On the zeros of component functions of the Riemann Xi approximates

Surl-Hee (Shirley) Ahn

A THESIS

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Master of Arts

2012

Supervisor of Thesis

Graduate Group Chairman

Acknowledgments

First, I would like to thank my thesis adviser Jim Haglund, who made it possible for me to write a math master's thesis at Penn. Without his help and support, I would not have been able to have this invaluable experience of doing math research. I have immensely enjoyed working on my given topic for the thesis and I wish that I had more time to explore the topic further. It was wonderful seeing the empirical evidence that supported the Riemann Hypothesis and I would love to see more work being done in this field. I also sincerely hope to see a proof of the Riemann Hypothesis in my lifetime.

Second, I would like to acknowledge Jerry Kazdan, Michael Pimsner, and Sneha Subramanian, who were my mentors in the math department. Without their help and guidance, I would have not been able to pass the math preliminary exam and solidify my understanding of undergraduate mathematics. I would like to extend my gratitude to Ponzy Lu and Srilata Gangulee, who made it possible for me to pursue two master degree programs in chemistry and mathematics at Penn.

Lastly, I am grateful for family and friends who have supported me throughout my five years at Penn.

Abstract

It has been proven that the conjecture on the monotonicity of the zeros of the Riemann Xi approximates implies the Riemann Hypothesis. To explore this conjecture further, we computed and came up with conjectures on the zeros of the component functions of the Riemann Xi approximates. That is, we looked at the zeros of incomplete gamma functions, hyperbolic gamma functions, and linear combinations of Φ_n's, which are functions that make up the Riemann Xi approximates. Specifically, we checked to see if there exist monotone zeros and interlacing between the zeros of the function and of its derivative.

Dedicated to my loving parents

Contents

1 The Riemann Hypothesis and the Riemann Xi function

One of the most famous unsolved problems in mathematics today is the Riemann Hypothesis. The German mathematician Georg Friedrich Bernhard Riemann (1826-1866) stated that all non-trivial zeros of the Riemann zeta function have $\mathrm{Re}(s) = \frac{1}{2}$.

To understand the hypothesis more clearly, we first looked at the Riemann zeta function. We initially defined the Riemann zeta function to be the Dirichlet series

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

where $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$. Note that when $\mathrm{Re}(s) \leq 1$, the series (1.1) diverges and when $\mathrm{Re}(s) > 1$, the series (1.1) converges and defines an analytic function in this half-plane region.

To extend the Riemann zeta function to the $\mathrm{Re}(s) \leq 1$ region, we used analytic continuation. That is, suppose we are given a pair of functions f and F analytic in regions Ω and Ω', respectively, with $\Omega \subset \Omega'$. If the two functions agree on the smaller set Ω, we say that F is an analytic continuation of f into the region Ω' [SS03]. The following theorem then guarantees that there can be only one such analytic continuation, since F is uniquely determined by f.

Theorem 1.1. *Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for*

all z in some non-empty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z) = g(z)$ throughout Ω [SS03].

Riemann proved that $\zeta(s)$ can be continued analytically to an analytic function over the whole complex plane, except for the point $s = 1$, where $\zeta(s)$ has a simple pole with residue 1. Hence, we had the following definition for the Riemann zeta function $\zeta(s)$.

Definition 1.2. The Riemann zeta function $\zeta(s)$ is the analytic continuation of the Dirichlet series (1.1) to the whole complex plane, except for the point $s = 1$ [BCRW08].

After defining the Riemann zeta function, we looked at its properties, particularly the location of its zeros.

Theorem 1.3. *The Riemann zeta function $\zeta(s)$ satisfies the following:*

1. *The only zeros of $\zeta(s)$ outside the strip $0 \leq Re(s) \leq 1$ are at the negative even integers, $-2, -4, -6, \ldots$. We call these the trivial zeros [SS03].*

2. *$\zeta(s)$ has no zeros on the line $Re(s) = 1$ [SS03].*

3. *The only pole of $\zeta(s)$ is at $s = 1$. The pole is simple with residue 1 [BCRW08].*

4. *The non-trivial zeros lie inside the region $0 \leq Re(s) \leq 1$ and are symmetric about both the vertical line $Re(s) = \frac{1}{2}$ and the real axis $Im(s) = 0$ [BCRW08].*

Finally, Riemann stated his famous conjecture about the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

Conjecture 1.4. *(The Riemann Hypothesis) All non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the line $Re(s) = \frac{1}{2}$.*

After formally covering the Riemann Hypothesis, we looked at related functions that closely relate to the Riemann zeta function. Specifically, we looked at the Riemann xi function, a modification of the Riemann zeta function that Riemann introduced in his 1859 paper. For any $s \in \mathbb{C}$, the Riemann xi function is defined as

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s). \tag{1.2}$$

This function is analytic everywhere in the complex plane and satisfies the following relation: $\xi(1-s) = \xi(s)$ for all $s \in \mathbb{C}$. In fact, the zeros of $\xi(s)$ are precisely the non-trivial zeros of $\zeta(s)$ [BCRW08]. Note that since $\xi(s)$ is a real-valued function when $\mathrm{Re}(s) = \frac{1}{2}$, we can calculate the zeros of $\zeta(s)$ that lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ by determining where the function changes sign [BCRW08]. Hence, we looked at the Riemann Xi (uppercase xi) function, which is precisely

$$\Xi(t) \equiv \xi(s), \tag{1.3}$$

where $s \equiv \frac{1}{2} + it$ and $t \in \mathbb{R}$. This function can also be defined as

$$\begin{aligned} \Xi(it) &\equiv \xi(\frac{1}{2} - t) = \xi(\frac{1}{2} + t) && (1.4) \\ &= \frac{1}{2}(\frac{1}{2} + t)(-\frac{1}{2} + t)\pi^{-1/4-t/2}\Gamma(\frac{t}{2} + \frac{1}{4})\zeta(t + \frac{1}{2}), && (1.5) \end{aligned}$$

which gives

$$\Xi(t) = \frac{1}{2}(\frac{1}{2} + it)(-\frac{1}{2} + it)\pi^{-1/4-it/2}\Gamma(\frac{it}{2} + \frac{1}{4})\zeta(it + \frac{1}{2}). \quad (1.6)$$

This function is analytic everywhere and has zeros with imaginary part between $-\frac{1}{2}$ and $\frac{1}{2}$ [BCRW08]. $\Xi(t)$ was the function originally considered and actually denoted $\xi(t)$ by Riemann himself [Edw01]. Riemann "conjectured" that all zeros α of $\Xi(t)$ are real, that is, all non-trivial zeros of the Riemann zeta function are of the form $\frac{1}{2} + i\alpha$ [BCRW08]. Since $\Xi(t)$ is even and real on the real line, stating that all zeros of $\Xi(t)$ in the first quadrant of the complex plane Q ($\mathrm{Re}(t) \geq 0$, $\mathrm{Im}(t) \geq 0$) are real is equivalent to the Riemann Hypothesis [Hag11].

As stated in [Hag11], the Riemann Xi function can be re-written as

$$\Xi(z) = \int_0^\infty \cos(zt)\phi(t)dt, \quad (1.7)$$

where we have only changed the variable from t to z for Ξ and $\phi(t) = \sum_{n=1}^{\infty} \phi_n(t)$

with

$$\phi_n(t) = \exp(-n^2\pi \exp(2t))(8\pi^2 n^4 \exp(\frac{9}{2}t) - 12\pi n^2 \exp(\frac{5}{2}t)). \tag{1.8}$$

Hence, $\Xi(z)$ can be simply expressed as an integral of an infinite sum of functions. Note that if a given family of approximates approach $\Xi(z)$ uniformly, and if we can prove that, within a certain sub-region of Q, all zeros of each element in the family are real, with the size of the sub-region expanding to eventually include all of Q as the approximates approach $\Xi(z)$, then this would also imply the Riemann Hypothesis [Hag11].

After looking at related functions that closely relate to the Riemann zeta function and arriving at statements that imply the Riemann Hypothesis, we looked at the statements further. First, we looked at the following function.

From [Hag11], a hyperbolic gamma function $G(z; a, b)$ is defined as

$$G(z; a, b) = 4\int_0^\infty \cos(2zu) \exp(2bu - a\exp(2u))du, \tag{1.9}$$

where $z \in \mathbb{C}$, $a, b \in \mathbb{R}$ with $a > 0$. Using change of variables ($t = a\exp(2u)$,

$dt = a\exp(2u)2du$, $du = dt/2t$), as stated in [Hag11], we can re-write this as

$$\begin{aligned} G(z;a,b) &= 2\int_a^\infty \exp(b\log(t/a)-t)\cos(z\log(t/a))\frac{dt}{t} && (1.10)\\ &= \int_a^\infty (t/a)^b\exp(-t)(\exp(iz\log(t/a)))+\exp(-iz\log(t/a))\frac{dt}{t} && (1.11)\\ &= \int_a^\infty \exp(-t)((t/a)^{b+iz}+(t/a)^{b-iz})\frac{dt}{t} && (1.12)\\ &= \frac{\Gamma(b+iz,a)}{a^{b+iz}}+\frac{\Gamma(b-iz,a)}{a^{b-z}}, && (1.13) \end{aligned}$$

where

$$\Gamma(z,a) = \int_a^\infty \exp(-t)t^{z-1}dt \tag{1.14}$$

is the (upper) incomplete gamma function. Note that for $a \in \mathbb{R}$, $a > 0$, $\Gamma(z,a)$ is analytic everywhere, so $G(z;a,b)$ is analytic everywhere as well [Hag11].

Using (1.7) and (1.10) as in [Hag11], $\Xi(z)$ can be re-written as

$$\Xi(z) = \sum_{n=1}^{\infty} 2\pi^2 n^4 G(z/2;n^2\pi,9/4) - 3\pi n^2 G(z/2;n^2\pi,5/4). \tag{1.15}$$

Note that since the family of approximates uniformly converge to $\Xi(z)$, we interchanged the integration and summation signs. Recall that proving all zeros of each element in the family of approximates to be real within a certain sub-region of Q implies the Riemann Hypothesis. Hence, we looked at the following Riemann Xi

approximates as defined in [Hag11]

$$\Xi_N(z) := \sum_{n=1}^{N} \Phi_n(z), \tag{1.16}$$

where

$$\Phi_n(z) := 2\pi^2 n^4 G(z/2; n^2\pi, 9/4) - 3\pi n^2 G(z/2; n^2\pi, 5/4). \tag{1.17}$$

Note that $\phi(t)$ from $\Xi(z)$ has been replaced by $\sum_{n=1}^{N} \phi_n(t)$ to obtain the Ξ approximates.

Interestingly, all of the computer runs done on Maple so far indicate that $\Xi_N(z)$ has monotonic zeros [Hag11]. That is, when we list the zeros by increasing real part, the imaginary parts of the zeros are monotone non-decreasing. Formally, if $\{\alpha_1, \alpha_2, \ldots\}$ are the zeros of $\Xi_N(z)$ in a region D numbered so that $\mathrm{Re}(\alpha_i) < \mathrm{Re}(\alpha_{i+1})$ for $i \geq 1$, then $\mathrm{Im}(\alpha_i) \leq \mathrm{Im}(\alpha_{i+1})$ [Hag11]. Note that we assumed $\Xi_N(z)$ has at most one zero on the intersection of any vertical line with D [Hag11]. Hence, [Hag11] provided the following conjecture and proposition.

Conjecture 1.5. *For $N \in \mathbb{N}$, $\Xi_N(Z)$ has monotonic zeros in Q [Hag11].*

Proposition 1.6. *Conjecture 1.6 implies the Riemann Hypothesis [Hag11].*

Before looking at the proof of Proposition 1.7, we recalled the argument principle first.

Theorem 1.7. *(Argument principle) Suppose f is meromorphic in an open set con-*

taining a circle C and its interior. If f has no poles and never vanishes on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \textit{(number of zeros of f inside C) - (number of poles of f inside C)}, \tag{1.18}$$

where the zeros and poles are counted with their multiplicities [SS03].

Then, we looked at Hardy's Theorem, which establishes the most basic/minimal necessary condition for the Riemann Hypothesis to be true.

Theorem 1.8. *(Hardy's Theorem) There are infinitely many zeros of $\zeta(s)$ on the line $Re(s) = \frac{1}{2}$.*

Proof of Prop 1.7 (from [Hag11]). The proof proves the contrapositive of the proposition. Assume that the Riemann Hypothesis is false and let $\tau = \sigma + it$ be the zero of $\Xi(z)$ in Q with minimal real part, among those zeros with positive imaginary part. By the argument principle, we have

$$\frac{1}{2\pi i}\int_{C_\epsilon} \frac{\Xi'(z)}{\Xi(z)}dz = 1, \tag{1.19}$$

where the integral is taken counterclockwise around a circle C_ϵ centered at τ, of sufficiently small radius ϵ, so no other zeros of $\Xi(z)$ are enclosed in C_ϵ. Recall that $\Xi(z)$ is analytic everywhere, so $\Xi(z)$ does not have poles. Next, we choose N sufficiently large so that $\Xi_N(z)$ has a real zero γ with $\gamma > 2\sigma$. We can do this since $\Xi_N(z)$ converges uniformly to $\Xi(z)$ in Q, both $\Xi(z)$ and $\Xi_N(z)$ are real on the real line,

and $\Xi(z)$ has infinitely many positive real zeros by Hardy's theorem. By assumption, $\Xi_N(z)$ has monotonic zeros in Q. Hence, $\Xi_N(z)$ has no non-real zeros in Q with real part less than 2σ. This implies

$$\frac{1}{2\pi i}\int_{C_\epsilon}\frac{\Xi_N'(z)}{\Xi_N(z)}dz = 0, \tag{1.20}$$

and so

$$\begin{aligned} 1 &= \frac{1}{2\pi i}\int_{C_\epsilon}\frac{\Xi'(z)}{\Xi(z)} - \frac{\Xi_N'(z)}{\Xi_N(z)}dz & (1.21)\\ &= \frac{1}{2\pi i}\int_{C_\epsilon}\frac{\Xi'(z)\Xi_N(z) - \Xi_N'(z)\Xi(z)}{\Xi(z)\Xi_N(z)}dz. & (1.22)\end{aligned}$$

On the closed and bounded set C_ϵ, $|\Xi(z)|$ is nonzero. Hence, $|\Xi(z)|$ must reach an absolute minimum on C_ϵ. Call this δ, which is positive since $|\Xi(z)| > 0$ for all $z \in C_\epsilon$. Since $\Xi_N(z)$ uniformly converges to $\Xi(z)$ as $N \to \infty$, the minimum of $|\Xi_N(z)|$ on C_ϵ must eventually be greater than $\delta/2$. Hence, for sufficiently large N, the modulus of the denominator of the integrand in (1.22) is bounded away from zero, but since $\Xi_N'(z)$ approaches $\Xi'(z)$ uniformly, the numerator approaches zero. Hence, the integral will also approach zero, a contradiction. Hence, $\Xi_N(z)$ does not have monotonic zeros in Q. $\square$

Remark 1.9. A weaker form of Conjecture 1.6, which still implies the Riemann Hypothesis, states that there are no non-real zeros of $\Xi_N(z)$ in Q whose real part is less than the largest real zero of $\Xi_N(z)$ [Hag11]. Computations indicate this weaker form

of Conjecture 1.6 is true at least for $N \leq 10$ [Hag11].

Taken together, we arrived that the conjecture on the monotonicity of the zeros of the Riemann Xi approximates implies the Riemann Hypothesis. To explore this conjecture further, we then looked at the zeros of incomplete gamma functions, hyperbolic gamma functions, and linear combinations of Φ_n's, which make up the Riemann Xi approximates.

2 Zeros of the Incomplete Gamma Function

Recall that the (upper) incomplete gamma function is of the form

$$\Gamma(z, a) = \int_a^\infty \exp(-t)t^{z-1}dt \tag{2.1}$$

where $a \in \mathbb{R}$ with $a > 0$. A linear combination of the incomplete gamma functions make up the hyperbolic gamma function, whose linear combinations make up the Riemann Xi approximates.

Since the Riemann Xi approximates were conjectured to have monotonic zeros, we wanted to compute the zeros of various incomplete gamma functions and check for monotonicity. In order to find the zeros, Maple's Analytic function was used under the RootFinding package. The domain was set to be from $0+0i$ to $50+250i$ so that Maple would be able to compute the zeros in a reasonable amount of time. From [Hag11], the zeros of $\Gamma(z, a)$ in Q are known to satisfy $y \sim \frac{2}{\pi}x \ln x$ as $|z| \to \infty$ [Hag11]. However, y should be at least twice the value of $\frac{2}{\pi}x \ln x$, since this approximation only holds for large values of z, so we rounded up $\frac{4}{\pi}x \ln x$ to the nearest integer when $x = 50$.

The zeros of various incomplete gamma functions were computed, where a ranged from 0.5 to 7 in intervals of 0.5. The zeros of each incomplete gamma function turned out to be monotonic and are listed for representative incomplete gamma functions in the Appendix section.

In addition, we computed the zeros of the derivatives of incomplete gamma func-

tions used previously to see if the zeros are monotonic. If so, then this would provide evidence that the monotonicity of zeros of incomplete gamma functions is a strong property. We also checked to see if there is any interlacing between the real parts, imaginary parts, and modulus of the zeros of the incomplete gamma function and of the zeros of its derivative. If so, then this would provide evidence that the zeros have Rolle's Theorem-like property. Recall that the Rolle's Theorem states that if $f(x)$ is a real-valued continuous function on $[a, b]$, differentiable on (a, b), and $f(a) = f(b) = 0$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. Hence, we wanted to see if the derivative's zeros would be in between the original function's zeros, since this is true for real-valued continuous functions by the Rolle's Theorem.

The zeros of each derivative turned out to be monotonic and are listed for the derivatives of representative incomplete gamma functions in the Appendix section. From $a = 0.5$ to $a = 2$, there was interlacing between the imaginary parts and between the modulus. From $a = 2.5$ to $a = 7$, there was interlacing between the real parts, between the imaginary parts, and between the modulus. The results are summarized in Table 1 below. The zeros of the incomplete gamma function and of its derivative were graphed together and representative graphs are shown in the Appendix section.

a	Real Parts	Imaginary Parts	Modulus
0.5	No	Yes	Yes
1	No	Yes	Yes
1.5	No	Yes	Yes
2	No	Yes	Yes
2.5	Yes	Yes	Yes
3	Yes	Yes	Yes
3.5	Yes	Yes	Yes
4	Yes	Yes	Yes
4.5	Yes	Yes	Yes
5	Yes	Yes	Yes
5.5	Yes	Yes	Yes
6	Yes	Yes	Yes
6.5	Yes	Yes	Yes
7	Yes	Yes	Yes

Table 1: Interlacing of zeros of $\Gamma(a, z)$ and of its derivative

Hence, we made the following conjectures.

Conjecture 2.1. *The zeros of the incomplete gamma function and of its derivative are monotonic.*

Conjecture 2.2. *Interlacing exists between the imaginary parts and between the modulus of the zeros of the incomplete gamma function and of the zeros of its derivative for all $a \in \mathbb{R}$ with $a > 0$. In addition, there exists an $\alpha \in \mathbb{R}$ such that when $a \geq \alpha$, where $2 < \alpha \leq 2.5$, interlacing also exists between the real parts of the zeros of the incomplete gamma function and of the zeros of its derivative.*

With the zeros of the incomplete gamma function, as well as the zeros of its derivative, being monotonic, the monotonicity of zeros appears to be a strong property of the incomplete gamma function. As a result, the empirical evidence in this section provides further support for the monotonicity of the zeros of the Riemann

Xi approximates, since linear combinations of incomplete gamma functions make up the hyperbolic gamma function, whose linear combinations make up the Riemann Xi approximates, as shown previously. In addition, the interlacing of zeros, which gets stronger as a increases, also provides us with a glimpse into the interesting properties of the incomplete gamma function. We repeated this procedure for hyperbolic gamma functions, detailed in the next section.

3 Zeros of the Hyperbolic Gamma Function

Recall that the hyperbolic gamma function is of the form

$$G(z;a,b) = \frac{\Gamma(b+iz,a)}{a^{b+iz}} + \frac{\Gamma(b-iz,a)}{a^{b-iz}} \tag{3.1}$$

where $z \in \mathbb{C}$, $a, b \in \mathbb{R}$ with $a > 0$. A linear combination of the hyperbolic gamma functions, which is a linear combination of the incomplete gamma functions, make up the Riemann Xi approximates.

Again, we wanted to compute the zeros of various hyperbolic gamma functions and check for monotonicity. The domain was set to be from $0.01 + 0i$ to $250 + 50i$ so that Maple would be able to compute the zeros in a reasonable amount of time. From [Hag11], the zeros of $G(z;a,b)$ in Q are known to satisfy $x \sim \frac{2}{\pi} y \ln y$ as $|z| \to \infty$ [Hag11]. However, x should be at least twice the value of $\frac{2}{\pi} y \ln y$, since this approximation only holds for large values of z, so we rounded up $\frac{4}{\pi} y \ln y$ to the nearest integer when $y = 50$.

The zeros of various hyperbolic gamma functions were computed, where a ranged from 1.5 to 3.5 in intervals of 0.5 and b ranged from -5 to 7 in intervals of 0.5. The zeros of each hyperbolic gamma function turned out to be monotonic and are listed for representative hyperbolic gamma functions in the Appendix section.

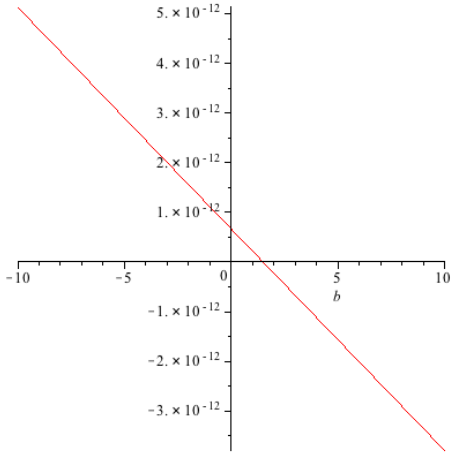
Although the zeros of the hyperbolic gamma functions in different regions of Q were all monotonic in several hundred computer runs, [Hag11] indicated that there

exist hyperbolic gamma functions with non-monotonic zeros. In [Hag11], it has been shown that as $z \to \infty$ along the positive real line, the hyperbolic gamma function can be approximated as

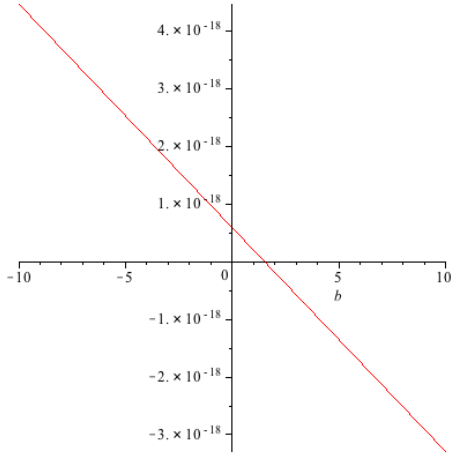
$$G(x; a, b) = \frac{2(a-b)}{e^a} \frac{1}{x^2} + O(\frac{1}{x^4}). \tag{3.2}$$

Hence for large x, we have $G(x; a, 2a) < 0$ and $G(x; a, 0) > 0$, so there must exist some value of b, $0 < b < 2a$ for which $G(x; a, b) = 0$ [Hag11]. If this is the case, then x will a real zero of $G(z; a, b)$, which occurs after many non-real zeros, so the zeros of $G(z; a, b)$ will not be not monotonic.

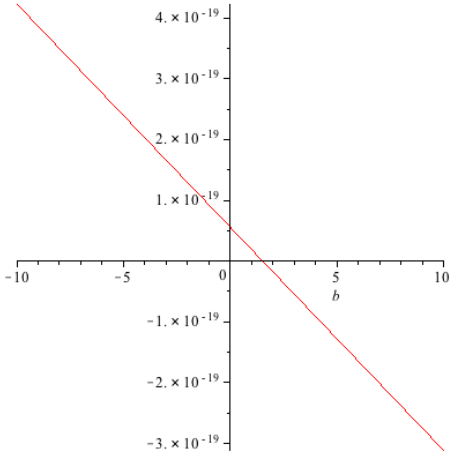
In order to find numerical solutions of b for which $G(x; a, b) = 0$, we plotted $G(x; a, b)$ with respect to b, where x was set to equal four arbitrary large real numbers $(10^6, 8^{10}, 9^{10}, 10^{10})$ and a was set to range from 1.5 to 3.5 in intervals of 0.5. The number of digits was set to 100 for accuracy of the values. From graphing $G(x; a, b)$, we found out that all graphs were linearly decreasing and $G(x; a, b) = 0$ only when $a \simeq b$. The following four graphs of $G(x; a, b)$ where $a = 1.5$ in Figure 1 illustrate the results.



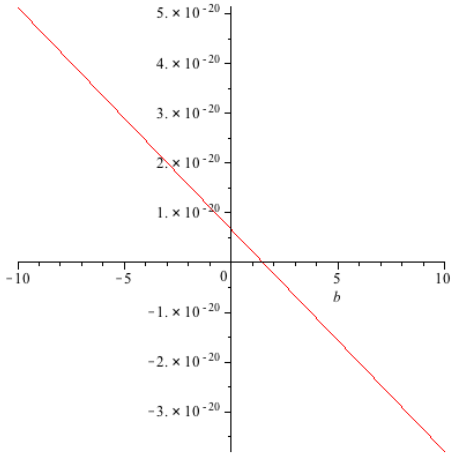
(a) $G(10^6; 1.5, b)$



(b) $G(8^{10}; 1.5, b)$



(c) $G(9^{10}; 1.5, b)$



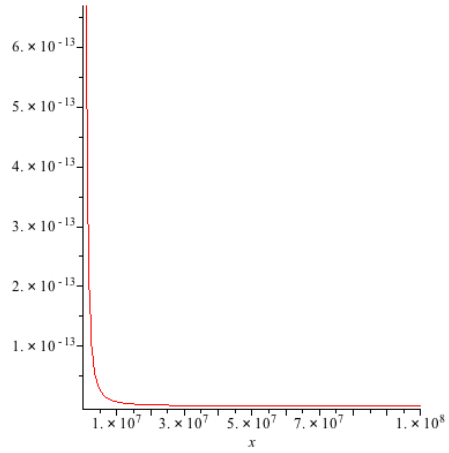
(d) $G(10^{10}; 1.5, b)$

Figure 1: Graphs of $G(x; 1.5, b)$ where x is real and large

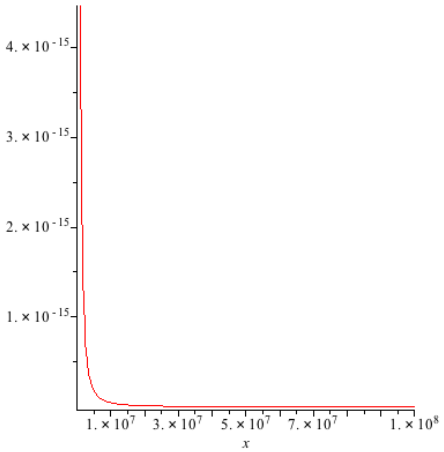
Next, we wanted to test whether we can really find a large x such that $G(x;a,b) = 0$ for any a and positive b. Hence, we plotted $G(x;a,b)$ with respect to x, where x ranged from 10^6 and a and b were set to equal arbitrary real positive numbers. Again, the number of digits was set to 100 for accuracy of the values. From graphing $G(x;a,b)$, we found out that none of the graphs had points such that $G(x;a,b) = 0$. Interestingly, the graph of $G(x;a,b)$, where x was large, resembled an exponentially decreasing curve when $b < a$ and resembled an asymptotically increasing curve when $b \geq a$. The following four graphs of $G(x;a,b)$ where $a = 1.5$ in Figure 2 illustrate this.

Finally, we wanted to test whether we can find a large x such that $G(x;a,b) = 0$ for any a and non-positive b. Hence, we plotted $G(x;a,b)$ with respect to x, where x ranged from 10^6 and a and b were set to equal arbitrary real positive numbers and non-positive numbers, respectively. Again, the number of digits was set to 100 for accuracy of the values, and there were no points such that $G(x;a,b) = 0$ for all graphs. Since $b < a$ in this case, the graph of $G(x;a,b)$, where x was large, resembled an exponentially decreasing curve, consistent with the previous results when $b < a$. The following four graphs of $G(x;a,b)$ where $a = 1.5$ in Figure 3 illustrate the results.

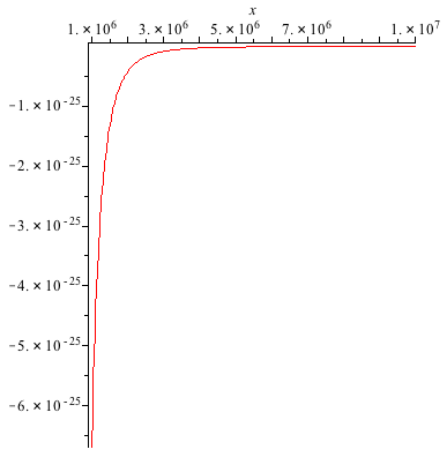
Taken together, these results indicated that $G(z;a,b)$ had a large real zero x only when $a \simeq b$. This implies that the hyperbolic gamma function has non-monotonic zeros when $a \simeq b$, since $G(x;a,b) = 0$ indicates the presence of a large real zero after many non-real zeros.



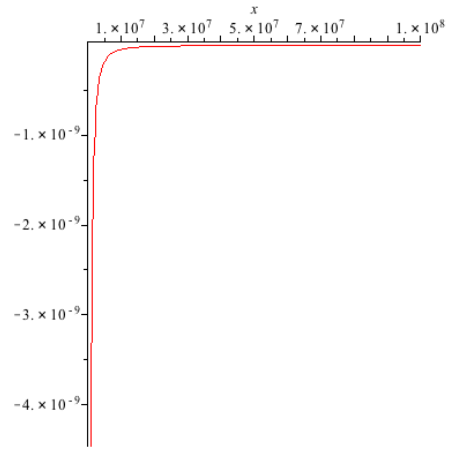
(a) $G(x; 1.5, 10^{-10})$



(b) $G(x; 1.5, 1.49)$

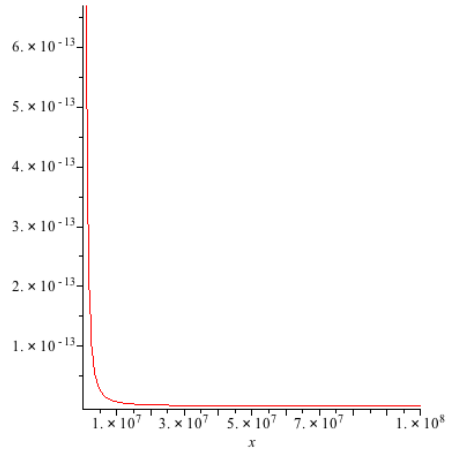


(c) $G(x; 1.5, 1.5)$

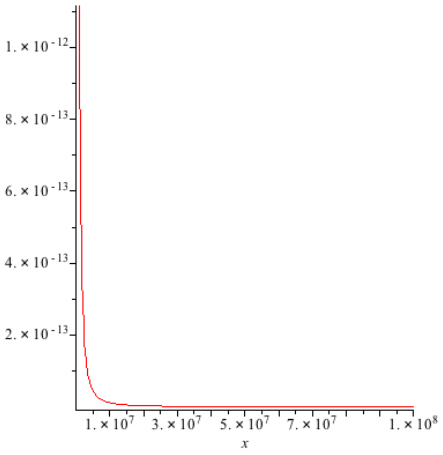


(d) $G(x; 1.5, 10^4)$

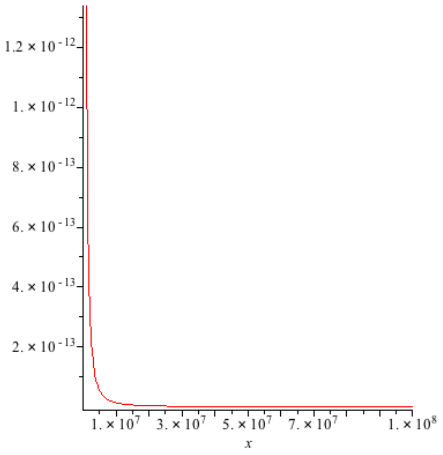
Figure 2: Graphs of $G(x; 1.5, b)$ where b is positive



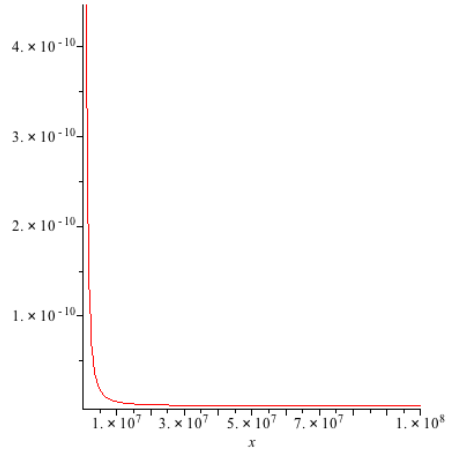
(a) $G(x; 1.5, 0)$



(b) $G(x; 1.5, -1)$



(c) $G(x; 1.5, -1.5)$



(d) $G(x; 1.5, -10^3)$

Figure 3: Graphs of $G(x; 1.5, b)$ where b is non-positive

Hence, we made the following conjecture.

Conjecture 3.1. *The zeros of the hyperbolic gamma function are monotonic when* $a \not\simeq b$.

Although the zeros were monotonic for tested hyperbolic gamma functions where $a = b$, the region in which the zeros were computed was nowhere near the large x values for which $G(x; a, b) = 0$.

In addition, we observed that the zero of $G(z; a, b)$ in Q with the smallest modulus was real when b was approximately equal to or greater than a. More specifically, the lower bound for b for which the zero with the smallest modulus is real approached a as a increased. For example, for $a = 1.5$, b had to be greater than $0.86a$ for the zero with the smallest modulus to be real, whereas for $a = 2$, b had to be greater than $0.9a$. The lower bounds were calculated by graphing $G(z; a, b)$ with respect to z, where z was a real number ranging from 0 to 10, a ranged from 1.5 to 3.5 in intervals of 0.5. and b was set to arbitrary real numbers. As b increased, the graph "lowered down" and crossed the horizontal axis after a certain point. The smallest crossing point corresponded to the zero with the smallest modulus for that particular a and b, which was real. The results are summarized in Table 2 below.

a	Lower bound for b
1.5	$0.86a$
2	$0.9a$
2.5	$0.924a$
3	$0.9367a$
3.5	$0.94857a$

Table 2: Lower bound for b for which the zero of $G(z; a, b)$ with the smallest

modulus is real

Furthermore, we computed the zeros of the derivatives of hyperbolic gamma functions used previously to see if the zeros are monotonic. If so, then this would provide evidence that the monotonicity of zeros of hyperbolic gamma functions is a strong property. However, the zeros of few derivatives were not able to be computed within two days (e.g. for $a = 1.5$ and $b = 4, 6, 6.5$). The zeros of each derivative turned out to be monotonic and are listed for the derivatives of representative hyperbolic gamma functions in the Appendix section. However, the zeros of the derivatives of hyperbolic gamma functions where $a \simeq b$ might not be monotonic, like the zeros of the original function, when computed in a bigger region. Hence, this needs to be tested to conclude that the derivatives of hyperbolic gamma functions where $a \simeq b$ have monotonic zeros or not.

We also checked to see if there is any interlacing between the real parts, imaginary parts, and modulus of the zeros of the hyperbolic gamma function and of the zeros of its derivative. Again, we wanted to see if the derivative's zeros would be in between the original function's zeros and have a Rolle's Theorem-like property. For most functions, including when $b \leq 0$ and $b < a$, there was interlacing between the real parts and between the modulus, but no interlacing between the imaginary parts. For rest of the functions, interlacing did not exist at all. The results are summarized in the Appendix section. The zeros of a hyperbolic gamma function and of its derivative were graphed together and representative graphs are shown in the Appendix section.

Hence, we made the following conjectures.

Conjecture 3.2. *The zeros of the derivative of hyperbolic gamma function are monotonic when* $a \not\approx b$.

Conjecture 3.3. *There exists an* $\alpha \in \mathbb{R}$ *such that when* $b \leq \alpha$, *where* $0 < \alpha < a$, *interlacing exists between the real parts and between the modulus of the zeros of the hyperbolic gamma function and of the zeros of its derivative for all* $a \in \mathbb{R}$ *with* $a > 0$.

With the zeros of the hyperbolic gamma function, as well as the zeros of its derivative, being monotonic when $a \not\approx b$, the monotonicity of zeros appears to be a strong property of the hyperbolic gamma function, like the incomplete gamma function. As a result, the empirical evidence in this section provides further support for the monotonicity of the zeros of the Riemann Xi approximates, since linear combinations of hyperbolic gamma functions make up the Riemann Xi approximates, as shown previously. It is interesting that there is no interlacing between the imaginary parts of zeros for the hyperbolic gamma function, even though the incomplete gamma function had it for all $a \in \mathbb{R}$ with $a > 0$. We repeated this procedure once more for linear combinations of Φ_n's, detailed in the next section.

4 Zeros of the Linear Combination of Φ_n's

Recall that the Riemann Xi approximate is of the form

$$\Xi_N(z) := \sum_{n=1}^{N} \Phi_n(z), \tag{4.1}$$

where

$$\Phi_n(z) := 2\pi^2 n^4 G(z/2; n^2\pi, 9/4) - 3\pi n^2 G(z/2; n^2\pi, 5/4). \tag{4.2}$$

Hence, the Riemann Xi approximate is simply a finite sum of Φ_n's. It has already been conjectured that the Riemann Xi approximate has monotonic zeros. Hence, we wanted to check whether linear combinations of Φ_n's, where not all of the Φ_n's have a coefficient of 1, have monotonic zeros.

To introduce the ultimate motivation for this problem, the following definition and proposition need to be defined.

Definition 4.1. A family $\{f_1(z), \ldots, f_k(z)\}$ of polynomials with real coefficients is called compatible if $\sum_{j=1}^{k} c_j f_j(z)$ has only real zeros whenever $c_j \in \mathbb{R}$, $c_j \geq 0$ for $1 \leq j \leq k$. It is called pairwise compatible if $\{f_i, f_j\}$ forms a compatible family for each pair $1 \leq i < j \leq k$ [Hag11].

Proposition 4.2. *Let $f_1, \ldots, f_k$ be polynomials with positive leading coefficients and all roots real. Then $f_1, \ldots, f_k$ being pairwise compatible is equivalent to $f_1, \ldots, f_k$ being compatible [CS07].*

In [Hag11], it has been questioned whether a similar statement holds if we replace real polynomials with positive leading coefficients and only real zeros, by even, real, entire functions having monotonic zeros in Q. However, we do not know whether any series of the form $\sum_{k=1}^{N} c_k \Phi_k(z)$, where $c_k \in \mathbb{R}$ with $c_k \geq 0$, even has monotonic zeros. Hence, this needed to be tested first in order to tackle the big question. From [Hag11], it has been advised that $c_1 = 1$ and $0 \leq c_i \leq 1$ for $i > 1$ to avoid non-monotonic zeros.

As a result, zeros of fourteen different linear combinations of Φ_n's with the above condition were computed and checked for monotonicity. The domain was set to be from $0.01 + 0i$ to $250 + 50i$ so that Maple would be able to compute the zeros in a reasonable amount of time. From [Hag11], the zeros of any function of the form $\sum_{k=1}^{N} u_k G(z; a_k, b_k)$, where $a_k, b_k \in \mathbb{R}$, $u_k \in \mathbb{C}$, $a_k > 0$, in Q are known to satisfy $x \sim \frac{2}{\pi} y \ln y$ as $|z| \to \infty$ [Hag11]. However, x should be at least twice the value of $\frac{2}{\pi} y \ln y$, since this approximation only holds for large values of z, so we rounded up $\frac{4}{\pi} y \ln y$ to the nearest integer when $y = 50$. The zeros of each linear combination of Φ_n's turned out to be monotonic and are listed in the Appendix section.

In addition, we attempted to compute the zeros of the derivatives of linear combinations of Φ_n's previously used to see if the zeros are monotonic. If so, then this would provide evidence that the monotonicity of zeros of liner combinations of Φ_n's is a strong property. However, Maple was not able to compute the zeros in a reasonable amount of time. As a result, an alternative method introduced in [Hag11] needs to

be used to compute the zeros of the derivatives. The alternative method is outlined as follows.

Recall that Riemann "conjectured" that all zeros α of $\Xi(z)$ are real. This is equivalent to the statement that the modulus of $\Xi(z)$ is monotone increasing along any vertical ray which starts at a point $x \geq 0$ on the non-negative real line and travels straight upward to $x + i\infty$ [Hag11]. To see why this is true, first recall a few definitions.

Definition 4.3. Let f be an entire function. If there exist a positive number ρ and constants $A, B > 0$ such that $|f(z)| \leq A \exp(B|z|^\rho)$ for all $z \in \mathbb{C}$, then we say that f has an order of growth $\leq \rho$. We define the order of growth of f as $\rho_f = \inf \rho$, where the infimum is over all $\rho > 0$ such that f has an order of growth $\leq \rho$ [SS03].

Definition 4.4. For each integer $k \geq 0$ we define canonical factors by $E_0(z) = 1 - z$ and $E_k(z) = (1 - z)\exp(z + z^2/2 + \cdots + z^k/k)$, for $k \geq 1$. The integer k is called the degree of the canonical factor [SS03].

Now recall Hadamard's factorization theorem.

Theorem 4.5. *Suppose f is entire and has growth order ρ_0. Let k be the integer so that $k \leq \rho_0 < k + 1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeros of f, then*

$$f(z) = \exp(P(z))z^m \prod_{n=1}^{\infty} E_k(z/a_n), \tag{4.3}$$

where P is a polynomial of degree $\leq k$, and m is the order of the zero of f at $z = 0$

[SS03].

Using Hadamard's factorization theorem, the modulus of $\Xi(z)$ can be written as

$$|\Xi(z)| = |\Xi(0)| \prod_i |1 - z^2/\alpha_i^2|, \tag{4.4}$$

where the α_i's are the positive real zeros of $\Xi(z)$ [Hag11]. Then as stated in [Hag11], we can take the partial derivative of equation (4.4) with respect to y, where $z = x+iy$, and get

$$\begin{aligned}
\frac{\partial}{\partial y}|\Xi(z)| &= |\Xi(0)| \frac{\partial}{\partial y} \exp(\sum_{i=1}^{\infty} \ln(1 - z^2/\alpha_i^2)) && (4.5)\\
&= |\Xi(0)| \exp(\sum_{i=1}^{\infty} \ln(1 - z^2/\alpha_i^2)) \frac{\partial}{\partial y} \sum_{i=1}^{\infty} \ln(1 - z^2/\alpha_i^2) && (4.6)\\
&= |\Xi(0)| \exp(\sum_{i=1}^{\infty} \ln(1 - z^2/\alpha_i^2)) \sum_{i=1}^{\infty} \frac{1}{1 - z^2/\alpha_i^2} (\frac{4y}{\alpha_i^2}(1 + \frac{y^2}{\alpha_i^2})). && (4.7)
\end{aligned}$$

Hence, equation (4.7) is positive for positive y and indeed, $|\Xi(z)|$ is monotone increasing along any vertical ray if the Riemann Hypothesis is true.

Now for a given function $F(z)$, analytic in Q, and α a non-negative real number, let $M(F, \alpha)$ denote the value of $y \geq 0$ where the function $|F(\alpha + iy)|$ is minimal [Hag11]. If this minimum occurs at more than one y, let $M(F, \alpha)$ denote the infimum of such y [Hag11]. We can calculate $M(F, \alpha)$ by calculating $|F(\alpha + iy)|$ at many closely spaced grid points y and then choose the y that gives the minimum [Hag11]. From [Hag11], it has been noted that the modulus of a sum of hyperbolic gamma

functions increases rapidly when y increases beyond a certain point.

In addition, let the set of pairs $\{(\alpha, M(F,\alpha))\}$ in Q denote the "M-curve" of F [Hag11]. When the M-curve is plotted on a grid using Maple, it resembles a continuous curve and has the property that the non-real zeros of F in Q occur at the same places as the local maxima of the M-curve [Hag11]. Hence, this is another method to calculate the zeros of the derivatives of the linear combinations of Φ_n's.

To support this method further, [Hag11] shows why the local maximums of the M-curve are linked to the zeros of F. Assume that $|F(\alpha+iy)|$, $y \geq 0$, is monotone decreasing in y until it reaches a certain minimum and then is monotone increasing afterwards [Hag11]. Let $F = u(x,y) + iv(x,y)$ and take the partial derivative of $|F(\alpha+iy)|^2 = u^2+v^2$ with respect to y [Hag11]. Then set the partial derivative equal to 0, since we are concerned with the M-curve, which consists of points $y \geq 0$ where $|F(\alpha+iy)|$ is minimal. Then we get

$$uu_y + vv_y = 0. \tag{4.8}$$

Now as stated in [Hag11], assume that the Implicit Function Theorem also holds. Then equation (4.8) implicitly defines y as a function of x [Hag11]. So we can take the derivative of equation (4.8) with respect to x and get

$$u_x u_y + uu_{yx} + (u_y u_y + uu_{yy})y' + v_x v_y + vv_{yx} + (v_y v_y + vv_{yy})y' = 0. \tag{4.9}$$

Now assume that $z = \alpha + iy$ is a zero of F. Then when we plug in z to F, we get $|F(\alpha + iy)|^2 = u^2 + v^2 = 0 \Rightarrow u = v = 0$. As a result, equation (4.9) simplifies to

$$u_x u_y + v_x v_y + (u_y^2 + v_y^2)y' = 0. \tag{4.10}$$

Finally as stated in [Hag11], use the Cauchy-Riemann equations ($u_x = v_y, u_y = -v_x$) and get

$$(u_y^2 + v_y^2)y' = 0. \tag{4.11}$$

Hence, we get that $y' = 0$, since $u_y = v_y = 0$ would imply that F is constant [Hag11]. Hence, since $y' = 0$, $z = \alpha + iy$ is a local maximum of the M-curve, as well as a zero of F.

Taken together, the M-curve method should be attempted to calculate the zeros of the derivatives of linear combinations of Φ_n's. Then we will be able to conclude that the derivatives of linear combinations of Φ_n's have monotonic zeros or not. Moreover, we will be able to check whether there is any interlacing between the real parts, imaginary parts, and modulus of the zeros of the linear combination of Φ_n's and of the zeros of its derivative. As a result, more work needs to be done on the linear combinations of Φ_n's. Ultimately, we hope that the findings and results from the linear combinations of Φ_n's will support the Riemann Xi approximates having monotonic zeros.

5 Appendix

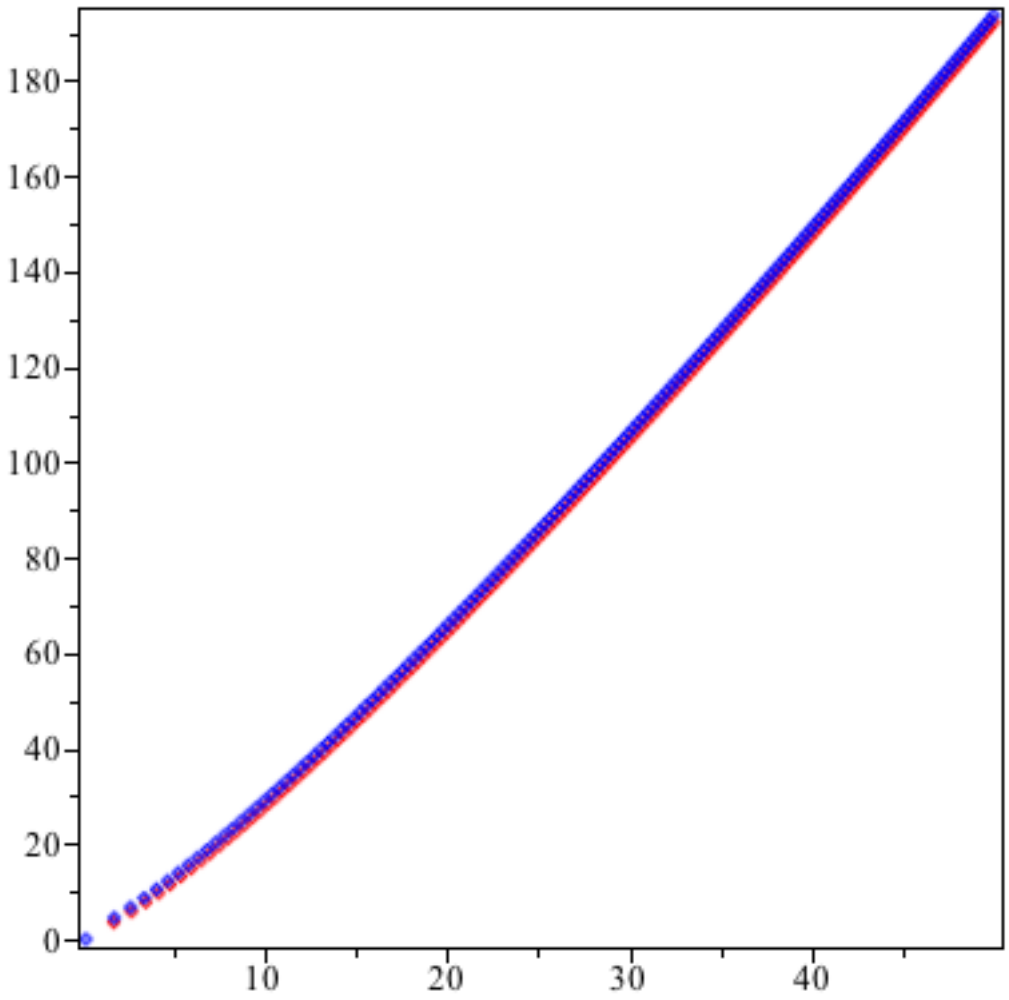


Figure 4: Zeros of $\Gamma(z, 0.5)$ in red and of its derivative in blue

Zeros of $\Gamma(z, 0.5)$ in rectangle with corners (0,0), (50,250).

```
1.746623662+3.509431123*I,
2.715124825+5.754064220*I,
3.502197402+7.753272095*I,
4.195286003+9.616964790*I,
4.827998540+11.38962033*I,
5.417532345+13.09492246*I,
5.974108640+14.74739962*I,
6.504400310+16.35678636*I,
7.013055325+17.93001392*I,
7.503468920+19.47224236*I,
7.978213290+20.98744659*I,
8.439294125+22.47877231*I,
8.888312450+23.94876428*I,
9.326571255+25.39951873*I,
9.755148440+26.83278909*I,
10.17494819+28.25006126*I,
10.58673815+29.65260872*I,
10.99117691+31.04153371*I,
11.38883477+32.41779849*I,
11.78020958+33.78224966*I,
12.16573920+35.13563714*I,
12.54581115+36.47862934*I,
12.92077061+37.81182532*I,
13.29092653+39.13576476*I,
13.65655690+40.45093599*I,
14.01791301+41.75778295*I,
14.37522277+43.05671049*I,
14.72869388+44.34808941*I,
15.07851620+45.63226029*I,
15.42486383+46.90953700*I,
15.76789696+48.18020957*I,
16.10776335+49.44454674*I,
16.44459968+50.70279815*I,
16.77853270+51.95519620*I,
17.10968019+53.20195780*I,
17.43815189+54.44328570*I,
17.76405022+55.67936985*I,
18.08747096+56.91038860*I,
18.40850385+58.13650960*I,
18.72723312+59.35789065*I,
19.04373797+60.57468075*I,
```

```
19.35809295+61.78702050*I,
19.67036838+62.99504295*I,
19.98063067+64.19887425*I,
20.28894259+65.39863375*I,
20.59536362+66.59443505*I,
20.89995010+67.78638605*I,
21.20275554+68.97458940*I,
21.50383076+70.15914300*I,
21.80322413+71.34014010*I,
22.10098168+72.51766985*I,
22.39714730+73.69181725*I,
22.69176287+74.86266385*I,
22.98486837+76.03028750*I,
23.27650201+77.19476290*I,
23.56670036+78.35616155*I,
23.85549843+79.51455220*I,
24.14292974+80.67000085*I,
24.42902645+81.82257080*I,
24.71381943+82.97232300*I,
24.99733831+84.11931615*I,
25.27961156+85.26360670*I,
25.56066657+86.40524905*I,
25.84052967+87.54429565*I,
26.11922624+88.68079715*I,
26.39678070+89.81480235*I,
26.67321666+90.94635840*I,
26.94855670+92.07551090*I,
27.22282287+93.20230390*I,
27.49603629+94.32678000*I,
27.76821742+95.44898045*I,
28.03938600+96.56894520*I,
28.30956114+97.68671290*I,
28.57876134+98.80232100*I,
28.84700448+99.91580585*I,
29.11430788+101.0272026*I,
29.38068833+102.1365456*I,
29.64616210+103.2438680*I,
29.91074493+104.3492019*I,
30.17445214+105.4525787*I,
30.43729857+106.5540287*I,
30.69929860+107.6535816*I,
30.96046624+108.7512661*I,
```

```
31.22081506+109.8471102*I,
31.48035828+110.9411412*I,
31.73910874+112.0333854*I,
31.99707889+113.1238687*I,
32.25428089+114.2126162*I,
32.51072654+115.2996524*I,
32.76642734+116.3850013*I,
33.02139449+117.4686860*I,
33.27563889+118.5507292*I,
33.52917116+119.6311532*I,
33.78200162+120.7099799*I,
34.03414044+121.7872290*I,
34.28559738+122.8629225*I,
34.53638205+123.9370800*I,
34.78650384 + 125.0097212 I,
35.03597184 + 126.0808653 I,
35.28479499 + 127.1505310 I,
35.53298199 + 128.2187365 I,
35.78054134 + 129.2854999 I,
36.02748148 + 130.3508387 I,
36.27381010 + 131.4147697 I,
36.51953555 + 132.4773101 I,
36.76466543 + 133.5384761 I,
37.00920732 + 134.5982838 I,
37.25316862 + 135.6567488 I,
37.49655659 + 136.7138866 I,
37.73937830 + 137.7697122 I,
37.98164068 + 138.8242404 I,
38.22335050 + 139.8774857 I,
38.46451449 + 140.9294622 I,
38.70513906 + 141.9801838 I,
38.94523062 + 143.0296641 I,
39.18479535 + 144.0779165 I,
39.42383950 + 145.1249540 I,
39.66236893 + 146.1707895 I,
39.90038954 + 147.2154356 I,
40.13790709 + 148.2589046 I,
40.37492720 + 149.3012087 I,
40.61145542 + 150.3423597 I,
40.84749716 + 151.3823693 I,
41.08305773 + 152.4212490 I,
41.31814236 + 153.4590100 I,
```

```
41.55275615 + 154.4956634 I,
41.78690412 + 155.5312200 I,
42.02059121 + 156.5656905 I,
42.25382224 + 157.5990854 I,
42.48660197 + 158.6314149 I,
42.71893506 + 159.6626893 I,
42.95082608 + 160.6929183 I,
43.18227952 + 161.7221118 I,
43.41329981 + 162.7502794 I,
43.64389127 + 163.7774306 I,
43.87405818 + 164.8035745 I,
44.10380471 + 165.8287204 I,
44.33313499 + 166.8528773 I,
44.56205305 + 167.8760539 I,
44.79056288 + 168.8982589 I,
45.01866838 + 169.9195010 I,
45.24637340 + 170.9397886 I,
45.47368171 + 171.9591298 I,
45.70059704 + 172.9775330 I,
45.92712303 + 173.9950060 I,
46.15326330 + 175.0115569 I,
46.37902136 + 176.0271933 I,
46.60440072 + 177.0419230 I,
46.82940479 + 178.0557534 I,
47.05403694 + 179.0686922 I,
47.27830050 + 180.0807464 I,
47.50219874 + 181.0919234 I,
47.72573486 + 182.1022303 I,
47.94891204 + 183.1116741 I,
48.17173339 + 184.1202616 I,
48.39420199 + 185.1279997 I,
48.61632060 + 186.1348952 I,
48.83809299 + 187.1409543 I,
49.05952130 + 188.1461839 I,
49.28060869 + 189.1505903 I,
49.50135799 + 190.1541798 I,
49.72177204 + 191.1569588 I,
49.94185358 + 192.1589332 I
```

Zeros of the derivative of $\Gamma(z, 0.5)$ in rectangle with corners (0,0), (50,250).

```
.2173610855,
1.776890301+4.480266640*I,
```

```
2.669524875+6.679777750*I,
3.418173563+8.643600690*I,
4.087332409+10.47969358*I,
4.703445928+12.23006990*I,
5.280668265+13.91684528*I,
5.827684785+15.55355158*I,
6.350301055+17.14926362*I,
6.852634620+18.71045747*I,
7.337736330+20.24196773*I,
7.807944120+21.74752936*I,
8.265098290+23.23010661*I,
8.710679420+24.69210392*I,
9.145900535+26.13550703*I,
9.571770840+27.56198085*I,
9.989141090+28.97293940*I,
10.39873669+30.36959704*I,
10.80118237+31.75300681*I,
11.19702087+33.12408968*I,
11.58672747+34.48365723*I,
11.97072125+35.83242944*I,
12.34937407+37.17104898*I,
12.72301781+38.50009261*I,
13.09195019+39.82008052*I,
13.45643953+41.13148403*I,
13.81672875+42.43473196*I,
14.17303861+43.73021592*I,
14.52557051+45.01829485*I,
14.87450878+46.29929878*I,
15.22002272+47.57353204*I,
15.56226820+48.84127619*I,
15.90138920+50.10279225*I,
16.23751901+51.35832285*I,
16.57078135+52.60809415*I,
16.90129132+53.85231715*I,
17.22915619+55.09118945*I,
17.55447619+56.32489615*I,
17.87734511+57.55361110*I,
18.19785088+58.77749795*I,
18.51607611+59.99671075*I,
18.83209848+61.21139500*I,
19.14599118+62.42168810*I,
19.45782327+63.62772015*I,
```

```
19.76766002+64.82961445*I,
20.07556316+66.02748790*I,
20.38159116+67.22145165*I,
20.68579949+68.41161135*I,
20.98824083+69.59806760*I,
21.28896523+70.78091630*I,
21.58802028+71.96024895*I,
21.88545153+73.13615280*I,
22.18130209+74.30871150*I,
22.47561331+75.47800490*I,
22.76842465+76.64410950*I,
23.05977382+77.80709855*I,
23.34969692+78.96704245*I,
23.63822851+80.12400860*I,
23.92540173+81.27806190*I,
24.21124834+82.42926460*I,
24.49579886+83.57767665*I,
24.77908257+84.72335575*I,
25.06112763+85.86635740*I,
25.34196113+87.00673520*I,
25.62160911+88.14454080*I,
25.90009669+89.27982390*I,
26.17744804+90.41263270*I,
26.45368648+91.54301365*I,
26.72883451+92.67101155*I,
27.00291382+93.79666990*I,
27.27594539+94.92003065*I,
27.54794946+96.04113445*I,
27.81894561+97.16002065*I,
28.08895279+98.27672735*I,
28.35798930+99.39129160*I,
28.62607288+100.5037492*I,
28.89322072+101.6141348*I,
29.15944945+102.7224823*I,
29.42477517+103.8288243*I,
29.68921360+104.9331926*I,
29.95277983+106.0356182*I,
30.21548861+107.1361311*I,
30.47735424+108.2347605*I,
30.73839059+109.3315348*I,
30.99861120+110.4264815*I,
31.25802902+111.5196277*I,
```

```
31.51665695+112.6109994*I,
31.77450732+113.7006221*I,
32.03159219+114.7885208*I,
32.28792327+115.8747214*I,
32.54351203+116.9592420*I,
32.79836954+118.0421111*I,
33.05250682+119.1233494*I,
33.30593389+120.2029787*I,
33.55866156+121.2810203*I,
33.81069968+122.3574952*I,
34.06205837+123.4324248*I,
34.31274615+124.5058255*I
34.56277332+125.5777203*I,
34.81214864+126.6481269*I,
35.06088096+127.7170639*I,
35.30897905+128.7845494*I,
35.55645102+129.8506011*I,
35.80330547+130.9152365*I,
36.04955033+131.9784723*I,
36.29519350+133.0403253*I,
36.54024268+134.1008116*I,
36.78470542+135.1599472*I,
37.02858906+136.2177476*I,
37.27190082+137.2742281*I,
37.51464774+138.3294035*I,
37.75683673+139.3832885*I,
37.99847454+140.4358974*I,
38.23956777+141.4872443*I,
38.48012290+142.5373428*I,
38.72014627+143.5862066*I,
38.95964408+144.6338487*I,
39.19862242+145.6802823*I,
39.43708724+146.7255199*I,
39.67504438+147.7695741*I,
39.91249957+148.8124572*I,
40.14945842+149.8541810*I,
40.38592643+150.8947575*I,
40.62190899+151.9341982*I,
40.85741140+152.9725144*I,
41.09243886+154.0097174*I,
41.32699645+155.0458180*I,
41.56108917+156.0808271*I,
```

```
41.79472193+157.1147551*I,
42.02789954+158.1476126*I,
42.26062673+159.1794096*I,
42.49290818+160.2101563*I,
42.72474834+161.2398625*I,
42.95615179+162.2685378*I,
43.18712289+163.2961919*I,
43.41766597+164.3228340*I,
43.64778527+165.3484735*I,
43.87748496+166.3731193*I,
44.10676934+167.3967807*I,
44.33564183+168.4194654*I,
44.56410701+169.4411831*I,
44.79216856+170.4619420*I,
45.01983031+171.4817504*I,
45.24709604+172.5006166*I,
45.47396944+173.5185485*I,
45.70045415+174.5355542*I,
45.92655377+175.5516416*I,
46.15227182+176.5668183*I,
46.37761176+177.5810920*I,
46.60257702+178.5944702*I,
46.82717096+179.6069602*I,
47.05139687+180.6185692*I,
47.27525803+181.6293046*I,
47.49875761+182.6391732*I,
47.72189881+183.6481821*I,
47.94468472+184.6563382*I,
48.16711839+185.6636481*I,
48.38920286+186.6701186*I,
48.61094108+187.6757561*I,
48.83233598+188.6805672*I,
49.05339044+189.6845582*I,
49.27410732+190.6877354*I,
49.49448939+191.6901051*I,
49.71453943+192.6916732*I,
49.93426014+193.6924460*I
```

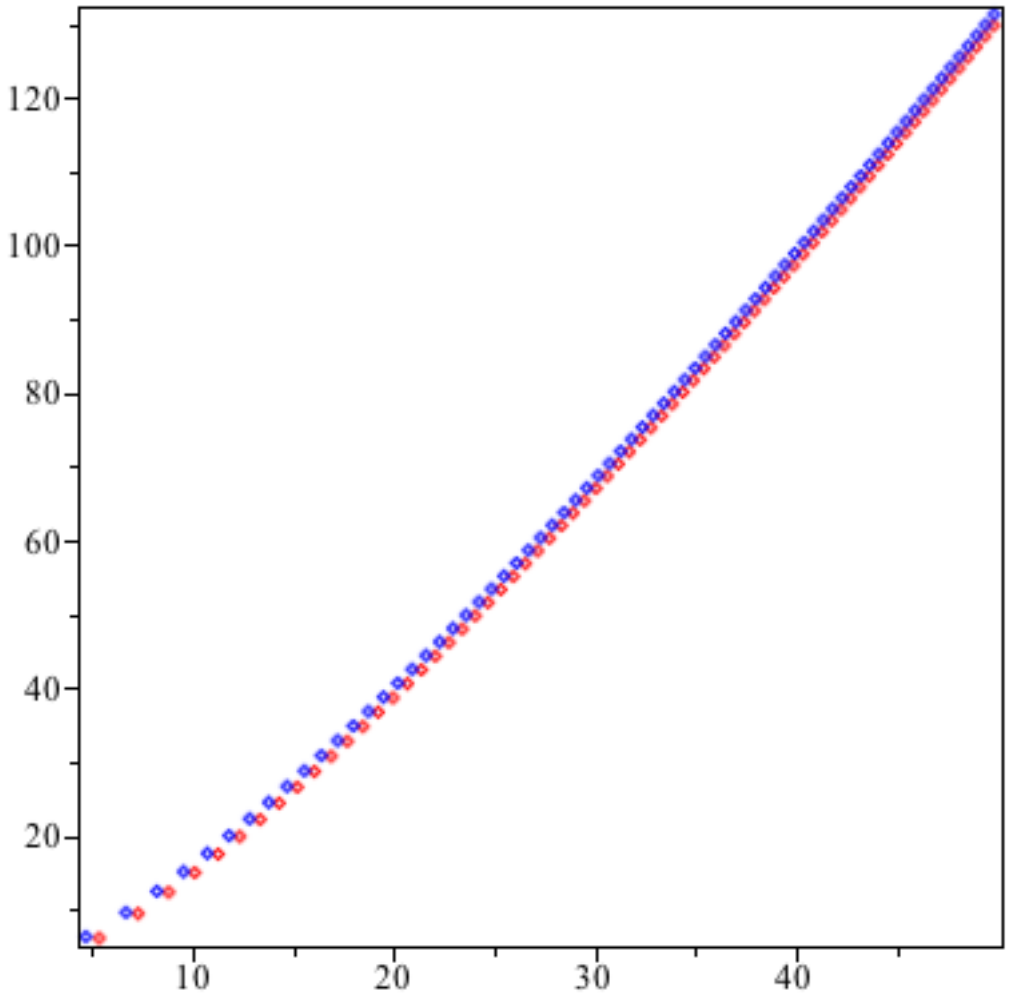


Figure 5: Zeros of $\Gamma(z, 2.5)$ in red and of its derivative in blue

Zeros of $\Gamma(z, 2.5)$ in rectangle with corners (0,0), (50,250).

```
5.336520625+6.105954720*I,
7.278037125+9.433072415*I,
8.791805325+12.32873262*I,
10.09279767+14.98879263*I,
11.26029108+17.49212126*I,
12.33389394+19.88044659*I,
13.33675256+22.17920907*I,
14.28378130+24.40535896*I,
15.18525827+26.57090361*I,
16.04862413+28.68474050*I,
16.87947200+30.75369458*I,
17.68213356+32.78314584*I,
18.46004534+34.77743088*I,
19.21598885+36.74011005*I,
19.95225355+38.67415217*I,
20.67075111+40.58206581*I,
21.37309765+42.46599502*I,
22.06067424+44.32779077*I,
22.73467238+46.16906508*I,
23.39612866+47.99123289*I,
24.04595174+49.79554487*I,
24.68494349+51.58311340*I,
25.31381594+53.35493350*I,
25.93320476+55.11189985*I,
26.54368038+56.85482055*I,
27.14575703+58.58442885*I,
27.73990026+60.30139255*I,
28.32653317+62.00632205*I,
28.90604170+63.69977735*I,
29.47877908+65.38227370*I,
30.04506957+67.05428640*I,
30.60521175+68.71625540*I,
31.15948129+70.36858875*I,
31.70813332+72.01166595*I,
32.25140460+73.64584055*I,
32.78951524+75.27144275*I,
33.32267039+76.88878165*I,
33.85106160+78.49814690*I,
34.37486806+80.09981060*I,
34.89425768+81.69402860*I,
35.40938814+83.28104210*I,
```

```
35.92040765+84.86107865*I,
36.42745580+86.43435325*I,
36.93066425+88.00106945*I,
37.43015731+89.56141995*I,
37.92605256+91.11558770*I,
38.41846132+92.66374640*I,
38.90748911+94.20606125*I,
39.39323608+95.74268950*I,
39.87579740+97.27378100*I,
40.35526358+98.79947865*I,
40.83172079+100.3199189*I,
41.30525117+101.8352322*I,
41.77593306+103.3455432*I,
42.24384112+104.8509714*I,
42.70904728+106.3516311*I,
43.17161948+107.8476320*I,
43.63162330+109.3390795*I,
44.08912142+110.8260745*I,
44.54417405+112.3087142*I,
44.99683874+113.7870920*I,
45.44717089+115.2612977*I,
45.89522370+116.7314179*I,
46.34104828+118.1975360*I,
46.78469382+119.6597323*I,
47.22620768+121.1180844*I,
47.66563542+122.5726670*I,
48.10302096+124.0235525*I,
48.53840687+125.4708105*I,
48.97183378+126.9145087*I,
49.40334123+128.3547122*I,
49.83296732+129.7914842*I
```

Zeros of the derivative of $\Gamma(z, 2.5)$ in rectangle with corners (0,0), (50,250).

```
4.673138861+6.313713520*I,
6.669823675+9.592500780*I,
8.213885745+12.46795224*I,
9.535105445+15.11617279*I,
10.71753639+17.61144725*I,
11.80286092+19.99380937*I,
12.81529823+22.28790950*I,
13.77038452+24.51027300*I,
14.67878843+26.67265548*I,
```

```
15.54821071+28.78379435*I,
16.38442603+30.85040699*I,
17.19189729+32.87779833*I,
17.97415889+34.87025042*I,
18.73406707+36.83128303*I,
19.47396960+38.76383400*I,
20.19582448+40.67038775*I,
20.90128516+42.55306926*I,
21.59176326+44.41371410*I,
22.26847552+46.25392176*I,
22.93247966+48.07509688*I,
23.58470216+49.87848147*I,
24.22596008+51.66518070*I,
24.85697846+53.43618345*I,
25.47840423+55.19237905*I,
26.09081759+56.93457120*I,
26.69474133+58.66348915*I,
27.29064851+60.37979725*I,
27.87896889+62.08410305*I,
28.46009428+63.77696380*I,
29.03438323+65.45889225*I,
29.60216466+67.13036190*I,
30.16374139+68.79181075*I,
30.71939290+70.44364515*I,
31.26937779+72.08624310*I,
31.81393592+73.71995685*I,
32.35329027+75.34511540*I,
32.88764858+76.96202665*I,
33.41720477+78.57097930*I,
33.94214021+80.17224450*I,
34.46262483+81.76607735*I,
34.97881813+83.35271820*I,
35.49087003+84.93239385*I,
35.99892171+86.50531875*I,
36.50310626+88.07169575*I,
37.00354937+89.63171710*I,
37.50036986+91.18556520*I,
37.99368022+92.73341325*I,
38.48358707+94.27542600*I,
38.97019159+95.81176030*I,
39.45358989+97.34256560*I,
39.93387337+98.86798445*I,
```

```
40.41112906+100.3881530*I,
40.88543957+101.9032012*I,
41.35688486+103.4132538*I,
41.82553959+104.9184297*I,
42.29147618+106.4188430*I,
42.75476359+107.9146032*I,
43.21546790+109.4058154*I,
43.67365235+110.8925804*I,
44.12937755+112.3749950*I,
44.58270168+113.8531527*I,
45.03368058+115.3271429*I,
45.48236786+116.7970521*I,
45.92881508+118.2629634*I,
46.37307184+119.7249572*I,
46.81518586+121.1831107*I,
47.25520309+122.6374986*I,
47.69316783+124.0881931*I,
48.12912278+125.5352638*I,
48.56310912+126.9787781*I,
48.99516661+128.4188011*I,
49.42533364+129.8553959*I,
49.85364731+131.2886235*I
```

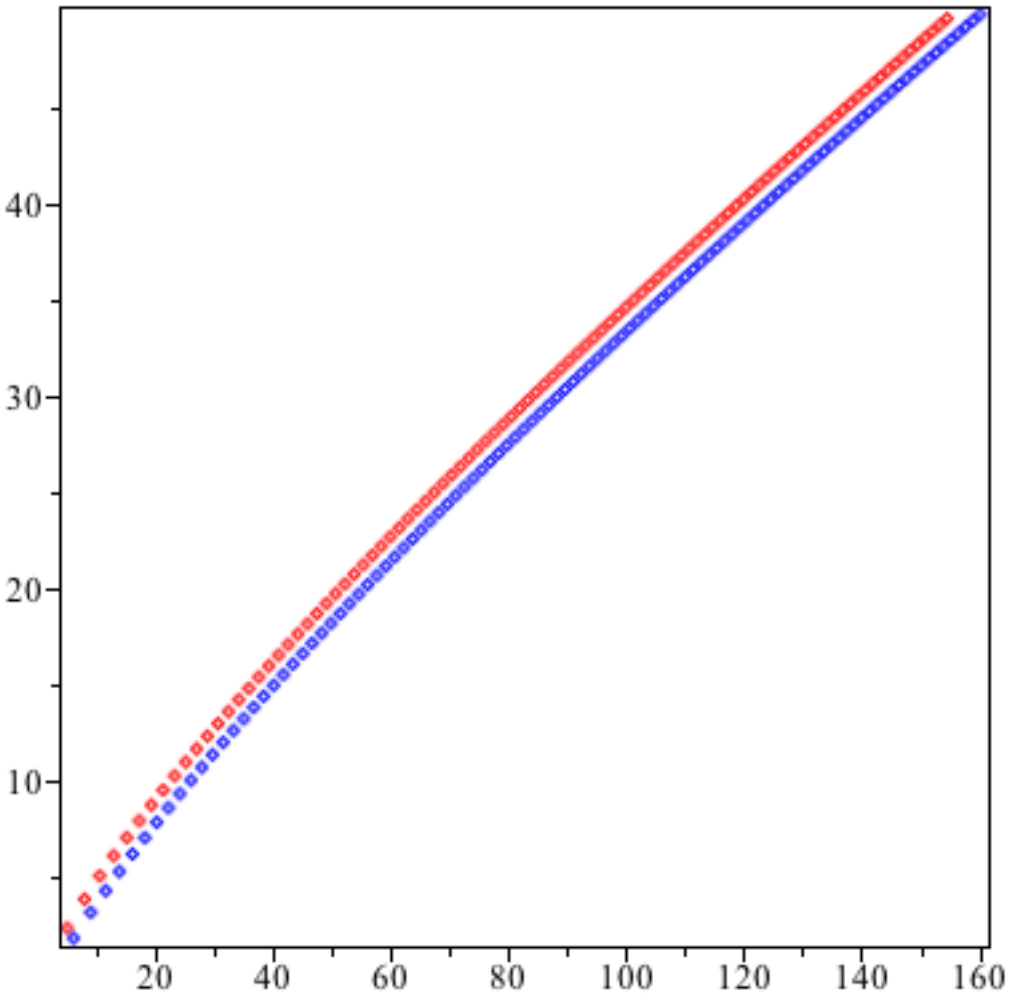


Figure 6: Zeros of $G(z; 1.5, 0.5)$ in red and of its derivative in blue

Zeros of $G(z; 1.5, 0.5)$ in rectangle with corners (0.01,0), (250,50).

```
4.930509536+2.344806407*I,
7.887533670+3.853753372*I,
10.45799459+5.059539400*I,
12.82456329+6.107220345*I,
15.05699185+7.054072105*I,
17.19142967+7.929322735*I,
19.24969926+8.750257600*I,
21.24626190+9.528105720*I,
23.19132932+10.27065308*I,
25.09245398+10.98356359*I,
26.95542222+11.67111202*I,
28.78479237+12.33662021*I,
30.58423722+12.98273137*I,
32.35677172+13.61159040*I,
34.10490997+14.22496688*I,
35.83077611+14.82434137*I,
37.53618656+15.41096800*I,
39.22270938+15.98592028*I,
40.89171068+16.55012582*I,
42.54438990+17.10439286*I,
44.18180755+17.64943082*I,
45.80490719+18.18586659*I,
47.41453313+18.71425743*I,
49.01144474+19.23510146*I,
50.59632825+19.74884611*I,
52.16980640+20.25589515*I,
53.73244660+20.75661448*I,
55.28476775+21.25133691*I,
56.82724590+21.74036634*I,
58.36031930+22.22398107*I,
59.88439240+22.70243686*I,
61.39983960+23.17596932*I,
62.90700830+23.64479618*I,
64.40622170+24.10911908*I,
65.89778095+24.56912526*I,
67.38196751+25.02498893*I,
68.85904480+25.47687253*I,
70.32925980+25.92492783*I,
71.79284455+26.36929691*I,
73.25001740+26.81011298*I,
74.70098415+27.24750116*I,
```

```
76.14593905+27.68157917*I,
77.58506570+28.11245793*I,
79.01853785+28.54024208*I,
80.44652025+28.96503051*I,
81.86916920+29.38691676*I,
83.28663320+29.80598946*I,
84.69905340+30.22233266*I,
86.10656430+30.63602615*I,
87.50929410+31.04714579*I,
88.90736505+31.45576376*I,
90.30089390+31.86194880*I,
91.68999240+32.26576642*I,
93.07476715+32.66727936*I,
94.45532055+33.06654715*I,
95.83175050+33.46362700*I,
97.20415105+33.85857357*I,
98.57261230+34.25143917*I,
99.93722095+34.64227396*I,
101.2980602+35.03112606*I,
102.6552102+35.41804158*I,
104.0087480+35.80306488*I,
105.3587478+36.18623856*I,
106.7052812+36.56760357*I,
108.0484170+36.94719934*I,
109.3882219+37.32506382*I,
110.7247600+37.70123358*I,
112.0580934+38.07574386*I,
113.3882821+38.44862867*I,
114.7153839+38.81992080*I,
116.0394550+39.18965193*I,
117.3605496+39.55785268*I,
118.6787202+39.92455260*I,
119.9940177+40.28978030*I,
121.3064914+40.65356344*I,
122.6161892+41.01592880*I,
123.9231574+41.37690230*I,
125.2274409+41.73650905*I,
126.5290835+42.09477340*I,
127.8281275+42.45171894*I,
129.1246142+42.80736855*I,
130.4185837+43.16174445*I,
131.7100747+43.51486818*I,
```

```
132.9991252+43.86676066*I,
134.2857719+44.21744224*I,
135.5700508+44.56693264*I,
136.8519966+44.91525110*I,
138.1316434+45.26241624*I,
139.4090242+45.60844625*I,
140.6841713+45.95335878*I,
141.9571160+46.29717104*I,
143.2278890+46.63989974*I,
144.4965202+46.98156121*I,
145.7630387+47.32217132*I,
147.0274729+47.66174552*I,
148.2898506+48.00029894*I,
149.5501988+48.33784625*I,
150.8085440+48.67440179*I,
152.0649120+49.00997956*I,
153.3193277+49.34459283*I,
154.5718170+49.67825642*I
```

Zeros of the derivative of $G(z; 1.5, 0.5)$ in rectangle with corners (0.01,0), (250,50).

```
6.012778145556205933700285+1.816685806631677557038486*I,
8.946717135741403483682750+3.163596264551669699246458*I,
11.482738612173969012015 04+4.288287138232281657955343*I,
13.819611724874590472236 29+5.284827502352804202776070*I,
16.027025092502362142060 33+6.195739474729819398477650*I,
18.140162449589581259117 35+7.043994725715278238941665*I,
20.180025654587641719767 76+7.843714560006929985126650*I,
22.160462710812055293712 05+8.604334340696817400336300*I,
24.091232945133786530787 72+9.332538027267450402140415*I,
25.979551216784515147345 28+10.033270563974575057688 98*I,
27.830947525599485806224 20+10.710315243722586320458 82*I,
29.649782085393580356968 80+11.366645152100998337809 46*I,
31.439571969703503043495 15+12.004648342535414958800 11*I,
33.203207828728123529828 00+12.626278213129474359803 60*I,
34.943103067382578085545 44+13.233157430027269227256 22*I,
36.661299728684005790063 83+13.826651845971652277334 77*I,
38.359545620998423229366 89+14.407924385289014662761 14*I,
40.039351761116262628116 08+14.977975167742317941931 92*I,
41.702035989153481545563 88+15.537671944408331134288 93*I,
43.348756647547834264454 98+16.087773564670238314409 36*I,
44.980538978055537602232 54+16.628948334038254611445 36*I,
46.598296087389468087887 09+17.161788562328890639520 27*I,
```

48.20284579798027217012371+17.68682222782857500642458*I,
49.79492433718799601975071+18.20452242815975116625877*I,
51.37519756645970452357507+18.71531511144842653948631*I,
52.94427027414996785982150+19.21958545618855566001801*I,
54.50269392817577390549750+19.71768317829972582969224*I,
56.05097319181600640806645+20.20992697839793528873712*I,
57.58957143744156323064610+20.69660829398262338841791*I,
59.11891544176952015268980+21.17799448515780495417718*I,
60.63939940755295748879625+21.65433155525388848236023*I,
62.15138842708114010156785+22.12584648692412309466583*I,
63.65522148009011362839455+22.59274925827263470987934*I,
65.15121404096363928529215+23.05523459112255162048668*I,
66.63966035620018527158970+23.51348347377703164389487*I,
68.12083544212535867959340+23.96766449292094118713877*I,
69.59499684406812962736870+24.41793500318093473247642*I,
71.06238619119056202149377+24.86444215795124008624277*I,
72.52323057548510767001445+25.30732382113348166387716*I,
73.97774377884218393306125+25.74670937622752716004104*I,
75.42612736832238760058393+26.18272044659041718786884*I,
76.86857167667129039450363+26.61547153853135105447311*I,
78.30525668255726569667285+27.04507061713872313597724*I,
79.73635280289000493881690+27.47161962326699189110154*I,
81.16202160780716123246123+27.89521493888897099436676*I,
82.58241646743378095841005+28.31594780699722132327388*I,
83.99768313827191585979835+28.73390471138016168261600*I,
85.40796029602440739564515+29.14916772087512947664328*I,
86.81338002076374110656940+29.56181480208842518585317*I,
88.21406823959689984198260+29.97192010405236966916714*I,
89.61014513132818180115610+30.37955421784615250390246*I,
91.00172549706591182919720+30.78478441382814982031835*I,
92.38891910024101899961130+31.18767485880210894650552*I,
93.77183097909334422926893+31.58828681515962882080351*I,
95.15056173432515186221565+31.98667882379967975931986*I,
96.52520779431223600661370+32.38290687241667186072783*I,
97.89586165999422134214660+32.77702455056694229030262*I,
99.26261213133128987611460+33.16908319276543069082570*I,
100.62554451700969218140800+33.55913201072634761562455*I,
101.98474082889887363686010+33.94721821574095231254888*I,
103.34027996260535639445620+34.33338713207972966462572*I,
104.69223786532969438392380+34.71768230221324895960828*I,
106.04068769211033575768990+35.10014558456406576494808*I,
107.38569951429931903440300+35.48081724442971699429344*I,

108.72734264106368456839 98+35.859736038652899008922 56*I,
110.06568137496414613075 19+36.236939294558231796031 57*I,
111.40077950190112820766 18+36.612462983624674573660 60*I,
112.73269821650786422639 18+36.986341790317874706846 66*I,
114.06149666332430546433 06+37.358609176466817390582 68*I,
115.38723203437453626090 64+37.729297441533506162875 92*I,
116.70995966076701723453 97+38.098437779092531328225 91*I,
118.02973309876304896203 83+38.466060329808829937639 46*I,
119.34660421071992104429 64+38.832194231176320221026 20*I,
120.66062324128017396135 88+39.196867664257068183760 92*I,
121.97183888914681746687 41+39.560107897639920110724 19*I,
123.28029837475583741659 54+39.921941328818854463526 34*I,
124.58604750413154754375 64+40.282393523174444416626 60*I,
125.88913072918700895395 46+40.641489250726579883348 60*I,
127.18959120471058832717 72+40.999252520812801017681 49*I,
128.48747084226052686150 59+41.355706614834090246716 56*I,
129.78281036117194337667 64+41.710874117198621325882 27*I,
131.07564933686481746353 29+42.064776944583651867085 60*I,
132.36602624662703318812 03+42.417436373626364147181 47*I,
133.65397851303336950891 60+42.768873067145913648318 17*I,
134.93954254514927473373 46+43.119107098991152123463 68*I,
136.22275377765724729880 88+43.468157977601377667769 76*I,
137.50364670803356368030 05+43.816044668360962078533 64*I,
138.78225493189385962875 72+44.162785614822756623812 54*I,
140.05861117661759730458 61+44.508398758869728439555 88*I,
141.33274733335367782453 51+44.852901559879283935297 31*I,
142.60469448750230921872 81+45.196311012950150534148 79*I,
143.87448294776166728105 26+45.538643666247475893932 40*I,
145.14214227382183344602 80+45.879915637517930414156 77*I,
146.40770130278291550744 52+46.220142629823033728631 72*I,
147.67118817436911132301 18+46.559339946535641474906 47*I,
148.93263035500572490479 98+46.897522505641500102611 39*I,
150.19205466082175436277 05+47.234704853384982445575 97*I,
151.44948727963661120181 12+47.570901177295535018405 46*I,
152.70495379198577255091 98+47.906125318628981227105 46*I,
153.95847919123668672680 00+48.240390784255616360549 24*I,
155.21008790284302515486 46+48.573710758024985365166 22*I,
156.45980380278237925180 21+48.906098111635339404130 76*I,
157.70765023521972047722 91+49.237565415034009722090 18*I,
158.95365002943635815803 14+49.568124946373306160903 33*I,
160.19782551606172718794 64+49.897788701545032598373 16*I

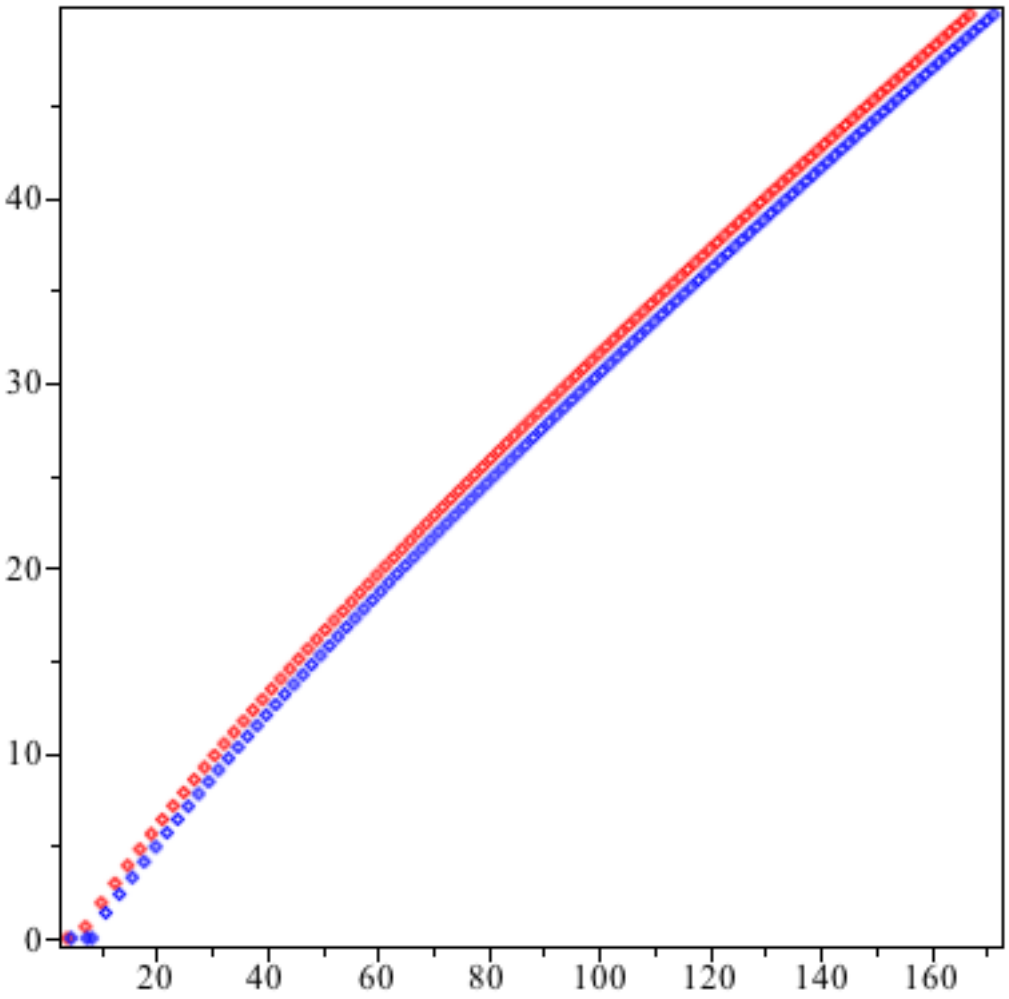


Figure 7: Zeros of $G(z; 1.5, 1.5)$ in red and of its derivative in blue

Zeros of $G(z; 1.5, 1.5)$ in rectangle with corners (0.01,0), (250,50).

```
3.937905687,
7.251513627+.6107439837*I,
10.12360981+1.903711076*I,
12.59124934+2.964286416*I,
14.87966733+3.919174854*I,
17.04985809+4.800045080*I,
19.13298468+5.625227920*I,
21.14784064+6.406456279*I,
23.10693326+7.151792462*I,
25.01914884+7.867066170*I,
26.89110279+8.556659971*I,
28.72789672+9.223965825*I,
30.53357592+9.871677515*I,
32.31141650+10.50197919*I,
34.06412296+11.11666447*I,
35.79396034+11.71723378*I,
37.50285552+12.30496601*I,
39.19245796+12.88093626*I,
40.86419832+13.44609427*I,
42.51932997+14.00124869*I,
44.15895444+14.54711765*I,
45.78405010+15.08433859*I,
47.39548780+15.61347150*I,
48.99405218+16.13501372*I,
50.58045060+16.64942390*I,
52.15531945+17.15710595*I,
53.71923580+17.65842243*I,
55.27274135+18.15371994*I,
56.81631710+18.64329210*I,
58.35041105+19.12743180*I,
59.87543670+19.60639366*I,
61.39176620+20.08042226*I,
62.89977160+20.54972471*I,
64.39974665+21.01448388*I,
65.89203190+21.47492077*I,
67.37690485+21.93120568*I,
68.85461480+22.38349109*I,
70.32544330+22.83194031*I,
71.78959795+23.27668059*I,
73.24731060+23.71787799*I,
74.69879350+24.15561319*I,
```

```
76.14424355+24.59004556*I,
77.58382005+25.02125740*I,
79.01771980+25.44936066*I,
80.44612175+25.87445412*I,
81.86918835+26.29667290*I,
83.28702460+26.71602009*I,
84.69979655+27.13264381*I,
86.10764730+27.54666567*I,
87.51071435+27.95804704*I,
88.90909710+28.36695130*I,
90.30292400+28.77339900*I,
91.69230585+29.17745205*I,
93.07736820+29.57923665*I,
94.45817960+29.97874025*I,
95.83488500+30.37607838*I,
97.20750330+30.77120792*I,
98.57619880+31.16435306*I,
99.94104950+31.55541870*I,
101.3020818+31.94447933*I,
102.6594611+32.33164194*I,
104.0131896+32.71684608*I,
105.3633836+33.10022927*I,
106.7101026+33.48179934*I,
108.0534227+33.86157431*I,
109.3934006+34.23971774*I,
110.7301175+34.61605250*I,
112.0636115+34.99077168*I,
113.3939414+35.36381285*I,
114.7212276+35.73529304*I,
116.0454060+36.10514746*I,
117.3666622+36.47358976*I,
118.6849925+36.84041339*I,
120.0004036+37.20590617*I,
121.3129930+37.56980410*I,
122.6228475+37.93230992*I,
123.9299068+38.29342338*I,
125.2342994+38.65318830*I,
126.5360900+39.01167600*I,
127.8352154+39.36874859*I,
129.1318420+39.72450698*I,
130.4258786+40.07901733*I,
131.7174605+40.43236320*I,
```

```
133.0066207+40.78437228*I,
134.2933320+41.13518979*I,
135.5777499+41.48485416*I,
136.8598452+41.83351800*I,
138.1395566+42.18072661*I,
139.4169953+42.52690826*I,
140.6921776+42.87187780*I,
141.9652842+43.21583448*I,
143.2360706+43.55864397*I,
144.5048026+43.90042138*I,
145.7714049+44.24125044*I,
147.0358727+44.58076756*I,
148.2983600+44.91953434*I,
149.5587354+45.25722820*I,
150.8171880+45.59401438*I,
152.0736012+45.92968392*I,
153.3280566+46.26424587*I,
154.5806022+46.59795682*I,
155.8313212+46.93092457*I,
157.0800599+47.26281420*I,
158.3269492+47.59376888*I,
159.5719638+47.92385924*I,
160.8152711+48.25319042*I,
162.0566793+48.58140666*I,
163.2965010+48.90895708*I,
164.5343684+49.23573476*I,
165.7705745+49.56130794*I,
167.0051230+49.88623883*I
```

Zeros of the derivative of $G(z; 1.5, 1.5)$ in rectangle with corners (0.01,0), (250,50).

```
4.667700946,
7.483808310,
8.407400255,
10.91195567+1.365745584*I,
13.39023297+2.369900146*I,
15.67830509+3.283486490*I,
17.84460858+4.132243178*I,
19.92240068+4.931563363*I,
21.93137142+5.691387905*I,
23.88444642+6.418626360*I,
25.79071631+7.118334843*I,
27.65689195+7.794350635*I,
```

```
29.48811465+8.449671525*I,
31.28846455+9.086710847*I,
33.06119737+9.707415100*I,
34.80902847+10.31341524*I,
36.53420862+10.90607470*I,
38.23865246+11.48655147*I,
39.92399502+12.05585095*I,
41.59165762+12.61482951*I,
43.24285997+13.16425792*I,
44.87876304+13.70478333*I,
46.50024609+14.23701387*I,
48.10821144+14.76146515*I,
49.70345400+15.27859999*I,
51.286657495517458164536 20+15.788865302981448470013 86*I,
52.858434893165702524531 95+16.292626085686756233324 08*I,
54.419373027093572762886 45+16.790235923740164760394 30*I,
55.969996365503514714331 25+17.282012065136097203316 54*I,
57.510785404452737165561 80+17.768244615352080463899 30*I,
59.042181804229404869647 15+18.249199712882106369967 78*I,
60.564592767566542647700 10+18.725122240121515226572 22*I,
62.078394793454821487354 40+19.196238150352661646806 48*I,
63.583936913103016593906 35+19.662756475465038462541 89*I,
65.081543493608487887369 05+20.124871066535356812110 41*I,
66.571516678566763914284 35+20.582762109602258410680 42*I,
68.054138522029012889267 90+21.036597451245654969420 59*I,
69.529672862073118720328 90+21.486533762440799798577 54*I,
70.998366972169344958602 65+21.932717564242894213499 42*I,
72.460453022032423063919 30+22.375286134898854756460 52*I,
73.916149374409229648510 00+22.814368314773453383923 34*I,
75.365661739989397566352 80+23.250085222860218924382 54*I,
76.809184209141524418552 80+23.682550896502061840820 66*I,
78.246900176312689286644 95+24.111872864178444927989 96*I,
79.678983170561268326292 45+24.538152659751499552057 57*I,
81.105597603726585127630 50+24.961486285344901786063 65*I,
82.526899446098079775230 90+25.381964629010767026604 34*I,
83.943036838071416697295 65+25.799673842484832553572 00*I,
85.354150645121399380520 60+26.214695683609592631158 54*I,
86.760374962443318409060 20+26.627107827395353233115 90*I,
88.161837574784466019140 30+27.036984149171402707175 12*I,
89.558660376280917853204 60+27.444394982838232594367 00*I,
90.950959754510886849175 75+27.849407356854408454927 27*I,
92.338846942458292668242 40+28.252085210267973615516 11*I,
```

93.722428341634944435017 90+28.652489590823684616944 40*I,
95.101805819225590870645 30+29.050678836936915871738 12*I,
96.477076981787682077448 00+29.446708745116918891236 98*I,
97.848335427749224834105 80+29.840632724241439068932 13*I,
99.215670980697102797903 30+30.232501937927437986623 12*I,
100.57916990522923684933 73+30.622365436105453240186 98*I,
101.938915106952411495891 0+31.010270276785101720235 70*I,
103.294986318039641753448 2+31.396261638893992624434 42*I,
104.647460269613347532606 0+31.780382926979829674849 10*I,
105.996410852090580391120 4+32.162675868484022350325 68*I,
107.341909264511764574788 0+32.543180604223224912951 06*I,
108.684024153772869946717 3+32.921935772651626451525 08*I,
110.022821744590918019053 6+33.298978588420452997240 20*I,
111.358365960952767308739 7+33.674344915701094339999 83*I,
112.690718539725970095757 8+34.048069336693742758032 67*I,
114.019939137047053312728 4+34.420185215703746893806 91*I,
115.346085428045913786721 2+34.790724759132452686058 17*I,
116.669213200414320422063 1+35.159719071697614668953 64*I,
117.989376442281078750681 5+35.527198209170072838315 32*I,
119.306627424815624180201 0+35.893191227887917743877 14*I,
120.621016779945135570319 2+36.257726231286474474350 85*I,
121.932593573537234724596 5+36.620830413661831947591 31*I,
123.241405374370552380581 2+36.982530101367071419290 95*I,
124.547498319188540281995 6+37.342850791623583008828 97*I,
125.850917174107578275208 7+37.701817189114704222219 56*I,
127.151705392628389195832 3+38.059453240515196722151 69*I,
128.449905170479789620264 8+38.415782167097644148161 02*I,
129.745557497505656677646 8+38.770826495545569968175 93*I,
131.038702206789490383823 4+39.124608087092820905103 01*I,
132.329378021195929332564 4+39.477148165099433020499 92*I,
133.617622597494886047876 2+39.828467341165700415968 38*I,
134.903472568221474431309 6+40.178585639878417717454 30*I,
136.186963581413487720513 5+40.527522522276193072147 98*I,
137.468130338357746363378 2+40.875296908114262474773 82*I,
138.747006629467077338391 6+41.221927197003320178622 31*I,
140.023625368400926275346 8+41.567431288491461218174 02*I,
141.298018624534566200483 3+41.911826601153363743222 21*I,
142.570217653874484819539 4+42.255130090746279036982 67*I,
143.840252928510745820898 0+42.597358267488207851824 81*I,
145.108154164690874676087 7+42.938527212509789502883 29*I,
146.373950349594067040466 8+43.278652593527884288212 95*I,
147.637669766879214745676 6+43.617749679785563129758 85*I,

148.89934002107535059400000+43.95583335630020603850000*I,
150.15898806087859342312480+44.29291813745863085439712*I,
151.41664020141549700964700+44.62901817999560569560898*I,
152.67232214552884098274680+44.96414729538972435525843*I,
153.92605900413832301006300+45.29831896170842719496580*I,
155.17787531572529492493000+45.63154633493191587742303*I,
156.42779506498760963493760+45.96384225978382519807672*I,
157.67584170070779096355000+46.29521928009476681748100*I,
158.92203815287508695630460+46.62568964872323698477350*I,
160.16640684909950250134700+46.95526533705687297550100*I,
161.40896973035361439190080+47.28395804411564192599530*I,
162.64974826607583784228950+47.61177920527724245180516*I,
163.88876346866682541563150+47.93874000064378627306224*I,
165.12603590740882612455200+48.26485136306769710025070*I,
166.36158572183610393951450+48.59012398585371068732322*I,
167.59543263458290134483540+48.91456833015287756228378*I,
168.82759596373392671667100+49.23819463206355295688505*I,
170.05809463470093562109600+49.56101290945350215429520*I,
171.28694719164765919824300+49.88303296851644914829712*I

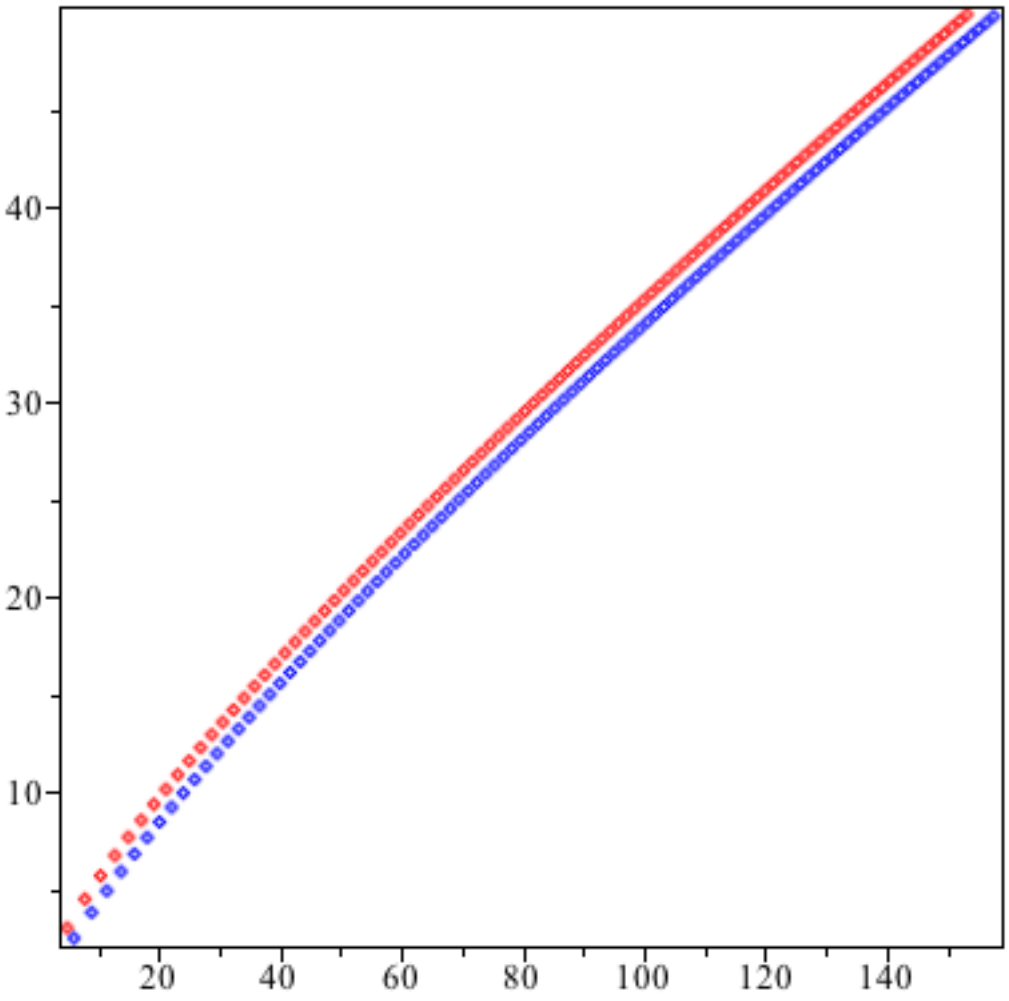


Figure 8: Zeros of $G(z; 1.5, 0)$ in red and of its derivative in blue

Zeros of $G(z; 1.5, 0)$ in rectangle with corners (0.01,0), (250,50).

```
4.918293784+3.035804313*I,
7.855077370+4.521631898*I,
10.42198413+5.713456405*I,
12.78793500+6.752038915*I,
15.02057324+7.692317805*I,
17.15549805+8.562501560*I,
19.21434510+9.379356290*I,
21.21150298+10.15381474*I,
23.15715338+10.89347941*I,
25.05883645+11.60389345*I,
26.92233411+12.28924827*I,
28.75220392+12.95280594*I,
30.55211959+13.59716552*I,
32.32509771+14.22443843*I,
34.07365422+14.83636825*I,
35.79991540+15.43441520*I,
37.50569934+16.01981687*I,
39.19257591+16.59363346*I,
40.86191282+17.15678160*I,
42.51491098+17.71006039*I,
44.15263221+18.25417156*I,
45.77602127+18.78973548*I,
47.38592353+19.31730387*I,
48.98309935+19.83737005*I,
50.56823580+20.35037729*I,
52.14195645+20.85672577*I,
53.70482945+21.35677819*I,
55.25737430+21.85086461*I,
56.80006770+22.33928643*I,
58.33334835+22.82231978*I,
59.85762130+23.30021842*I,
61.37326135+23.77321625*I,
62.88061630+24.24152940*I,
64.38000975+24.70535809*I,
65.87174325+25.16488828*I,
67.35609855+25.62029299*I,
68.83333930+26.07173359*I,
70.30371280+26.51936095*I,
71.76745140+26.96331618*I,
73.22477365+27.40373170*I,
74.67588560+27.84073190*I,
```

```
76.12098165+28.27443380*I,
77.56024560+28.70494767*I,
78.99385145+29.13237758*I,
80.42196400+29.55682187*I,
81.84473980+29.97837358*I,
83.26232740+30.39712081*I,
84.67486825+30.81314731*I,
86.08249685+31.22653234*I,
87.48534155+31.63735142*I,
88.88352475+32.04567636*I,
90.27716325+32.45157560*I,
91.66636890+32.85511436*I,
93.05124850+33.25635491*I,
94.43190445+33.65535672*I,
95.80843475+34.05217664*I,
97.18093350+34.44686910*I,
98.54949095+34.83948620*I,
99.91419380+35.23007788*I,
101.2751255+35.61869198*I,
102.6323660+36.00537451*I,
103.9859925+36.39016961*I,
105.3360793+36.77311970*I,
106.6826981+37.15426557*I,
108.0259177+37.53364649*I,
109.3658048+37.91130028*I,
110.7024237+38.28726334*I,
112.0358364+38.66157079*I,
113.3661030+39.03425650*I,
114.6932814+39.40535317*I,
116.0174277+39.77489234*I,
117.3385963+40.14290450*I,
118.6568396+40.50941912*I,
119.9722086+40.87446470*I,
121.2847527+41.23806877*I,
122.5945197+41.60025811*I,
123.9015560+41.96105847*I,
125.2059066+42.32049490*I,
126.5076152+42.67859166*I,
127.8067242+43.03537226*I,
129.1032750+43.39085953*I,
130.3973074+43.74507558*I,
131.6888605+44.09804192*I,
```

```
132.9779722+44.44977938*I,
134.2646792+44.80030820*I,
135.5490176+45.14964816*I,
136.8310220+45.49781836*I,
138.1107266+45.84483738*I,
139.3881644+46.19072333*I,
140.6633676+46.53549383*I,
141.9363678+46.87916602*I,
143.2071955+47.22175660*I,
144.4758806+47.56328180*I,
145.7424523+47.90375750*I,
147.0069390+48.24319908*I,
148.2693686+48.58162159*I,
149.5297680+48.91903973*I,
150.7881637+49.25546777*I,
152.0445816+49.59091964*I,
153.2990470+49.92540899*I
```

Zeros of the derivative of $G(z; 1.5, 0)$ in rectangle with corners (0.01,0), (250,50).

```
6.055175040+2.509891054*I,
8.941093605+3.831161188*I,
11.46529262+4.941189375*I,
13.79725980+5.928524430*I,
16.00222937+6.832893300*I,
18.11404312+7.676147400*I,
20.15316722+8.471859295*I,
22.13319801+9.229158615*I,
24.06376327+9.954542980*I,
25.95200354+10.65283606*I,
27.80340436+11.32773810*I,
29.62229812+11.98216304*I,
31.41218386+12.61845545*I,
33.17594023+13.23853581*I,
34.91597245+13.84400140*I,
36.63431684+14.43619804*I,
38.33271724+15.01627269*I,
40.01268180+15.58521243*I,
41.67552630+16.14387433*I,
43.32240764+16.69300834*I,
44.95434998+17.23327532*I,
46.57226566+17.76526120*I,
48.17697191+18.28948864*I,
```

```
49.76920466+18.80642598*I,
51.34962947+19.31649513*I,
52.91885085+19.82007802*I,
54.47742030+20.31752098*I,
56.02584230+20.80914012*I,
57.56458030+21.29522450*I,
59.09406100+21.77603930*I,
60.61467870+22.25182864*I,
62.12679850+22.72281782*I,
63.63075950+23.18921523*I,
65.12687710+23.65121428*I,
66.61544565+24.10899466*I,
68.09674040+24.56272383*I,
69.57101880+25.01255810*I,
71.03852255+25.45864368*I,
72.49947890+25.90111761*I,
73.95410153+26.34010846*I,
75.40259250+26.77573704*I,
76.84514185+27.20811730*I,
78.28192965+27.63735652*I,
79.71312650+28.06355612*I,
81.13889390+28.48681190*I,
82.55938540+28.90721465*I,
83.97474680+29.32485036*I,
85.38511685+29.73980071*I,
86.79062760+30.15214324*I,
88.19140520+30.56195171*I,
89.58756970+30.96929642*I,
90.97923610+31.37424430*I,
92.36651410+31.77685914*I,
93.74950880+32.17720202*I,
95.12832095+32.57533115*I,
96.50304690+32.97130226*I,
97.87377925+33.36516875*I,
99.24060690+33.75698166*I,
100.6036151+34.14679006*I,
101.9628859+34.53464088*I,
103.3184983+34.92057934*I,
104.6705283+35.30464877*I,
106.0190490+35.68689078*I,
107.3641310+36.06734563*I,
108.7058422+36.44605182*I,
```

```
110.0442484+36.82304652*I,
111.3794130+37.19836562*I,
112.7113972+37.57204363*I,
114.0402600+37.94411392*I,
115.3660594+38.31460903*I,
116.6888488+38.68355887*I,
118.0086840+39.05099460*I,
119.3256157+39.41694492*I,
120.6396942+39.78143798*I,
121.9509690+40.14450082*I,
123.2594866+40.50615982*I,
124.5652930+40.86644042*I,
125.8684327+41.22536745*I,
127.1689488+41.58296456*I,
128.4668834+41.93925521*I,
129.7622771+42.29426182*I,
131.0551694+42.64800618*I,
132.3455992+43.00050960*I,
133.6336035+43.35179254*I,
134.9192188+43.70187516*I,
136.2024809+44.05077677*I,
137.4834238+44.39851648*I,
138.7620815+44.74511247*I,
140.0384865+45.09058274*I,
141.3126710+45.43494463*I,
142.5846657+45.77821518*I,
143.8545012+46.12041074*I,
145.1222070+46.46154758*I,
146.3878120+46.80164116*I,
147.6513442+47.14070692*I,
148.9128312+47.47875958*I,
150.1722998+47.81581368*I,
151.4297764+48.15188350*I,
152.6852862+48.48698274*I,
153.9388544+48.82112481*I,
155.1905054+49.15432294*I,
156.4402632+49.48658992*I,
157.6881510+49.81793834*I
```

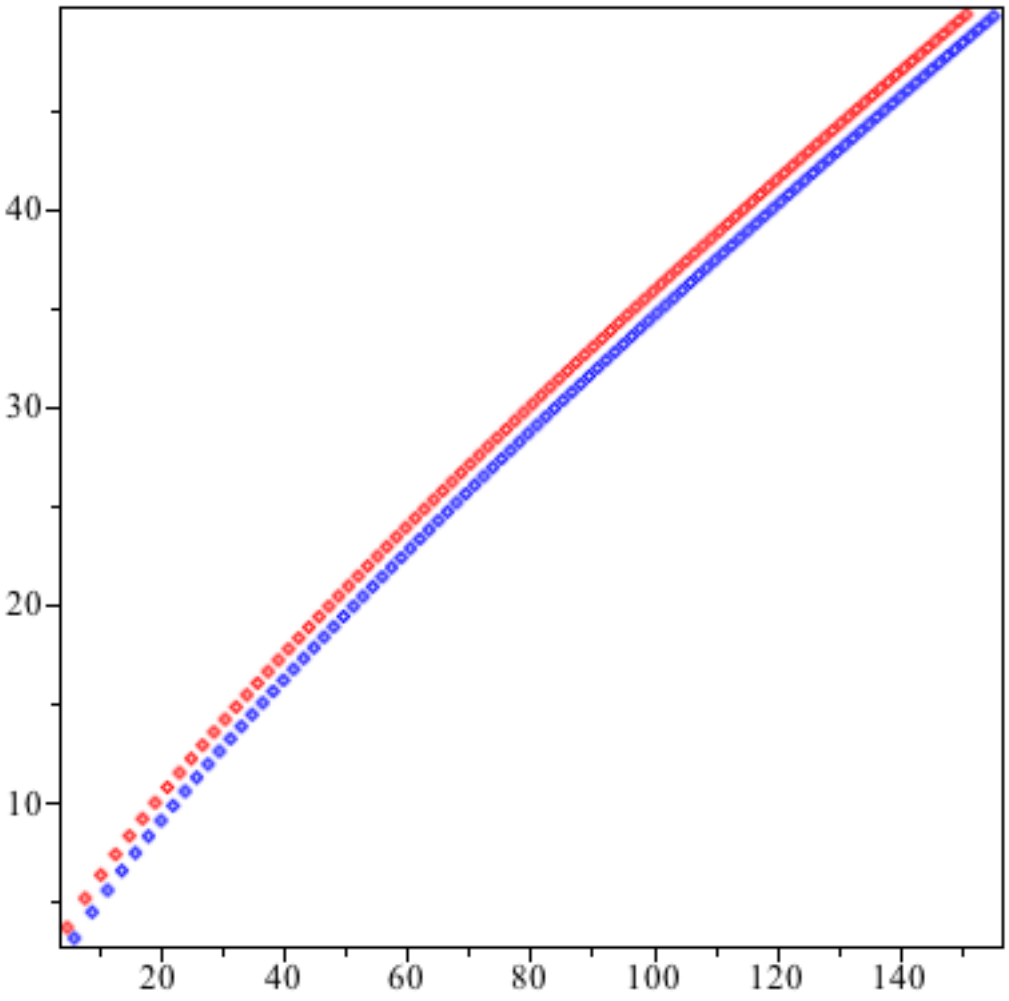


Figure 9: Zeros of $G(z; 1.5, -0.5)$ in red and of its derivative in blue

Zeros of $G(z; 1.5, -0.5)$ in rectangle with corners (0.01,0), (250,50).

4.91857154724811691011710 3+3.65424668283771192678198 4*I,
7.84189681485152979135410 0+5.13309042724547050961327 0*I,
10.40436799197687719272643+6.31828612464122302203307 0*I,
12.76846632387278753130505+7.35187526308279788077191 5*I,
15.00024993276077965837001+8.28828554187123317404448 5*I,
17.13478234398162827614478+9.15536213044309677408052 5*I,
19.19347599671980026620459+9.96964659628581804129338 0*I,
21.19061298518832014703360+10.74192894200378114915656*I,
23.13631921629433654103878+11.47971689491351098863834*I,
25.03810359424844276724184+12.18848807848128992246964*I,
26.90172962761340986521853+12.87238700647490326901014*I,
28.73174362271471160075646+13.53464119665221361804970*I,
30.53181220599742547077000+14.17782357344041854205977*I,
32.30494741769529281972066+14.80402539148552197593028*I,
34.05366220130566089322250+15.41497439967318570792248*I,
35.78008102725542222794108+16.01211807757056762187121*I,
37.48602059705773897407634+16.59668380397990930580197*I,
39.17305000730512853395031+17.16972333013752552123057*I,
40.84253645399869111189665+17.73214629550954965502210*I,
42.49568053194073053592385+18.28474592004035246371258*I,
44.13354390073242894703790+18.82821899868761900724375*I,
45.75707125366195385569459+19.36318167281124386090557*I,
47.36710796880176718860344+19.89018202174441988102998*I,
48.96441444215488402663527+20.40971022600187072796210*I,
50.54967785+20.92220685*I,
52.12352180+21.42806967*I,
53.68651460+21.92765929*I,
55.23917590+22.42130392*I,
56.78198245+22.90930331*I,
58.31537310+23.39193213*I,
59.83975305+23.86944282*I,
61.35549715+24.34206811*I,
62.86295345+24.81002306*I,
64.36244550+25.27350695*I,
65.85427505+25.73270484*I,
67.33872395+26.18778898*I,
68.81605590+26.63892003*I,
70.28651840+27.08624815*I,
71.75034380+27.52991390*I,
73.20775075+27.97004914*I,
74.65894535+28.40677772*I,

```
76.10412215+28.84021622*I,
77.54346500+29.27047446*I,
78.97714790+29.69765612*I,
80.40533585+30.12185914*I,
81.82818540+30.54317624*I,
83.24584515+30.96169524*I,
84.65845655+31.37749945*I,
86.06615425+31.79066798*I,
87.46906665+32.20127605*I,
88.86731615+32.60939523*I,
90.26101965+33.01509372*I,
91.65028895+33.41843652*I,
93.03523100+33.81948572*I,
94.41594815+34.21830057*I,
95.79253855+34.61493778*I,
97.16509625+35.00945158*I,
98.53371155+35.40189391*I,
99.89847125+35.79231455*I,
101.2594587+36.18076126*I,
102.6167540+36.56727986*I,
103.9704343+36.95191436*I,
105.3205741+37.33470708*I,
106.6672448+37.71569870*I,
108.0105155+38.09492836*I,
109.3504529+38.47243379*I,
110.6871213+38.84825128*I,
112.0205827+39.22241588*I,
113.3508971+39.59496136*I,
114.6781226+39.96592031*I,
116.0023153+40.33532422*I,
117.3235295+40.70320350*I,
118.6418178+41.06958753*I,
119.9572311+41.43450475*I,
121.2698188+41.79798266*I,
122.5796288+42.16004786*I,
123.8867074+42.52072616*I,
125.1910997+42.88004248*I,
126.4928494+43.23802106*I,
127.7919990+43.59468536*I,
129.0885898+43.95005812*I,
130.3826616+44.30416144*I,
131.6742536+44.65701674*I,
```

```
132.9634036+45.00864484*I,
134.2501486+45.35906596*I,
135.5345242+45.70829974*I,
136.8165655+46.05636529*I,
138.0963064+46.40328118*I,
139.3737801+46.74906546*I,
140.6490188+47.09373572*I,
141.9220539+47.43730905*I,
143.1929162+47.77980212*I,
144.4616355+48.12123116*I,
145.7282409+48.46161195*I,
146.9927609+48.80095990*I,
148.2552233+49.13929000*I,
149.5156552+49.47661693*I,
150.7740831+49.81295490*I
```

Zeros of the derivative of $G(z; 1.5, -0.5)$ in rectangle with corners (0.01,0), (250,50).

```
6.094213420704737735160435+3.118635463969787968256242*I,
8.950657216792105729802910+4.438288561925345058770253*I,
11.463997016102192468924 22+5.543072863694853382048090*I,
13.790593229925108678679 50+6.526108066952662232230220*I,
15.992467648754039989777 12+7.427035193497844865119147*I,
18.102330703097956587775 48+8.267473956570995408589015*I,
20.140152148568503928151 11+9.060828510747485526047075*I,
22.119278176961239001986 54+9.816114185779634445462570*I,
24.049197637607083291052 30+10.539749944644864696082 13*I,
25.936968298062449741063 94+11.236503783555100736525 83*I,
27.788023598744938020148 78+11.910035464917698506402 82*I,
29.606661701211636889801 04+12.563228698470161213945 26*I,
31.396358349282129886904 00+13.198404950889035345455 70*I,
33.159975846312249738204 38+13.817466784626660613302 90*I,
34.899907675086387450848 36+14.421997243971643960318 96*I,
36.618181587566350062542 20+15.013330747185892251047 52*I,
38.316534961104595714637 42+15.592604893147224067180 21*I,
39.996471085713701803365 69+16.160799121478320440209 46*I,
41.659302005435005292485 02+16.718764094583466381733 86*I,
43.306181668100475014358 02+17.267244391020146623240 60*I,
44.938131953123876804348 21+17.806896285524104754933 82*I,
46.556063375082088321237 76+18.338301858942641342916 46*I,
48.160791745576708530351 08+18.861980325355629606359 88*I,
49.753051724378816576330 94+19.378397220460952095856 18*I,
51.333507946403191352249 65+19.887971925986456441592 00*I,
```

52.902764238041671924578 95+20.391083884991427990761 24*I,
54.461371311973678737534 55+20.888077776690273984364 00*I,
56.009833238808739839180 65+21.379267856532317671822 60*I,
57.548612926819223202664 30+21.864941620793008691490 68*I,
59.078136790814125742502 95+22.345362920174771766071 55*I,
60.598798753212313541540 80+22.820774620636217715371 72*I,
62.110963691324957009726 45+23.291400889594991596749 00*I,
63.614970422432134843083 05+23.757449170170499344548 89*I,
65.111134300771809337791 20+24.219111894090693345056 60*I,
66.599749486840495784987 45+24.676567974441283667446 42*I,
68.081090938545431965754 80+25.129984111969011639668 58*I,
69.555416165088908490611 05+25.579515942705423207959 76*I,
71.022966777513277273142 70+26.025309049911530872347 14*I,
72.483969864217367168305 55+26.467499859498415886483 95*I,
73.938639215187794122093 50+26.906216434957513465574 56*I,
75.387176414954112435817 10+27.341579185286196865713 33*I,
76.829771821206461979629 30+27.773701497302792374537 50*I,
78.266605443477154133539 55+28.202690302019657992314 40*I,
79.697847734180692060590 35+28.628646583312455336148 36*I,
81.123660302548942773786 50+29.051665835932304405593 20*I,
82.544196560525223049364 75+29.471838478910803481173 86*I,
83.959602308441582931325 10+29.889250229570588239292 38*I,
85.370016267256376622443 55+30.303982442647882089929 36*I,
86.775570563241083559666 50+30.716112418435550109452 47*I,
88.176391170249364811826 15+31.125713683347055732255 35*I,
89.572598314055596776112 45+31.532856245868425602123 42*I,
90.964306842696693669195 00+31.937606830494596694036 18*I,
92.351626566275193288174 60+32.340029091928297298144 98*I,
93.734662569271198349735 85+32.740183811545619433582 02*I,
95.113515498055788520315 35+33.138129077895828244226 59*I,
96.488281825990593981652 15+33.533920452798041584255 90*I,
97.859054098230383775053 10+33.927611124419467084372 36*I,
99.225921158111940531948 50+34.319252048564948071926 26*I,
100.58896835680827087778 23+34.708892079272325592497 33*I,
101.94827774774821225687 82+35.096578089689779507213 61*I,
103.30392826714425547974 77+35.482355084107510758620 63*I,
104.65599590183295231614 25+35.866266301924874522303 92*I,
106.00455384551010623208 03+36.248353314253676400918 20*I,
107.34967264433490974027 10+36.628656113787359070613 08*I,
108.69142033278146261629 52+37.007213198503003820233 97*I,
110.02986256053111197948 32+37.384061649707396542553 97*I,
111.36506271112344942558 44+37.759237204888956770943 64*I,

112.69708201301642049749938+38.13277432579332352843362*I,
114.02597964364584495333308+38.50470626210116243292562*I,
115.35181282702084472607000+38.87506511105172509717479*I,
116.67463692534347752881000+39.24388187332435236218236*I,
117.99450552509762358943760+39.61118650546203154443670*I,
119.31147051801330124889200+39.97700796909591561064855*I,
120.62558217727759688712110+40.34137427720705862820947*I,
121.93688922933185103362120+40.70431253764122842938873*I,
123.24543892156626461827720+41.06584899407427168605920*I,
124.55127708619734273336860+41.42600906460890717396285*I,
125.85544820059028396308850+41.78481737816881551309914*I,
127.15499544426729226835890+42.14229780884230707054467*I,
128.45296075282360702717500+42.49847350831553326868378*I,
129.74838486895561310110860+42.85336693652402718681125*I,
131.04130739078952787367790+43.20699989064119941307724*I,
132.33176681768470749507130+43.55939353251317038874194*I,
133.61980059367242922955970+43.91056841464089855350774*I,
134.90544514867896557192390+44.26054450480288233578189*I,
136.18873593767075710279180+44.60934120940470038187267*I,
137.46970747784941536748800+44.95697739563524329789224*I,
138.74839338401505549314350+45.30347141250362351938604*I,
140.02482640220799129286020+45.64884111082537584828786*I,
141.29903844173105203124080+45.99310386222163329483026*I,
142.57106060564763598054630+46.33627657719043956142142*I,
143.84092321984404403820300+46.67837572230520352018633*I,
145.10865586073858545152700+47.01941733659147888243133*I,
146.37428738171437070085460+47.35941704712973383141666*I,
147.63784593834756201024100+47.69839008392853358956493*I,
148.89935901249810205654620+48.03635129410956929423598*I,
150.15885343532555215556390+48.37331515544320712686824*I,
151.41635540928861170310590+48.70929578927068247452115*I,
152.67189052918313304634740+49.04430697284670700311222*I,
153.92548380226996497981040+49.37836215113407559611896*I,
155.17715966754073077286320+49.71147444807984039886036*I

Table 3: Interlacing of zeros of $G(z; 1.5, b)$ and of its derivative

b	Real Parts	Imaginary Parts	Modulus
-5	Yes	No	Yes
-4.5	Yes	No	Yes
-4	Yes	No	Yes
-3.5	Yes	No	Yes
-3	Yes	No	Yes
-2.5	Yes	No	Yes
-2	Yes	No	Yes
-1.5	Yes	No	Yes
-1	Yes	No	Yes
-0.5	Yes	No	Yes
0	Yes	No	Yes
0.5	Yes	No	Yes
1	Yes	No	Yes
1.5	No	No	No
2	Yes	No	Yes
2.5	Yes	No	Yes
3	No	No	No
3.5	Yes	No	Yes
4	–	–	–
4.5	No	No	No
5	No	No	No
5.5	No	No	No
6	–	–	–
6.5	–	–	–
7	No	No	No

Table 4: Interlacing of zeros of $G(z; 2, b)$ and of its derivative

b	Real Parts	Imaginary Parts	Modulus
-5	Yes	No	Yes
-4.5	Yes	No	Yes
-4	Yes	No	Yes
-3.5	Yes	No	Yes
-3	Yes	No	Yes
-2.5	Yes	No	Yes
-2	Yes	No	Yes
-1.5	Yes	No	Yes
-1	Yes	No	Yes
-0.5	Yes	No	Yes
0	Yes	No	Yes
0.5	Yes	No	Yes
1	Yes	No	Yes
1.5	Yes	No	Yes
2	Yes	No	Yes
2.5	Yes	No	Yes
3	Yes	No	Yes
3.5	No	No	No
4	No	No	No
4.5	Yes	No	Yes
5	Yes	No	Yes
5.5	No	No	No
6	No	No	No
6.5	Yes	No	Yes
7	No	No	No

Table 5: Interlacing of zeros of $G(z; 2.5, b)$ and of its derivative

b	Real Parts	Imaginary Parts	Modulus
0	Yes	No	Yes
0.5	Yes	No	Yes
1	Yes	No	Yes
1.5	Yes	No	Yes
2	Yes	No	Yes
2.5	Yes	No	Yes
3	Yes	No	Yes
3.5	Yes	No	Yes
4	Yes	No	Yes
4.5	No	No	No
5	No	No	No
5.5	Yes	No	Yes
6	Yes	No	Yes
6.5	No	No	No
7	No	No	No

Table 6: Interlacing of zeros of $G(z; 3, b)$ and of its derivative

b	Real Parts	Imaginary Parts	Modulus
0	Yes	No	Yes
0.5	Yes	No	Yes
1	Yes	No	Yes
1.5	Yes	No	Yes
2	Yes	No	Yes
2.5	Yes	No	Yes
3	Yes	No	Yes
3.5	Yes	No	Yes
4	Yes	No	Yes
4.5	Yes	No	Yes
5	Yes	No	Yes
5.5	No	No	No
6	Yes	No	Yes
6.5	Yes	No	Yes
7	Yes	No	Yes

Table 7: Interlacing of zeros of $G(z; 3.5, b)$ and of its derivative

b	Real Parts	Imaginary Parts	Modulus
0	Yes	No	Yes
0.5	Yes	No	Yes
1	Yes	No	Yes
1.5	Yes	No	Yes
2	Yes	No	Yes
2.5	Yes	No	Yes
3	Yes	No	Yes
3.5	Yes	No	Yes
4	Yes	No	Yes
4.5	Yes	No	Yes
5	Yes	No	Yes
5.5	Yes	No	Yes
6	No	No	No
6.5	No	No	No
7	Yes	No	Yes

Zeros of $\Phi_1(z) + 0.1\Phi_2(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.05399773051498293782360,
20.704913291390744019669l2+2.6168448664523400627330420*I,
26.123371343963482227408480+7.0324781485019695467275200*I,
31.556198597161749177191680+10.639678623120234599881320*I,
36.774524489154987537253380+13.673935987603488293994440*I,
41.779783615952609052176320+16.357759601613895172835830*I,
46.605583945360036772618300+18.802253639943082223948230*I,
51.28133970+21.07020087*I,
55.82997811+23.20092762*I,
60.26925938+25.22096602*I,
64.61314132+27.14921718*I,
68.87278790+28.99971130*I,
73.05728437+30.78320189*I,
77.17413840+32.50815184*I,
81.22963952+34.18136412*I,
85.22911935+35.80840254*I,
89.17714710+37.39389060*I,
93.07767478+38.94171735*I,
96.93415065+40.45519570*I,
100.7496044+41.93717241*I,
104.5267156+43.39011464*I,
108.2678706+44.81618052*I,
111.9752060+46.21726886*I,
115.6506419+47.59506093*I,
119.2959142+48.95105541*I
```

Zeros of $\Phi_1(z) + 0.5\Phi_2(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.089022200465230546879 34,
21.126196754376008539827 43+2.167809862399044817261300*I,
26.499255612414659008992 69+6.518830488201067516712930*I,
31.862818882976443466247 64+10.14336746433991744098012*I,
37.040618396572342887241 22+13.19772765785699083108784*I,
42.019183634688933009744 05+15.89946490084455881257690*I,
46.825904611783113785099 67+18.35899602252905948556528*I,
51.48714365+20.63964558*I,
56.02425940+22.78124952*I,
60.45410138+24.81073198*I,
64.79006670+26.74728875*I,
69.04294488+28.60516290*I,
73.22156696+30.39527466*I,
77.33325752+32.12621185*I,
81.38417205+33.80487628*I,
85.37954140+35.43690962*I,
89.32385810+37.02700166*I,
93.22101067+38.57908837*I,
97.07439923+40.09652760*I,
100.8870152+41.58220032*I,
104.6615056+43.03860450*I,
108.4002298+44.46792244*I,
112.1053006+45.87207420*I,
115.7786228+47.25276093*I,
119.4219142+48.61149393*I,
123.0367398+49.94963297*I
```

Zeros of $\Phi_1(z) + 0.9\Phi_2(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.12540279204329297838269,
22.06024626629065768582840+.60618400820072142407869 40*I,
27.53884211673404487188027+5.15198867174526113482420 7*I,
32.71014677011416973696234+8.80850459527855462564791 5*I,
37.77790937484626471097612+11.90936008245477481533350*I,
42.68217959113845045177974+14.65533585860379427503230*I,
47.43554602566259471613 72+17.153287127116063683084 80*I,
52.05612020+19.46695909*I,
56.56095710+21.63714420*I,
60.96437863+23.69163755*I,
65.27818550+25.65028656*I,
69.51214950+27.52788542*I,
73.67436573+29.33574212*I,
77.77164703+31.08276078*I,
81.80976600+32.77609674*I,
85.79368880+34.42160532*I,
89.72766933+36.02412397*I,
93.61542830+37.58772657*I,
97.46023565+39.11587714*I,
101.2649633+40.61156436*I,
105.0321707+42.07734953*I,
108.7641444+43.51548360*I,
112.4629346+44.92794733*I,
116.1303883+46.31648341*I,
119.7681929+47.68266510*I,
123.3778470+49.02785824*I
```

Zeros of $\Phi_1(z) + 0.95\Phi_2(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.13005217571049725343824,
21.3786682188487650276107 8,
23.3332439091832661796858 7,
27.9930557273391513562877 0+4.5836489368550933682844 08*I,
33.0777222365109883929811 8+8.2440439414103265293772 50*I,
38.0989309666315891811347 0+11.3615761505780848135319 6*I,
42.9708492267167363722018 3+14.1244403439592230997146 5*I,
47.7008286946627034159655 6+16.6376908663691540396020 8*I,
52.30353035+18.96478334*I,
56.79417375+21.14672842*I,
61.18595615+23.21158270*I,
65.49002785+25.17946198*I,
69.71566370+27.06532218*I,
73.87068145+28.88064326*I,
77.96164925+30.63445708*I,
81.99416110+32.33399442*I,
85.97304930+33.98519285*I,
89.90250925+35.59298711*I,
93.78615740+37.16148063*I,
97.62721980+38.69420856*I,
101.4284894+40.19413687*I,
105.1925158+41.66391734*I,
108.9215435+43.10581510*I,
112.6175914+44.52182683*I,
116.2824866+45.91372360*I,
119.9178737+47.28304363*I,
123.5253008+48.63127754*I,
127.1060884+49.95961580*I
```

Zeros of $\Phi_1(z) + 0.1\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.045439582468775860324 84,
20.625346049212362519425 71+2.697151806087224635627440*I,
26.056166986133444106404 06+7.125359662760955155036280*I,
31.501431437822429482478 93+10.729150321378560557391 79*I,
36.727022834702046958523 92+13.759614049555006218265 64*I,
41.737034868795254947928 68+16.440127309151238021709 81*I,
46.566228742610848149074 98+18.881869582257028208775 45*I,
51.24456568+21.14750392*I,
55.79525380+23.27625694*I,
60.23621437+25.29458581*I,
64.58150488+27.22133530*I,
68.84235660+29.07049587*I,
73.02789948+30.85279237*I,
77.14567365+32.57666365*I,
81.20199115+34.24889227*I,
85.20220355+35.87503108*I,
89.15089300+37.45969003*I,
93.05202295+39.00675062*I,
96.90904962+40.51951690*I,
100.7250090+42.00082743*I,
104.5025878+43.45314750*I,
108.2441776+44.87862916*I,
111.9519160+46.27916597*I,
115.6277294+47.65643840*I,
119.2733554+49.01193992*I
```

Zeros of $\Phi_1(z) + 0.5\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.04543959702460918817180,
20.625346222374964005332 06+2.697151661078025645802802*I,
26.056167196354061775796 21+7.125359483353992231633490*I,
31.501431676113119178107 87+10.729150122672906705612 98*I,
36.727023098540099960034 25+13.759613823627657802526 06*I,
41.737035149991483816194 02+16.440127052782978515286 29*I,
46.566229033139099434988 08+18.881869296126729782829 16*I,
51.24456645+21.14750427*I,
55.79525408+23.27625610*I,
60.23621447+25.29458511*I,
64.58150532+27.22133525*I,
68.84235687+29.07049568*I,
73.02789980+30.85279148*I,
77.14567383+32.57666344*I,
81.20199175+34.24889243*I,
85.20220375+35.87503058*I,
89.15089385+37.45968888*I,
93.05202345+39.00675008*I,
96.90904948+40.51951573*I,
100.7250091+42.00082713*I,
104.5025878+43.45314688*I,
108.2441774+44.87862796*I,
111.9519160+46.27916506*I,
115.6277298+47.65643777*I,
119.2733561+49.01193918*I
```

Zeros of $\Phi_1(z) + 0.9\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.045439611580442749206 02,
20.625346395537600392885 42+2.697151516068796252272386*I,
26.056167406574742027228 94+7.125359303946993756290220*I,
31.501431914403898054664 41+10.729149923967213224308 83*I,
36.727023362378272584272 50+13.759613597700259698009 36*I,
41.737035431187861702151 94+16.440126796414652578638 27*I,
46.566229323667524950814 40+18.881869009996343070622 82*I,
51.24456637+21.14750355*I,
55.79525415+23.27625600*I,
60.23621490+25.29458463*I,
64.58150563+27.22133448*I,
68.84235734+29.07049534*I,
73.02789990+30.85279128*I,
77.14567405+32.57666258*I,
81.20199196+34.24889172*I,
85.20220405+35.87503000*I,
89.15089344+37.45968911*I,
93.05202360+39.00674944*I,
96.90905007+40.51951597*I,
100.7250098+42.00082709*I,
104.5025886+43.45314706*I,
108.2441780+44.87862828*I,
111.9519167+46.27916550*I,
115.6277294+47.65643691*I,
119.2733566+49.01193960*I
```

Zeros of $\Phi_1(z) + 0.95\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.045439613399921960731 26,
20.625346417182932395352 28+2.697151497942640440319802*I,
26.056167432852331458959 32+7.125359281521116447125940*I,
31.501431944190251684770 81+10.729149899128998752694 35*I,
36.727023395358052573279 73+13.759613569459331441202 01*I,
41.737035466337419415710 20+16.440126764368607165681 03*I,
46.566229359983590390842 40+18.881868974230038523967 12*I,
51.24456660+21.14750350*I,
55.79525461+23.27625620*I,
60.23621480+25.29458439*I,
64.58150562+27.22133473*I,
68.84235760+29.07049585*I,
73.02790007+30.85279127*I,
77.14567412+32.57666282*I,
81.20199162+34.24889190*I,
85.20220435+35.87503054*I,
89.15089375+37.45968918*I,
93.05202332+39.00674980*I,
96.90905002+40.51951582*I,
100.7250098+42.00082785*I,
104.5025884+43.45314666*I,
108.2441774+44.87862811*I,
111.9519164+46.27916513*I,
115.6277302+47.65643762*I,
119.2733561+49.01193960*I
```

Zeros of $\Phi_1(z) + 0.1\Phi_2(z) + 0.1\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.05399773418755034466230,
20.704913338860828938918 74+2.616844825965611206674141*I,
26.123371402300471141531 32+7.032478098532344406975476*I,
31.556198663240144567464 02+10.639678567791546573701 95*I,
36.774524562305879705280 20+13.673935924745151853105 79*I,
41.779783693920410127528 98+16.357759530339290139996 50*I,
46.605584025929413918733 15+18.802253560435223405409 90*I,
51.28133956+21.07020050*I,
55.82997844+23.20092714*I,
60.26925967+25.22096656*I,
64.61314128+27.14921788*I,
68.87278797+28.99971094*I,
73.05728466+30.78320188*I,
77.17413933+32.50815283*I,
81.22963957+34.18136395*I,
85.22911905+35.80840212*I,
89.17714735+37.39389068*I,
93.07767473+38.94171703*I,
96.93415150+40.45519616*I,
100.7496048+41.93717310*I,
104.5267162+43.39011519*I,
108.2678707+44.81618044*I,
111.9752060+46.21726793*I,
115.6506417+47.59506047*I,
119.2959150+48.95105538*I
```

Zeros of $\Phi_1(z) + 0.1\Phi_2(z) + 0.2\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.05399773786011776639767,
20.704913386330916505750 80+2.616844785478879949760620*I,
26.123371460637464883828 30+7.032478048562716492916476*I,
31.556198729318546830007 73+10.639678512462855451522 18*I,
36.774524635456781085979 05+13.673935861886811538745 07*I,
41.779783771888222675751 33+16.357759459064679945596 70*I,
46.605584106498804477960 98+18.802253480927357745587 28*I,
51.28133960+21.07020026*I,
55.82997876+23.20092794*I,
60.26925993+25.22096660*I,
64.61314153+27.14921761*I,
68.87278792+28.99971076*I,
73.05728475+30.78320185*I,
77.17413892+32.50815204*I,
81.22963955+34.18136446*I,
85.22911915+35.80840218*I,
89.17714645+37.39388984*I,
93.07767580+38.94171785*I,
96.93415100+40.45519538*I,
100.7496049+41.93717285*I,
104.5267156+43.39011386*I,
108.2678720+44.81618108*I,
111.9752066+46.21726845*I,
115.6506416+47.59506036*I,
119.2959148+48.95105482*I
```

Zeros of $\Phi_1(z) + 0.5\Phi_2(z) + 0.5\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.08902221953875717720683,
21.126197147045882958775 75+2.167809477973057268501037*I,
26.499256134788988752287 67+6.518830031877912285567943*I,
31.862819472019062231804 28+10.143366959794764210291 33*I,
37.040619048137901941469 56+13.197727087115973768115 64*I,
42.019184329361085862120 63+15.899464256289238129248 65*I,
46.825905330317287493939 05+18.358995305553919456738 97*I,
51.48714348+20.63964522*I,
56.02425935+22.78124805*I,
60.45410380+24.81073310*I,
64.79006690+26.74728712*I,
69.04294573+28.60516210*I,
73.22156768+30.39527250*I,
77.33325836+32.12621022*I,
81.38417333+33.80487597*I,
85.37954206+35.43690894*I,
89.32385833+37.02700000*I,
93.22101060+38.57908714*I,
97.07440018+40.09652722*I,
100.8870153+41.58219964*I,
104.6615063+43.03860408*I,
108.4002304+44.46792129*I,
112.1053015+45.87207240*I,
115.7786245+47.25275960*I,
119.4219149+48.61149127*I,
123.0367384+49.94962946*I
```

Zeros of $\Phi_1(z) + 0.5\Phi_2(z) + 0.95\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.089022367049314930646 2,
21.126197500448934661298 08+2.167809131989472510141997*I,
26.499256604926219710077 12+6.518829621186868656515712*I,
31.862820002157891495557 82+10.143366505703897279701 69*I,
37.040619634547536394893 22+13.197726573448775095320 47*I,
42.019184954566808288224 14+15.899463676189078368834 41*I,
46.825905976998961115200 46+18.358994660275808478805 06*I,
51.48714383+20.63964428*I,
56.02426005+22.78124754*I,
60.45410452+24.81073226*I,
64.79006775+26.74728610*I,
69.04294553+28.60515840*I,
73.22156905+30.39527181*I,
77.33325862+32.12620937*I,
81.38417380+33.80487441*I,
85.37954262+35.43690778*I,
89.32385850+37.02699822*I,
93.22101193+38.57908773*I,
97.07440083+40.09652577*I,
100.8870162+41.58219880*I,
104.6615065+43.03860300*I,
108.4002300+44.46791994*I,
112.1053017+45.87207261*I,
115.7786230+47.25275740*I,
119.4219166+48.61149250*I,
123.0367396+49.94962908*I
```

Zeros of $\Phi_1(z) + 0.2\Phi_2(z) + 0.1\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.06263333033717458644054,
20.792453687036544693300 30+2.527195910558330219737487*I,
26.198560491534920157384 72+6.928938817633301664380444*I,
31.617495754383561781491 42+10.539841044341329808235 28*I,
36.827698523455894113108 54+13.578270073701925105894 74*I,
41.827632320947923504315 97+16.265758605109845396117 31*I,
46.649629231681973497706 16+18.713308604900542850624 78*I,
51.32249140+20.98382859*I,
55.86883300+23.11675257*I,
60.30623280+25.13869802*I,
64.64853565+27.06862288*I,
68.90683255+28.92060356*I,
73.09015730+30.70542638*I,
77.20598096+32.43158124*I,
81.26056585+34.10588858*I,
85.25922540+35.73393233*I,
89.20651190+37.32034470*I,
93.10636653+38.86902857*I,
96.96222630+40.38330328*I,
100.7771131+41.86602160*I,
104.5537007+43.31965893*I,
108.2943705+44.74637766*I,
112.0012532+46.14808096*I,
115.6762668+47.52645439*I,
119.3211446+48.88299783*I
```

Zeros of $\Phi_1(z) + 0.95\Phi_2(z) + 0.5\Phi_3(z)$ in rectangle with corners (0.01,0), (250,50).

```
14.13005219566626258257954,
23.3332510731368144452822,
27.99306090894274240308300+4.583644205971031946122978*I,
33.07772791106543955947734+8.244038643757668850342223*I,
38.09893723213515652968370+11.361570253514049242854 91*I,
42.97085591362847867937990+14.124433776295175891311 35*I,
47.70083563459427905780926+16.637683636246918708078 04*I,
52.30353603833716565597965+18.964767670325155706857 53*I,
56.79417383962384497471800+21.146720565539264205609 04*I,
61.18596823966500289600705+23.211584153992883458386 86*I,
65.49004187817750170618910+25.179464031734089720545 18*I,
69.71568633742799937358055+27.065323144763469976821 78*I,
73.87069517299449124771405+28.880641190671593092174 14*I,
77.96165311298997073344600+30.634447273439777368281 45*I,
81.99417044461233775922020+32.333991790223201200215 43*I,
85.97306750296842268334940+33.985199625364817397518 72*I,
89.90251925008387023826190+35.592985462476512461012 06*I,
93.78616884440681810335020+37.161479080972809964179 24*I,
97.62721743840364925692815+38.694190080218224376477 60*I,
101.42849574536407670574 70+40.194130767805171741872 58*I,
105.19252156356353283404 96+41.663909476005785227624 27*I,
108.92154640823658104513 94+43.105802546222542542989 10*I,
112.61759362747224520205 43+44.521810643159068992359 39*I,
116.28248980439636914239 27+45.913703369255576084713 30*I,
119.91789082247537381535 08+47.283055015042962446423 28*I,
123.52530365404984866339 52+48.631273504207190840172 35*I,
127.10610469492069398172 96+49.959624050634007911831 62*I
```

References

[BCRW08] P. Borwein, S. Choi, B. Rooney, and A Weirathumuller. The Riemann Hypothesis. CMS Books in Mathematics, Springer, New York, NY, 2008.

[CS07] M. Chudnovsky and P. Seymour. The roots of the independence polynomial of a claw free graph. J. Combin. Theory Ser. Bm 97(3):350-357, 2007.

[Edw01] H.M. Edwards. Riemann's Zeta Function. Dover Publications Inc., Mineola, NY, 2001.

[Hag11] J. Haglund. Some conjectures on the zeros of approximates to the Riemann Xi-function and incomplete gamma functions. Cent. Eur. J. Math., 9(2):302-318, 2011.

[SS03] E.M. Stein and R. Shakarchi. Complex Analysis. Princeton Lectures in Analysis, Princeton University Press, Princeton, NJ, 2003.