c_{k,2n}(B,q)$ satisfy the same recursion for shifted Ferrers boards with at least one cell in the last column of B_{2n}, so case (2) of the theorem is taken care of.

The final case that we have to deal with to show that the algebraic definition and combinatoric interpretation of the q-hit numbers coincide is for a board B that has no squares in the last column of B_{2n}. Our method of proof for case (3) again amounts to showing that $f^a_{k,2n}(B,q)$ and $f^c_{k,2n}(B,q)$ satisfy a recursion with the same initial conditions. To prove this case, we first need a few preliminary theorems. We begin by giving the following analogue of Theorem 3.1.6 for $f^c_{k,2n}(B,q)$.

Theorem 3.3.4. *Let $B \subseteq B_{2n}$ have no squares in the last column of B_{2n} and k a positive integer. Then*

$$f^c_{k,2n}(B,q) = \sum_{i=1}^{2n-1} q^{2n-i-1} f^c_{k,2n-2}(B/\overline{(i,2n)},q).$$

Proof. We can partition $F_{k,2n}(B)$ into $\cup_{i=1}^{2n-1} F^{(i,2n)}_{k,2n}(B)$ since B contains no squares in the last column on B_{2n} and any perfect matching must contain some edge with vertex $2n$ as an endpoint. Note that since $(i,2n) \notin B$, there exists a 1:1 correspondence between $F^{(i,2n)}_{k,2n}(B)$ and $F_{k,2n-2}(B/\overline{(i,2n)})$. For a rook placement $p \in F^{(i,2n)}_{k,2n}(B)$, the rook $\mathbf{r}$ on $(i,2n)$ p_m-cancels all squares $(j,2n)$ with $j < i$ since all squares above $(i,2n)$ are not in B. Consequently, there are $2n-i-1$ uncanceled squares in the last column on B_{2n} with respect to p. Hence,

$$\sum_{p \in F^{(i,2n)}_{k,2n}(B)} q^{t_B(p)} = q^{2n-i-1} \sum_{F_{k,2n-2}(B/\overline{(i,2n)})} q^{t_{B/\overline{(i,2n)}}(p)} = q^{2n-i-1} f^c_{k,2n-2}(B/\overline{(i,2n)},q),$$

and the theorem follows. □

Haglund and Remmel also derived the following two recursions for boards that have no squares in the final column of B_{2n}, whose proofs can be found in [5].

Theorem 3.3.5. *Let $B \subseteq B_{2n}$ be a shifted Ferrers board with no squares in the last column of B_{2n}, (i,r) the southernmost square in the rightmost column of B, and k a positive integer. Then the following hold:*

(1) $m_k(B/\overline{(r,2n)},q) = [r-2k]m_{k-1}(B/\overline{(i,r)},q) + q^{r-2-2k}m_k(B/\overline{(i,2r)},q)$

(2) $\sum_{j=1}^{2n-1} q^{2n-j-1}m_k(B/\overline{(j,2n)},q) = [2n-1-2k]m_k(B,q) - (q^{2n-1}-q^{2n-3-2k})m_{k+1}(B,q).$

These two theorems along with two other lemmas, which we will not present here but can be found in [5], yield the final piece of the puzzle necessary to prove case (3).

Theorem 3.3.6. *Let $B \subseteq B_{2n}$ be a shifted Ferrers board and k a positive integer. Then*

$$f^a_{k,2n}(B,q) = \sum_{i=1}^{2n-1} q^{2n-i-1} f^a_{k,2n-2}(B/\overline{(i,2n)},q).$$

Thus, Theorems 3.3.4 and 3.3.6 combine to show that the $f^a_{k,2n}(B,q)$ and $f^c_{k,2n}(B,q)$ satisfy the same recursion, so Theorem 3.3.1 has now been proved for all shifted Ferrers boards in B_{2n}.

This chapter has presented some of the most recent results in rook theory for matchings, much of which was developed by Haglund and Remmel in [5]. We now switch gears and examine some results on the zeros of polynomials associated with matchings of graphs.

Chapter 4

Zeros of Rook Polynomials

Over the last few decades, due to the work of Nijenhuis, Haglund, Ono, Wagner, and others, much has been discovered about the zeros of polynomials that arise in the study of rook theory. We start this chapter by presenting in Section 4.1 some of the fundamental theorems and conjectures in this area followed by some recent results and conjectures in Section 4.2 on the zeros of polynomials associated with weighted matchings of graphs.

4.1 Fundamental Results and Conjectures

In this section, we need a slightly broader definition of rook placements than we did in previous chapters. Now we will be placing rooks on an n x n array of nonnegative real numbers, instead of just an array of zeros and ones. We give this new definition below along with definitions of rook and hit polynomials. Unless otherwise specified, in this chapter $A = (a_{ij})$ will refer to an n x n matrix with nonnegative real entries, and J_n will stand for the n x n matrix of all ones.

Definition 4.1.1. A *k-element rook placement* on an n x n matrix $A = (a_{ij})$ is an arrangements of k non-attacking rooks on the entries of A (i.e. no two rooks are in the same row or column). We define the *weight* of such a rook placement to be the product of the entries in the squares on which a rook lies, and we define the *k-th rook number*, denoted $m_k(A)$, to be the sum of the weights over all k-element rook placements on A. We let $m_0(A) = 1$.

Definition 4.1.2. For an n x n matrix $A = (a_{ij})$, we define the *rook polynomial of A*, denoted $M(z; A)$, to be $\sum_{k=0}^{n} m_k(A)z^k$, and the *hit polynomial of A*, denoted $T(z; A)$, to be $\sum_{k=0}^{n} m_k(A)(n-k)!(z-1)^k$.

A Ferrers board in this context is a matrix of zeros and ones such that if a square (i, j) is 1, then all squares to the right and above are also 1. Similarly, a shifted Ferrers board (known as a threshold graph to graph theorists) is an n x n matrix of zeros and ones that is weakly decreasing across rows and down columns.

We now present some fundamental results about the zeros of rook and hit polynomials, starting with a theorem proved by Nijenhuis in [8].

Theorem 4.1.3. *Let A be an n x n matrix with nonnegative real entries. Then all of the zeros of the rook polynomial $M(z; A)$ are real.*

In particular, the rook polynomials of any board are real. More recently, Haglund, Ono, and Wagner went on to prove in [4] an analogous result for the hit polynomials of Ferrers boards, which does not hold for boards in general.

Theorem 4.1.4. *Let A be a Ferrers board. Then all of the zeros of the hit polynomial $T(z;A)$ are real.*

There is an intrinsic interplay between the rook and hit polynomial of a matrix and its permanent. Recalling that the permanent of an n x n matrix $A = (a_{ij})$, denoted $per(A)$, is defined as $\sum_{\sigma \in S_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{n,\sigma(n)}$, it is easy to see that $m_n(A) = per(A)$ and that $m_k(A)$ is the sum of all k x k permanental minors of A. Moreover, we see by expanding $per((z-1)A + J_n)$ in powers of $z-1$ that $per((z-1)A+J_n) = T(z;A)$. Since $per((z-1)A + J_n)$ has real zeros if and only if $per(zA + J_n)$ has real zeros, Theorem 4.1.4 tells us that for a Ferrers board A, $per(zA + J_n)$ has only real zeros. This is a special case of the following conjecture presented in [4].

The Monotone Column Permanent (MCP) Conjecture: Let A be an n x n matrix with real entries, which weakly increase down columns (i.e. $a_{i,j} \leq a_{i+1,j}$). Then all of the zeros of $per(zA + J_n)$ are real.

As a segue into the next section, which examines the zeros of polynomials associated with threshold graphs, we conclude this section with some graph theoretic terminology and a fundamental result, which was later discovered to yield Nijenhuis' theorem as a consequence.

Definition 4.1.5. An n x n matrix $G = (g_{ij})$ with nonnegative real entries is called a *weighted graph* if it is strictly upper triangular. If for all i, j, $g_{ij} \in \{0, 1\}$, we call G a *graph.*

Sometimes we will express a weighted graph G as a pair (V, E) of vertices and weighted edges instead of from the matrix perspective. The logic here is that G can be identified with the graph on n vertices with weight $g_{i,j}$ between vertices i and j. For example, the board B_{2n}, an upper trinagular matrix of all ones, corresponds to the complete graph on $2n$ vertices, K_{2n}. Recall from Section 3.1 that a matching is a pairwise vertex disjoint subset of the edges of a graph, and a perfect matching is a matching such that every vertex is the endpoint of some edge in the matching. We now define some terms associated with matchings of graphs.

Definition 4.1.6. Let $G = (V, E)$ be a weighted graph. We define the *weight* of a matching m of G to be the product of the weights of the edges in m and the *k-th matching number*, denoted $mat_k(G)$ to be the sum of these weights over all matchings in G with k edges. We define the *matching polynomial* of G to be $\sum_k mat_k(G) z^k$, and we let $mat_0(G) = 1$.

For even n, the function $mat_{n/2}(G)$ is known as the *Hafnian* of G and is denoted $Hf(G)$. This corresponds to the sum over the weights of all perfect matchings in G. We now give a sample calculation of the matching polynomial of a weighted graph.

Example 4.1.7. Consider the weighted graph $G = (V, E)$, where $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$ with weights $g_{1,2} = 7, g_{1,3} = 3, g_{2,4} = 5$, and $g_{3,4} = 4$. Both the matrix representation and pictorial version of this weighted graph are given below. The reader can verify that the matching polynomial of G is $1 + (7 + 3 + 5 + 4)z + (7 * 4 + 3 * 5)z^2 = 1 + 19z + 43z^2$.

$$\begin{pmatrix} 0 & 7 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 4.1: $G = (g_{ij})$

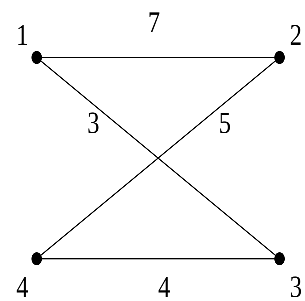

Figure 4.2: $G = (V, E)$

In [6], Heilmann and Lieb showed that the matching polynomial of a simple graph (i.e. a graph with no cycles) with nonnegative real edge weights has only real zeros. For bipartite graphs, the matching polynomial reduces to the rook polynomial, so Theorem 4.1.3 follows from the Heilmann-Lieb theorem. We turn now to some recent results and conjectures of Haglund regarding the zeros of polynomials associated with shifted Ferrers boards.

4.2 Zeros of Polynomials for Threshold Graphs

We begin this section by presenting a "K_n-version" of the hit polynomial for weighted graphs as presented by Haglund in [2]. In this section, we let $n!!$ represent the product of the odd integers less than or equal to n.

Definition 4.2.1. Let G be a weighted graph on n vertices. We define the K_n*-hit polynomial* of G, denoted $Q(z;G)$, to be $\sum_{k\geq 0} mat_k(G)(z-1)^k(n-2k)!!$.

For n even (if odd, just add an isolated vertex to make n even), $Q(z;G) = Hf((z-1)G + J_n)$, following from the fact that a k-edge matching of K_n can be extended to a perfect matching of K_n in $(n-2k)!!$ ways.

Recalling the definitions of a corner square and of the shifted Ferrers board $G/\overline{(i,j)}$ (see pages 14 and 18), we now arrive at a theorem proved by Haglund in [2], which yields the subsequent corollary about the zeros of $Q(z;G)$ for a threshold graph G.

Theorem 4.2.2. *Let G be a shifted Ferrers board, (i,j) the right-most corner square of G, and n an even natural number. Then $Hf(G+zJ_n)$ has only real zeros, which are interlaced by $Hf(G/\overline{(i,j)} + zI_{n-2})$.*

Corollary 4.2.3. *Let G be a threshold graph. Then $Q(z;G)$ has only real zeros.*

This corollary turns out to be a special case of a conjecture also presented in [2]. Before we give this conjecture, we need to define the notions of a hook and monotonicity. The k-th hook of K_n is the set of all squares (i, j) of K_n such that either $i = k$ or $j = k$, and in general the k-th hook of a graph G is the set of all squares that are both in the k-th hook of K_n and also in G. If we travel along a hook of G, we say we are moving in the positive direction if we are moving upward or to the left. We now define the notion of monotonicity of an upper triangular matrix with respect to a graph G.

Definition 4.2.4. Let W be an upper triangular array of real numbers, G a graph on n vertices, and h a hook of G. Then we say W is *monotone with respect to* h if the values of the corresponding squares in W are weakly increasing as we traverse h in a positive direction. If for each square (i, j) of G it is the case that W is monotone with respect to one of the two hooks of G containing (i, j), then we say that W is *G-monotone*.

In light of Corollary 4.2.3 and the MCP conjecture, Haglund posed the following conjecture about K_n-monotone arrays.

Conjecture 4.2.5. *Let $W = (w_{ij})$ be a K_n-monotone array with nonnegative real entries, where n is an even natural number. Then $Hf(zW + J_n)$ has only real zeros.*

Haglund argued in [2] that the special case of Conjecture 4.2.5 where $w_{ij} \in \{0, 1\}$ follows from Corollary 4.2.3 by showing that in such instances W is isomorphic to a threshold graph. Conjecture 4.2.5 has been proved true for $n \leq 5$ and at the time of the writing of this paper is open for $n \geq 6$. We conclude this section by presenting an even more general conjecture that combines Conjecture 4.2.5 and the MCP Conjecture.

Conjecture 4.2.6. *Let $W = (w_{ij})$ be a G-monotone array, where G is either K_n or the complete bipartite graph $K_{n/2,n/2}$, with n an even natural number. Then the polynomial*

$$\sum_{PM} \prod_{(i,j)\in PM} (zw_{ij} + 1)$$

has only real zeros, where PM is the set of all perfect matchings of G.

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