Three Faces of the Delta Conjecture

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The Algebraic Side

Let $X_n = \{x_1, \ldots, x_n\}$, $Y_n = \{y_1, \ldots, y_n\}$ be sets of variables. Let

$$\mathrm{DR}_n = \mathbb{C}[X_n, Y_n]/\{\sum_i x_i^a y_i^b : a, b \geq 0, a + b > 0\}$$

be the ring of diagonal coinvariants. S_n acts "diagonally" on DR_n by permuting the X and Y variables in the same way.

Example: $n = 2$

Cosets $\{1, x_1, y_1\}$ form a basis for DR_2, so $\mathrm{Hilb}(\mathrm{DR}_2) = 1 + q + t$.

The identity in S_2 acts by fixing all the cosets, while $\sigma = (12)$ fixes 1 and sends $\{x_1, y_1\}$ to $\{x_2, y_2\}$. Since $x_1 + x_2 = 0 = y_1 + y_2$, $x_2 = -x_1, y_2 = -y_1$. Hence the coset 1 corresponds to the trivial character, while x_1, y_1 correspond to the sign character, and the bigraded character of DH_2 is $s_2 + (q + t)s_{1,1}$.

The Symmetric Function Side

Let Δ'_f be a linear operator defined via

$$\Delta'_f \tilde{H}_\mu(X;q,t) = f[B_\mu - 1]\tilde{H}_\mu(X;q,t),$$

where $B_\mu = \sum_{s\in\mu} q^{\mathrm{coarm}(s)} t^{\mathrm{coleg}(s)}$. For example $B_{3,2} = 1 + q + q^2 + t + tq$.
Haiman proved that the bigraded character of DR_n under the diagonal action is given by

$$\Delta'_{e_{n-1}} e_n(X) = \sum_{\mu\vdash n} \frac{T_\mu \tilde{H}_\mu(X;q,t) M B_\mu \prod'_{s\in\mu}(1-q^{\mathrm{coarm}(s)})(1-t^{\mathrm{coleg}(s)})}{\prod_{s\in\mu}(t^{\mathrm{leg}(s)} - q^{\mathrm{arm}(s)+1})(q^{\mathrm{arm}(s)} - t^{\mathrm{leg}(s)+1})}$$

where $M = (1-q)(1-t)$ and $T_\mu = t^{n(\mu)} q^{n(\mu')}$, with $n(\mu) = \sum_i (i-1)\mu_i$.

The Combinatorial Side

Given a Dyck path π and a word parking function P (a filling of the squares just to the right of North steps of π with cars , i.e. integers between 1 and n, strictly increasing up columns), let a_i be the number of area squares in the ith row (from the bottom). Cars in rows (i, j) with $i < j$ form an inversion pair if either $a_i = a_j$ and $car_i < car_j$, or $a_i = a_j + 1$ and $car_i > car_j$. Let d_i be the number of inversion pairs (i, j) with $i < j$. Furthermore, we call a car at the bottom of a column a *valley*, and say the valley is *moveable* if, when we slide the car one square to the left, the result is still a word parking function, i.e we still have strict decrease down columns. For example, in Figure 1, cars $1, 2$ and 8 (in rows $5, 6$ and 8) are moveable, but cars 4 and 3 in rows 1 and 2 are not.

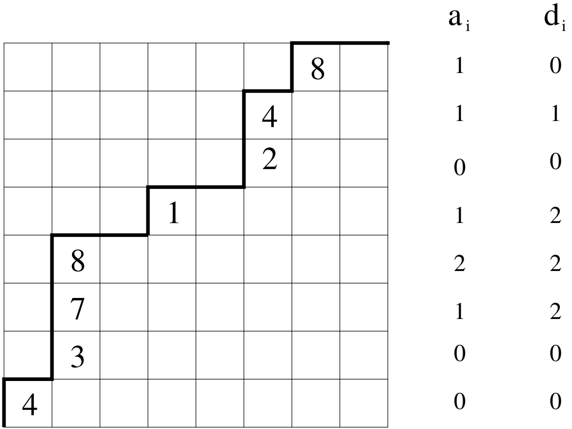


Figure: A word parking function with area $= 6$. There are dinv (i, j)-row pairs $(7, 8), (5, 7), (5, 8), (4, 5), (4, 7), (3, 6), (3, 8)$, so dinv $= 7$. The total weight is $x_1 x_2 x_3 x_4^2 x_7 x_8^2 q^7 t^6$.

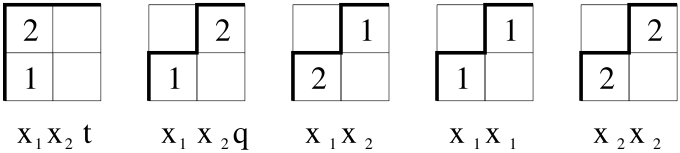


Figure: The various word parking functions when $n = 2$, together with their x, q, t weights.

Theorem (Carlsson-Mellit, 2015)

$$\Delta'_{e_{n-1}} e_n = \sum_{P \in \mathrm{WP}(n)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} x^P.$$

where the sum is over all word parking functions P on n cars.

Still Open: Find a combinatorial expression for the Schur expansion of the right-hand-side above.

Corollary (Conjectured by H., Loehr in 2002)

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} \prod_{i=1}^{n-1} [w_i(\sigma)]_q.$$

Let $w_i(\sigma)$ equal the number of w_j which are in σ_i's run and larger than σ_i, or in the next run to the right and smaller than σ_i.

Example

$$\sigma = 25713846 \to 257|138|46|0$$
$$(w_1, w_2, \ldots, w_8) = (3, 3, 2, 2, 1, 2, 2, 1).$$

Theorem (Carlsson-Oblomkov, 2018)

A monomial basis for DR_n is given by a certain family of cosets, one for each $\sigma \in S_n$. The contribution to $\mathrm{Hilb}(\mathrm{DR}_n)$ of monomials associated to σ is $t^{\mathrm{maj}(\sigma)} \prod_{i=1}^{n-1} [w_i(\sigma)]_q$.

Examples

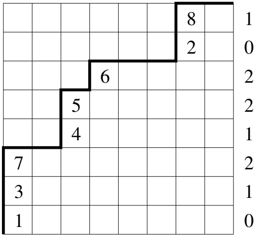
$$\sigma = 25713846 \to y_1 y_2 y_3 \times y_1 y_2 y_3 y_4 y_5 y_6$$

$$(1 + x_2 + x_2^2)(1 + x_5 + x_5^2)(1 + x_7)(1 + x_1)(1 + x_8)(1 + x_4)$$

$$\text{Set all } x_i = 0; \sum_{\sigma \in S_n} \prod_{k \in \mathrm{Des}} y_1 y_2 \cdots y_k \to \text{Garsia-Stanton basis}$$

$$\text{Set all } y_i = 0; \sigma = (12 \cdots n) : (w_1, w_2, \ldots) = (n, n-1, \ldots) \to$$

$$(1 + x_1 + \ldots x_1^{n-1}) \cdots (1 + x_{n-2} + x_{n-2}^2)(1 + x_{n-1}) \to \text{Artin basis.}$$



The Delta Conjecture (H., Remmel, Wilson, 2015)

$$\Delta'_{e_{k-1}} e_n = \sum_{P \in \mathrm{WP}(n)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{a_i > a_{i-1}} \left(1 + z/t^{a_i}\right)\Big|_{z^{n-k}}$$

$$= \sum_{P \in \mathrm{WP}(n)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{\text{movable valleys}} \left(1 + z/q^{d_i+1}\right)\Big|_{z^{n-k}}$$

Let Π be an ordered set partition of $\{1, 2, \ldots, n\}$, and let $\sigma = \sigma(\Pi)$ be the ordering of the blocks of Π which minimizes maj. For example, if $\Pi = \{\{2,3,5\}, \{1,6,7,9\}, \{4,8\}\}$, then $\sigma(\Pi) = 235679148$, and $\text{minimaj}(\Pi) = \text{maj}(\sigma) = 6$. Next form σ^* by marking every number which is not leftmost (in minimaj order) from its block;

$$\sigma^* = 23^*5^*67^*9^*1^*48^*.$$

Now construct the vector $(w_1(\Pi), w_2(\Pi), \ldots)$ by first isolating the unmarked elements of σ^*, map them to a permutation, and apply previous rule:

$$264 \to 132 \to 13\big|2\big|0 \to (1,1,1).$$

For marked elements σ_i^*, w_i equals the number of unmarked elements smaller than σ_i in its run plus the number of unmarked elements which are larger in the previous run.

$$\sigma^* = \{23^*5^*\}\{67^*9^*1^*\}\{48^*\} \to (1,1,1,1,2,2,2,1,1).$$

Theorem H.-Sergel, 2018

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$$\sum_{P \in \mathrm{PF}(n)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{\text{movable valleys}} (1 + z/q^{d_i+1})\Big|_{z^{n-k}} =$$

$$\sum_{\substack{\Pi \\ k \text{ blocks}}} t^{\mathrm{minimaj}(\Pi)} \prod_{i=1}^{n} [w_i(\Pi)]_q.$$

Open Question: Is there an analogue involving the rise version of the Delta Conjecture?

A module for the Delta Conjecture

M. Zabrocki has recently introduced a module whose bigraded character is conjecturally equal to the combinatorial and symmetric function sides of the Delta Conjecture. Let $\Theta_n = \{\theta_1, \ldots, \theta_n\}$ be a set of anticommuting variables, i.e. $\theta_i\theta_j = -\theta_j\theta_i, 1 \le i \le j \le n$. Note this implies $\theta_i^2 = 0$. Let X_n, Y_n be two sets of commuting variables, which also commute with the θ_i. Set

$$\mathrm{TR}_n = \mathbb{C}[X_n, Y_n, \Theta_n]/\{\sum_i x_i^a y_i^b \theta_i^c : a, b, c \ge 0, a+b+c > 0, c \le 1\}.$$

S_n acts on TR_n diagonally by permuting the x_i, y_i, θ_i in the same way. Then Zabrocki conjectures that the tri-graded character of this action is given by

$$\sum_{k=1}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n,$$

where q, t give the grading in the x and y variables and z the grading in the θ variables.