THEOREMS AND CONJECTURES INVOLVING ROOK POLYNOMIALS WITH ONLY REAL ZEROS

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Abstract. Let $A = (a_{ij})$ be a real $n \times n$ matrix with non-negative entries which are weakly increasing down columns. If $B = (b_{ij})$ is the $n \times n$ matrix where $b_{ij} := a_{ij} + z$, then we conjecture that all of the roots of the permanent of B, as a polynomial in z, are real. Here we establish several special cases of the conjecture.

1. Introduction and Statement of Results

Throughout $A = (a_{ij})$ will denote an $n \times n$ matrix. A placement of rooks on the squares of A is non-attacking if no two rooks are in the same column, and no two are in the same row. We define the *weight* of such a placement to be the product of the entries in A which are under the rooks, and we define the kth rook number $r_k(A)$ to be the sum of these weights over all non-attacking placements of k rooks on A. Furthermore by convention $r_0(A) := 1$. When $n = 2$ these rook numbers are

$$r_2(A) = a_{11}a_{22} + a_{12}a_{21}, \quad r_1(A) = a_{11} + a_{12} + a_{21} + a_{22}, \quad \text{and} \quad r_0(A) = 1.$$

Note that $r_n(A)$ equals $\mathrm{per}(A)$, the permanent of A, defined as

$$\mathrm{per}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma_i}.$$

If each a_{ij} is zero or one, then A is called a *board*, and $r_k(A)$ is the number of placements of k rooks on the non-zero entries of A.

We call $R(z; A) := \sum_{k=0}^{n} r_k(A) z^k$ the *rook polynomial* of A. Nijenhuis [Nij] proved that if the a_{ij} are non-negative real numbers, then all the roots of $R(z; A)$ are real. One application is that the rook numbers $r_k(A)$ are log-concave (i.e. $r_i(A)^2 \geq r_{i-1}(A) r_{i+1}(A)$). This follows from the fact that if $f(z) = \sum_k b_k z^k$ has only real roots , then the b_k are log-concave. In fact more is true, it turns out that [p. 52, HLP]

$$b_k^2 \geq b_{k-1} b_{k+1} \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right).$$

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E. Bender noted that Nijenhuis' result follows from the Heilmann-Lieb Theorem [HeLi] which asserts that all of the roots of matching polynomials of simple graphs with non-negative weights are real (the Heilmann-Lieb Theorem has been extensively generalized in Theorem 3.3 of [Wag3]). To see this consider the complete bipartite graph G from n vertices on top to n vertices below, where the edge between vertex i above and vertex j below is assigned weight a_{ij}. Given a *matching* M with k edges (i.e. any selection of k edges no two of which share a common vertex) one obtains a rook placement by placing a rook on a_{ij} in A if and only if M contains the edge connecting i above to j below. So the weighted matching polynomial of G is just the rook polynomial of A.

Now we briefly describe various combinatorial investigations from which it is known that all of the roots of certain *natural* polynomials are real. A multiset is a set in which elements can occur more than once. A permutation σ of a multiset L of positive integers is a linear list $\sigma_1\sigma_2\cdots$ of the elements of L, and a *descent* of such a σ is a positive integer i such that $\sigma_i > \sigma_{i+1}$. Let $N_k(\mathbf{v})$ denote the number of permutations of the multiset

$$\mathbf{v} := \{1^{v_1}2^{v_2}\cdots m^{v_m}\}$$

(i^{v_i} denotes v_i copies of i) with exactly $k-1$ descents. Simion proved that all the roots of

$$\sum_{k\geq 1} N_k(\mathbf{v})z^k \tag{1}$$

are real [Sim], and consequently that the $N_k(\mathbf{v})$ are log-concave. For example, if $\mathbf{v} = \{1^3 2^1\}$, the 4 permutations are

$$1112 \qquad 1121 \qquad 1211 \qquad 2111,$$

and so (1) is $3z^2+z$. Simion's result proves a special case of the famous Neggers-Stanley "Poset Conjecture" [Bre], [Wag2], which says that a certain polynomial associated to a labelled poset has only real zeros.

The *hit polynomial* $T(z;A)$ of A is

$$T(z;A) = \sum_{k=0}^{n} k!\, r_{n-k}(A)(z-1)^{n-k}.$$

If A is a board, then its coefficient of z^k is called the k^{th} *hit number*, the number of ways of placing n non-attacking rooks on A where exactly k rooks lie on non-zero entries. In their seminal paper on rook theory [KaRi] (see also [Rio]), Riordan and Kaplansky showed that

$$\prod_{i=1}^{m} v_i! \sum_{k\geq 1} N_k(\mathbf{v})z^{k-1}$$

is the hit polynomial for the "Simon Newcomb board" (see Figure 1). This is the matrix where the entries in the first v_1 columns are all zero, and the next v_2 columns have ones in the bottom v_1 rows and zeros above, etc. Therefore Simion's result implies that all the roots of these polynomials are real, and of course that the hit numbers of these boards are log-concave.

Using Laguerre's Theorem, that if $\sum_k b_k z^k$ has only real roots then $\displaystyle\sum_k \frac{b_k}{k!} z^k$ also has only real roots ([Lag]; see also [Th. 11, GaWa]), one can deduce that if $T(z;A)$ has only real roots, then so too does $R(z;A)$. The converse of Laguerre's Theorem is false, and in particular the hit polynomials of boards do not generally have only real roots. However we will show that this is true for Ferrers boards.

Definition 1. *Let A be a board with the property that if $a_{ij} = 1$, then $a_{st} = 1$ whenever $i \leq s \leq n$ and $j \leq t \leq n$. Then A is called a Ferrers board.*

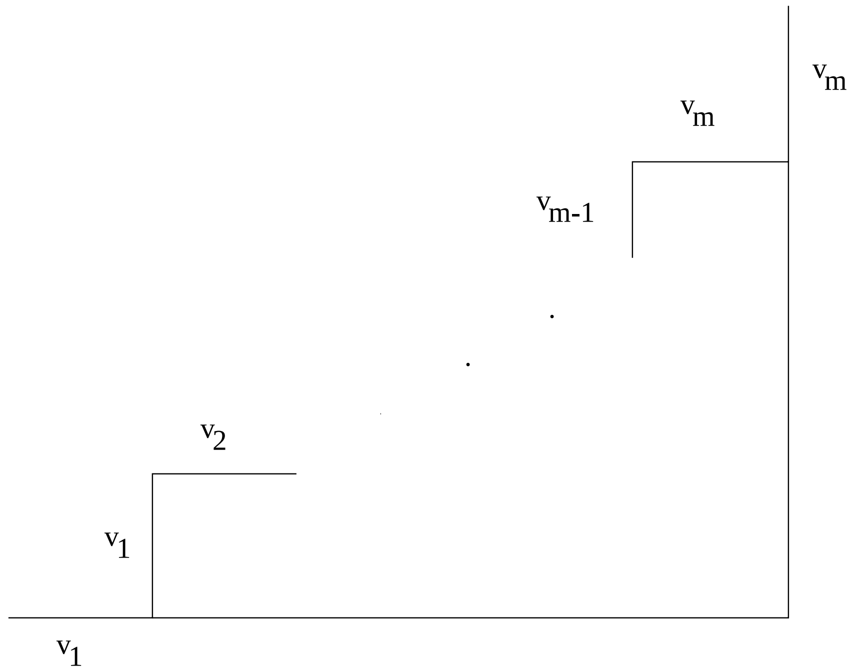

FIGURE 1. The Simon Newcomb board associated to the multiset $\{1^{v_1} \cdots m^{v_m}\}$.

Theorem 1. *All the roots of the hit polynomial of a Ferrers board are real.*

An r-descent of a multiset permutation σ is a value of i such that $\sigma_i - \sigma_{i+1} \geq r$. Let $N_k(\mathbf{v}, r)$ be the number of permutations of the multiset $\mathbf{v} = \{1^{v_1} \cdots m^{v_m}\}$ containing exactly $k-1$ r-descents. For example, if $\mathbf{v} = \{1^1 2^1 3^1\}$, then the permutations are

$$123 \qquad 132 \qquad 213 \qquad 231 \qquad 312 \qquad 321.$$

Of these, 231 and 312 have one 2-descent, so $N_2(\{1^1 2^1 3^1\}, 2) = 2$.

The numbers $N_k(\mathbf{v}, r)$ are the hit numbers for special Ferrers boards, ones which generalize the Simon Newcomb boards [Hag1],[Hag2]. Thus Theorem 1 implies the following, which reduces to Simion's result when $r = 1$.

Corollary 1. *Let r be a positive integer, and let $\mathbf{v}$ be a multiset of positive integers. Then all the roots of*

$$\sum_k N_k(\mathbf{v}, r) z^k$$

are real.

It is natural to seek a weighted version of Theorem 1 analogous to the Heilmann-Lieb theorem.

Definition 2. *Let A be a real $n \times n$ matrix. We call A a Ferrers matrix if $a_{ij} \leq a_{st}$ whenever $i \leq s \leq n$ and $j \leq t \leq n$.*

Conjecture 1. *If A is a Ferrers matrix, then all the roots of the hit polynomial $T(z;A)$ are real.*

Note that if $a_{ij} \in \{0,1\}$, then Conjecture 1 holds by Theorem 1. In section 2 we prove Theorem 1 and also the next result.

Theorem 2. *Conjecture 1 is true if $n \le 3$.*

Let J_n denote the $n \times n$ matrix, all of whose entries equal one. When expanded out in powers of z, $\mathrm{per}(zJ_n + A)$ becomes

$$\sum_{k=0}^{n} k! r_{n-k}(A) z^k.$$

Conjecture 1 is thus equivalent to the assertion that $\mathrm{per}(zJ_n + A)$ has all real zeros whenever A is a Ferrers matrix.

Expressing Conjecture 1 (and also Conjecture 2 below) in terms of permanents has several advantages. One implication is that if $z > -\min_{ij} a_{ij}$ (resp. $z < -\max_{ij} a_{ij}$) all the entries of $zJ_n + A$ are positive (resp. negative) in which case $\mathrm{per}(zJ_n + A)$ cannot be zero. Hence if A is Ferrers all real roots of $\mathrm{per}(zJ_n + A)$ are in the interval $[-a_{nn}, -a_{11}]$. It is also easy to see that in Conjecture 1 we can assume without loss of generality that

$\underline{C1}$ $\quad 0 \le a_{ij} \le 1$ for all $1 \le i, j \le n$.

Less trivially, we can develop recursions by expanding the permanent in terms of permanental minors, a method which plays a crucial role in the broadest special case of Conjecture 1 we can prove. We require some additional hypotheses. Define $a_{0j} := 0$ for all $0 \le j \le n+1$, $a_{i0} := 0$ for all $0 \le i \le n+1$, $a_{n+1,j} := 1$ for all $1 \le j \le n+1$, and $a_{i,n+1} := 1$ for all $1 \le i \le n+1$. Also define conditions

$\underline{C2}$ $\quad$ For all $1 \le r < s \le n$ and $0 \le t < u < v \le n+1$:

$$(a_{ru} - a_{rt})(a_{sv} - a_{su}) \le (a_{rv} - a_{ru})(a_{su} - a_{st}),$$

and

$\underline{C3}$ $\quad$ For all $0 \le t < u < v \le n+1$ and $1 \le r < s \le n$:

$$(a_{ur} - a_{tr})(a_{vs} - a_{us}) \le (a_{vr} - a_{ur})(a_{us} - a_{ts}).$$

Theorem 3. *If A is a Ferrers matrix satisfying $\underline{C1}$, $\underline{C2}$, and $\underline{C3}$, then per$(zJ_n + A)$ has all its zeros in the interval $[-1, 0]$.*

In section 3 we prove Theorem 3 using mathematical induction and the method of interlacing roots. By the same method, we also prove that Conjecture 1 holds for any $n \times n$ Ferrers matrix A provided $a_{i,n} \le a_{i+1,1}$ for all $1 \le i \le n-1$.

By running a program written in Maple which constructs matrices using a random number generator, the authors and E. R. Canfield have verified the following stronger form of Conjecture 1 holds for over 100,000 matrices of sizes ranging from 3×3 to 13×13.

Conjecture 2. *If A is a matrix with real entries where $a_{ij} \leq a_{sj}$ when $s \geq i$, then all the roots of the hit polynomial $T(z;A)$ are real.*

Since permuting the columns of a matrix doesn't change the rook numbers, the $a_{ij} \in \{0,1\}$ case of Conjecture 2 also follows from Theorem 1.

Example 1. If $a_{ij} := q_j$, then $\mathrm{per}(zJ_n + A) = n!(z+q_1)(z+q_2)\cdots(z+q_n)$. Thus the set of polynomials with only real roots and leading coefficient $n!$ are all examples of the phenomenon asserted by Conjectures 1 and 2. Furthermore Conjecture 1 is equivalent to the claim that for any Ferrers matrix A there exists another matrix A' which is constant down columns and *rook equivalent* to A (i.e. $r_k(A) = r_k(A')$ for $0 \leq k \leq n$).

2. The Tableau Method

We now turn to the proofs of Theorems 1 and 2.

Proof of Theorem 1: Goldman, Joichi, and White [GJW] proved that

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k}(A) = \prod_{i=1}^{n}(x+c_i-i+1), \tag{2}$$

where c_i is the number of ones in (the height of) the ith column of the Ferrers board A. This is equivalent to

$$\sum_{k=0}^{n} \binom{x+k}{n} t_k(A) = \prod_{i=1}^{n}(x+c_i-i+1), \tag{3}$$

where $t_k(A)$ is the coefficient of z^k in $T(z;A)$. The result now follows easily by [Bre, p. 43].

Theorem. *(Brenti) Let $f(x) = \sum_{k=0}^{n}\binom{x+k}{n}b_k$ be a polynomial with all real zeros, with smallest root $\lambda(f)$ and largest root $\Lambda(f)$. If all the integers in the intervals $[\lambda,-1]$ and $[0,\Lambda]$ are also roots of f, then all the roots of $\sum_{k=0}^{n} b_k x^k$ are real.* □

Proof of Theorem 2: For a 2×2 matrix A, the discriminant of $\mathrm{per}(zJ_2 + A)$ is

$$(a_{11}+a_{12}+a_{21}+a_{22})^2 - 8(a_{11}a_{22}+a_{12}a_{21})$$

$$= a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 + 2a_{11}a_{12} + 2a_{11}a_{21} + 2a_{12}a_{22} + 2a_{21}a_{22} - 6a_{11}a_{22} - 6a_{12}a_{21}.$$

The roots will all be real if and only if this expression is non-negative. We now describe a method by which this becomes visibly evident if A is Ferrers.

By replacing z by $z - \min_{ij} a_{ij}$ we can assume without loss of generality that $a_{ij} \geq 0$. Since A is Ferrers, either

$$0 \leq a_{11} \leq a_{12} \leq a_{21} \leq a_{22} \tag{4}$$

or

$$0 \leq a_{11} \leq a_{21} \leq a_{12} \leq a_{22}, \tag{5}$$

or both. If (4) holds, replace a_{11} by w_1, a_{12} by w_1+w_2, a_{21} by $w_1+w_2+w_3$, and a_{22} by $w_1+w_2+w_3+w_4$. The inequalities in (4) then imply that all the w_i are non-negative. The discriminant now becomes

$$w_4^2 + w_2^2 + 4w_3^2 + 6w_4w_2 + 4w_4w_3 + 4w_2w_3.$$

Note that this is w-monomial positive, i.e. the coefficient of every possible monomial in the w_i is non-negative. If (5) holds, replace a_{11} by w_1, a_{21} by $w_1 + w_2$, etc. By symmetry, the discriminant will again be w-monomial positive. Thus Conjecture 1 holds if $n = 2$.

To the set of inequalities in (4), we can associate the array

$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$$

and to (5) the array

$$\begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}.$$

These arrays are special cases of what are known as standard tableau of square $n \times n$ shape, that is $n \times n$ Ferrers matrices the set of whose entries equals the set $\{1, 2, \ldots, n^2\}$. In the second tableau the 2 in square $(2, 1)$ indicates a_{21} is the second-smallest a_{ij}. More generally, we can clearly break up the set of $n \times n$ Ferrers matrices into subsets, according to how the a_{ij} are related to one another, with a standard tableau of square $n \times n$ shape associated to each subset. The square containing k in such a tableau will indicate the kth-smallest a_{ij}, which we can then parameterize as $w_1 + \ldots + w_k$, with w_i non-negative. Note that if some of the a_{ij} are equal, there is some ambiguity in selecting a representative tableau, but different tableau choices will have the effect of simply replacing certain w_i by other w_i in the discriminant, which will not affect monomial positivity.

Since a cubic polynomial has only real roots if and only if its discriminant is non-negative, to complete the proof of Theorem 2 it suffices to show that for all possible standard tableaux in the shape of a 3×3 square, the discriminant of $T(z; A)$, when expressed in terms of the w_i, is w-monomial positive. This has been verified via a long Maple calculation (here the discriminant of $T(z; A)$ typically consists of over 1500 monomials in the w_i). $\square$

A similar argument proves the $n = 2$ case of Conjecture 2; by Theorem 2 we can assume that A is of the form

$$A := \begin{bmatrix} w_1 & w_1 + w_2 \\ w_1 + w_2 + w_3 + w_4 & w_1 + w_2 + w_3 \end{bmatrix},$$

in which case the discriminant of $\mathrm{per}(A + zJ_n)$ equals

$$(w_2 - w_4)^2 + 4w_2w_3 + 4w_3^2 + 4w_3w_4.$$

Although not monomial positive, it is still clearly positive.

The $n = 3$ case of Conjecture 2 is still open, as the discriminant of $T(z; A)$ for a typical matrix A (parameterized to reflect the column monotone condition) consists of over a thousand monomials, many of which have negative coefficients.

3. Interlacing Arguments

Before presenting the proof of Theorem 3, we consider three particularly interesting special cases.

$\underline{\mathrm{U}}$ $\quad a_{ij} \in \{0, 1\}$ for all $1 \le i, j \le n$.

$\underline{\Pi}$ $\quad$ There exist positive real numbers p_i for $1 \le i \le n$ and q_j for $1 \le j \le n$ such that

$$a_{ij} = p_i q_j \text{ for all } 1 \le i, j \le n.$$

$\underline{\Sigma}$ $\quad$ There exist positive real numbers p_i for $1 \le i \le n$ and q_j for $1 \le j \le n$ such that

$$a_{ij} = p_i + q_j \text{ for all } 1 \le i, j \le n.$$

Lemma 1. *If A is a Ferrers matrix satisfying $\underline{\mathrm{U}}$, then A satisfies $\underline{C1}$, $\underline{C2}$, and $\underline{C3}$.*

Proof : Condition $\underline{C1}$ is a trivial consequence of $\underline{\mathrm{U}}$. The hypothesis A being Ferrers and $\underline{\mathrm{U}}$, and the conclusion $\underline{C2}$ and $\underline{C3}$ are symmetric under transposition $A \mapsto A^{\top}$; hence we need only prove $\underline{C2}$. Fix any $1 \le r < s \le n$ and $0 \le t < u < v \le n+1$; there are ten cases for the 2-by-3 submatrix

$$\begin{bmatrix} a_{rt} & a_{ru} & a_{rv} \\ a_{st} & a_{su} & a_{sv} \end{bmatrix},$$

leading to eight cases for the differences

$$\begin{bmatrix} a_{ru} - a_{rt} & a_{rv} - a_{ru} \\ a_{su} - a_{st} & a_{sv} - a_{su} \end{bmatrix}.$$

In each case the inequality $\underline{C2}$ is easily seen to hold. $\square$

Theorem 3 therefore implies Theorem 1.

Lemma 2. *If A is Ferrers and satisfies $\underline{C1}$ and $\underline{\Pi}$, then A satisfies $\underline{C2}$ and $\underline{C3}$.*

Proof : Again by symmetry, we only need to prove $\underline{C2}$, so fix any $1 \le r < s \le n$ and $0 \le t < u \le v \le n+1$; we separate four cases for

$$\begin{bmatrix} a_{rt} & a_{ru} & a_{rv} \\ a_{st} & a_{su} & a_{sv} \end{bmatrix}.$$

If $1 \le t$ and $v \le n$ then $a_{ij} = p_i q_j$ for all these entries of A, and $\underline{C2}$ reduces to

$$p_r(q_u - q_t)p_s(q_v - q_u) \le p_r(q_v - q_u)p_s(q_u - q_t),$$

which is trivial. If $1 \le t$ and $v = n+1$ then $\underline{C2}$ reduces to

$$p_r(q_u - q_t)(1 - p_s q_u) \le p_s(q_u - q_t)(1 - p_r q_u),$$

which (since A Ferrers implies $q_t \le q_u$) reduces to $p_r \le p_s$, which again follows if A is Ferrers. If $0 = t$ and $v \le n$ then $\underline{C2}$ reduces to

$$p_r q_u p_s(q_v - q_u) \le p_s q_u p_r(q_v - q_u),$$

which is trivial. Finally, if $0 = t$ and $v = n+1$ then $\underline{C2}$ reduces to

$$p_r q_u(1 - p_s q_u) \le p_s q_u(1 - p_r q_u),$$

which (since $q_u \ge 0$), by $\underline{C1}$ reduces to $p_r \le p_s$ which follows when A is Ferrers. $\square$

Corollary 2. *If A is a Ferrers matrix satisfying $\underline{C1}$ and $\underline{\Pi}$, then $per(zJ_n + A)$ has all its zeros in the interval* $[-1, 0]$.

One of the strongest theorems involving polynomials having only real zeros is Corollary 3 below, which Szegö proved under the more general assumption that the roots of $f(z)$ are all real but need not all be non-positive [Sze], [PoSz,Problem 154]. Our proof of the weaker result is quite different from his proof, which uses Grace's Apolarity Theorem [Gra], [Vle].

Corollary 3. *(Szegö) If $f(z) := \sum_{k=0}^{n} b_k z^k$ and $g(z) := \sum_{k=0}^{n} d_k z^k$ are polynomials of degree n all of whose roots are non-positive real numbers, then the polynomial*

$$\sum_{k=0}^{n} b_k d_k k!(n-k)! z^k$$

has only real roots.

Proof : Let $f(z) := b_n(z+p_1)(z+p_2)\cdots(z+p_n)$ and $g(z) := d_n(z+q_1)(z+q_2)\cdots(z+q_n)$, where $0 \leq p_1 \leq \ldots \leq p_n$ and $0 \leq q_1 \leq \ldots \leq q_n$. Let $e_k(X)$ denote the kth elementary symmmetric function in the set of variables $X := \{x_1, x_2, \ldots, x_n\}$. Then if $a_{ij} := p_i q_j$, $\mathrm{per}(zJ_n + A)$ becomes

$$\sum_{k=0}^{n} z^k k!(n-k)! e_{n-k}(P) e_{n-k}(Q),$$

which has only real zeros by Corollary 2. Since $e_{n-k}(P) = b_k/b_n$ and $e_{n-k}(Q) = d_k/d_n$, the thesis follows. □

Remark 1 : The case where $a_{ij} := p_i q_j$ with $p_1 \leq p_2 \leq \ldots \leq p_n$ and $0 \leq q_1 \leq q_2 \leq \ldots \leq q_n$ doesn't follow from Corollary 2 since if some of the p_i are negative A need not be Ferrers. However, this matrix will still be weakly increasing down columns, and so Szegö's Theorem is equivalent to this special case of Conjecture 2.

Lemma 3. *If A is a Ferrers matrix satisfying $\underline{\Sigma}$, then A satisfies $\underline{C2}$ and $\underline{C3}$.*

Proof : Yet again, by symmetry, we need only prove $\underline{C2}$, so fix any $1 \leq r < s \leq n$ and $0 \leq t < u < v \leq n+1$; we again separate four cases for

$$\begin{bmatrix} a_{rt} & a_{ru} & a_{rv} \\ a_{st} & a_{su} & a_{sv} \end{bmatrix}.$$

If $1 \leq t$ and $v \leq n$ then $a_{ij} = p_i + q_j$ for all these entries of A, and $\underline{C2}$ reduces to

$$(q_u - q_t)(q_v - q_u) \leq (q_v - q_u)(q_u - q_t),$$

which is trivial. If $1 \leq t$ and $v = n+1$ then $\underline{C2}$ reduces to

$$(q_u - q_t)(1 - p_s - q_u) \leq (q_u - q_t)(1 - p_r - q_u),$$

which (given that A is Ferrers implies $q_t \leq q_u$) reduces to $p_r \leq p_s$, and this also follows when A is Ferrers. If $0 = t$ and $v \leq n$ then $\underline{C2}$ reduces to

$$(p_r + q_u)(q_v - q_u) \leq (p_s + q_u)(q_v - q_u),$$

which reduces to $p_r \leq p_s$ and follows from A being Ferrers. Finally, if $0 = t$ and $v = n+1$ then $\underline{C2}$ reduces to

$$(p_r + q_u)(1 - p_s - q_u) \leq (p_s + q_u)(1 - p_r - q_u),$$

which reduces to $p_r \leq p_s$ and follows if A is Ferrers. □

Corollary 4. *If A is a Ferrers matrix satisfying $\underline{C1}$ and $\underline{\Sigma}$, then all the roots of $per(zJ_n + A)$ are in the interval* $[-1, 0]$.

To prove Theorem 3 we need some lemmas about polynomials with only real zeros. A polynomial $P \in \mathbb{R}[z]$ is *standard* if either $P \equiv 0$ or the leading coefficient of P is positive. Let $P, Q \in \mathbb{R}[z]$ be two such polynomials, of degrees p and q respectively. Let the zeros of P be

$$\xi_1 \leq \cdots \leq \xi_p$$

and let the zeros of Q be

$$\theta_1 \leq \cdots \leq \theta_q.$$

We say that P *alternates left of* Q if $p = q$ and

$$\xi_1 \leq \theta_1 \leq \cdots \leq \xi_p \leq \theta_p.$$

This is denoted $P \langle\!\langle Q$.

We say that P *interlaces* Q if $q = p + 1$ and

$$\theta_1 \leq \xi_1 \leq \cdots \leq \theta_p \leq \xi_p \leq \theta_{p+1}.$$

This is denoted $P \dagger Q$.

Furthermore if $P \langle\!\langle Q$ or $P \dagger Q$, then we use the notation $P \prec Q$. Moreover by convention if P has only real zeros, then $P \langle\!\langle 0$, $0 \langle\!\langle P$, $P \dagger 0$, and $0 \dagger P$.

Lemmas 4 and 5 can be proved using the techniques from section 3 of [Wag1].

Lemma 4. *Let $P, Q, S \in \mathbb{R}[z]$ be standard, with all zeros real and in the interval $[\alpha, \beta]$, and with $S \not\equiv 0$.*

(a) $P \langle\!\langle Q$ *if and only if* $(z - \alpha)Q \langle\!\langle (z - \beta)P$.

(b) *If* $S \prec P$ *and* $S \prec Q$ *then* $S \prec P + Q$.

(c) *If* $P \prec S$ *and* $Q \prec S$ *then* $P + Q \prec S$.

(d) *If* $P \prec Q$ *then* $P \prec P + Q \prec Q$.

Lemma 5. *Let $P_1, \ldots, P_m \in \mathbb{R}[z]$ be standard, with only real non-positive zeros, and such that $P_i \not\equiv 0$ for all $1 \leq i \leq m$. If $P_1 \prec \cdots \prec P_m$ and $P_1 \prec P_m$ then $P_h \prec P_i$ for all $1 \leq h < i \leq m$.*

Lemma 6. *Let $P_1, \ldots, P_m \in \mathbb{R}[z]$ be standard, with only real non-positive zeros, and such that $P_h \prec P_i$ for all $1 \leq h < i \leq m$. For any $c_1, \ldots, c_m \geq 0$, $P_1 \prec c_1P_1 + \cdots + c_mP_m \prec P_m$.*

Proof : We may reduce to the case that $c_i > 0$ and $P_i \not\equiv 0$ for all $1 \leq i \leq m$. Since $P_1 \prec c_iP_i$ for all $1 \leq i \leq m$, Lemma 4(b) (and induction on m) show that $P_1 \prec c_1P_1 + \cdots + c_mP_m$. Similar reasoning shows that $c_1P_1 + \cdots + c_mP_m \prec P_m$. □

Lemma 7. *Let $P_1, \ldots, P_m \in \mathbb{R}[z]$ be standard, with only real non-positive zeros, and such that $P_h \prec P_i$ for all $1 \leq h < i \leq m$. For $1 \leq i \leq m$ let $S'_i := P_1 + \cdots + P_i$ and $S_i := P_i + \cdots + P_m$. Then*

$$S'_1 \prec \cdots \prec S'_m = S_1 \prec \cdots \prec S_m \qquad \textit{and} \qquad S'_1 \prec S_m.$$

Proof : Since $P_1 \prec P_2$, Lemma 4 (d) implies that $S'_1 \prec S'_2 \prec P_2$. Assume, inductively, that $S'_1 \prec \cdots \prec S'_i \prec P_i$ where $i < m$. By the hypothesis and Lemma 6, $S'_i \prec P_{i+1}$; thus Lemma 4 (d) implies that $S'_i \prec S'_{i+1} \prec P_{i+1}$. By induction we conclude that $S'_1 \prec \cdots \prec S'_m$. A similar argument shows that $S_1 \prec \cdots \prec S_m$. Finally, $S'_1 = P_1 \prec P_m = S_m$ is part of the hypothesis. □

Lemma 8. *Let $P_1, \ldots, P_m \in \mathbb{R}[z]$ be standard, with only real non-positive zeros, and such that $P_h \prec P_i$ for all $1 \le h < i \le m$. Let $b_1, \ldots, b_m \ge 0$ and $c_1, \ldots, c_m \ge 0$ be such that $b_i c_{i+1} \le c_i b_{i+1}$ for all $1 \le i \le m-1$. Then $c_1 P_1 + \cdots + c_m P_m \prec b_1 P_1 + \cdots + b_m P_m$.*

Proof : We may reduce to the case that $b_i > 0$ and $c_i > 0$ for all $1 \le i \le m$ by a limiting argument. The inequalities imply that $b_h c_i \le c_h b_i$ for all $1 \le h < i \le m$, and thus we may reduce to the case that $P_i \not\equiv 0$ for all $1 \le i \le m$. Replacing P_i by P_i/c_i for $1 \le i \le m$ we reduce further to the case $c_1 = \cdots c_m = 1$ and $b_1 \le \cdots \le b_m$. For $1 \le i \le m$ let $S_i := P_i + \cdots + P_m$. Lemmas 5 and 7 imply that $S_h \prec S_i$ for all $1 \le h < i \le m$. Now, with $b_0 := 0$, Lemma 6 implies that

$$S_1 \prec \sum_{i=1}^{m} (b_i - b_{i-1}) S_i \prec S_m,$$

proving the result. $\square$

Proof of Theorem 3: We proceed by induction on n, dividing the induction hypothesis into two parts, I(n) and II(n).

I(n): For a square Ferrers matrix A of side $m \le n$ satisfying $\underline{C1}$, $\underline{C2}$, and $\underline{C3}$, $\mathrm{per}(zJ_m + A)$ has all zeros in the interval $[-1, 0]$.
For an m-by-m matrix A and $1 \le i, j \le m$ let $T_{ij}A$ denote the $(m-1)$-by-$(m-1)$ submatrix of A obtained by deleting the i-th row and j-th column of A, and let $Q_{ij} := \mathrm{per}(zJ_{m-1} + T_{ij}A)$.

II(n): For a square Ferrers matrix A of side $m \le n$ satisfying $\underline{C1}$, $\underline{C2}$, and $\underline{C3}$, both of the following conditions hold.

$\underline{C4}$ For any $1 \le r \le m$: $Q_{rj} \langle\!\langle Q_{rk}$ for all $1 \le j < k \le m$.

$\underline{C5}$ For any $1 \le c \le m$: $Q_{hc} \langle\!\langle Q_{ic}$ for all $1 \le h < i \le m$.

The bases of induction, I(1) and II(1), are trivial. The next cases, I(2) and II(2), also follow easily. We divide the induction step into two parts: I(n) and II(n) imply II$(n+1)$, and I(n) and II$(n+1)$ imply I$(n+1)$.

For the first part of the induction step, assume I(n) and II(n), and let A be an $(n+1) \times (n+1)$ Ferrers matrix satisfying $\underline{C1}$, $\underline{C2}$, and $\underline{C3}$. By the symmetry $A \mapsto A^\top$ it suffices to prove $\underline{C4}$. Fix a row $1 \le r \le n+1$ and two columns $1 \le j < k \le n+1$. By I(n) we know that both Q_{rj} and Q_{rk} have all their zeros in $[-1, 0]$; we must show that $Q_{rj} \langle\!\langle Q_{rk}$. Notice that for all $1 \le i \le n$, $T_{i,k-1}T_{rj}A = T_{ij}T_{rk}A$, and define $P_i := \mathrm{per}(zJ_{n-1} + T_{ij}T_{rk}A)$ for all $1 \le i \le n$. By II(n) we have $P_h \langle\!\langle P_i$ for all $1 \le h < i \le n$. Also, defining

$$b_i := \begin{cases} a_{ij} & \text{if } 1 \le i \le r-1, \\ a_{i+1,j} & \text{if } r \le i \le n, \end{cases}$$

and

$$c_i := \begin{cases} a_{ik} & \text{if } 1 \le i \le r-1, \\ a_{i+1,k} & \text{if } r \le i \le n, \end{cases}$$

we have

$$Q_{rj} = \sum_{i=1}^{n} (z + c_i) P_i \quad \text{and} \quad Q_{rk} = \sum_{i=1}^{n} (z + b_i) P_i.$$

For $1 \le i \le n$ let $F_i := P_i + \cdots + P_n$, and let $F_{n+1} :\equiv 0$.

For $1 \leq i \leq n$, the polynomial P_i is standard, not identically zero, and has all its zeros in the interval $[-1, 0]$, by I(n). Thus we see that

$$(z+1)P_1 \lll \cdots \lll (z+1)P_n \lll zP_1 \lll \cdots \lll zP_n.$$

Thus, for any $1 \leq g \leq n$ we have

$$(z+1)P_g \lll \cdots \lll (z+1)P_n \lll zP_1 \lll \cdots \lll zP_g \quad \text{and} \quad (z+1)P_g \lll zP_g. \tag{6}$$

Apply Lemmas 5 and 7 to (6) to see that

$$S'_1 \prec S'_2 \prec \cdots \prec S'_{n+1} \prec S_1 \prec \cdots \prec S_{n+1} \quad \text{and} \quad S'_1 \prec S_{n+1},$$

in which case S'_i is the sum of the first i terms of (6), and S_i is the sum of the last $n+2-i$ terms of (6).

From Lemma 5 applied to this sequence, we see in particular that $S'_n \prec S_2$. But

$$S'_n = (z+1)P_g + \ldots + (z+1)P_n + zP_1 + \ldots + zP_{g-1} = zF_1 + F_g$$

and

$$S_2 = (z+1)P_{g+1} + \ldots + (z+1)P_n + zP_1 + \ldots + zP_g = zF_1 + F_{g+1}.$$

Also, one sees directly that

$$zF_1 + F_1 = (z+1)F_1 \lll zF_1 = zF_1 + F_{n+1}.$$

From Lemma 5 we conclude that

$$zF_1 + F_g \lll zF_1 + F_h \qquad \text{for all} \qquad 1 \leq g < h \leq n+1.$$

Putting $c_0 := 0$ and $c_{n+1} := 1$ and $b_0 := 0$ and $b_{n+1} := 1$ we have

$$Q_{rj} = \sum_{g=1}^{n+1} (c_g - c_{g-1})(zF_1 + F_g) \quad \text{and} \quad Q_{rk} = \sum_{g=1}^{n+1} (b_g - b_{g-1})(zF_1 + F_g).$$

Condition $\underline{C3}$ implies that for all $1 \leq g \leq n$,

$$(b_g - b_{g-1})(c_{g+1} - c_g) \leq (c_g - c_{g-1})(b_{g+1} - b_g).$$

Lemma 8 now implies that $Q_{rj} \lll Q_{rk}$, completing the first part of the induction step.

For the second part of the induction step assume I(n) and II$(n+1)$, and let A be an $(n+1) \times (n+1)$ Ferrers matrix satisfying $\underline{C1}$, $\underline{C2}$, and $\underline{C3}$. Let $P_i := \mathrm{per}(zJ_n + T_{i1}A)$ for $1 \leq i \leq n+1$; each P_i has all its zeros in the interval $[-1, 0]$, by I(n), and $P_h \lll P_i$ for all $1 \leq h < i \leq n+1$, by II$(n+1)$. Moreover,

$$\mathrm{per}(zJ_{n+1} + A) = \sum_{i=1}^{n+1} (z + a_{i1})P_i.$$

If A is Ferrers and $\underline{C1}$ holds we have $0 \leq a_{i1} \leq \cdots \leq a_{n+1,1} \leq 1$, so for all $1 \leq i \leq n+1$ we have $a_{i1}(1-a_{i+1,1}) \leq a_{i+1,1}(1-a_{i1})$, where $a_{01} := 0$ and $a_{n+2,1} := 1$. By Lemmas 6 and 8 we have

$$P_1 \lll \sum_{i=1}^{n+1}(1-a_{i1})P_i \lll \sum_{i=1}^{n+1} a_{i1}P_i \lll P_{n+1}.$$

Since the zeros of these polynomials are all in the interval $[-1,0]$, Lemma 4 (a) implies that

$$(z+1)\sum_{i=1}^{n+1} a_{i1}P_i \lll z\sum_{i=1}^{n+1}(1-a_{i1})P_i.$$

Applying Lemma 4 (d) to this we obtain

$$(z+1)\sum_{i=1}^{n+1} a_{i1}P_i \lll \sum_{i=1}^{n+1}(z+a_{i1})P_i \lll z\sum_{i=1}^{n+1}(1-a_{i1})P_i,$$

and so $\mathrm{per}(zJ_{n+1}+A)$ has all its zeros in the interval $[-1,0]$. This completes the proof. □

We can also show Conjecture 1 is true if the underlying tableau is the trivial staircase

$$\begin{array}{cccc} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & & & \vdots \\ n^2-n+1 & & \dots & n^2 \end{array}$$

Theorem 4. *Conjecture 1 is true provided the tableau associated to A is the one above, i.e. if $a_{1,n} \leq a_{2,1}$, $a_{2,n} \leq a_{3,1}$, $\ldots$, $a_{n-1,n} \leq a_{n,1}$. Furthermore, under these assumptions, with $P(z) := per(zJ_n + A)$,*

$$\begin{aligned} &P(z) > 0 \text{ for } z > -a_{11},\ P(z) < 0 \text{ for } -a_{21} < z < -a_{1n}, \\ &\ldots, (-1)^n P(z) < 0 \text{ for } -a_{n1} < z < -a_{n-1,n},\ (-1)^n P(z) > 0 \text{ for } z < -a_{nn}. \end{aligned} \tag{7}$$

Proof : By induction on n. First assume that the inequalities on the a_{ij} are strict, and note that (7) then guarantees that $P(z)$ has n distinct roots in $[-a_{nn}, -a_{11}]$. Let $P_{ij}(z)$ denote the permanent of the matrix obtained by deleting row i and column j of $zJ_n + A$. By expanding in minors we have

$$P(z) = (a_{11}+z)P_{11}(z) + (a_{12}+z)P_{12}(z) + \ldots + (a_{1n}+z)P_{1n}(z). \tag{8}$$

Applying the induction hypothesis to the P_{1j} we get

$$P_{11}(z) > 0 \text{ for } z > -a_{22},\ P_{12}(z) > 0 \text{ for } z > -a_{21}, \ldots, P_{1n}(z) > 0 \text{ for } z > -a_{21}.$$

By (8) this implies

$$P(z) > 0 \text{ for } z > -a_{11}, \text{ and } P(z) < 0 \text{ for } -a_{21} < z < -a_{1n}.$$

Similarly using

$$P_{11}(z) < 0 \text{ for } -a_{32} < z < -a_{2n}, \ldots, P_{1n}(z) < 0 \text{ for } -a_{31} < z < -a_{2,n-1}$$

we get

$$P(z) > 0 \text{ for } -a_{31} < z < -a_{2n}.$$

Continuing in this manner we easily verify the rest of (7). The case where some of the a_{ij} are equal follows from a simple continuity argument using the argument principle. □

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