A New Interpretation of the Cocharge Statistic

Kendra Ann Nelsen

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Contents

Abstract

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Kendra Ann Nelsen

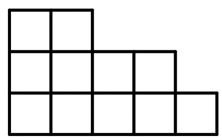
Advisor: James Haglund

In this thesis we will start by introducing the combinatorial formula for the Macdonald polynomial. We then show how cocharge arises naturally from this formula. Expanding these methods to compositions, we see how an alternative description of cocharge arises naturally from the combinatorial statistics used in the formula for the Macdonald polynomial. The final chapter of the thesis outlines the proof of Lascoux and Schützenberger's famous theorem that expands the Kostka-Foulkes polynomial in terms of charge.

1 Preliminaries

1.1 Tableaux

A *partition* $\mu = (\mu_1, \ldots, \mu_k)$ of n is a tuple of weakly decreasing, nonzero parts such that $\mu_1 + \cdots + \mu_\ell = n$. The length of the partition $\ell(\mu) = k$ is equal to the number of parts of μ, and the size of the partition $|\mu| = n$ is the sum of the parts. Identify a partition μ with the tableau that has μ_i boxes in the i^{th} row. For the partition $\mu = (5, 4, 2)$ we identify μ with the following tableau



. (1.1)

Equivalently, the *diagram* of μ is $dg(\mu) = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ | i <= \ell(\mu), j <= \mu_i\}$. Abusing notation, we refer to $dg(\mu)$ as μ. We partially order partitions μ and ν of n by saying $\mu \leq \nu$ if $\mu_1 + \cdots + \mu_k \leq \nu_1 + \cdots \nu_k$ for all k.

Let $\mu' = (\mu'_1, \ldots, \mu'_j)$ where μ'_i is equal to the number of boxes in the i^{th} column of μ. In the above example, when $\mu = (5, 4, 2)$ we have $\mu' = (3, 3, 2, 2, 1)$. A *filling* of a partition μ is a map $\sigma : \mu \to \mathbb{Z}_+$ that assigns a positive integer to each of the boxes in the partition μ. A *super filling* of a partition μ is a map $\sigma : \mu \to \mathbb{Z}_+ \bigcup \mathbb{Z}_-$ that assigns a non-zero integer to each of the boxes in the partition μ. Denote negative numbers with a bar overtop, i.e. negative 4 is written $\overline{4}$. For example, a filling and a super filling of the partition $\mu = (5, 4, 2)$ are:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 4 \\ \cline{1-4} 2 & 3 & 4 & 1 \\ \hline 1 & 3 & 2 & 3 & 6 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline 2 & \bar{6} \\ \cline{1-4} 3 & \bar{2} & 4 & \bar{1} \\ \hline 1 & \bar{2} & \bar{2} & 5 & 3 \\ \hline \end{array}.$$

The *reading word* of a tableau is found by reading the entries of the tableau left to right, top to bottom. Denoted $w(\sigma)$, the reading word of the above filling is 14234113236. If $u = (1,5)$ is the box in the first row, fifth column of the above partition μ, then $\mu(u) = 6$ is the filling of the box u. Define the weight of a filling as a tuple λ where λ_i is the number of i's that appear in the filling. Denote the weight as $wt(\mu) = (\lambda_1, \cdots, \lambda_k)$ so that in the above filling, we have $wt(\mu) = (3,2,3,2,0,1)$. Similarly, define the weight of a super filling as a pair of tuples (λ, ρ) where λ_i is the number of i's and ρ_i is the number of $\bar{i}$'s appearing in the super filling. The above super filling has weight $wt(\sigma) = (\lambda, \rho)$ where $\lambda = (1,1,2,1,1)$ and $\rho = (1,3,0,0,0,1)$.

We can *standardize* filling $\sigma(\mu)$ of weight $\lambda = (\lambda_1, \cdots, \lambda_k)$ in the following way:

1. In reading order (from left to right, top to bottom) replace the first entry containing a 1 with a 1, the second entry containing a 1 with a 2, $\cdots$, the λ_1^{th} entry containing a 1 with a λ_1.

2. In reading order replace the first entry containing a 2 with a $\lambda_1 + 1$, the second entry containing a 2 with a $\lambda_1 + 2$, $\cdots$, the λ_2^{th} entry containing a 2 with a $\lambda_1 + \lambda_2$.

3. Repeat the same process for entries containing 3 through λ_k

Standardization produces a new filling $\tilde{\sigma}(\mu)$ with weight $wt(\tilde{\sigma}) = 1^{|\mu|}$ (i.e. a filling with entries consisting of one 1, one 2, ..., one $|\mu|$). Below we illustrate standardizing a filling:

$\sigma(\mu)$

1	4			
2	3	4	1	
1	3	2	3	6

,

$\tilde{\sigma}(\mu)$

1	9			
4	6	10	2	
3	7	5	8	11

.

A super filling $\sigma(\mu)$ can be *standardized* using a similar algorithm. First, one must fix a complete order on the *super alphabet* $\mathbb{Z}_+ \bigcup \mathbb{Z}_-$. Two orderings that we will use later are:

$$(a) \quad 1 < \overline{1} < 2 < \overline{2} < 3 < \overline{3} \cdots,$$

$$(b) \quad 1 < 2 < 3 < \ldots < \overline{3} < \overline{2} < \overline{1}.$$

For the purposes of explaining how to standardize a super filling, we will use ordering (a). Let $\sigma(\mu)$ be a super filling with $wt(\sigma) = (\lambda, \rho)$.

1. In reading order (from left to right, top to bottom) replace the first entry containing a 1 with a 1, the second entry containing a 1 with a 2, $\cdots$, the λ_1^{th} entry containing a 1 with a λ_1.

2. In backwards reading order (from right to left, bottom to top) replace the first entry containing a $\overline{1}$ with a λ_1+1, the second entry containing a $\overline{1}$ with a λ_1+2, $\cdots$, the ρ_1^{th} entry containing a $\overline{1}$ with a $\lambda_1+\rho_1$.

3. Repeat the same process for entries containing $2, \overline{2}, 3, \overline{3}, \ldots$.

Below we illustrate standardizing a super filling:

$$\begin{array}{cc} \sigma(\mu) & \tilde{\sigma}(\mu) \\ \begin{array}{|c|c|c|c|c|} \hline 2 & \overline{6} \\ \cline{1-2} 3 & \overline{2} & 4 & \overline{1} \\ \cline{1-4} 1 & \overline{2} & \overline{2} & 5 & 3 \\ \hline \end{array}, & \begin{array}{|c|c|c|c|c|} \hline 3 & 11 \\ \cline{1-2} 7 & 6 & 9 & 2 \\ \cline{1-4} 1 & 5 & 4 & 10 & 8 \\ \hline \end{array}. \end{array}$$

To standardize using an alternate ordering, go from the smallest letter in the ordering to the largest letter in the ordering replacing positive letters in reading order and negative letters in backwards reading order.

Let $\mu = (\mu_1, \ldots, \mu_\ell)$ and $\lambda = (\lambda_1, \cdots, \lambda_k)$ be partitions. A *semi-standard tableau* of shape μ and weight λ is a filling of the boxes of μ with λ_1 1's, $\cdots$, λ_m m's such that:

1. the values in the columns decrease strictly from top to bottom,

2. the values in the rows increase weakly from left to right.

Let $\mu = (5, 4, 2)$ and $\lambda = (3, 2, 3, 2, 0, 1)$. A possible semi-standard tableau T of shape μ (denoted $shp(T) = \mu$) is:

3	4			
2	2	3	4	
1	1	1	3	6

We will use the following notation:

$$SSYT(\lambda) = \{\text{semi-standard tableau T with } shp(T) = \lambda \text{ and weight not specified}\},$$

$$SSYT(\lambda, \mu) = \{\text{semi-standard tableau T with } shp(T) = \lambda \text{ and } wt(T) = \mu\}.$$

1.2 Statistics on Tableaux

The goal of this section is to define two combinatorial statistics, $inv(\sigma)$ and $maj(\sigma)$ of a filling σ, that are used in Haglund's combinatorial formula for the Macdonald Polynomial [4].

A *descent* of a filling σ is a pair of entries $(\mu(u), \mu(v))$, with $\mu(u) > \mu(v)$, where u is the box directly above v in the Ferris diagram of μ. Define

$$Des(\sigma) = \{u \in \mu |\ (\mu(u), \mu(v)) \text{ is a descent pair }\}.$$

For example, with respect to the ordering $1 < \overline{1} < 2 < \overline{2} < \cdots$, the filling below has three descents which correspond to the boxes denoted with a subscript d on the entry:

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 4_d \\
\cline{1-2}
2_d & 3 & 4_d & 1 \\
\cline{1-4}
1 & 3 & 2 & 3 & 6 \\
\hline
\end{array}$$

Equivalently, $Des(\sigma) = \{(2,1),(2,3),(3,2)\}$. Observe that for all semi-standard tableau, entries not in the first row are descents by definition. We can expand the definition of the descent set to include super fillings. First, fix a complete ordering on $\mathbb{Z}_+ \bigcup \mathbb{Z}_-$, and let u be the box directly above v in μ. A *descent* of a super filling σ is a pair of entries $(\mu(u), \mu(v))$, with $\mu(u) > \mu(v)$, or $\mu(u) = \mu(v) \in \mathbb{Z}_-$. For example, the super filling below has four descents which correspond to the boxes denoted with a subscript d on the entry:

$$\begin{array}{|c|c|c|c|c|}
\hline
2 & \bar{6}_d \\
\cline{1-2}
3_d & \bar{2}_d & 4_d & \bar{1} \\
\cline{1-4}
1 & \bar{2} & \bar{2} & 5 & 3 \\
\hline
\end{array}$$

Equivalently, $Des(\sigma) = \{(1,2),(2,2),(2,3),(3,2)\}$. It is important to observe that if σ is a filling or super filling, and $\tilde{\sigma}$ is its standardization, $Des(\sigma) = Des(\tilde{\sigma})$.

The *leg* (*arm*) of a box $u \in \mu$ is the number of boxes strictly above (to the right of) u and in the same column (row) as u. If $u = (1,2)$ in the partition $\mu = (5,4,2)$ below, then $leg(u) = 2$ and $arm(u) = 3$ as denoted in the following diagram:

	ℓ			
	ℓ			
	u	a	a	a

.

Two boxes $u, v \in \mu$ are said to *attack* one another if either of the following hold true:

1. boxes u and v are in the same row, i.e. $u = (i, j)$ and $v = (i, k)$, or

2. box u is in the row directly above v and in any column strictly to the right of v, i.e. $u = (i, k)$ and $v = (i - 1, j)$ where $j < k$.

Given a filling σ, a pair of attacking boxes $u, v \in \mu$ where $\sigma(u) > \sigma(v)$ form an *inversion pair* if $\sigma(u)$ proceeds $\sigma(v)$ in reading order. Two different inversion pairs are illustrated on the figures below, with boxes u and v denoted with subscripts on their corresponding entries:

1	4			
2_u	3	4	1_v	
1	3	2	3	6

,

1	4_u			
2_v	3	4	1	
1	3	2	3	6

.

Denote the set of all inversion pairs of a filling as $Inv(\sigma)$. Equivalently,

$$Inv(\sigma) = \{(u, v) | \sigma(u) > \sigma(v) \text{ is an inversion pair } \}.$$

In the above filling, there eight inversions giving $|Inv(\sigma)| = 8$. We can extend the definition of inversions to a super filling σ. Given a super filling σ, a pair of attacking boxes $u, v \in \mu$ where $\sigma(u) > \sigma(v)$ or $\sigma(u) = \sigma(v) \in \mathbb{Z}_-$ form an *inversion pair* if $\sigma(u)$ proceeds $\sigma(v)$ in reading order. Using the ordering $1 < \overline{1} < 2 < \overline{2} < \cdots$, two different inversion pairs are illustrated on the figures below, with boxes u and v denoted with subscripts on their corresponding entries:

$$\begin{array}{|c|c|c|c|c|}\cline{1-2} 2 & \overline{6} \\ \cline{1-4} 3 & \overline{2} & 4 & \overline{1} \\ \hline 1 & \overline{2}_u & \overline{2}_v & 5 & 3 \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|c|c|c|}\cline{1-2} 2 & \overline{6} \\ \cline{1-4} 3 & \overline{2} & 4 & \overline{1}_u \\ \hline 1_v & \overline{2} & \overline{2} & 5 & 3 \\ \hline \end{array}\,.$$

Again, it is important to observe that if σ is a filling or super filling, and $\tilde{\sigma}$ is its standardization, $Inv(\sigma) = Inv(\tilde{\sigma})$.

Definition 1.1. The *generalized major index statistic* of a filling (or super filling) σ is:

$$maj(\sigma) = \sum_{u \in Des(\sigma)} 1 + leg(u). \tag{1.2}$$

Definition 1.2. The *inversion statistic* of a filling (or super filling) σ is:

$$inv(\sigma) = |Inv(\sigma)| - \sum_{u \in Des(\sigma)} arm(u). \tag{1.3}$$

From our previous observations, it is clear that if σ is a filling or super filling, and $\tilde{\sigma}$ is its standardization, $maj(\sigma) = maj(\tilde{\sigma})$ and $inv(\sigma) = inv(\tilde{\sigma})$.

As originally remarked in [2], $inv(\sigma)$ is always non-negative. The statement can be proven by counting *counter-clockwise triples.* A *triple* is a set of three boxes

$u, v, w \in \mu$ such that $u = (i+1, j)$, $v = (i, j)$, and $w = (i+1, k)$ where $k > j$. Equivalently, the boxes have the below configuration in a partition μ:

$$\begin{array}{|c|}\hline u \\ \hline v \\ \hline \end{array} \qquad \begin{array}{|c|}\hline w \\ \hline \end{array} \\ .$$

Let x, y, and z be the fillings of the boxes u, v, and w respectively after standardizing the triple (then $x, y, z \in \{1, 2, 3\}$). Then $u, v, w \in \mu$ form a *counter-clockwise triple* if one of the following is true: $x < y < z$, $y < z < x$, or $z < x < y$. Pictorially, u, v, w form a counter-clockwise triple if while reading the values from smallest to largest, we move in a counterclockwise arc:

$$\begin{array}{|c|}\hline x \\ \hline y \\ \hline \end{array} \;\circlearrowleft\; \begin{array}{|c|}\hline z \\ \hline \end{array} \quad . \tag{1.4}$$

For example, $x = 1$, $y = 2$, and $z = 3$ form a counter-clockwise triple, where $x = 1$, $y = 3$, and $z = 2$ do not. Triples are defined when u and w are boxes in the first row by placing a box v directly below u and filling v with ∞. As before, standardize and consider the relationship between the entries x, y, and z to determine if the triple is counter-clockwise.

Observe that each attacking pair of boxes in μ belong to a unique triple. Fix $u \in \mu$. For each triple (u, v, w) in which u occupies the box in the top left corner, let (x, y, z) be the standardized entries of the boxes. Suppose $u \in Des(\sigma)$ (i.e. $x > y$), then it follows that $x > z$ and/or $z > y$. By investigation, we see that (u, v, w) forms a counter-clockwise triple if and only if both $x > z$ and $z > y$ are true. Further, if both $x > z$ and $z > y$ true, then both pairs (u, w) and (w, v) are inversions, thus contributing two to $|Inv(\sigma)|$. Since there are exactly $arm(u)$ triples in which u occupies the box in the top left corner, this particular counter-clockwise triple contributes $(2 - 1) = 1$ to the calculation of $inv(\sigma)$. Now suppose exactly one of $x > z$ and $z > y$ is true, giving a clockwise triple. Exactly one of the pairs (u, w) and (w, v) is an inversion, thus contributing one to $|Inv(\sigma)|$. Again, since there are exactly $arm(u)$ triples in which u occupies the box in the top left corner, this clockwise triple contributes $(1 - 1) = 0$ to the calculation of $inv(\sigma)$.

Now suppose $u \notin Des(\sigma)$ (i.e. $x < y$), then it follows that $x < z$ and/or $z < y$. By investigation, we see that (u, v, w) forms a counter-clockwise triple if and only if exactly one of $x > z$ or $z > y$ is true. Further, if exactly one is true, then exactly one of the pairs (u, w) and (w, v) is an inversion, thus contributing one to $|Inv(\sigma)|$. Since $u \notin Des(\sigma)$, this counter-clockwise triple contributes $(1-0) = 1$ to the calculation of $inv(\sigma)$. Now suppose both $x < z$ and $z < y$ are true, giving a clockwise triple. Neither of the pairs (u, w) and (w, v) are inversions, thus contribute zero to $|Inv(\sigma)|$. Again, since $u \notin Des(\sigma)$, this clockwise triple contributes $(0 - 0) = 0$ to the calculation of $inv(\sigma)$. It is not difficult to see that standardizing the triples does not effect the calculations. It is now proven that

$$inv(\sigma) = |Inv(\sigma)| - \sum_{u \in Des(\sigma)} arm(u) = \sum_{cct \in \mu} 1, \tag{1.5}$$

where the last sum is over counter-clockwise triples in μ. Therefore $inv(\sigma) \geq 0$ for all σ.

1.3 Plethystic Notation

Macdonald polynomials are expressed in *plethystic notation* and the proof that Haglund's formula is a combinatorial version of the Macdonald polynomial requires the use of plethystic notation. In this subsection, we will introduce this notation and give simple examples to clarify. This section is summarized from [3].

Recall, for non-negative k the symmetric function

$$p_k(x_1, x_2, \ldots) = p_k = \sum_{i \geq 1} x_i^k \tag{1.6}$$

and for any non-negative tuple λ the *power-sums*

$$p_\lambda(x_1, x_2, \ldots) = p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}, \tag{1.7}$$

where $\ell(\lambda)$ is the length of the partition. For example, $p_{3,1} = (\sum x_i^3)(\sum x_i)$. It is well-known that the power-sums p_λ form a basis for the ring of symmetric functions denoted $\bigwedge$ [7].

Definition 1.3. Let $F(x_1, x_2, x_3, \cdots)$ be a formal power series of rational functions, and $X = (x_1, x_2, x_3, \ldots)$ be a set of variables. Define the *plethystic substitution of F* *into* p_k as

$$p_k[F] = F(x_1^k, x_2^k, x_3^k, \ldots), \tag{1.8}$$

and the *plethystic substitution of* X *into* p_k as

$$p_k[X] = p_k(X) = p_k(x_1, x_2, x_3, \ldots) \tag{1.9}$$

For example, if $F(x_1, x_2, x_3, \cdots) = \sum_{i\geq 1} x_i$, then $p_4[F] = \sum_{i\geq 1} x_i^4 = p_4[X]$. In other words, we treat $X = (x_1, x_2, x_3, \ldots)$ as $X = x_1 + x_2 + x_3 + \cdots$. Note the following:

1. If $-F = \sum -x_i$, then $p_k[-F] = -F[x_1^k, x_2^k, x_3^k, \ldots] = \sum -x_i^k = -p_k[F]$. More generally, $p_k[-F] = -p_k[F]$ for all F.

2. Let $Y = (-x_1, -x_2, -x_3, \ldots)$. Then $p_k[Y] = (-1)^k x_1^k + (-1)^k x_2^k + (-1)^k x_3^k + \cdots$. For clarity, define $p_k[\epsilon X] = (-1)^k x_1^k + (-1)^k x_2^k + (-1)^k x_3^k + \cdots$.

3. Let $r \in \mathbb{R}$. Then $p_k[rX] = p_k(rx_1, rx_2, rx_3, \ldots) = (rx_1)^k + (rx_2)^k + (rx_3)^k + \cdots = r^k p_k[X]$.

4. Let $Y = (y_1, y_2, y_3, \ldots)$. Then $p_k[X - Y] = \sum_i x_i^k - y_i^k = p_k[X] - p_k[Y]$, since $X - Y = x_1 + x_2 + x_3 + \cdots - y_1 - y_2 - y_3 - \cdots$.

5. Let t be a variable. Then $p_k[X(1-t)] = p_k[X - tX] = \sum_i x_i^k - x_i^k t^k = \sum_i x_i^k (1 - t^k)$.

In the natural way, extend plethystic substitution to p_λ for any non-negative tuple λ as

$$p_\lambda[F] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[F] = \prod_{i=1}^{\ell(\lambda)} F(x_1^{\lambda_i}, x_2^{\lambda_i}, x_3^{\lambda_i}, \ldots).$$

For example,

$$p_{3,1}[X(1-t)] = p_3[X(1-t)]p_1[X(1-t)] = \big(\sum x_i^3(1-t^3)\big)\big(\sum x_i(1-t)\big).$$

Definition 1.4. Let $F(x_1, x_2, x_3, \cdots)$ be a formal power series of rational functions. For a general symmetric function f with expansion $f = \sum_\lambda c_\lambda p_\lambda$ for $c_\lambda \in \mathbb{R}$, define the *plethystic substitution of F into f* as

$$f[F] = \sum_\lambda c_\lambda p_\lambda[F] = \sum_\lambda c_\lambda \prod_i^{\ell(\lambda)} p_{\lambda_i}[F]. \tag{1.10}$$

2 Macdonald Polynomials and Haglund's Combinatorial Formula

Definition 2.1. The *Macdonald polynomials* $\tilde{H}_\mu[Z; q, t]$ are the basis of the ring of symmetric functions defined and characterized by the following axioms:

$$\tilde{H}_\mu[X(q-1); q, t] = \sum_{\rho \leq \mu'} c_{\rho\mu}(q, t) m_\rho, \tag{2.1}$$

$$\tilde{H}_\mu[X(t-1); q, t] = \sum_{\rho \leq \mu} d_{\rho\mu}(q, t) m_\rho, \tag{2.2}$$

$$< \tilde{H}_\mu, s_{(n)} >= 1, \tag{2.3}$$

for suitable coefficients $c_{\rho\mu}, d_{\rho\mu} \in \mathbb{Q}(q, t)$ [4].

2.1 Haglund's Combinatorial Formula

Definition 2.2. Let μ be a partition. Define Haglund's combinatorial formula as

$$C_\mu(x; q, t) = \sum_{\sigma:\mu \to \mathbb{Z}_+} q^{inv(\sigma)} t^{maj(\sigma)} x^\sigma, \tag{2.4}$$

where $x^\sigma = \prod_u x_{\sigma(u)}$.

We will now mention a few results and definitions that will be needed to prove the equivalence of Macdonald polynomials to Haglund's combinatorial formula.

Theorem 2.3. *The polynomial $C_\mu(x; q, t)$ is symmetric in the variables x.*

Two proofs of this result can be found in [4].

Recall, the power-sums $p_\lambda(x)$, the monomial symmetric functions $m_\lambda(x)$, the elementary symmetric functions $e_\lambda(x)$, the complete homogeneous symmetric functions $h_\lambda(x)$, and the Schur functions $s_\lambda(x)$ are all bases for the ring of symmetric functions $\bigwedge$. Let ω be the involutory automorphism of $\bigwedge$ such that

$$\omega(e_\lambda(x)) = h_\lambda(x); \qquad \omega(h_\lambda(x)) = e_\lambda(x);$$

$$\omega(p_\lambda(x)) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda(x); \qquad \omega(s_\lambda(x)) = s_{\lambda'}(x).$$

Definition 2.4. The *superization* of a general symmetric function $f(x)$ is $\tilde{f}(x,y) = \omega_Y f[X+Y]$, where ω_Y acts on $f[X+Y]$ as a symmetric function in the y variables only.

More explicitly, let f be a symmetric function with expansion $f[X] = \sum_\lambda c_\lambda p_\lambda[X]$ for some constants c_λ. Then define

$$f[X+Y] = \sum_\lambda c_\lambda p_\lambda[X+Y] = \sum_\lambda c_\lambda (p_\lambda[X] + p_\lambda[Y]),$$

giving

$$\omega_Y f[X+Y] = \omega_Y \sum_\lambda c_\lambda (p_\lambda[X] + p_\lambda[Y]) = \sum_\lambda c_\lambda (p_\lambda[X] + \omega p_\lambda[Y]).$$

Hence, $\tilde{f}(x,y) = \sum_\lambda c_\lambda (p_\lambda[X] + (-1)^{|\lambda|-\ell(\lambda)} p_\lambda[Y])$.

Proposition 2.5. *Superization of Haglund's combinatorial formula gives*

$$\tilde{C}_\mu(x,y;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+\cup\mathbb{Z}_-} q^{inv(\sigma)} t^{maj(\sigma)} z^\sigma, \tag{2.5}$$

where $z_i = x_i$ *for* i *positive,* $z_{\bar{i}} = y_i$ *for* $\bar{i}$ *negative.*

The proof of this proposition can be found in [4].

2.2 The Equivalence

Theorem 2.6. *Haglund's combinatorial formula is equal to the Macdonald polynomial,*

$$\tilde{H}_\mu(x : q, t) = C_\mu(x; q, t).$$

Our exposition of Theorem 2.6 closely follows that of Haglund, Haiman, and Loehr in [4]. We will break the proof down into three steps, each step showing that that C_μ satisfies one of the three axioms of the Macdonald polynomials, i.e. (2.1), (2.2) , and (2.3).

Proof that $C_\mu(x; q, t)$ satisfies axiom (2.2)

By definition of superization, we have that $\tilde{f}(x, y) = \omega_Y f[X + Y]$, which gives

$$\begin{aligned}\tilde{f}(x, -y) = \omega_Y f[X + \epsilon Y] &= \sum_\lambda c_\lambda(p_\lambda[X] + \omega p_\lambda[\epsilon Y]) \\ &= \sum_\lambda c_\lambda(p_\lambda[X] + (-1)^{\lambda_i - 1}(-1)^{\lambda_i} p_\lambda[Y]) \\ &= \sum_\lambda c_\lambda(p_\lambda[X] - p_\lambda[Y]) = f[X - Y].\end{aligned}$$

Applying this to C_μ, we have $\tilde{C}_\mu(qx, -x; q, t) = C_\mu[X(q-1); q, t]$. Recalling Proposition 2.5

$$C_\mu[X(q-1); q, t] = \sum_{\sigma:\mu\to\mathbb{Z}_+\cup\mathbb{Z}_-} q^{inv(\sigma)} t^{maj(\sigma)} z^\sigma,$$

where $z_i = qx_i$ for i positive, $z_{\bar{i}} = -x_i$ for $\bar{i}$ negative. After simplification,

$$C_\mu[X(q-1); q, t] = \sum_{\sigma:\mu\to\mathbb{Z}_+\cup\mathbb{Z}_-} (-1)^{m(\sigma)} q^{p(\sigma)+inv(\sigma)} t^{maj(\sigma)} x^{|\sigma|},$$

where $x^{|\sigma|} = \prod_{u\in\mu} x_{|\sigma(u)|}$, and $p(\sigma)$ $(m(\sigma))$ is the number of positive (negative) entries in the filling σ. Note that $C_\mu(x; q, t)$ is defined with respect to any total ordering of $\mathbb{Z}_+ \cup \mathbb{Z}_-$. For this proof, it is convenient to use the ordering $1 < \bar{1} < 2 < \bar{2}, \ldots$.

Following [4] we want to construct an involution Υ on fillings $\sigma : \mu \to \mathbb{Z}_+ \cup \mathbb{Z}_-$ which cancels out all terms involving x^ρ where $\rho > \mu'$. Since $m_\rho = \sum_{(i_1,\ldots,i_{\ell(\rho)})} x_{i_1}^{\rho_1} \cdots x_{i_{\ell(\rho)}}^{\rho_{\ell(\rho)}}$, this will then prove that $C_\mu[X(q-1); q, t] = \sum_{\rho\le\mu'} c_{\rho\mu}(q, t) m_\rho$, for suitable coefficients. To construct Υ, first let $\hat{u}, \hat{v}$ be an attacking pair such that:

1. $x = |\sigma(\hat{u})| = |\sigma(\hat{v})|$ where α is the smallest integer in $\sigma(\mu)$ that occurs in such an attacking pair, and

2. In reading order, let $\hat{v}$ be the last box in such an attacking pair, and $\hat{u}$ be the last box attacking $\hat{v}$.

Note that at most one such attacking pair exists. Define

$$\Upsilon(\sigma) = \begin{cases} \sigma, & \text{if } \nexists \text{ an attacking pair } (\hat{u}, \hat{v}) \text{ as described above,} \\ \hat{\sigma} & \text{else, where } \hat{\sigma}(\hat{u}) = \overline{\sigma(\hat{u})},\ \hat{\sigma}(w) = \sigma(w)\ \forall\ w \neq \hat{u}. \end{cases}$$

In other words, Υ fixes the entry in each box, unless there exists a pair $(\hat{u}, \hat{v}) \in \sigma$ as described above, in which case Υ switches the sign of the entry in $\hat{u}$. Since $(\hat{u}, \hat{v})$ only depends on $|\sigma|$, it is easy to see that $\Upsilon\Upsilon(\sigma) = \sigma$ and $x^{|\sigma|} = x^{|\Upsilon\sigma|}$ for all σ. Suppose $\Upsilon(\sigma) \neq \sigma$, then we claim $(-1)^{m(\sigma)} = -(-1)^{m(\Upsilon\sigma)}$ but $p(\sigma) + inv(\sigma) = p(\Upsilon\sigma) + inv(\Upsilon\sigma)$, and $maj(\sigma) = maj(\Upsilon\sigma)$. Formally,

$$C_\mu[X(q-1); q, t] = \sum_{\Upsilon(\sigma)=\sigma} (-1)^{m(\sigma)} q^{p(\sigma)+inv(\sigma)} t^{maj(\sigma)} x^{|\sigma|}. \tag{2.6}$$

To prove equation (2.6), assume $\Upsilon(\sigma) \neq \sigma$. Note that Υ switches the sign of the entry in $\hat{u}$ but fixes all other entries, which gives $(-1)^{m(\sigma)} = -(-1)^{m(\Upsilon\sigma)}$. Because $\Upsilon(\sigma) \neq \sigma$, there is a unique attacking pair $(\hat{u}, \hat{v})$ such that $|\sigma(\hat{u})| = |\sigma(\hat{v})|$ (as in the definition of Υ). Since one of $\sigma(\hat{u})$ and $\Upsilon\sigma(\hat{u})$ is positive, without loss of generality, we can assume that $\sigma(\hat{u})$ is positive (i.e. $p(\Upsilon\sigma) = p(\sigma) - 1$). Let $\sigma(\hat{u}) = x$, which gives $\Upsilon\sigma(\hat{u}) = \overline{x}$.

A key ingredient in showing Υ fixes $maj(\sigma)$ and $p(\sigma) + inv(\sigma)$ is that $Des(\sigma) = Des(\Upsilon(\sigma))$. To show $Des(\sigma) = Des(\Upsilon(\sigma))$ it is enough to consider the entries above and below $\hat{u}$. Assume $\hat{u} \in \mu$ so that there exists boxes w_a and w_b above and below $\hat{u}$ respectively. Let $\sigma(w_a) = \Upsilon\sigma(w_a) = a$ and $\sigma(w_b) = \Upsilon\sigma(w_b) = b$:

$$\begin{array}{|c|} \hline a \\ \hline x \\ \hline b \\ \hline \end{array} \;\xrightarrow[\Upsilon]{}\; \begin{array}{|c|} \hline a \\ \hline \overline{x} \\ \hline b \\ \hline \end{array}.$$

From the definition of $(\hat{u}, \hat{v})$, it follows that $\hat{v}$ precedes w_b in reading order and $(\hat{v}, w_b)$ is an attacking pair. Since $\hat{v}$ is the last box in reading order in an attacking pair with both entries equal to x, then $|b| \neq x$. Using $1 < \overline{1} < 2 < \overline{2}, \ldots$, we get $b < x \Leftrightarrow b < \overline{x}$, i.e. $\hat{u} \in Des(\sigma)$ if and only if $\Upsilon\hat{u} \in Des(\Upsilon\sigma)$. Now consider the box w_a above $\hat{u}$ containing a. If $a \leq x$, then $a < \overline{x}$, thus $w_a \notin Des(\sigma) \bigcup Des(\Upsilon\sigma)$. Else if $a > x$, then $a \geq \overline{x}$, thus, since $\overline{x}$ is negative, $w_a \in Des(\sigma) \bigcap Des(\Upsilon\sigma)$. Therefore $Des(\sigma) = Des(\Upsilon(\sigma))$. Simply by the definition of the generalized major index, this implies $maj(\sigma) = maj(\Upsilon\sigma)$.

As previously noted, $p(\Upsilon\sigma) = p(\sigma) - 1$. Therefore to prove $p(\sigma) + inv(\sigma) = p(\Upsilon\sigma) + inv(\Upsilon\sigma)$, it is sufficient to show $inv(\Upsilon\sigma) = inv(\sigma) + 1$. From $Des(\sigma) = Des(\Upsilon(\sigma))$, we have $\sum_{u \in Des(\sigma)} arm(u) = \sum_{u \in Des(\Upsilon\sigma)} arm(u)$. Thus, if $|Inv(\Upsilon\sigma)| = |Inv(\sigma)| + 1$, then $inv(\Upsilon\sigma) = inv(\sigma) + 1$. The only inversion pairs in $Inv(\Upsilon\sigma)$ that could possibly differ from inversion pairs in $Inv(\sigma)$ must include $\hat{u}$. Let $w \in \mu$ attack $\hat{u}$ and precede $\hat{u}$ in reading order. If $\sigma(w) \leq x$, then $\sigma(w) < \overline{x}$, thus $(w, \hat{u}) \notin Inv(\sigma) \bigcup Inv(\Upsilon\sigma)$. Else if $\sigma(w) > x$, then $\sigma(w) \geq \overline{x}$, thus since $\overline{x}$ is negative, $(w, \hat{u}) \in Inv(\sigma) \bigcap Inv(\Upsilon\sigma)$. On the other hand, let $w \neq \hat{v}$ attack $\hat{u}$ and follow $\hat{u}$ in reading order. Using a similar argument, it is clear that $|\sigma(w)|$ must equal x to give an inversion to one of σ or $\Upsilon\sigma$, but not the other. Further by the definition of $\hat{v}$, w must precede $\hat{v}$ in reading order. Hence, w lies between $\hat{u}$ and $\hat{v}$ in reading order which is a contradiction to the definition of $(\hat{u}, \hat{v})$. Therefore if w follows $\hat{u}$ in reading order then $(w, \hat{u}) \notin$

$Inv(\sigma)\bigcup Inv(\Upsilon\sigma)$. Finally consider $(\hat{u},\hat{v})$. Since $\hat{u}$ precedes $\hat{v}$ in reading order and $\sigma(\hat{u}) = x > 0$, $(\hat{u},\hat{v}) \notin Inv(\sigma)$. On the other hand, since $\Upsilon\sigma(\hat{u}) = \overline{x} < 0$, $(\hat{u},\hat{v}) \in Inv(\Upsilon\sigma)$. Hence $|Inv(\Upsilon\sigma)| = |Inv(\sigma)| + 1$ and so $p(\sigma) + inv(\sigma) = p(\Upsilon\sigma) + inv(\Upsilon\sigma)$ as desired.

It is now proved that: $(-1)^{m(\sigma)} = -(-1)^{m(\Upsilon\sigma)}$ but $p(\sigma) + inv(\sigma) = p(\Upsilon\sigma) + inv(\Upsilon\sigma)$, and $maj(\sigma) = maj(\Upsilon\sigma)$, which implies

$$C_\mu[X(q-1);q,t] = \sum_{\Upsilon(\sigma)=\Upsilon} (-1)^{m(\sigma)} q^{p(\sigma)+inv(\sigma)} t^{maj(\sigma)} x^{|\sigma|}. \tag{2.7}$$

With the above statement proved, we near the end of the proof that $C_\mu(x;q,t)$ satisfies axiom (2.1). Let ρ be a partition and $\sigma : \mu \to \mathbb{Z}_+ \bigcup \mathbb{Z}_-$ be such that $\Upsilon\sigma = \sigma$ and $wt(|\sigma|) = \rho$ which gives $x^{|\sigma|} = x^\rho$. By the definition of weight, $\rho_1 + \cdots + \rho_j$ is the total number of entries in σ with absolute value at most j. Since $\Upsilon\sigma = \sigma$, for all attacking pairs $u, v \in \sigma$ we have that $|\sigma(u)| \neq |\sigma(v)|$. Hence in every row of μ there is at most one entry filled with x or $\overline{x}$ for all $x \in \mathbb{Z}_+$. This implies that $\rho_1 + \cdots + \rho_j$ cannot exceed $\sum_i min(\mu_i, j) = \mu'_1 + \cdots + \mu'_j$. Therefore $\rho \leq \mu'$, which completes the proof that $C_\mu(x;q,t)$ satisfies the first axiom (2.1) for the Macdonald polynomials.

□

We outline the proof below that $C_\mu(x;q,t)$ satisfies the second axiom, but leave the details to be found in [4].

Proof that $C_\mu(x;q,t)$ satisfies axiom (2.2)

Applying $\tilde{f}(x,-y) = \omega_Y f[X-Y]$ and Proposition 2.5 to the second axiom (2.2), and simplifying gives:

$$C_\mu[X(t-1);q,t] = \sum_{\sigma:\mu\to\mathbb{Z}_+\cup\mathbb{Z}_-} (-1)^{m(\sigma)} q^{inv(\sigma)} t^{p(\sigma)+maj(\sigma)} x^{|\sigma|},$$

where $x^{|\sigma|} = \prod_{u\in\mu} x_{|\sigma(u)|}$, $p(\sigma)$, and $m(\sigma)$ are defined as before. For this proof, it is convenient to use the ordering $1 < 2 < 3 < \cdots < \overline{3} < \overline{2} < \overline{1}$. We construct an involution Ω on fillings $\sigma : \mu \to \mathbb{Z}_+ \cup \mathbb{Z}_-$ which cancels out all terms involving x^ρ where $\rho > \mu$, which will complete the proof of axiom (2.2).

To construct Ω, first define $\hat{u}$ to be a box in μ such that:

1. $x = |\sigma(\hat{u})|$ be the smallest integer in $\sigma(\mu)$ that occurs such that $x < i$ where $\hat{u} = (i,j) \in \mu$,

2. Let $\hat{u}$ be the first cell in reading order that satisfies the above condition.

Note that at most one such box $\hat{u}$ exists. Define

$$\Omega(\sigma) = \begin{cases} \sigma, & \text{if } \nexists \text{ a box } \hat{u} \text{ as described above,} \\ \hat{\sigma} & \text{else, where } \hat{\sigma}(\hat{u}) = \overline{\sigma(\hat{u})},\ \hat{\sigma}(w) = \sigma(w)\ \forall\ w \neq \hat{u}. \end{cases}$$

In other words, Ω fixes the entry in each box, unless there exists a box $\hat{u} \in \sigma$ as described above, in which case then Ω switches the sign of the entry in $\hat{u}$. Since $\hat{u}$ only depends on $|\sigma|$, it is easy to see that $\Omega\Omega(\sigma) = \sigma$ and $x^{|\sigma|} = x^{|\Omega\sigma|}$ for all σ. Suppose, $\Omega(\sigma) \neq \sigma$, then we claim $(-1)^{m(\sigma)} = -(-1)^{m(\Omega\sigma)}$ but $inv(\sigma) = inv(\Omega\sigma)$, and $p(\sigma) + maj(\sigma) = p(\Omega\sigma) + maj(\Omega\sigma)$. More formally,

$$C_\mu[X(q-1); q, t] = \sum_{\Omega(\sigma)=\sigma} (-1)^{m(\sigma)} q^{inv(\sigma)} t^{p(\sigma)+maj(\sigma)} x^{|\sigma|}. \tag{2.8}$$

The bulk of the remaining part of the proof centers on proving the above equation. The proof can be read in [4].

Assuming equation (2.8) has been proven, observe that if $\Omega(\sigma) = \sigma$, then all entries x with $|x| \leq j$ occur in rows 1 through j. If ρ is a partition and $x^{|\sigma|} = x^\rho$, then $\rho_1 + \cdots + \rho_j \leq \mu_1 + \ldots + \mu_j$ for all j, which means $\rho \leq \mu$. This now completes the proof that $C_\mu(x; q, t)$ satisfies the second axiom (2.2) for the Macdonald polynomials.

□

Proof that $C_\mu(x; q, t)$ satisfies axiom (2.3)

Recall, that $s_{(n)} = h_n$ and $< m_\mu, h_\nu >= \delta_{\mu\nu}$ giving

$$1 =< \tilde{H}_\mu, s_{(n)} >=< \tilde{H}_\mu, h_n >= c_{n\mu}(q, t), \tag{2.9}$$

and

$$1 =< \tilde{H}_\mu, s_{(n)} >=< \tilde{H}_\mu, h_n >= d_{n\mu}(q, t), \tag{2.10}$$

where $c_{n\mu}(q, t)$ coefficient of $m_n(x) \in \tilde{H}_\mu$ in axiom (2.1) and $d_{n\mu}(q, t)$ coefficient of $m_n(x) \in \tilde{H}_\mu$ in axiom (2.2). Let σ be the filling $\sigma : \mu \to 1$ which has $maj(\sigma) = inv(\sigma) = 0$ since there are no descents or inversions with a filling of all $1's$. By the definition of C_μ, this gives that the coefficient of $x^\sigma = m_n(x)$ is 1. Hence, $< C_\mu, s_{(n)} >= 1$ as desired.

□

Theorem 2.6 that Haglund's Combinatorial Formula is in fact the Macdonald polynomial has now been proven.

3 Macdonald polynomials and Cocharge

In this section, we will show that when $q = 0$ the Macdonald polynomial can be expanded in terms of the charge statistic defined by Lascoux and Schützenberger.

Proposition 3.1. *Specializing the Macdonald polynomial by setting $q = 0$ gives*

$$C_\mu(x;0,t) = \tilde{H}_\mu(x;0,t) = \sum_\lambda \big(\sum_{T \in SSYT(\lambda,\mu)} t^{cc(T)} s_\lambda(x) \big), \tag{3.1}$$

where $cc(T)$ is the cocharge of T. The sum is over all semistandard tableaux T of shape λ and weight μ [4].

We will begin with the definition of charge (cocharge) and follow with a lemma that will be use to prove Proposition 3.1.

3.1 Defining Charge and Cocharge

A *word* is a sequence of positive integers, referred to as *letters*. The *weight* of a word w is $wt(w) = (\nu_1, \ldots, \nu_n)$ if there are ν_1 one's, ν_2 two's, ..., ν_n n's in the word. A word is said to have *partitioned weight* ν if $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$. A *standard word* is a word with $wt(w) = 1^n$, i.e. a permutation. Let $w = c_1c_2\cdots c_n$ be a standard word. The *descent set* of w^{-1} is $Des(w^{-1}) = \{i \in w | i \text{ occurs before } i+1 \text{ in } w\}$. We will define the charge and cocharge of a standard word first and then extend this definition to all words of partitioned weight.

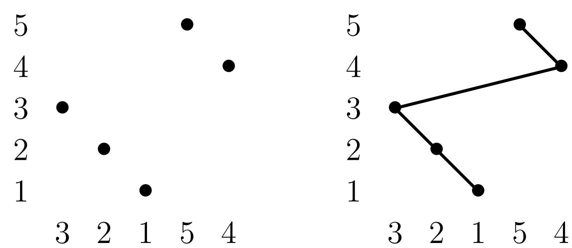
Definition 3.2. Let $w = c_1c_2\cdots c_n$ be a standard word. The charge and cocharge of w are defined respectively:

$$ch(w) = \sum_{i \notin Des(w)} (n-i),$$

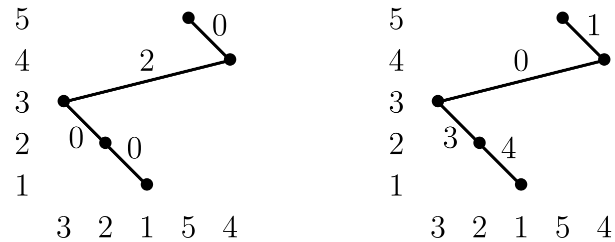
and

$$cc(w) = \sum_{i \in Des(w)} (n-i).$$

Graphically, charge and cocharge of w can be calculated by plotting a point above each letter $c \in w$ at height c. Then, connecting the points starting from the letter equal to 1 and moving to the letter equal to n. For $w = 32154$:

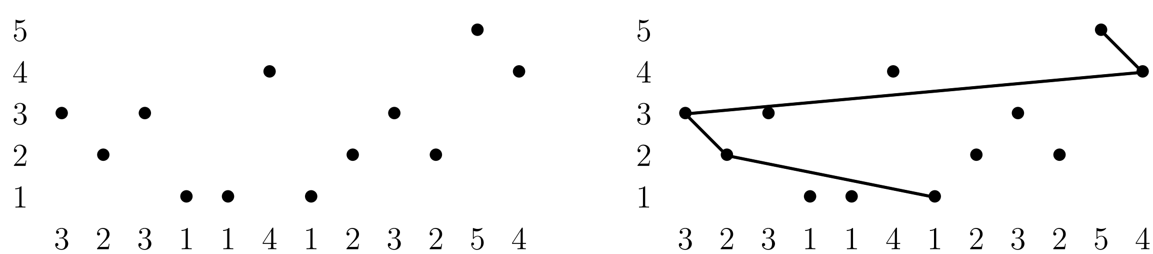


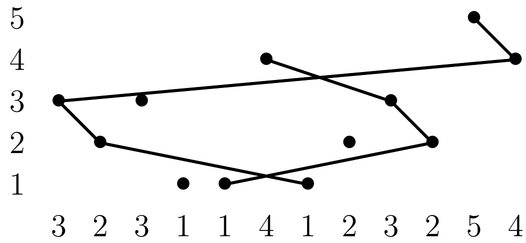
We will refer to edges with their left endpoint higher than their right endpoint as *left edges*, and edges with their right endpoint higher than their left endpoint as *right edges*. To calculate charge, label the left edges with a 0, and the right edges with the number of edges above it, including itself. Charge of w is the sum of the labels, giving $ch(32154) = 2$. To calculate cocharge, just interchange the labels on the left and right edges to get $cc(32154) = 8$. Observe that for all standard words w, we have that $ch(w) + cc(w) = \binom{n}{2}$.

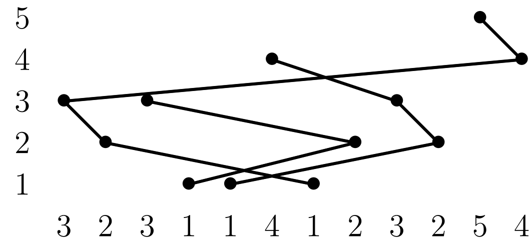


$$ch(w) = 0 + 0 + 2 + 0 = 2 \qquad cc(w) = 4 + 3 + 0 + 1 = 8$$

If $\nu = (\nu_1, \ldots, \nu_n)$ is the partitioned weight of a word w, then w has ν_1 *standard subwords* which we use the graphical representation of the word to define. First plot the letters of w as before. The *first standard subword* w_1 of w is defined from w by moving from right to left through w selecting the first 1, selecting the first 2 to the left of the chosen 1, selecting the first 3 to the left of the chosen 2, ... If after selecting $i-1$, there is no i to the left of $i-1$, then select the the rightmost i, and continue looking for $i+1$ as above. The process terminates with i when the letter $i+1$ is not represented in w. To find the $n+1^{th}$ *standard subword*, w_{n+1}, repeat the same process while ignoring letters in the 1^{st} through n^{th} standard subwords. For $w = 323114123254$, the letters are plotted below and the subwords are drawn one-by-one.







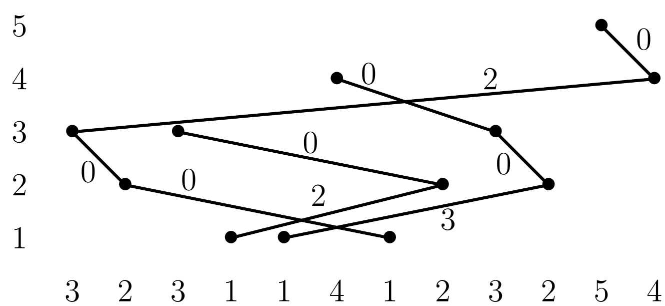
Definition 3.3. Let w be a word of partitioned weight with standard subwords $w_1, \ldots, w_n$. Then the charge and cocharge of w are

$$ch(w) = \sum_{j=1}^{n} ch(w_j) = \sum_{j=1}^{n} \sum_{i \notin Des(w_j)} (n-i), \tag{3.2}$$

and

$$cc(w) = \sum_{j=1}^{n} cc(w_j) = \sum_{j=1}^{n} \sum_{i \in Des(w_j)} (n-i). \tag{3.3}$$

Graphically, we label the edges just as before and sum over all edges to get $ch(w) = 7$ or $cc(w) = 12$. The graph for charge is shown below:



$$\begin{aligned} ch(w) &= ch(w_1) + ch(w_2) + ch(w_3) \\ &= 2 + 3 + 2 = 7 \end{aligned}$$

Definition 3.4. Let σ be a filling of a partition μ with $n = |\mu|$. Label the boxes of μ as $u_1 = (i_1, j_1), u_2 = (i_2, j_2), \ldots, u_n = (i_n, j_n)$ such that:

1. $\sigma(u_1) \geq \sigma(u_2) \geq \cdots \geq \sigma(u_n)$,

2. For each chain of constants $\sigma(u_k) = \cdots = \sigma(u_\ell)$, list the boxes in backwards reading order, i.e. bottom to top, right to left.

Define the *cocharge word* of σ as the list of row indices of the cells u_k in the order described above, which gives $cw(\sigma) = i_1 i_2 \cdots i_n$.

For example, for the following filling of $\mu = (5, 4, 2)$ we get:

$$\begin{array}{|c|c|lll}\hline 1 & 4 \\ \hline 2 & 3 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{1} \\ \hline 1 & 3 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline \end{array} \qquad \begin{array}{rl} u_1 \cdots u_n = & 6 \;\; 4 \;\; 4 \;\; 3 \;\; 3 \;\; 3 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 1 \\ cw(\sigma) = & 1 \;\; 2 \;\; 3 \;\; 1 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 1 \;\; 2 \;\; 3 \,\cdot \end{array}$$

We will refer to the word $u_1 \cdots u_n$ as the *filling word* of σ, denoted $f(\sigma)$. For the example above, $f(\sigma) = 6443332211$. The filling word is always in non-increasing order.

3.2 Relating Macdonald Polynomials and Cocharge

The following lemma will be needed for the proof of Proposition 3.1.

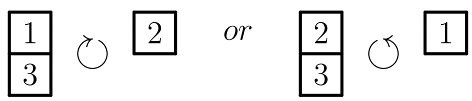
Lemma 3.5. *Fix a partition μ and a sequence of multisets $\beta_1, \beta_2, \ldots$ with $|\beta_i| = \mu_i$. There exists exactly one counter-clockwise free filling $\sigma : \mu \to \mathbb{Z}_+$ such that the weight of the i^{th} row of $\sigma(\mu)$ is equal to β_i.*

Let $\mu = (4, 4, 1)$ and $\beta_1 = \{1, 1, 3, 6\}$, $\beta_2 = \{1, 2, 4, 4\}$, and $\beta_3 = \{3\}$. We claim that the unique tableau with zero counter-clockwise triples with the i^{th} row filled by β_i is:

3			
2	4	4	1
1	1	3	6

.

Proof. As observed in [4], we need to fill μ without counter-clockwise triples. Consider boxes $u, w \in \mu_1$ such that u is to the left of w. These boxes are part of a triple with box v containing ∞ added directly below u. After standardization, there are two options for the entries of this triple:



Since the second option gives a counter-clockwise triple and standardization occurs in reading order, the first row must be filled in non-decreasing order (from left to right). Fill boxes one a time from left to right, bottom to top. Assume all boxes up to $u = (i, j)$ have been filled where $i > 1$. Let $\tilde{\beta}_i$ be the subset of β_i that have not yet been used to fill μ_i. Let $v = (i-1, j)$, the box directly below u, be filled with y. Pick x from β_i to fill u. Either $x \leq y$ or $x > y$. If $x \leq y$, then all boxes w to the right of u in row i must be filled with a $z \in \beta_i$ where $x \leq z \leq y$ to avoid counter-clockwise triples. Alternatively, if $x > y$ then all boxes w to the right of u in row i must be filled with a $z \in \beta_i$ where $z \notin (y, x)$ to avoid counter-clockwise triples. Hence box u must be filled with the smallest value $x \in \beta_i$ such that $x > y$ else if one does not exist the smallest value in β_i period. This gives a "no-choice" algorithm to construct the desired filling. □

We are now ready to prove Proposition 3.1 as first proven in [4].

Proof. First, we claim that if σ is a filling such that $inv(\sigma) = 0$, then $maj(\sigma) = cc(cw(\sigma))$. Look at the entries of the first column of μ, i.e. $u_{k_1} = (1,1), \ldots, u_{k_\ell} = (\ell, 1)$. By construction of the cocharge word, u_{k_1} corresponds to a 1 in the cocharge word, ..., u_{k_ℓ} corresponds to a ℓ in the cocharge word. Further, since $inv(\sigma) = 0$, σ has zero counter-clockwise triples, which (by the "no-choice" algorithm shown in the proof of Lemma 3.5) gives that the bottom row of μ is non-increasing, and so the entry in the first column is the smallest in row one. Recall from Definition 3.4 chains of constants are recorded in backwards reading order, the box u_{k_1} corresponds to rightmost 1 in $cw(\sigma)$. By the proof of Lemma 3.5, the entry in $u_{k_i} = (i, 1)$ for $i > 1$ is either

1. the smallest entry in row i greater than $\sigma((i-1,1))$, if one exists; or
2. the smallest entry in row i entirely.

This then implies that i_{k_i} corresponds to either

1. the rightmost i to the left of $i_{k_{i-1}}$ in $cw(\sigma)$, if one exists, or
2. the rightmost i in $cw(\sigma)$ entirely.

Let $w = cw(\sigma)$, then that the first standard subword w_1 of w in the definition of cocharge of $cw(\sigma)$ contains the exact same letters of w, i.e. i_{k_1} through i_{k_ℓ}. Further, it is clear that there is a descent in $\sigma(\mu)$ exactly when there exists a smallest entry in row i greater than $\sigma((i-1,1))$ which means that i_{k_i} is the rightmost i to the left of $i_{k_{i-1}}$ in $cw(\sigma)$. This exactly corresponds to the right edges in the definition of $cc(cw(\sigma))$. It is also clear that for a descent u $(leg(u)+1)$ is exactly equal to the *the number of edges above the right edge including itself*, i.e. $(n-i)$, from the definition of $cc(cw(\sigma))$. Hence we have proven that

$$cc(w_1) = \sum_{u=(i,1)\in Des(\sigma)} 1 + leg(u). \tag{3.4}$$

Selecting the second subword w_1 from $w = cw(\sigma)$ will be the same as restricting σ to the diagram obtained by deleting the first column of μ. Denote this filling as σ_1. When we delete the first column, it remains that $inv(\sigma_1) = 0$, thus we have that the claim follows by induction.

As proved in [5], if $P(w)$ denotes the RSK insertion tableau of w, then $cc(w) = cc(P(w))$ for every word with partition content. Let $\beta(\sigma)$ be the multiset of pairs the filling of an entry and its row index $(\sigma(u), i)$ where $u = (i,j) \in \mu$, and observe that specifying $\beta(\sigma)$ is equivalent to giving the multisets $\beta_1, \ldots, \beta_{\ell(\sigma)}$ as in Lemma 3.5. Again by Lemma 3.5 for a fixed partition μ, the map $\sigma \to \beta(\sigma)$ is a bijection from the set of fillings with $inv(\sigma) = 0$ (counter-clockwise free fillings), to multisubsets of $\mathbb{Z}_+ \times \mathbb{Z}_+$. Observe that the first index in $\beta(\sigma)$ comes from $f(\sigma)$ (the filling word), while the second index comes from $cw(\sigma)$ (the cocharge word). Applying RSK to $\beta(\sigma)$, ordering the pairs in $\beta(\sigma)$ as in the filling word from largest to smallest, creates the pair $(P(\sigma), Q(\sigma))$ of semistandard tableau of the same shape λ for some λ. It follows that $P(\sigma) = P(cw(\sigma))$, and $x^{\sigma} = x^{Q(\sigma)}$. Since the map $\sigma \rightarrow (P(\sigma), Q(\sigma))$ is a bijection from counter-clockwise free fillings σ to pairs $(P(\sigma), Q(\sigma))$ of semistandard tableaux of the same shape such that P has weight μ, we can conclude that:

$$\begin{aligned}\tilde{H}_{\mu}(x;0,t) &= \sum_{inv(\sigma)=0} t^{mag(\sigma)} x^{\sigma} \\ &= \sum_{\lambda} \Big(\sum_{P \in SSYT(\lambda,\mu)} t^{cc(P)} \Big) \Big(\sum_{Q \in SSYT(\lambda)} x^{Q} \Big) \\ &= \sum_{\lambda} \Big(\sum_{T \in SSYT(\lambda,\mu)} t^{cc(T)} s_{\lambda}(x) \Big),\end{aligned}$$

which completes the proof of Proposition 3.1 as desired.

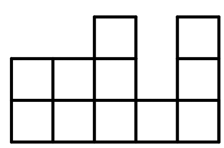
□

4 Beyond Partitions

In the previous section, we saw an interesting relationship between cocharge and the generalized major index statistic on fillings σ with $inv(\sigma) = 0$, (counter-clockwise free fillings). In this section we explore the possibility that a similar relationship may exist when looking at fillings of compositions.

4.1 Compositions

We will begin by defining compositions and then expand the definition of inversion triples of a partition to a composition. A *composition* $\nu' = (\nu'_1, \ldots, \nu'_k)$ of n is a tuple of nonzero parts such that $\nu'_1 + \cdots + \nu'_k = n$. Identify a composition with the tableau that has ν'_i boxes in the i^{th} column. For the composition $\nu' = (2, 2, 3, 1, 3)$, we identify ν' with the following tableau



. (4.1)

Whereas ν'_i is the number of boxes in column i, let ν_i be the number of boxes in row i. For (4.1), we have $\nu_1 = 5$, $\nu_2 = 4$, and $\nu_3 = 2$. We can think of a composition as a permutation of the columns of a permutation. For example, if $\mu = (5, 4, 2)$, then the composition $\nu' = (2, 2, 3, 1, 3)$ is equal to $w(\mu') = w((\mu'_1, \mu'_2, \mu'_3, \mu'_4, \mu'_5)) = (\mu'_3, \mu'_5, \mu'_1, \mu'_5, \mu'_4) = \nu'$ where $w \in S_{\mu_1}$ is the permutation $w = (13)(254)$ and acts on the subscripts of μ. Explicitly, $w(\mu') = w((3, 3, 2, 2, 1)) = (2, 2, 3, 1, 3) = \nu'$.

The definition of fillings/super fillings of compositions is analogous to that of fillings/superfillings of partitions. For a filling σ of the composition ν' there are two types of inversion triples. The first type (counter-clockwise triples) comes from the original definition of a counter-clockwise triple in a filling of a partition. Recall, a *triple* is a set of boxes $u, v, w \in \nu'$ such that $u, v \in \nu'_i$ and $w \in \nu'_j$ where $i < j$ with the below configuration. After standardization x, y, z are the fillings of the boxes u, v, w respectively:

$$\begin{array}{ccc} \boxed{x} & \quad & \boxed{z} \\ \boxed{y} & & \end{array} .$$

Then $u, v, w \in \nu'$ form a *counter-clockwise composition triple* if both of the following are true:

1. $x < y < z$, $y < z < x$, or $z < x < y$, and

2. $\nu'_i \geq \nu'_j$, i.e. the column containing u and v is taller than or equal to the column containing w.

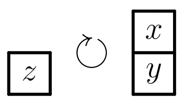
Alternatively, define a *composition triple* as a set of three boxes $u, v, w \in \nu'$ such that $u, v \in \nu'_i$ and $w \in \nu'_j$ where $i > j$ with the below configuration. After standardization x, y, z are the fillings of the boxes u, v, w respectively:

$$\begin{array}{ccc} & \quad & \boxed{x} \\ \boxed{z} & & \boxed{y} \end{array} .$$

The boxes $u, v, w \in \nu'$ form a *clockwise composition triple* if both of the following are true:

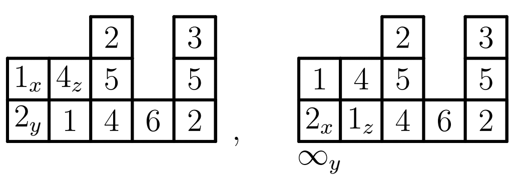
1. $x < y < z$, $y < z < x$, or $z < x < y$, and
2. $\nu'_i < \nu'_j$, i.e. the column containing u and v is strictly higher than the column containing w.

Pictorially, u, v, w form a clockwise composition triple if while reading the values from smallest to largest, we move in a clockwise arc:



(4.2)

For clarity, we give examples of both counter-clockwise triples and clockwise composition triples. Let σ be the filling of the composition $\nu' = (2, 2, 3, 1, 3)$ illustrated in (4.1). There are exactly two counter-clockwise composition triples in the filling and they are denoted by placing subscripts x, y, and z on the entries.



(4.3)

In addition, there are exactly two clockwise composition triples again denoted by placing subscripts x, y, and z on the entries:

		2_x		3
1_z	4	5_y		5
2	1	4	6	2

,

		2		3_x
1_z	4	5		5_y
2	1	4	6	2

.

For convenience, we will refer to counter-clockwise and clockwise composition triples as *inversion triples*, and specify the type only when necessary.

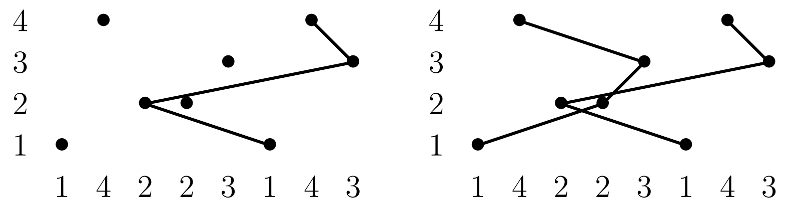
Definition 4.1. Let σ be a filling of a composition ν'. Define $inv(\sigma)$ as the number of inversion triples in $\sigma(\nu')$.

4.2 A New Look at Cocharge

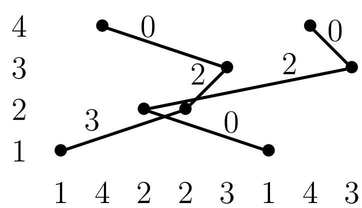
4.2.1 An Alternative Description of Cocharge

Let w be a word of partitioned weight $\lambda = (\lambda_1, \ldots, \lambda_k)$. Previously, to calculate cocharge we partitioned w into $n = \lambda_1$ standard subwords $w_1, \ldots, w_n$, and then defined $ch(w)$ to be the sum of the charge of the subwords. In this section, we will partition w into n new standard subwords $\hat{w}_1, \ldots, \hat{w}_n$, and claim that the sum of the charge of the individual subwords is also $ch(w)$.

To define the new subwords $\hat{w}_1, \ldots, \hat{w}_n$, begin with the largest letter to appear in w. Since $\ell(\lambda) = k$, k is the largest letter in w. Connect the rightmost k to the leftmost $k-1$ to appear to the right of k, if one exists, otherwise connect k to the leftmost $k-1$. Continue in this fashion until there are no smaller letters. The first standard subword $\hat{w}_1$ is comprised of the letters selected above. To select the i^{th} standard subword $\hat{w}_i$, first select subwords $\hat{w}_1$ through $\hat{w}_{i-1}$. Then let j be the largest unselected letter. Pick the rightmost j and connect it to the leftmost unselected $j-1$ to the right of j, if one exists, otherwise connect it to the overall leftmost unselected $j-1$. Continue until there are no smaller letters. For example, let $w = 14223143$, then we have:



Let $\hat{ch}(w) = ch(\hat{w}_1) + \cdots + ch(\hat{w}_n)$. Continuing with the example,



$$\hat{ch}(w) = 2 + 5 = 7.$$

Theorem 4.2. *[5] Let w be a word. Then statistic* charge *is equal to $\hat{ch}(w)$, i.e. $ch(w) = \hat{ch}(w)$.*

Assuming Theorem 4.2, there is a similar description for cocharge. Let $\hat{cc}(w) = cc(\hat{w}_1) + \cdots + cc(\hat{w}_n)$. We claim that $\hat{cc}(w) = cc(w)$. Recall, that for all words w of weight 1^m, $ch(w) + cc(w) = \binom{m}{2}$. Observe that no matter what method is used partitioning w into standard subwords, for each $1 \leq m \leq k$ there will still be the same number of subwords with weight 1^m. Hence if we partition a word into standard subwords, graph it and label each edge with the number of edges above it including itself, the sum of the labels will always be $S = m_1\binom{1}{2} + \cdots + m_n\binom{n}{2}$, where m_i is the number of standard subwords of length i. By construction of $\hat{cc}$ and $\hat{ch}$, it follows that $\hat{cc}(w) = S - \hat{ch}(w) = S - ch(w)$ implying that $\hat{cc}(w) = cc(w)$.

To prove Theorem 4.2, Killpatrick used the following theorem by Lascoux and Shützenberger:

Theorem 4.3. *[6]* Charge *is the unique function from words to non-negative integers such that:*

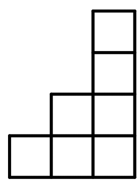
1. $ch(\emptyset) = 0$,
2. $ch(w) = ch(\alpha w)$ *for* $\alpha \in S_k$
3. *if* $x \neq 1$ *and the word* wx *has partitioned weight, then* $ch(wx) = ch(xw) + 1$
4. *if* $w1^m$ *is a word of partitioned weight* λ *where* $\lambda_1 = m$, $ch(w1^m) = ch(w)$
5. *if words* w *and* $\tilde{w}$ *are Knuth equivalent then* $ch(w) = ch(\tilde{w})$.

A proof to this theorem can be found in [5]. In proving Theorem 4.2, Killpatrick proved that $\hat{ch}$ also satisfies properties 1 through 5. By uniqueness, this proved that $\hat{ch}$ is in fact charge.

4.2.2 Cocharge and Compositions

We are primarily interested in developing a relationship between the generalized major index of fillings with $inv(\sigma) = 0$ and cocharge. In this process, we also show how the description of cocharge as $\hat{cc}$ arises naturally from compositions. We will follow the methods developed in Chapter 3 to analyze maj and cc.

Let ν' be a composition such that $\nu'_1 < \ldots < \nu'_k$. One such composition that meets these requirements is $\nu' = (1, 2, 4)$:



. (4.4)

The following lemma is similar to Lemma 3.5.

Lemma 4.4. *Fix a composition ν' where $\nu'_1 < \ldots < \nu'_k$ and a sequence of multisets $\beta_1, \beta_2, \ldots$ with $|\beta_i| = \nu_i$. There exists exactly one filling $\sigma : \nu' \to \mathbb{Z}_+$ such that $inv(\sigma) = 0$ and the weight of the i^{th} row of $\sigma(\nu')$ is equal to β_i.*

Proof. Regardless of the choice of σ, $\sigma(\nu')$ contains no counter-clockwise triples. Let n be the number of rows in ν'. Again since $\nu'_1 < \ldots < \nu'_k$, $|\beta_n| = 1$ forcing the filling of the entry in top row. Now, fill the boxes column by column from top to bottom, right to left. Let $v = (i, j)$ be the next box to fill. Let $\tilde{\beta}_i \subseteq \beta_i$ be the numbers in β_i that have yet to be used to fill ν. If $|\tilde{\beta}_i| = 1$, fill v with the only unused letter in β_i.

On the other hand, if $|\tilde{\beta}_i| > 1$, then let u be the box directly above v and $x = \sigma(u)$ (if there is no box u above v, then by construction of ν, $|\tilde{\beta}_i| = 1$). First suppose v is filled with $y \in \tilde{\beta}_i$ such that $y < x$. To ensure that x, y are not part of a clockwise composition triple, all z remaining in β_i must not be in the interval (y, x), else if $z \in (y, x)$ then z would eventually fill a box in row i to the left of v and a clockwise composition triple would be created since $y < z < x$. Hence if v is filled with $y < x$, then y must be the largest letter in β_i that is strictly less than x.

Next, suppose v is filled with $y \in \tilde{\beta}_i$ such that $y \geq x$. To ensure that x and y are not part of a clockwise composition triple, all z remaining in β_i must not be less than x, else if $z < x$ then z would eventually fill a box in row i to the left of v and a clockwise composition triple would be created since $z < x \leq y$ (after standardization this would translate to $z < x < y$). Hence if v is filled with $y \geq x$, then there does not exist any letter $z \in \beta_i$ such that $z < x$ and y must be the largest letter in β_i. This gives a "no-choice" algorithm to construct the desired filling. □

An example of a filling of $\nu' = (2, 4, 6)$ with $\beta_1 = \{1, 3, 4\}$, $\beta_2 = \{1, 5\}$, $\beta_3 = \{3\}$, $\beta_4 = \{5\}$ that has no triples of either kind is:

		5
		3
	5	1
2	4	6

.

The *cocharge word* of a composition is defined the same as for a partition as in

Definition 3.4. For example, we have:

$$\begin{array}{|c|c|c|} \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 \\ \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 \\ \cline{2-3} \multicolumn{1}{c|}{} & 5 & 1 \\ \hline 2 & 4 & 6 \\ \hline \end{array} \qquad \begin{array}{rccccccc} u_1 \cdots u_n = & 6 & 5 & 5 & 4 & 3 & 2 & 1 \\ cw(\sigma) = & 1 & 2 & 4 & 1 & 3 & 1 & 2\,. \end{array} \tag{4.5}$$

Proposition 4.5. *Let ν' be a composition such that $\nu'_1 < \cdots < \nu_k$. Let σ be a filling of ν' such that $inv(\nu') = 0$. Then*

$$maj(\sigma) = cc(cw(\sigma)), \tag{4.6}$$

where $maj(\sigma)$ is the generalized major index statistic defined just as it is for a partition.

Proof. First we want to translate the filling σ into a graph so that each column of ν' becomes its own subword. To begin, plot the cocharge word $cw(\sigma)$ as we did in the definition of charge. To create the desired graph, follow the algorithm for constructing a filling ν' with no counter-clockwise or clockwise composition triples given in the proof of Lemma 4.4. Let n be the number of rows in ν' and m be the number of columns in ν'. Name the boxes in the rightmost column in ν' as $u_{k_1} = (1, m)$, $u_{k_2} = (2, m)$, ..., $u_{k_n} = (n, m)$, so $u_{k_i} = (i, m)$ is the box in the i^{th} row and last column m. By construction of the cocharge word, the box u_{k_i} corresponds to an i in the cocharge word. Since the parts of ν' are strictly increasing, u_{k_n} corresponds to the only $n \in cw(\sigma)$. To create the subword that corresponds to the last column, we will start at the unique $n \in cw(\sigma)$. Next observe that by the proof of Lemma 4.4, the entry in $u_{k_i} = (i, m)$ for $i < n$ is either

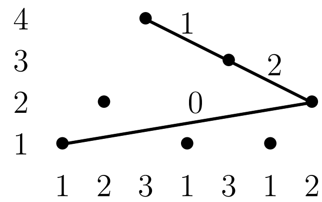
1. the largest entry in row i smaller than $\sigma(u_{k_{i+1}})$, if one exists; or
2. the largest entry in row i entirely.

This then implies that the $i \in cw(\sigma)$ that corresponds to u_{k_i} is either

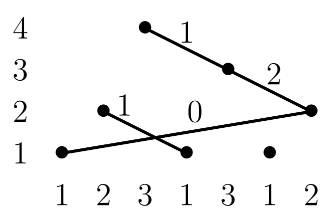
1. the leftmost i to the right the $i+1 \in cw(\sigma)$ that corresponds to $u_{k_{i+1}} \in \sigma$, if one exists; or
2. the leftmost i in $cw(\sigma)$ entirely.

We select our first subword in $cw(\sigma)$ using the above reasoning. In other words, start with the unique largest letter $cw(\sigma)$. Connect i to the leftmost $i-1 \in cw(\sigma)$ to the right of i, if one exists; otherwise, connect i to the leftmost $i-1$ in $cw(\sigma)$ entirely.

Observe that the descents in the final column of ν' correspond exactly to the *left edges* in the graph (as used in the definition of charge, a left edge is one that the left endpoint is higher than the right endpoint). Thus, to achieve summand of $1 + leg(u)$ for a descent u as in the definition of $maj(\sigma)$, we label the left edges with the number of edges above it including itself, and the right edges with a zero. From figure (4.5) we have $cw(\sigma) = 1241312$ which gives:



To complete the construction, note that there is nothing special about the final column of ν' or the first subword selected in $cw(\sigma)$. To continue, ignore the last column of ν' (the composition that remains still has strictly increasing column heights and inversion statistic equal to zero), and ignore the first subword selected from $cw(\sigma)$ (the word that remains still contains a unique largest letter since $\nu'_i < \nu'_{i+1}$). We are now free to choose the second subword and label its edges in the same fashion. Iterate this process until all letters have been used. Observe that $\hat{cc}(\sigma)$ is the sum of all edges. It is clear that $\hat{cc}(\sigma) = maj(\sigma)$. In our example, this gives:



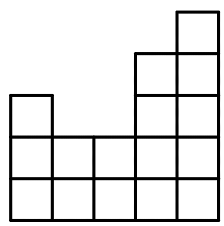
$$\hat{cc}(\sigma) = 1 + 2 + 0 + 1 = 4$$

As noted earlier, it follows easily from Theorem 4.2 that $cc(w) = \hat{cc}(w)$ for all words w. Hence, it is proven that $maj(\sigma) = cc(cw(\sigma))$ for all fillings σ of a composition with strictly increasing column heights, and we have also shown that Killpatrick's description of charge arises naturally as the generalized major index of fillings of ν_i with no inversion triples where $\nu'_i < \nu'_{i+1}$.

$\square$

4.2.3 A Conjecture

Another specialized composition that we will begin to analyze is $\nu' = (\nu'_1, \ldots, \nu'_k)$ where $\nu'_1 \geq \ldots \geq \nu'_i < \nu'_{i+1} < \ldots < \nu'_k$ and $\nu'_1 < \nu'_{i+1}$ for some fixed $1 \leq i \leq k$. In other words, $(\nu'_1, \ldots, \nu'_i)$ forms a partition and $(\nu'_{i+1}, \ldots, \nu'_k)$ forms a strictly increasing composition such that the height of the tallest column of the partition is strictly smaller than the height of the shortest column of the composition. For example, $\nu' = (3, 2, 2, 4, 5)$ satisfies the conditions for $i = 3$:

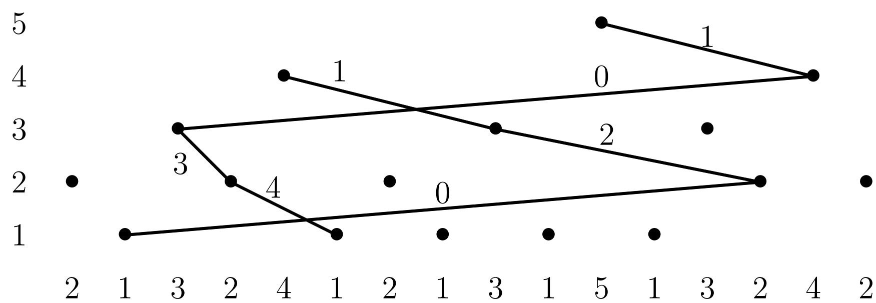


Following the algorithms laid out in the proof of Lemmas 3.5 and 4.4, it is not difficult to see that for each sequence of multisets $\beta_1, \beta_2, \ldots$ with $|\beta_i| = \nu_i$ there exists exactly one counter-clockwise and clockwise composition free filling $\sigma : \nu' \to \mathbb{Z}_+$ such that the weight of the i^{th} row of $\sigma(\nu')$ is equal to β_i. To see this, one first fills columns ν'_{i+1} through ν'_k as in Lemma 4.4 (fill top to bottom, right to left). After these columns are filled, with the remaining letters we fill columns ν'_1 through ν'_i as in Lemma 3.5 (fill the remaining entries left to right, bottom to top). If $\beta_1 = \{3, 4, 5, 7, 9\}$, $\beta_2 = \{1, 2, 6, 8, 10\}$, $\beta_3 = \{3, 5, 9\}$, $\beta_4 = \{2, 8\}$, and $\beta_5 = \{4\}$ then the unique counter-clockwise and clockwise composition free filling is:

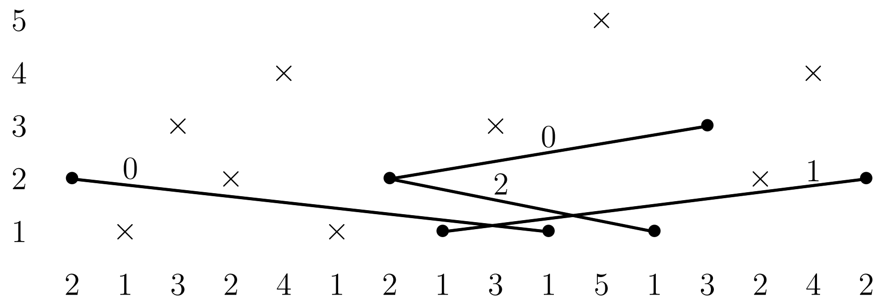
				4
			8	2
3			5	9
6	10	1	2	8
3	4	5	9	7

. (4.7)

Moreover, we can now construct a graph where we pick the first $k-i$ subwords as we did in Proposition 4.5 and then the last i subwords as we did in the original definition of charge 3.3. We will refer to the first $k-i$ subwords selected as $\hat{w}_1, \ldots, \hat{w}_{k-i}$ and the final i subwords selected as $w_{k-i+1}, \ldots, w_k$. For filling in figure (4.7), the following illustration is the graph of the two subwords corresponding to columns four and five and label edges as in the calculation of $\hat{cc}$.



Next, ignoring the letters already selected, we graph the subwords corresponding to the first three columns of figure (4.7) and label edges as in the calculation of cc. For clarity, we will use leave off the edges of the subwords selected above and denote the letters used in those subwords with a $\times$ instead of a $\bullet$.



It is not difficult to see that the left edges correspond to descents and the labels correspond to $1 + leg(u)$ as in the calculation of the generalized major index. Denote the sum of all edges of the above graph as $\tilde{cc}(w)$, i.e. $\tilde{cc}(w) = \hat{cc}(\hat{w}_1) + \cdots + \hat{cc}(\hat{w}_{k-i}) + cc(w_1) + \cdots + cc(w_i) = cc(\hat{w}_1) + \cdots + cc(\hat{w}_{k-i}) + cc(w_{k-i+1}) + \cdots + cc(w_k)$. Hence, for our example we have that the generalized major index is the sum of all the labels on the above graphs, i.e. $maj(\sigma) = 11 + 3 = 14 = \tilde{cc}(w)$. On the other hand, calculating $cc(cw(\sigma))$ also gives 14, but, for this example, the graphs are not identical.

This leads us to a conjecture. It appears that for compositions $\nu' = (\nu'_1, \ldots, \nu'_k)$ where $\nu'_1 \geq \ldots \geq \nu'_i < \nu'_{i+1} < \ldots < \nu'_k$ and $\nu'_1 < \nu'_{i+1}$ for some fixed i, we have $maj(\sigma) = cc(cw(\sigma))$ where σ is a counter-clockwise and clockwise composition free filling.

Let $inv(\sigma) = 0$ and σ be a filling of a composition ν' with $\nu'_1 \geq \ldots \geq \nu'_i < \nu'_{i+1} < \ldots < \nu'_k$ and $\nu'_1 < \nu'_{i+1}$ for some fixed $1 \leq i \leq k$. To prove $maj(\sigma) = cc(cw(\sigma))$, first define $\tilde{ch}$ of a word w by selecting the subwords in the same fashion as in $\tilde{cc}$, but labeling opposite edges. It is sufficient to prove that the statistic $\tilde{ch}$ satisfies Theorem

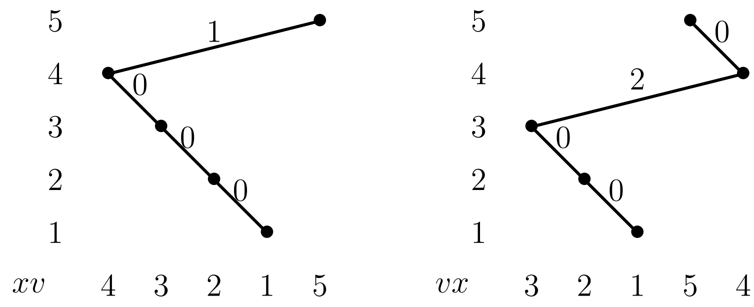
4.3 on words $w = cw(\sigma)$. We will first prove that the statistic $\tilde{ch}$ satisfies property (4) of Theorem 4.3.

Lemma 4.6. *Let $w = cw(\sigma)$ where $inv(\sigma) = 0$ and σ is a filling of a composition ν' with $\nu'_1 \geq \ldots \geq \nu'_i < \nu'_{i+1} < \ldots < \nu'_k$ and $\nu'_1 < \nu'_{i+1}$ for some fixed $1 \leq i \leq k$. Let $w = vx$ be a word of partitioned weight and x a letter not equal to one. Then $\tilde{ch}(vx) = \tilde{ch}(xv) + 1$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be the weight of vx. We will prove this statement in two cases: $x \neq m$ and $x = m$.

First assume $x \neq m$. Let x be in the j^{th} subword picked. Visualizing the graph, it is easy to see that in the word xv, x is still in the j subword chosen. Moreover, all edges not connected to x will remain the same, thus not changing their labels. On the other hand, the right edge connecting some $x - 1$ to x and the left edge connecting x to some $x + 1$ in vx have now been switched. Instead, there is a left edge connecting some $x - 1$ to x and a right edge connecting x to some $x + 1$ in xv. Hence the label on the right edge connected to x in xv is one less than the the label on the right edge connected to x in vx. Since all other edges remained unchanged, $\tilde{cc}(vx) = \tilde{cc}(xv) + 1$.

Below is a simple example where $x = 4$ and $v = 3215$.



The case where $x \in vx_j$ for some j is analogous.

Now assume $x = m$. By construction of vx and by definition of cocharge word, x is the unique largest letter in vx and xv. Therefore, x is in the first subword chosen. If i is equal to 1, then $\tilde{cc} = \hat{cc}$ and we are done.

Assume $i > 1$. It is not difficult to see that x will pair with the same $m - 1$ in both vx and xv. Therefore, all subwords in xv consist of the same letters as they did in vx. Hence, the charge of all subwords not containing x remains the same, but the charge of the subword containing x will decrease by exactly one, as desired. $\square$

The only significant step left is to prove that if w and $\tilde{w}$ are Knuth equivalent, then $\tilde{ch}(w) = \tilde{ch}(\tilde{w})$. We say that two words w and $\tilde{w}$ are differ by a *Knuth relation* if either of the following holds:

1. $x \leq y < z$ and $w = a_1 \cdots a_m xzy a_{m+1} \cdots a_n$ and $\tilde{w} = a_1 \cdots a_m zxy a_{m+1} \cdots a_n$, or

2. $x < y \leq z$ and $w = a_1 \cdots a_m yxz a_{m+1} \cdots a_n$ and $\tilde{w} = a_1 \cdots a_m yzx a_{m+1} \cdots a_n$.

Further, we say that two words w and $\tilde{w}$ are *Knuth equivalent* if there is a chain of words $w^1 \cdots w^n$ such that w^i and w^{i+1} differ by a Knuth relation and $w = w^i$ and $w^n = \tilde{w}$. With tedious methods, we have proven that if w and $\tilde{w}$ differ by the first type of Knuth relation, then $\tilde{ch}(w) = \tilde{ch}(\tilde{w})$, but we leave the details to the reader. The second Knuth relation has yet to be verified.

More generally, let w be a word of weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then select the first $1 \geq i \leq \lambda_1$ subwords as in the construction of $\hat{ch}$, denoted $\hat{w}_1, \ldots, \hat{w}_i$. Then from the remaining unselected letters, select the last $\lambda_1 - i$ subwords as in the construction of ch, denoted $w_1, \ldots, w_{\lambda_1 - i}$. It appears that the following may hold $ch(w) = ch(\hat{w}_1) + \cdots + ch(\hat{w}_i) + ch(w_1) + \cdots + ch(w_{\lambda_1 - i})$. It appears that the above statement is more difficult to prove, namely part 3 of Theorem 4.3 no longer appears trivial to prove.

5 Kostka-Foulkes Polynomials

In this section, we introduce the Kostka-Foulkes polynomials and outline the proof of Lascoux and Schützenberger's theorem that expands the Kostka-Foulkes polynomial in terms of charge. Our exposition directly follows that of [8].

5.1 Introductory Information

Define the set of all *partitions* as

$$\mathcal{P} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

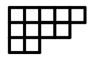
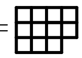
Let δ be the *staircase* partition $(n-1, n-2, \ldots, 2, 1, 0)$, and $\varepsilon_i \in \mathbb{Z}^n$ be basis vectors of $\mathbb{R}^n$ where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the i^{th} position. Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates. If $w \in S_n$ let $(-1)^{\ell(w)}$ be the sign of the permutation w, and let w act on polynomials in the variables $x_1, \ldots, x_n$ by permuting the variables. For $r \in \mathbb{Z}_{\geq 0}$ define the *homogeneous symmetric functions* $h_r = h_r(x_1, \ldots, x_n)$ by

$$\prod_{i=1}^{n} \frac{1}{1 - x_i z} = \sum_{r \geq 0} h_r(x_1, \ldots, x_n) z^r$$

and $h_r = 0$ if $r \in \mathbb{Z}_{<0}$. If $\lambda = (\lambda_1, \cdots, \lambda_n)$ where $\lambda_i \in \mathbb{Z}$, define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_n}$. For each pair $1 \leq i < j \leq n$ define the *raising operator* $R_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$R_{ij}\mu = \mu + \varepsilon_i - \varepsilon_j \text{ and define } (R_{i_1j_1} \cdots R_{i_kj_k}) s_\mu = s_{R_{i_1j_1} \cdots R_{i_kj_k}\mu},$$

for pairs $i_\ell < j_\ell$. Graphically, R_{ij} takes a single box of the j^{th} row and attaches the box to the end of the i^{th} row of $\mu \in \mathbb{Z}^n$. If $\lambda = (2,4,5)$ then $R_{13}\lambda = R_{13}(2,4,5) = (3,4,4)$.

R_{13}  $=$ 

For $\mu \in \mathbb{Z}^n$, define the *Schur function* as

$$s_\mu = \prod_{1 \le i < j \le n} (1 - R_{ij}) h_\mu.$$

Let $w \in S_n$ and $\mu \in \mathbb{Z}^n$. The *straightening law* for Schur functions is

$$s_\mu = (-1)^{\ell(w)} s_{w \circ \mu}, \text{ where } w \circ \mu = w(\mu + \delta) - \delta.$$

Using the straightening law, we see that composing the operation of raising operators on Schur functions s_λ should be avoided. For example, if $n = 2$ and s_1 denotes the transposition in the symmetric group S_2 then, by the straightening law, $s_{(0,1)} = -s_{s_1((0,1)+(1,0))-(1,0)} = -s_{(0,1)}$ giving that $s_{(0,1)} = 0$ and so

$$R_{12}(R_{12}s_{(-1,2)}) = R_{12}s_{(0,1)} = R_{12}0 = 0, \quad \text{whereas } R_{12}^2 s_{(-1,2)} = s_{(1,0)} = x_1 + x_2.$$

Definition 5.1. For $\mu \in \mathbb{Z}^n$ define the *Hall-Littlewood polynomials* as

$$Q_\mu = \left(\prod_{1 \le i < j \le n} \frac{1}{1 - tR_{ij}} \right) s_\mu.$$

Definition 5.2. For partitions μ and λ we define the *Kostka-Foulkes polynomials* $K_{\lambda\mu}(t)$ by

$$Q_\mu = \sum_{\lambda \in \mathcal{P}} K_{\lambda\mu}(t) s_\lambda.$$

5.2 Words and Tableaux

To follow the notation from [8], let

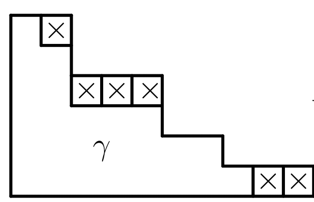
$$B(\lambda) = SSYT(\lambda) = \{\text{semi-standard tableau T with } shp(T) = \lambda\},$$

$$B(\lambda)_\mu = SSYT(\lambda, \mu) = \{\text{semi-standard tableau T with } shp(T) = \lambda \text{ and } wt(T) = \mu\},$$

$$B(\mathcal{S})_\mathcal{W} = \{\text{semi-standard tableau T with } shp(T) \in \mathcal{S} \text{ and } wt(T) \in \mathcal{W}\}.$$

Let λ, and γ be a partitions such that $\gamma_i \leq \lambda_i$ for all i. Then, λ/γ consists of the boxes in λ that are not in γ. The length of the skew shape λ/γ is $\ell(\lambda/\gamma) = |\lambda| - |\gamma|$. The skew shape λ/γ is a *horizontal strip* if it contains at most one box in each column. For example, λ/γ consists of the boxes containing $\times$:

$\lambda =$



where $\ell(\lambda/\gamma) = 6$. (5.1)

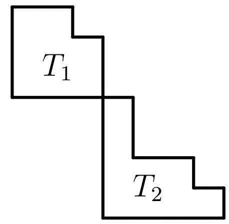
For partitions μ and γ and a nonnegative integer r let

$$\gamma \otimes (r) = (r) \otimes \gamma = \{\text{partitions } \lambda | \lambda/\gamma \text{ is a horizontal strip of length } r\},$$

$$(B(r) \otimes B(\gamma))_\mu = \{\text{pairs } v \otimes T \big| v \in B(r), T \in B(\gamma),\ wt(v) + wt(T) = \mu\},$$

$$(B(\gamma) \otimes B(r))_\mu = \{\text{pairs } T \otimes v \big| v \in B(r), T \in B(\gamma),\ wt(v) + wt(T) = \mu\}.$$

Let T_1 and T_2 be tableaux. Define $T_1 * T_2$ as the *jeu de taquin* reduction of:



.

If x is a letter and T is a tableau, define the *column insertion* of x into T as $x * T$, and the *row insertion* of x into T as $T * x$.

As in [8], the following lemma gives tableau versions of the Pieri rule [7].

Lemma 5.3. *Let γ, μ, and τ be partitions and $r, s \in \mathbb{Z}_{\geq 0}$. Then there are bijections*

$$\begin{array}{ccc} (B(r) \otimes B(\gamma))_\mu & \longleftrightarrow & B(\gamma \otimes (r))_\mu \\ v \otimes T & \longrightarrow & v * T, \\ (B(\gamma) \otimes B(s))_\tau & \longleftrightarrow & B(\gamma \otimes (s))_\tau \\ T \otimes u & \longrightarrow & T * u. \end{array}$$

Let $B(\mathcal{P})_{\geq} = \bigcup_{1 \leq i \leq n} B(\mathcal{P})_{\geq i}$, where

$$B(\mathcal{P})_{\geq i} = \left\{ \text{semi-standard tableaux } b \;\middle|\; \begin{array}{c} wt(b) = (0, \ldots, 0, \mu_i, \cdots, \mu_n) \\ \text{and} \quad \mu_i \geq \cdots \geq \mu_n \geq 0 \end{array} \right\}.$$

Let i^k be the unique semi-standard tableau of shape (k) and weight $(0, \ldots, 0, k, 0, \ldots, 0)$, where the k appears in the i^{th} entry.

Using the notation from this section we restate Theorem 4.3 from [6]. *Charge* is the unique function $ch : B(\mathcal{P})_{\geq} \to \mathbb{Z}_{\geq 0}$ such that:

1. $ch(\emptyset) = 0$,

2. $ch(w) = ch(\alpha w)$ for $\alpha \in S_k$

3. if $T \in B(\mathcal{P})_{\geq i+1}$ and $T * i^{\mu_i} \in B(\mathcal{P})_{\geq i}$ then $ch(T * i^{\mu_i}) = ch(T)$,

4. if $T \in B(\mathcal{P})_{\geq i}$ and x is a letter not equal to i then $ch(x * T) = ch(T * x) + 1$, and

5. if words w and $\tilde{w}$ are Knuth equivalent then $ch(w) = ch(\tilde{w})$.

5.3 Kostka-Foulkes and Charge

Theorem 5.4. *[6] For partitions λ and μ,*

$$K_{\lambda\mu}(t) = \sum_{b \in B(\lambda)_\mu} t^{ch(b)},$$

where the sum is over all semi-standard tableaux b of shape λ and weight μ.

Versions of this proof can be found in [9], [1] and [8]. Below we will outline the proof found in [8].

Proof. The proof is by induction on n. For the induction assumption, suppose that equation (5.4) holds for all partitions $\mu = (\mu_1, \ldots, \mu_n)$. Then it will be sufficient to prove that for all partitions $(\mu_0, \mu) = (\mu_0, \mu_1, \ldots, \mu_n)$, $Q_{(\mu_0,\mu)}$ has the expansion

$$Q_{(\mu_0,\mu)} = \sum_{\substack{\nu \in \mathcal{P} \\ p \in B(\nu)_{(\mu_0,\mu)}}} t^{ch(p)} s_\nu. \tag{5.2}$$

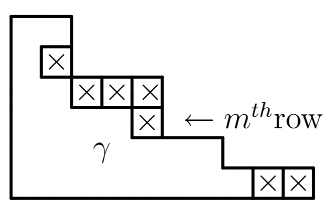
By the definition of the Hall-Littlewood polynomials, we have

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \left(\prod_{0\le i<j\le n} \frac{1}{1-tR_{ij}} \right) s_{(\mu_0,\mu)} \\ &= \left(\prod_{1\le j\le n} \frac{1}{1-tR_{0j}} \right) \left(\prod_{1\le i<j\le n} \frac{1}{1-tR_{ij}} \right) s_{(\mu_0,\mu)} \\ &= \left(\prod_{1\le j\le n} \frac{1}{1-tR_{0j}} \right) \sum_{\lambda\in\mathcal{P}} K_{\lambda\mu}(t) s_{(\mu_0,\lambda)}. \end{aligned}$$

As noted, it is usually necessary to avoid composing raising operators, but with the particular product of raising operators below, it was proven in [8] that they can in fact be composed. Thus, by composition of the raising operators:

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\lambda\in\mathcal{P}} K_{\lambda\mu}(t) \sum_{r\in\mathbb{Z}_{\ge 0}} t^r \sum_{\substack{k\in\mathbb{Z}^n \\ k_1+\cdots+k_n=r}} R_{01}^{k_1}\cdots R_{0n}^{k_n} s_{(\mu_0,\lambda)} \qquad (5.3) \\ &= \sum_{\lambda\in\mathcal{P}} K_{\lambda\mu}(t) \sum_{r\in\mathbb{Z}_{\ge 0}} t^r \sum_{\substack{k\in\mathbb{Z}^n \\ k_1+\cdots+k_n=r}} s_{(\mu_0+r,\lambda-k)}, \end{aligned}$$

where $\gamma = \lambda - (k_1, \ldots, k_n)$. Suppose that λ/γ is not a horizontal strip. Let m the smallest integer so that $\lambda_m - k_m < \lambda_{m+1}$, i.e. the index of the first row that violates the definition of a horizontal strip. For the example below we have $\gamma = \lambda - (2, 0, 1, 3, 1, 0)$.

$\lambda =$ 

Now, let $s_m \in S_n$ be the transposition that switches m and $m+1$. Let $\tilde{\gamma} = s_m \circ \gamma$, so that by straightening we have $s_{(\mu_0+r,\gamma)} = -s_{(\mu_0+r,\tilde{\gamma})}$ where $\tilde{\gamma}_i = \gamma_i$ for $i \ne m, m+1$,

$\tilde{\gamma}_m = \gamma_{m+1} - 1$, and $\tilde{\gamma}_{m+1} = \gamma_m + 1$. Thus $\lambda/\tilde{\gamma}$ is not a horizontal strip since $\tilde{\gamma}_m = \gamma_m - 1 = \lambda_{m+1} - k_{m+1} - 1 < \lambda_{m+1}$. It is not difficult to see that $\tilde{\tilde{\gamma}} = \gamma$ and so the pairing of γ and $\tilde{\gamma}$ provides a cancellation in equation 5.3). Therefore

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\lambda \in \mathcal{P}} K_{\lambda\mu}(t) \sum_{r \in \mathbb{Z}_{\geq 0}} t^r \sum_{\substack{\gamma \in \mathcal{P} \\ \lambda \in \gamma \otimes (r)}} s_{(\mu_0+r,\gamma)}, \\ &= \sum_{\gamma,r} \sum_{\substack{\gamma \in \mathcal{P} \\ \lambda \in \gamma \otimes (r)}} t^r K_{\lambda\mu}(t) s_{(\mu_0+r,\gamma)}. \end{aligned}$$

First by the induction assumption, and then applying the first bijection from Lemma 5.3:

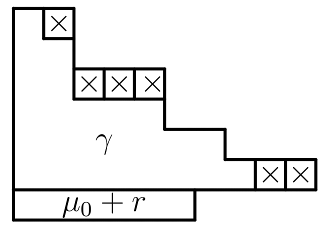
$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\gamma,r} \sum_{\substack{\gamma \in \mathcal{P} \\ \lambda \in \gamma \otimes (r)}} \sum_{b \in B(\lambda)_\mu} t^r t^{ch(b)} s_{(\mu_0+r,\gamma)}, \\ &= \sum_{\gamma,r} \sum_{b \in B(\gamma \otimes (r))_\mu} t^{r+ch(b)} s_{(\mu_0+r,\gamma)}, \\ &= \sum_{\gamma,r} \sum_{v \otimes T \in (B(r) \otimes B(\gamma))_\mu} t^{r+ch(v*T)} s_{(\mu_0+r,\gamma)}. \end{aligned}$$

By applying properties (3) and (4) of charge respectively, it follows that

$$\begin{aligned} Q_{(\mu_0,\mu)} &= \sum_{\gamma,r} \sum_{v \otimes T \in (B(r) \otimes B(\gamma))_\mu} t^{r+ch(v*T*0^{\mu_0})} s_{(\mu_0+r,\gamma)}, \\ &= \sum_{\gamma,r} \sum_{v \otimes T \in (B(r) \otimes B(\gamma))_\mu} t^{ch(T*0^{\mu_0}*v)} s_{(\mu_0+r,\gamma)}. \end{aligned}$$

Fix $v \otimes T \in (B(r) \otimes B(\gamma))_\mu$ and let p and ν be the tableau and shape of $T * 0^{\mu_0} * v$ respectively, i.e. $p = T * 0^{\mu_0} * v$ and $\nu = shp(T * 0^{\mu_0} * v)$. In other words, ν is the tableau that is obtained by inserting $0^{\mu_0} * v$ into T. Since $T \in B(\gamma)$, $\ell(\nu/\gamma) = r + \mu_0$.

Let $\nu_0 = \mu + 0 + r$. Then we have

$(\nu_0, \nu) =$ 

Let d be such that $\mu_0 + r + d > \nu_d$ but $\mu_0 + r + d - 1 \leq \nu_{d-1}$. In other words, add an increasing staircase, $(0, 1, 2, \ldots)$, to the end of ν_0. Let d be the index of the first row in ν that is strictly shorter than the staircase.

The proof now breaks into three parts:

1. $d > 1$ and $(\mu_0 + r, \lambda) = (s_0 \cdots s_{d-3} s_{d-2} s_{d-3} \cdots s_0) \circ (\mu_0 + r, \lambda)$, or

2. $d > 1$ and $(\mu_0 + r, \lambda) \neq (s_0 \cdots s_{d-3} s_{d-2} s_{d-3} \cdots s_0) \circ (\mu_0 + r, \lambda)$,

3. $d = 1$.

In the end, we see that the terms from *case 3* contribute to the formula, the terms from *case 1* are identically zero, and with the terms from *case 2* we can construct a bijection that cancels all terms.

Case 1: Assume $d > 1$ and $(\mu_0 + r, \lambda) = (s_0 \cdots s_{d-3} s_{d-2} s_{d-3} \cdots s_0) \circ (\mu_0 + r, \lambda)$. It follows directly from straightening that $s_{(\mu_0 + r, \gamma)} = 0$.

Case 2: The difficulty of the proof lies in *case 2*. To create this bijection, one has to look at the bumping paths that arise when $0^{\mu_0} * v$ is inserted into T.

Case 3: Assume $d = 1$. Since $\mu_0 + r + 1.\nu_1$ and $\nu \in \otimes(\mu_0 + r)$ the horizontal strip ν/λ must have its boxes in each of the first $\mu_0 + r$ columns, which implies $\nu = (\mu_0 + r, \gamma)$. Row uninsertion of the horizontal strip ν/γ from p (using the second bijection from Lemma 5.3) recovers the pair $T \otimes (\mu_0 * v)$ and shows that $0^{\mu_0} * v$ is the first row of p. From this we can conclude:

$$
\begin{aligned}
Q_{(\mu_0,\mu)} &= \sum_{\gamma,r} \sum_{v \otimes T \in (B(r) \otimes B(\gamma))_\mu} t^{ch(T * 0^{\mu_0} * v)} s_{(\mu_0 + r, \gamma)} \\
&= \sum_{\substack{\nu \in \mathcal{P} \\ p \in B(\nu)_{(\mu_0,\mu)}}} t^{ch(p)} s_\nu
\end{aligned}
$$

as desired.

$\square$

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