A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants

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Abstract

A special case of Haiman's identity [Invent. Math. **149** (2002), pp. 371–407] for the character of the quotient ring of diagonal coinvariants under the diagonal action of the symmetric group yields a formula for the bigraded Hilbert series as a sum of rational functions in q, t. In this paper we show how a summation identity of Garsia and Zabrocki for Macdonald polynomial Pieri coefficients can be used to transform Haiman's formula for the Hilbert series into an explicit polynomial in q, t with integer coefficients. We also provide an equivalent formula for the Hilbert series as the constant term in a multivariate Laurent series.

1 Introduction

Let $X_n = \{x_1, \ldots, x_n\}, Y_n = \{y_1, \ldots, y_n\}$ be two sets of variables and let

$$\mathrm{DR}_n = \mathbb{C}[X_n, Y_n]/\big\langle\{\sum_i x_i^h y_i^k, \forall h, k \geq 0, h+k > 0\}\big\rangle \tag{1}$$

be the quotient ring of diagonal coinvariants. Let ∇ be the linear operator defined on the modified Macdonald polynomial basis $\{\tilde{H}_\mu(X_n; q, t)\}$, where $\mu \vdash n$ (i.e. μ is a partition of n), by

$$\nabla \tilde{H}_\mu(X_n; q, t) = T_\mu \tilde{H}_\mu(X_n; q, t), \tag{2}$$

where $T_\mu = t^{n(\mu)} q^{n(\mu')}$ and $n(\mu) = \sum_i (i-1)\mu_i$. The symmetric group acts "diagonally" on a polynomial $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ by $\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ and this action extends to DR_n. Haiman [Hai02] proved an earlier conjecture of Garsia and Haiman [GH96] that the Frobenius series of this action is given by $\nabla e_n(X_n)$, where e_n is the nth elementary symmetric function in a set of variables. (The Frobenius series is obtained by starting with the character and mapping the irreducible S_n-character χ^λ to the Schur function s_λ.) Since the Frobenius series of DR_n is given by ∇e_n, the Hilbert series $\mathrm{Hilb}(\mathrm{DR}_n)$ is given by $\langle \nabla e_n, h_1^n \rangle$ (See [Hag08, p. 24] for an explanation of why. Here $\langle , \rangle$ is the Hall scalar product, with respect to which the Schur functions are orthonormal, and $h_1(X) = \sum_i x_i$.) This results in a formula for $\mathrm{Hilb}(\mathrm{DR}_n)$ as an explicit sum of rational functions in q, t, described in detail in the next section. A corollary

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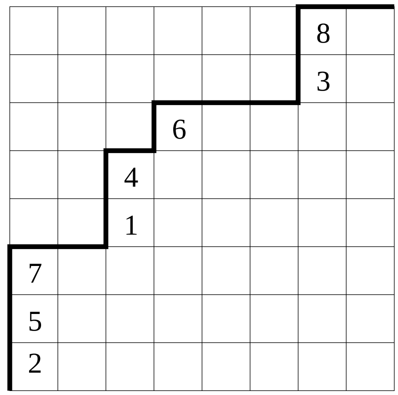

Figure 1: A parking function with area = 9 and dinv = 6

of this formula is that $\dim(\mathrm{DR}_n) = (n+1)^{n-1}$. See also [Hai94] and [Ber09] for background on this problem. We mention that many articles in the literature refer to the space of diagonal harmonics DH_n, which is known to be isomorphic to DR_n, and so $\mathrm{Hilb}(\mathrm{DH}_n) = \mathrm{Hilb}(\mathrm{DR}_n)$.

A Dyck path is a lattice path in the first quadrant of the xy-plane from $(0,0)$ to (n,n) consisting of unit north N and east E steps which never goes below the diagonal $x = y$. A parking function is a placement of the integers $1, 2, \ldots, n$ (called "cars") just to the right of the N steps of a Dyck path, so there is strict decrease down columns. An open conjecture of Loehr and the author expresses $\mathrm{Hilb}(\mathrm{DR}_n)$ as a positive sum of monomials, one for each parking function. To a given parking function π, we associate two statistics $\mathrm{area}(\pi)$ and $\mathrm{dinv}(\pi)$. The area statistic is defined as the number of squares strictly below π and strictly above the diagonal. The dinv statistic is the number of pairs of cars which form either "primary" or "secondary" inversions. Pairs of cars form a primary inversion if they are in the same diagonal, with the larger car in a higher row. Pairs form a secondary inversion if they are in successive diagonals, with the larger car in the outer diagonal and in a lower row. For example, for the parking function in Figure 1, car 8 forms primary inversions with cars 1 and 5, while car 5 forms a secondary inversion with car 3. The set of inversion pairs for this parking function is $\{(6,4), (7,1), (8,1), (8,5), (5,3), (3,2)\}$, so dinv = 6 while area = 9.

Conjecture 1 *[HL05], [Hag08, Chap. 5]*

$$Hilb(DR_n) = \sum_{\pi} q^{dinv(\pi)} t^{area(\pi)}, \tag{3}$$

where the sum is over all parking functions on n cars.

Remark 1 *In a recent preprint, Armstrong [Arm10] introduces a hyperplane arrangement model for $Hilb(DR_n)$ involving a pair of hyperplane arrangements with a statistic associated to each one. See also [AR]. He gives a bijection with parking functions which sends his pair of hyperplane arrangement statistics to $(area', bounce)$, another pair of statistics which Haglund and Loehr showed have the same distribution over parking functions as $(dinv, area)$.*

In this article we use a plethystic summation formula of Garsia and Zabrocki for Macdonald Pieri coefficients to show how $\langle \nabla e_n, h_1^n \rangle$ can be expressed as an element of $\mathbb{Z}[q,t]$. The most

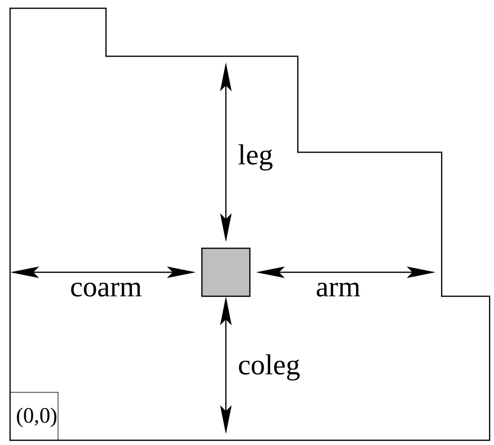

Figure 2: The leg, coleg, arm, and coarm of a square

elegant way of expressing our result is to say that $\mathrm{Hilb}(\mathrm{DR}_n)$ is the coefficient of $z_1 z_2 \cdots z_n$ in a certain multivariate Laurent series (see (54)). We are currently unable to see how our result implies a positive formula such as Conjecture 1, but are hopeful that further work will lead to such applications.

2 Background Material

For $\mu \vdash n$, and s a square of the Ferrers diagram of μ, let $l(s), a(s), l'(s), a'(s)$ denote the leg, arm, coleg, coarm, respectively, of s, i.e. the number of squares above s, to the right of s, below s, and to the left of s, as in Figure 2. Furthermore let

$$M = (1-q)(1-t), \quad B_\mu = \sum_{s \in \mu} t^{l'} q^{a'}, \quad \Pi_\mu = \prod_{\substack{s \in \mu \\ s \neq (0,0)}} (1 - t^{l'} q^{a'}), \quad w_\mu = \prod_{s \in \mu} (q^a - t^{l+1})(t^l - q^{a+1}). \tag{4}$$

The known expansion

$$e_n(X) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu(X;q,t) M \Pi_\mu B_\mu}{w_\mu} \tag{5}$$

then implies

$$\nabla e_n(X) = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu(X;q,t) M \Pi_\mu B_\mu}{w_\mu}. \tag{6}$$

Letting $F_\mu = \langle \tilde{H}_\mu, h_1^n \rangle$, by taking the scalar product of both sides of (6) with respect to h_1^n we get

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\mu \vdash n} \frac{T_\mu F_\mu M \Pi_\mu B_\mu}{w_\mu}. \tag{7}$$

Let $^\perp$ be the operation on symmetric functions which is adjoint to multiplication with respect to the Hall scalar product, i.e. for any symmetric functions f, g, h,

$$\langle f^\perp g, h\rangle = \langle g, fh\rangle. \tag{8}$$

If $\mu \vdash n$ and $\nu \vdash n-1$, then $\nu \to \mu$ means ν is obtained from μ by removing some corner square of μ, and $\mu \leftarrow \nu$ means μ is obtained from ν by adding a single square to the Ferrers shape of ν. Define generalized skew Pieri coefficients $c^{f^\perp}_{\mu,\nu}(q,t)$ and Pieri coefficients $d^f_{\mu,\nu}(q,t)$ by the formulas

$$f^\perp \tilde{H}_\mu(X;q,t) = \sum_{\substack{\nu \\ \nu\to\mu}} c^{f^\perp}_{\mu,\nu}(q,t)\tilde{H}_\nu(X;q,t) \tag{9}$$

$$f\tilde{H}_\nu(X;q,t) = \sum_{\substack{\mu \\ \mu\leftarrow\nu}} d^f_{\mu,\nu}(q,t)\tilde{H}_\mu(X;q,t). \tag{10}$$

Many of the identities in this paper are expressed using plethystic notation, defined as follows. If $p_k(X) = \sum_i x_i^k$ is the kth power sum, then for any expression E, the plethystic substitution of E into p_k is obtained by replacing all indeterminates in E by their kth powers. We denote this by $p_k[E]$, so for example

$$p_k[X(1-t)] = p_k(X)(1-t^k). \tag{11}$$

For any symmetric function $f(X)$, we define $f[E]$ by first expressing f as a polynomial in the p_k, then replacing each p_k by $p_k[E]$.

The $c^{f^\perp}_{\mu,\nu}$ and the $d^f_{\mu,\nu}$ are related via [GH02, (3.5)]

$$c^{f^\perp}_{\mu,\nu} w_\nu = d^{\omega f[X/M]}_{\mu,\nu} w_\mu, \tag{12}$$

where ω is the linear operator on symmetric functions satisfying $\omega s_\lambda = s_{\lambda'}$. Note $d^{\omega h_1[X/M]}_{\mu,\nu} = d^{h_1[X]}_{\mu,\nu}/M$. We abbreviate $c^{h_1\perp}_{\mu,\nu}(q,t)$ by $c_{\mu,\nu}$ and $d^{h_1[X/M]}_{\mu,\nu}(q,t)$ by $d_{\mu,\nu}$.

A special case of Macdonald's Pieri formulas [Mac95, Section 6.6] gives an expression for $d_{\mu,\nu}$ as a quotient of factors of the form $(t^a q^b - t^c q^d)$, where a, b, c, d have simple combinatorial descriptions. Garsia found a simplification in this formula, which Garsia and Zabrocki used to obtain the $k = 1$ case of the following summation formula [GZ05]. The proof of the result for general k appears in [BGHT99] and [Gar10].

$$\sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu} T^k = \begin{cases} 1/M & \text{if } k = 0, \\ (-1)^{k-1} e_{k-1}[MB_\nu - 1]/M & \text{if } k \geq 1, \end{cases} \tag{13}$$

where throughout this article T is an abbreviation for T_μ/T_ν. Eq. (13) is closely related to a corresponding summation formula involving the $c_{\mu,\nu}$ [GT96, Theorem 2.2].

As an exercise in plethystic notation, we show that (13) implies the following.

Lemma 1

$$\sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu}(1-T)T^k = \begin{cases} 0 & \textit{if } k = 0, \\ (-1)^{k-1} e_k[MB_\nu]/M & \textit{if } k \geq 1 \end{cases}. \tag{14}$$

Proof. If $k = 0$ the lemma follows immediately from (13), so assume $k \geq 1$. We begin with the "addition formula" (see [Hag08, pp. 19–22])

$$e_k[X - Y] = \sum_{j=0}^{k} e_j[X] e_{k-j}[-Y]. \tag{15}$$

This implies that

$$e_{k-1}[MB_\nu - 1] = \sum_{j=0}^{k-1} e_j[MB_\nu] e_{k-1-j}[-1]. \tag{16}$$

Now for any symmetric function f of homogeneous degree n,

$$f[-X] = (-1)^n \omega f(X), \tag{17}$$

so in particular $e_j[-1] = (-1)^j h_j[1] = (-1)^j$. Using this, the right-hand-side of (16) simplifies to

$$\sum_{j=0}^{k-1} e_j[MB_\nu](-1)^{k-1-j}. \tag{18}$$

Thus

$$\begin{aligned}
\sum_{\substack{\mu \\ \mu \leftarrow \nu}} Md_{\mu,\nu}(1-T)T^k &= \sum_{\substack{\mu \\ \mu \leftarrow \nu}} Md_{\mu,\nu}T^k - \sum_{\substack{\mu \\ \mu \leftarrow \nu}} Md_{\mu,\nu}T^{k+1} && (19)\\
&= (-1)^{k-1} \sum_{j=0}^{k-1} e_j[MB_\nu](-1)^{k-1-j} + (-1)^{k-1} \sum_{j=0}^{k} e_j[MB_\nu](-1)^{k-j} && (20)\\
&= (-1)^{k-1} e_k[MB_\nu]. && (21)
\end{aligned}$$

□

The following simple fact will be useful later.

Lemma 2

$$(-1)^{k-1} e_k[M]/M = \frac{t^k - q^k}{t - q} \qquad k \geq 1. \tag{22}$$

Proof. Begin by setting $X = 1 - q$, $Y = 1 - t$ in the Cauchy identity

$$e_k[XY] = \sum_{\lambda \vdash k} s_\lambda[X] s_{\lambda'}[Y]. \tag{23}$$

Using the well-known fact that for any $z \in \mathbb{C}$, $\lambda \vdash n$,

$$s_\lambda[1 - z] = \begin{cases} (-z)^r(1-z) & \text{if } \lambda = (n-r, 1^r), \quad 0 \leq r \leq n-1, \\ 0 & \text{else} \end{cases} \tag{24}$$

we get

$$e_k[M] = \sum_{r=0}^{k-1} (-q)^r (1-q)(-t)^{k-1-r}(1-t), \tag{25}$$

which is equivalent to (22). □

3 A New Recursive Procedure to Generate the Hilbert Series

By definition we have

$$e_1^\perp \tilde{H}_\mu(X;q,t) = \sum_{\substack{\nu \\ \nu\to\mu}} c_{\mu,\nu} \tilde{H}_\nu(X;q,t). \tag{26}$$

Taking the scalar product of both sides with respect to h_1^{n-1} we get

$$\langle e_1^\perp \tilde{H}_\mu, h_1^{n-1}\rangle = \langle \tilde{H}_\mu, e_1 h_1^{n-1}\rangle = F_\mu = \sum_{\substack{\nu \\ \nu\to\mu}} c_{\mu,\nu} F_\nu. \tag{27}$$

Plugging this recurrence for the F_μ into (7) yields

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\mu\vdash n} \frac{T_\mu M \Pi_\mu B_\mu}{w_\mu} \sum_{\substack{\nu \\ \nu\to\mu}} c_{\mu,\nu} F_\nu \tag{28}$$

$$= \sum_{\nu\vdash n-1} F_\nu M \sum_{\substack{\mu \\ \mu\leftarrow\nu}} \frac{B_\mu \Pi_\mu c_{\mu,\nu} T_\mu}{w_\mu}. \tag{29}$$

Now from (4) we see

$$B_\mu = B_\nu + T, \qquad \Pi_\mu = \Pi_\nu (1-T). \tag{30}$$

Using this and the $f = e_1$ case of (12) in (29) we get

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu\vdash n-1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu} (B_\nu + T)(1-T)T. \tag{31}$$

By (14) this implies

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu\vdash n-1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \left(\frac{e_1[MB_\nu]}{M} \frac{e_1[MB_\nu]}{M} - \frac{e_2[MB_\nu]}{M} \right). \tag{32}$$

(Although $e_1[MB_\nu]/M$ can be expressed more simply as $e_1[B_\nu]$, leaving (32) in the above form will prove more useful in the sequel.)

We now iterate the argument; first re-index the sum in (32) as a sum over $\mu \vdash n-1$, and replace F_μ by $\sum_{\nu\to\mu} c_{\mu,\nu} F_\nu$. Then write B_μ as $B_\nu + T$ as before, and reverse summation to get

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu\vdash n-2} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \tag{33}$$

$$\times \sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu} (1-T)T \left(\frac{e_1[M(B_\nu+T)]}{M} \frac{e_1[M(B_\nu+T)]}{M} - \frac{e_2[M(B_\nu+T)]}{M} \right).$$

Note that by (15), for $k \geq 1$

$$(-1)^{k-1} \frac{e_k[M(B_\nu+T)]}{M} = b_k + T^k a_k + \sum_{j=1}^{k-1} -M b_j T^{k-j} a_{k-j}, \tag{34}$$

where we have abbreviated $(-1)^{j-1}e_j[M]/M$ by a_j and $(-1)^{j-1}e_j[MB_\nu]/M$ by $b_j = b_j(\nu)$. Here we have used the fact that $e_k[MT]/M = T^k e_k[M]/M$ (since for any expression $p_j[XT] = T^j p_j[X]$). Note also that $a_1 = 1$. The inner sum in (33) thus becomes

$$\sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu}(1-T)T\left((b_1+Ta_1)^2 + b_2a_1 + T^2a_2 - Mb_1Ta_1\right) \tag{35}$$

$$= b_1^3 + 2b_1a_1b_2 + a_1^2b_3 + a_1b_2b_1 - Mb_1a_1^2b_2 + a_1a_2b_3 \tag{36}$$

by (14).

Let

$$A_1 = b_1 \tag{37}$$

$$A_2 = b_1^2 + b_2a_1 \tag{38}$$

$$A_3 = b_1^3 + 2b_1a_1b_2 + a_1^2b_3 + a_1b_2b_1 - Mb_1a_1^2b_2 + a_1a_2b_3. \tag{39}$$

The above discussion implies

Theorem 1 *For $p \in \mathbb{N}$, $1 \le p \le n$,*

$$Hilb(DR_n) = \sum_{\nu \vdash n-p+1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} A_p, \tag{40}$$

where $A_p = A_p(\nu)$ is a certain polynomial in the a_i, b_i. Moreover, A_p can be calculated recursively from A_{p-1} by the following procedure. First replace each b_k in A_{p-1} by $b_k + T^k a_k - \sum_{j=1}^{k-1} Mb_jT^{k-j}a_{k-j}$. Then multiply the resulting expression out to form a polynomial in T, say

$$\sum_j c_j T^j. \tag{41}$$

Finally, replace T^j by b_{j+1}, i.e.

$$A_p = \sum_j c_j b_{j+1}. \tag{42}$$

(We replace T^j by b_{j+1} since, after multiplying the expression above out to get $\sum c_jT^j$, we still have another factor of T coming from the outer sum. Applying (14) replaces T^{j+1} by b_{j+1}.)

We now give a non-recursive expression for A_p. Let Q_n denote the set of all $n \times n$ upper-triangular matrices C of nonnegative integers which satisfy

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = 1, \quad \text{for each } j,\ 1 \le j \le n. \tag{43}$$

For example,

$$Q_1 = \{\begin{bmatrix}1\end{bmatrix}\} \tag{44}$$

$$Q_2 = \{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\} \tag{45}$$

$$Q_3 = \{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&2&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&1&1\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&2\\0&0&3\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\0&0&2\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&1\\0&0&3\end{bmatrix}\}. \tag{46}$$

Geometrically, the condition (43) says that for all j, if we add all the entries of C in the jth row together, and then subtract all the entries in the jth column above the diagonal, we get 1. Note that these conditions imply that each row of C must have at least one positive entry.

For $C \in Q_n$, let $\mathrm{Pos}(C)$ denote the multiset of positive entries in C, and $\mathrm{pos}(C)$ its cardinality.

Theorem 2 *For $1 \le p \le n$ and A_p, b_j, a_j as above,*

$$A_p = \sum_{C \in Q_p} (-M)^{pos(C)-n} \prod_{c_{ii} \in Pos(C)} b_{c_{ii}} \prod_{\substack{c_{ij} \in Pos(C) \\ i<j}} a_{c_{ij}}. \tag{47}$$

Proof. Given $C \in Q_n$, note that each row of C must have at least one positive entry, and so $\mathrm{pos}(C) - n$ can be viewed as the number of pairs of positive entries in the same row, with no other positive entries between them. Let C' denote the element of Q_{n-1} obtained by adding each of the entries in the last column of C to the diagonal, then removing the last column and last row, i.e.

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } j \neq i \\ c_{ii} + c_{in} & \text{if } j = i \end{cases} \qquad 1 \le i, j \le n-1. \tag{48}$$

For example, if

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \tag{49}$$

then

$$C' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{50}$$

The various elements $C \in Q_n$ for which C' is a fixed element W of Q_{n-1} can be obtained by the following procedure. First choose an integer k for each $1 \le i \le n-1$ such that $0 \le k \le w_{ii}$, and set $c_{ii} = w_{ii} - k$, $c_{in} = k$. Then let $c_{nn} = 1 + \sum_{j=1}^{n-1} c_{jn}$ and $c_{ij} = w_{ij}$ for $i < j \le n-1$. In this procedure, if $0 < k < w_{ii}$ the number of positive entries in row i increases by one, or equivalently we get a new pair of positive entries with no positive entry between them. It is now easy to see this process mirrors the recursive description of the A_p described above, and the result follows by induction on n. $\square$

Corollary 1

$$Hilb(DR_n) = \sum_{C \in Q_n} (-M)^{pos(C)-n} \prod_{\substack{c_{ij} \in Pos(C) \\ 1 \le i \le j \le n}} [c_{ij}]_{q,t}, \tag{51}$$

where $[k]_{q,t} = (t^k - q^k)/(t-q)$ is the q,t-analog of the integer k.

Proof. Letting $p = n$ in (40), the sum over μ contains only one term, namely $\mu = (1)$. Now if $\mu = (1)$, (22) shows $b_j(\mu) = a_j(\mu) = [j]_{q,t}$. From (4) one sees that $B_{(1)} = 1, \Pi_{(1)} = 1, w_{(1)} = M$, and so (40) reduces to (51). □

Example 1 *The weights associated to the elements of Q_3, listed in the same left-to-right order as in (46) are*

$$1, \quad t+q, \quad t+q, \quad -M(t+q), \quad (t+q)(t^2+qt+q^2), \quad t+q, \quad t^2+qt+q^2. \tag{52}$$

Thus Hilb(DR_3) is the sum of these terms, namely

$$1 + 2q + 2t + 2q^2 + 3qt + 2t^2 + q^3 + q^2t + qt^2 + t^3. \tag{53}$$

The sequence $1, 2, 7, 40, 357, 4820, \ldots$ consisting of the cardinalities of the sets $Q_1, Q_2, Q_3, \ldots$ form entry $A008608$ in Sloane's on-line encyclopedia of integer sequences. In fact, it was comparing the number of monomials in A_n for small n with sequences in Sloane's encyclopedia that led the author to the discovery of the non-recursive expression for the A_n in terms of the elements of Q_n. The sequence was introduced to Sloane's list by Glenn Tesler, who in a private conversation with the author said they arose in unpublished work of Tesler's from the late 1990's on plethystic expressions for Macdonald's $D_{n,r}$ operators. Although Tesler doesn't recall any further details about this work, we will refer to elements of Q_n as "Tesler matrices".

The explicit formula (51) for Hilb(DR_n) can be formulated as a constant term identity.

Corollary 2 *For $n \geq 1$, Hilb(DR_n) is the coefficient of $z_1 z_2 \cdots z_n$ in*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \leq i < j \leq n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}. \tag{54}$$

Proof. Let

$$f(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}. \tag{55}$$

Expanding $f(x)$ as a Taylor's series in x about $x = 0$ gives

$$f(x) = 1 - M(x[1]_{q,t} + x^2[2]_{q,t} + x^3[3]_{q,t} + \cdots + x^k[k]_{q,t} + \cdots). \tag{56}$$

Selecting a term of the form $-M(z_i/z_j)^k[k]_{q,t}$ in $f(z_i/z_j)$ corresponds to setting $c_{ij} = k$ in an associated matrix $C \in Q_n$, while selecting a term of the form $-M(z_i)^k[k]_{q,t}$ corresponds to setting $c_{ii} = k$. The condition (43) for the matrix in (51) translates into taking the coefficient of $z_1 z_2 \cdots z_n$ in (54). □

Remark 2 *Let*

$$\Omega(X) = \sum_{n=0}^{\infty} h_n[X], \tag{57}$$

where $h_n = s_n$ is the complete homogeneous symmetric function. Then (54) can be written as

$$\frac{1}{(-M)^n}\Omega(-MZ_n)\prod_{1\le i<j\le n}\frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}, \tag{58}$$

where $Z_n = \{z_1,\dots,z_n\}$. The argument in the proof of Corollary 2 shows moreover that for $1 \le p \le n$,

$$A_p(\mu) = \frac{1}{(-M)^p}\Omega\left[-MB_\mu Z_p\right]\prod_{1\le i<j\le p}\frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}\Big|_{z_1z_2\cdots z_p}. \tag{59}$$

Here $|_{z_1\cdots z_p}$ means "take the coefficient of $z_1\cdots z_p$".

Remark 3 *Instead of using the recurrence for F_μ from (27), we could apply $e_1^\perp$ to ∇e_n and use (26) to get*

$$e_1^\perp\nabla e_n = \sum_{\nu\vdash n-1}\frac{T_\nu\tilde{H}_\nu M\Pi_\nu}{w_\nu}\left(\frac{e_1[MB_\nu]}{M}\frac{e_1[MB_\nu]}{M}-\frac{e_2[MB_\nu]}{M}\right). \tag{60}$$

More generally, for any integer p, $0 \le p \le n-1$ we have

$$(e_1^\perp)^p\nabla e_n = \sum_{\nu\vdash n-p}\frac{T_\nu\tilde{H}_\nu M\Pi_\nu}{w_\nu}A_{p+1}(\nu). \tag{61}$$

For any partition λ, as usual let $h_\lambda = \prod_i s_{\lambda_i}$. Then $\langle\nabla e_n, h_\lambda\rangle$ is the coefficient of the monomial symmetric function m_λ in the expansion of ∇e_n into monomials. To try and prove this is in $\mathbb{Z}[q,t]$ by the method of this section we could start by writing $h_k^\perp\nabla e_n$ as a sum over partitions of $n-k$, for k an integer in the range $1 \le k \le n-1$. We would then need a formula like (14) with $d_{\mu,\nu} = d_{\mu,\nu}^{e_1[X/M]}$ replaced by $d_{\mu,\nu}^{e_k[X/M]}$, but currently no such formula is known.

4 The m-parameter

The formula ∇e_n for the Frobenius series of DR_n is a special case of a more general result (also due to Haiman [Hai02]) which says that for any positive integer m, $\nabla^m e_n$ is the Frobenius series of a certain S_n-module $\mathrm{DR}_n^{(m)}$. Hence, from (6) we have

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \langle\nabla^m e_n, h_1^n\rangle \tag{62}$$

$$= \sum_{\substack{\mu\\ \mu\vdash n}}\frac{T_\mu^m F_\mu M\Pi_\mu B_\mu}{w_\mu}. \tag{63}$$

Using the recursive formula for F_μ, we can express this as a sum over partitions ν of $n-1$ as before. The only difference is that in the inner sum in (31), $(1-T)T$ gets replaced by $(1-T)T^m$, and similarly in the outer sum T_ν gets replaced by T_ν^m, i.e.

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \sum_{\nu\vdash n-1}\frac{T_\nu^m F_\nu M\Pi_\nu}{w_\nu}\sum_{\mu\leftarrow\nu}d_{\mu,\nu}(B_\nu+T)(1-T)T^m. \tag{64}$$

It follows that for any $1 \le p \le n$,

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \sum_{\mu \vdash n-p+1} \frac{T_\nu^m F_\nu M \Pi_\nu}{w_\nu} A_p^{(m)}, \tag{65}$$

where $A_p^{(m)} = A_p^{(m)}(\mu)$ is a polynomial in the b_j, a_j as before. We have $A_1^{(m)}(\mu) = b_1$, and for $p > 1$, we can construct $A_p^{(m)}$ recursively by the following procedure. First, replace each b_k in $A_{p-1}^{(m)}$ by $b_k + T^k a_k + \sum_{j=1}^{k-1} -M b_j T^{k-j} a_{k-j}$. Then, multiply the resulting expression out to form a polynomial in T say

$$\sum_j c_j T^j. \tag{66}$$

Finally, replace T^j by b_{j+m}, i.e.

$$A_p^{(m)} = \sum_j c_j b_{j+m}. \tag{67}$$

In terms of the Tesler matrices, we want the "hook sums" to equal $(1, m, m, \ldots, m)$ instead of $(1, 1, \ldots, 1)$. To be precise, define $Q_n^{(m)}$ to be the set of upper-triangular matrices C of nonnegative integers satisfying

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = \begin{cases} 1 & \text{if } j = 1, \\ m & \text{if } 2 \le j \le n \end{cases}. \tag{68}$$

We get the following extensions of the results in the previous section.

Theorem 3 *For $1 \le p \le n$, $m \ge 1$, and $A_p^{(m)}$, b_j, a_j as above,*

$$A_p^{(m)} = \sum_{C \in Q_p^{(m)}} (-M)^{pos(C)-n} \prod_{c_{ii} \in Pos(C)} b_{c_{ii}} \prod_{\substack{c_{ij} \in Pos(C) \\ i<j}} a_{c_{ij}}. \tag{69}$$

Furthermore, the special case $p = n$ of (65) reduces to

$$Hilb(DR_n^{(m)}) = \sum_{C \in Q_n^{(m)}} (-M)^{pos(C)-n} \prod_{\substack{c_{ij} \in Pos(C) \\ 1 \le i \le j \le n}} [c_{ij}]_{q,t}. \tag{70}$$

Corollary 3 *For $n \ge 1$, $Hilb(DR_n^{(m)})$ is the coefficient of $z_1 z_2^m z_3^m \cdots z_n^m$ in*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \le i < j \le n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}. \tag{71}$$

5 Conjectures and Open Questions

5.1 Tesler matrices with more general hook sums

In general the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ in (71) is not a positive polynomial in q,t, but Maple calculations suggest it is positive if the α_i are positive and nondecreasing.

Conjecture 2 *For $n \geq 1$ and α the reverse of a partition (so $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$)*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \leq i<j \leq n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)} \Big|_{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}} \in \mathbb{N}[q,t]. \tag{72}$$

Equivalently, the weighted sum over Tesler matrices with hook sums $\alpha_1, \ldots, \alpha_n$ is in $\mathbb{N}[q,t]$.

We now evaluate the $q=0$ case of Conjecture 2. To make the statement more compatible with the classical theory of symmetric functions, we first reverse α to form a partition μ, and also reverse the z variables, resulting in the statement

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \leq i<j \leq n} \frac{(1-z_j/z_i)(1-qtz_j/z_i)}{(1-qz_j/z_i)(1-tz_j/z_i)} \Big|_{z_1^{\mu_1} z_2^{\mu_2} \cdots z_n^{\mu_n}} \in \mathbb{N}[q,t]. \tag{73}$$

The Hall-Littlewood polynomial $Q_\mu(X;t)$ can be defined [Mac95, p. 209-211] as

$$Q_\mu(X;t) = \Omega[(1-t)XZ_n] \prod_{1 \leq i<j \leq n} \frac{(1-z_j/z_i)}{(1-tz_j/z_i)} \Big|_{z_1^{\mu_1} z_2^{\mu_2} \cdots z_n^{\mu_n}}. \tag{74}$$

Using this and (17) it follows that when $q=0$, (73) reduces to $Q_\mu[-1;t]/(t-1)^n$. From [Mac95, Exercise 3, p. 226] we have

$$Q_\mu\left[\frac{1-z}{1-t};t\right] = t^{n(\mu)} \prod_{i=1}^{n} (1-t^{1-i}z). \tag{75}$$

Letting $z=1/t$ and simplifying we get

$$\frac{Q_\mu[-1;t]}{(t-1)^n} = [n]!_t t^{n(\mu)+|\mu|-\binom{n+1}{2}}, \tag{76}$$

where $|\mu| = \sum_i \mu_i$. This verifies Conjecture 2 when $q=0$.

Remark 4 *The argument proving Theorem 3 shows that if $\alpha_1 = 1$, the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ in (72) can be obtained by starting with $\nabla^{\alpha_2} e_n$, applying $e_1^\perp$, then applying $\nabla^{\alpha_3-\alpha_2}$, applying $e_1^\perp$ again, then applying $\nabla^{\alpha_4-\alpha_3}$, etc. The author doesn't know if these polynomials have a representation-theoretic interpretation for $\alpha \neq (1,m,m,\ldots,m)$.*

5.2 Parking functions and Tesler matrices

In trying to show that (51) implies (3), a natural idea is to try and identify subsets of Tesler matrix terms from (51) which correspond to subsets of terms from (3). It is easy to see that the portion of (3) involving those parking functions where car n is in the bottom row equals (3) with n replaced by $n-1$, which conjecturally equals $\mathrm{Hilb}(\mathrm{DR}_{n-1})$. Similarly, if we restrict (51) to those matrices C with $c_{11}=1$, we get $\mathrm{Hilb}(\mathrm{DR}_{n-1})$. Maple calculations indicate the more general fact that the restriction of (51) to those matrices C with $c_{1i}=1$ equals the portion of (3) where the car in the bottom row is n if $i=1$ and $i-1$ if $i>1$. For example, the terms in (51) corresponding to the three matrices with $c_{1,2}=1$ are

$$t+q, \quad -M(t+q), \quad (t+q)(t^2+qt+t^2) \tag{77}$$

and the sum of these three terms reduces to $t^2+2tq+q^2+t^3+t^2q+tq^2+q^3$. One checks that this equals the sum of $q^{\mathrm{dinv}}t^{\mathrm{area}}$ over all parking functions on 3 cars with car 1 in the bottom row.

5.3 A refinement of the q,t-positivity

Note that $[k]_{q,t}$ can be expressed as $s_{k-1}(\{q,t\})$, i.e. the $(k-1)$st complete homogeneous symmetric function evaluated in the set of variables $\{q,t\}$. Also, $-M=t+q-1-qt$ equals $s_1-1-s_{1,1}$, also in the set of variables $\{q,t\}$. In (51) we can substitute in these Schur function formulations for $[k]_{q,t}$ and $-M$, multiply everything out using the Pieri rule for Schur function multiplication, and thereby obtain a formula for $\mathrm{Hilb}(\mathrm{DR}_n)$ in terms of Schur functions in the set of variables $\{q,t\}$. If we then cancel terms of the form s_λ where λ has more than two parts (which becomes zero since our set of variables has only two elements) it appears that the resulting expression is Schur-positive. For example, for $n=3$ the terms from (52) become

$$1, \quad s_1, \quad s_1, \quad (s_1-1-s_{1,1})s_1, \quad s_1s_2, \quad s_1, \quad s_2, \tag{78}$$

and the sum of these equals $1+2s_1+2s_2+s_{1,1}+s_2-s_{1,1,1}+s_3$. Since $s_{1,1,1}(\{q,t\})=0$, we can remove this leaving

$$\mathrm{Hilb}(\mathrm{DR}_3)=1+2s_1+2s_2+s_{1,1}+s_2+s_3. \tag{79}$$

F. Bergeron [Ber09, p.196] has previously conjectured a stronger statement, namely that

$$\mathrm{Hilb}(\mathrm{DR}_n)=\sum_{\sigma\in S_n} h_{\lambda(\sigma)}(\{q,t\}), \tag{80}$$

i.e. that for each permutation on n elements, there is some way of defining a partition $\lambda(\sigma)$ such that the sum of the $h_{\lambda(\sigma)}$ gives $\mathrm{Hilb}(\mathrm{DR}_n)$. Here $h_\lambda=\prod_i s_{\lambda_i}$ as before. When $n=3$, the expansion is

$$\mathrm{Hilb}(\mathrm{DR}_3)=1+2h_1+h_2+h_{1,1}+h_3, \tag{81}$$

in agreement with (79). Bergeron further conjectures that these sums have the remarkable property that if we evaluate them in the set of variables $\{q_1,q_2,\ldots,q_k\}$ we get the Hilbert series of diagonal coinvariants in k sets of variables, for any $k\geq 1$. We hope that further study of how the cancellation in identity (51) results in positivity will lead to progress on the $k=2$ case of Bergeron's conjecture.

5.4 Other questions

Conjecture 1 has a more general form [HHL+05] which gives a parking function model for $\langle \nabla^m e_n, h_\lambda \rangle$ for any positive integer m and any partition λ. Armstrong's hyperplane arrangement model [Arm10] includes a parameter m which gives a conjectured combinatorial expression for $\langle \nabla^m e_n, h_1^n \rangle$ for any $m \in \mathbb{Z}$. It would be interesting to know if a formula like (54) exists for $\langle \nabla^m e_n, h_\lambda \rangle$ for $m \in \mathbb{Z}$ and λ a partition.

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