A polynomial expression for the character of diagonal harmonics

A. Garsia
Department of Mathematics
University of California at San Diego, La Jolla, CA 92093-0112
garsia@math.ucsd.edu
and
J. Haglund
Department of Mathematics
University of Pennsylvania, Philadelphia, PA 19104-6395
jhaglund@math.upenn.edu

January 17, 2015

The first author was supported by NSF grant DMS-1068883. The second author was supported by NSF grant DMS-1200296 MR Subject Classifications: Primary: 05A15; Secondary 05E05,05A19
Keywords: character of diagonal harmonics, Macdonald polynomials, Tesler matrices

Abstract

Based on his study of the Hilbert scheme from algebraic geometry, Haiman [Invent. Math. **149** (2002), pp. 371–407] obtained a formula for the character of the space of diagonal harmonics under the diagonal action of the symmetric group, as a sum of Macdonald polynomials with rational coefficients. In this paper we show how Haiman's formula, combined with identities involving plethystic symmetric function operators, yields a new formula for this character. Our formula doesn't involve any mention of Macdonald polynomials, and the coefficients are visibly polynomials (not rational functions), although they are not manifestly positive. Our formula can be expressed as either a sum of weighted Tesler matrices, or as the constant term in a multivariate Laurent series.

1 Introduction

Let $X_n = \{x_1, \ldots, x_n\}, Y_n = \{y_1, \ldots, y_n\}$ be two sets of variables and let

$$\mathrm{DH}_n = \{f \in \mathbb{C}[X_n, Y_n] : \sum_i \partial_{x_i}^h \partial_{y_i}^k f = 0, \forall h, k \geq 0, h + k > 0\} \tag{1}$$

be the space of diagonal harmonics. There is a natural bigrading of DH_n, by decomposing it into subspaces of homogeneous bi-degree in the X and Y variables. Let ∇ be the linear operator defined on the modified Macdonald polynomial basis $\{\tilde{H}_\mu(X_n; q, t)\}$, where $\mu \vdash n$ (i.e. μ is a partition of n), by

$$\nabla \tilde{H}_\mu(X_n; q, t) = T_\mu \tilde{H}_\mu(X_n; q, t), \tag{2}$$

where $T_\mu = t^{n(\mu)} q^{n(\mu')}$ and $n(\mu) = \sum_i (i-1)\mu_i$, with μ' the conjugate partition. The symmetric group acts "diagonally" on a polynomial $f(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathrm{DH}_n$ by $\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$ and this action respects the bigrading. The Frobenius characteristic is a symmetric function whose coefficients are polynomials in q, t which reflect the bigrading, and where the Schur function $s_\lambda(X)$

corresponds to the irreducible S_n-character χ^λ. Haiman [Hai02] proved an earlier conjecture of the first author and Haiman [GH96] that the Frobenius characteristic is given by $\nabla e_n(X)$, where e_n is the nth elementary symmetric function in a set of variables.

In [Hag11] the second author showed how Haiman's formula for the Frobenius characteristic, combined with summation formulas of the first author and Zabrocki for generalized Pieri coefficients, can be used to obtain a new formula for the bigraded Hilbert series of DH_n. The formula is a polynomial expression in the two parameters q, t, and is expressed in terms of special types of $n \times n$ matrices called "Tesler matrices". It has an equivalent formulation as the constant term in a multivariate Laurent series. In this article we derive a related result for the character of DH_n, by a slightly different method; we combine Haiman's formula with identities involving the D_n operators from [BGHT99]. Our formula for ∇e_n does not imply, nor is it implied, by the Hilbert series formula from [Hag11], although it is similar in that it can also be expressed in terms of Tesler matrices, and is visibly a polynomial in q, t. Like the Hilbert series formula, it remains an open question how to start with the Tesler matrix sum and cancel the negative terms to obtain a positive combinatorial formula, along the lines of the conjectured formula for ∇e_n in [HHL$^+$05]. We should mention that part of our motivation for studying the D_n operators and their connection to ∇e_n was an attempt on our part to obtain a combinatorial proof using these operators of an interesting formula for the q, t-Catalan sequence in terms of Tesler matrices recently obtained by Gorsky and Negut [GN13]. (The q, t-Catalan sequence $C_n(q, t)$ was originally defined in [GH96] as $\langle \nabla e_n, s_{1^n} \rangle$, i.e. the coefficient of the sign character for DH_n). Their proof involves a residue calculation and geometric ingredients connected to the Hilbert scheme. In the end we were unable to prove their result using our methods (if one takes the coefficient of s_{1^n} in our formula for ∇e_n one gets a formula for $C_n(q, t)$ in terms of Tesler matrices which is different from theirs).

2 A Constant Term Expression For ∇e_n

Let $p_k(X) = \sum_i x_i^k$ be the kth power sum. If $E(t_1, \ldots, t_n)$ is any expression involving indeterminates $t_1, \ldots, t_n$, we let $p_k[E] = E(t_1^k, \ldots, t_n^k)$ denote the *plethystic substitution* of E into p_k. For example,

$$p_k\left[X(1-t)\right] = p_k(X)(1-t^k),$$

and $p_k[z_1 + \ldots + z_r] = p_k(\{z_1, \ldots, z_r\})$. More generally, if F is any symmetric function, then by $F[E]$ we mean the result of first expressing F as a polynomial in the power sums p_k, then replacing each p_k by $p_k[E]$. Readers unfamiliar with plethysm can consult [Hag08, pp. 19-22] for a more detailed discussion.

Let D_k, $k \geq 0$, be the linear operator on symmetric functions defined as

$$D_k F\left[X\right] = F\left[X + \frac{M}{z}\right] \Omega\left[-zX\right]\Bigg|_{z^k}, \tag{3}$$

where $M = (1-q)(1-t)$ throughout, and $|_{z^k}$ means "take the coefficient of z^k in". Here

$$\Omega(X) = \sum_{n=0}^{\infty} h_n(X) = \prod_{i=1}^{\infty} \frac{1}{(1-x_i)}, \tag{4}$$

where $h_n = s_n$ is the complete homogeneous symmetric function. The operators D_k were studied extensively in [BGHT99]; we will make use of the following results from that paper.

Theorem 1 *[BGHT99]*

$$D_0 \tilde{H}_\mu = (1 - MB_\mu(q,t))\tilde{H}_\mu \tag{5}$$

$$D_k e_1 - e_1 D_k = M D_{k+1} \tag{6}$$

$$\nabla e_1 \nabla^{-1} = -D_1 \tag{7}$$

Here $B_\mu(q,t) = \sum_{i=1}^{\ell(\mu)} t^{i-1} \frac{q^{\mu_i}-1}{q-1}$, *with* $\ell(\mu)$ *the number of parts of* μ.

We now proceed to develop our new polynomial expression for ∇e_n. As a first step we express D_n in terms of D_0.

Lemma 1

$$D_n = \frac{1}{M^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e_1^{n-k} D_0 e_1^k. \tag{8}$$

Proof. By induction on n, the case $n = 0$ being trivial. From (6),

$$\begin{aligned} M D_{n+1} &= D_n e_1 - e_1 D_n \qquad (9)\\ &= \left(\frac{1}{M^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e_1^{n-k} D_0 e_1^k \right) e_1 - e_1 \left(\frac{1}{M^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} e_1^{n-k} D_0 e_1^k \right)\\ &= \frac{1}{M^n} \sum_{k=0}^{n+1} e_1^{n+1-k} D_0 e_1^k \left(\binom{n}{k-1} (-1)^{n+1-k} - \binom{n}{k} (-1)^{n-k} \right)\\ &= \frac{1}{M^n} \sum_{k=0}^{n+1} e_1^{n+1-k} D_0 e_1^k \binom{n+1}{k} (-1)^{n+1-k}. \end{aligned}$$

□

Corollary 1

$$\nabla e_n = \frac{1}{(-M)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_1^{n-k} D_0 D_1^k \, \mathbf{1}, \tag{10}$$

where the operators on the right-hand-side of (10) are applied to the constant function $\mathbf{1}$*.*

Proof. Note that

$$D_n \mathbf{1} = \Omega[-zX] \Big|_{z^n} = (-1)^n e_n \tag{11}$$

since $h_n = \omega e_n$, and if f is of homogeneous degree n, $(-1)^n f[-X] = \omega f$. Thus by (8),

$$\begin{aligned} \nabla e_n &= (-1)^n \nabla D_n \nabla^{-1} \mathbf{1} \qquad (12)\\ &= \frac{1}{(-M)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \nabla e_1^{n-k} D_0 e_1^k \nabla^{-1} \mathbf{1}\\ &= \frac{1}{(-M)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\nabla e_1 \nabla^{-1})^{n-k} (\nabla D_0 \nabla^{-1}) (\nabla e_1 \nabla^{-1})^k \mathbf{1} \qquad (13)\\ &= \frac{1}{(-M)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_1^{n-k} D_0 D_1^k \mathbf{1}, \qquad (14) \end{aligned}$$

where in the last step we have used (7), and also the fact that by (5), D_0 commutes with ∇. □

Definition 1 *Let*

$$[k]_{q,t} = (t^k - q^k)/(t-q) = t^{k-1} + t^{k-2}q + \ldots + tq^{k-2} + q^{k-1} \qquad k \in \mathbf{N} \tag{15}$$

$$g(z) = \frac{(1-z)(1-qtz)}{(1-qz)(1-tz)} = \Omega\left[-Mz\right] \tag{16}$$

$$Z_n = \{z_0, z_1, \ldots, z_n\}$$

$$Z'_n = \{z_1, \ldots, z_n\}$$

$$G(Z_n) = \prod_{0 \le i < j \le n} g(z_i/z_j) \tag{17}$$

$$G(Z'_n) = \prod_{1 \le i < j \le n} g(z_i/z_j). \tag{18}$$

Note the identity (in the ring of formal power series)

$$g(z) = 1 - M(z + [2]_{q,t}z^2 + [3]_{q,t}z^3 + [4]_{q,t}z^4 + \ldots), \tag{19}$$

which implies

$$G(Z_n) = \Omega[-M \sum_{0 \le i < j \le n} z_i/z_j] \tag{20}$$

$$= \prod_{0 \le i < j \le n} 1 - M((z_i/z_j) + [2]_{q,t}(z_i/z_j)^2 + [3]_{q,t}(z_i/z_j)^3 + [4]_{q,t}(z_i/z_j)^4 + \ldots). \tag{21}$$

We will refer to $-M([k]_{q,t}(z_i/z_j)^k$, $k \ge 1$, from the expansion of $g(z_i/z_j)$ in powers of z_i/z_j, as a "(z_i/z_j)-domino". Let LHS and RHS be abbreviations for "left-hand-side" and "right-hand-side"; a given term in the RHS above will be a product of various dominoes. To such a product P we associate a graph $R(P)$ with vertex set $\{z_0, \ldots, z_n\}$ by including an edge from z_i to z_j in $R(p)$ if and only if there is a (z_i/z_j) domino in P. We say $R(p)$ is connected if there is a path in $R(p)$ from any given vertex to any other vertex. Let $\tilde{G}(Z_n)$ denote the sum of all terms P in the expansion in the RHS of (21) whose graph $R(P)$ is connected. Note that all terms in $\tilde{G}(Z_n)$ must involve at least n dominoes; the first domino creates an edge connecting two vertices, and each successive domino which connects to the graph already constructed adds at most one new vertex.

We now state our main result.

Theorem 2

$$\nabla e_n(X) = \frac{e_n[XZ_n]}{(-M)^n} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} z_k \right) G(Z_n) \Big|_{z_0 z_1 \cdots z_n}. \tag{22}$$

Furthermore,

$$\nabla e_n(X) = \frac{e_n[XZ'_n]}{(-M)^n} \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} z_k \right) \tilde{G}(Z_n) \Big|_{z_0 z_1 \cdots z_n}. \tag{23}$$

Corollary 2 *For any partition λ of a positive integer n,*

$$\langle \nabla e_n, s_\lambda \rangle \in \mathbf{Z}[q,t], \tag{24}$$

where $\langle , \rangle$ is the Hall scalar product, with respect to which the Schur functions are orthonormal.

Proof. The well-known Cauchy identity (dual form) says

$$e_n[XZ'_n] = \sum_{\lambda \vdash n} s_\lambda(X) s_{\lambda'}(Z'_n). \tag{25}$$

As noted above, each term in $\tilde{G}(Z_n)$ involves at least n dominoes, and each domino has a coefficient divisible by $-M$. Thus these terms are all divisible by the factor $(-M)^n$ occurring in the denominator of the RHS of (23), and everything else is visibly a polynomial in q, t. □

In [BGHT99] there is a proof, using manipulations of the D_k operators, that ∇ applied to any Schur function has polynomial coefficients. That proof is rather indirect though, and doesn't yield any specific formulas like (23) for those coefficients.

We now prove Theorem 2.

Proof. From the definition of D_i it follows that

$$D_0 D_1^k \mathbf{1} = D_0 \Omega\left[-Z'_k X\right] G(Z'_k)\Big|_{z_1\cdots z_k} \tag{26}$$

$$= \Omega\left[-Z'_k(X + M/z) - Xz)\right]\Big|_{z^0} G(Z'_k)\Big|_{z_1\cdots z_k}. \tag{27}$$

Thus

$$D_1^{n-k} D_0 D_1^k \mathbf{1} = D_1^{n-k}\Omega\left[-Z'_k(X + M/z) - Xz)\right]\Big|_{z^0} G(Z'_k)\Big|_{z_1\cdots z_k} \tag{28}$$

$$= \Omega\left[-Z'_k(X + \sum_{i=k+1}^{n} M/z_i) - Z'_k M/z - z(X + \sum_{i=k+1}^{n} M/z_i)\right]\Big|_{z^0} G(Z'_k)\Big|_{z_1\cdots z_k} \times$$

$$\Omega\left[-(Z'_n - Z'_k)X\right] G(Z'_n - Z'_k)\Big|_{z_{k+1}\cdots z_n}$$

$$= \Omega\left[-Z'_n X\right] \Omega\left[-Z'_k M/z - Xz - \sum_{i=k+1}^{n} Mz/z_i\right] G(Z'_n)\Big|_{z_1\cdots z_n}\Big|_{z^0}$$

$$= \Omega\left[-Z'_n X\right] \Omega\left[-Z'_k M/z - \sum_{i=k+1}^{n} Mz/z_i\right] \Omega\left[-Xz\right] G(Z'_n)\Big|_{z_1\cdots z_n}\Big|_{z^0}. \tag{29}$$

We now re-index the variables in (29), first replacing z_i by z_{i-1} for $1 \le i \le k$, and then replacing z by z_k, resulting in

$$z_k D_1^{n-k} D_0 D_1^k \mathbf{1} = \Omega\left[-Z_n X\right] G(Z_n)\Big|_{z_0\cdots z_n}. \tag{30}$$

Combining (30) with (10) we get

$$\nabla e_n(X) = \frac{1}{(-M)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} z_k D_1^{n-k} D_0 D_1^k \mathbf{1} \tag{31}$$

$$= \Omega\left[-Z_n X\right] \frac{1}{(-M)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} z_k G(Z_n)\Big|_{z_0\cdots z_n}. \tag{32}$$

Now all the terms in $G(Z_n)$ are of total degree zero in the z_i, and since we are taking the coefficient of $z_0 \cdots z_n$, only terms of homgeneous degree n in the z_i occurring in $\Omega\left[-Z_n X\right]$ contribute, i.e. $h_n\left[-Z_n X\right] = (-1)^n e_n\left[Z_n X\right]$. Equation (22) now follows.

The following lemma will allow us to obtain (23) from (22).

Lemma 2 *Let λ be a partition of n, and m_λ the corresponding monomial symmetric function. Then*

$$m_\lambda(Z_n)\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} z_k\right) G(Z_n)\Big|_{z_0\cdots z_n} = m_\lambda(Z'_n)\left(\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k} z_k\right) \tilde{G}(Z_n)\Big|_{z_0\cdots z_n}. \tag{33}$$

Proof. Say we choose the factor $\binom{n}{k}(-1)^{n-k}z_k$ from the middle term on the LHS above, and a product of dominoes P from $G(Z_n)$. Assume none of the dominoes in P involve z_k. Then in the LHS above we cannot use any terms in $m_\lambda(Z_n)$ that involve z_k, since we are looking for the coefficient of $z_0 \cdots z_k \cdots z_n$. Let $Z_{n,k}$ be the alphabet Z_n with z_k removed. Clearly

$$m_\lambda(Z_{n,k})G(Z_{n,k})\Big|_{z_0\cdots\hat{z_k}\cdots z_n} \tag{34}$$

is independent of k, and so the portion of the LHS of (33) where no dominoes involve z_k equals

$$\begin{aligned}\sum_{k=0}^{n} m_\lambda(Z_{n,k})\binom{n}{k}(-1)^{n-k}z_k G(Z_{n,k})\Big|_{z_0\cdots z_k\cdots z_n} &= m_\lambda(Z_{n,k})G(Z_{n,k})\Big|_{z_0\cdots\hat{z_k}\cdots z_n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} \\ &= 0.\end{aligned} \tag{35}$$

Next assume at least one of the dominoes in P involves z_k, and also that $R(P)$ is not connected. Let R_k be the connected component of $R(P)$ containing vertex k, and P_k the portion of P corresponding to R_k. Say there are α vertices in R_k less than k, and β vertices greater than k, where α, β are fixed nonnegative integers satisfying $0 \le \alpha, \beta,\ \alpha+\beta < n$. To get a nonzero contribution to the LHS of (33), we need to find a term ζ in $m_\lambda(Z_n)$ with the following property: for each z_i-variable in z_kP_k, the z_i power in ζ, when added to the z_i-power in z_kP_k, equals 1. After doing this the portion of ζ involving variables not in z_kP_z will be a term ζ' in a monomial symmetric function for some partition π where $\pi \subset \lambda$. Letting W be the set of variables obtained by starting with Z_n and removing all variables occurring in z_kP_k, we need to multiply our contribution from the R_k term above by

$$m_\pi(W)G(W)\Big|_{\prod_{z_i\in W} z_i}. \tag{36}$$

Note the value of (36) depends only on π and $\alpha+\beta$, not on k or what the actual α vertices less than k or the β vertices larger than k are.

There are $\binom{k}{\alpha}$ ways to choose α numbers from the set $\{0, 1, \ldots, k-1\}$, and $\binom{n-k}{\beta}$ ways to choose β numbers from the set $\{k+1, \ldots, n\}$. It follows that the contribution to the LHS of (33) from all terms with k occurring in a component with α vertices less than k and β vertices larger than k is divisible by a factor of

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{k}{\alpha}\binom{n-k}{\beta}. \tag{37}$$

Now for fixed n, $\binom{k}{\alpha}\binom{n-k}{\beta}$ can be viewed as a polynomial in k, of total degree $\alpha+\beta$. It is well-known that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m = 0 \qquad 0 \le m < n, \tag{38}$$

which shows (37) equals zero (since $R(P)$ is not connected, $\alpha+\beta < n$). Thus we can replace $G(Z_n)$ in the LHS of (33) by $\tilde{G}(Z_n)$. Furthermore, in any connected term P the power of z_0 must be greater than zero, so to contribute to the LHS of (33), P must use exactly one domino involving z_0, and the terms $m_\lambda(Z_n)$ and $\binom{n}{k}(-1)^{n-k}z_k$ cannot use z_0 at all. □

Since any Schur function is an integral sum of the m_λ, (23) follows from (33). □

Example 1 *It is well-known that $\langle \nabla e_n, s_n\rangle = 1$, but to deduce that directly from the definition of ∇ requires knowledge of Macdonald polynomial theory. In this example we show how to deduce this directly from (23).*

Note the coefficient of $S_n(X)$ in $e_n[XZ'_n]$ equals $s_{1^n}(Z'_n) = z_1 z_2 \cdots z_n$. In any term in $\tilde{G}(Z_n)$, the variable z_n must occur in at least one domino, and there to a negative power. Thus we must choose $k = n$ in the middle factor in the RHS of (23), and

$$\langle \nabla e_n, s_n \rangle = \frac{1}{(-M)^n} \tilde{G}(Z_n) \Big|_{z_0 z_n^{-1}}. \tag{39}$$

If P is a term in $\tilde{G}(Z_n)$ which contributes to the RHS of (39), then z_{n-1} must occur in at least one domino, and to a total power of 0. It can only be the upper parameter of a domino whose lower parameter is z_n, and there is only one of these. Thus (z_{n-1}/z_n) is one domino in P, and z_{n-1} occurs in one other domino, as a lower parameter. Iterating this argument shows P is the product of the n dominoes $(z_0/z_1)(z_1/z_2)\cdots(z_{n-1}/z_n)$, which has a total coefficient of $(-M)^n$.

3 Tesler Matrices

As described in [Hag11], a *Tesler matrix* C of size n is an $n \times n$ upper-triangular matrix of nonnegative integers which satisfies

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = 1, \quad \text{for each } j,\ 1 \le j \le n. \tag{40}$$

Let Q_n denote the set of all $n \times n$ Tesler matrices. For example,

$$Q_1 = \{\begin{bmatrix}1\end{bmatrix}\} \tag{41}$$

$$Q_2 = \{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\} \tag{42}$$

$$Q_3 = \{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&2&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&1&1\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&2\\0&0&3\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&1\\0&0&3\end{bmatrix}\}. \tag{43}$$

Geometrically, the condition (40) says that for all j, if we add all the entries of C in the jth row together, and then subtract all the entries in the jth column above the diagonal, we get 1. Let $\mathrm{pos}'(C)$ denote the number of positive, off-diagonal entries of C. For each Tesler matrix C, associate a graph $B(C)$ on the vertex set $\{1, 2, \ldots, n\}$ by including an edge from i to j in $B(C)$ if and only if $c_{ij} > 0$. We say C is *prime* if and only if $B(C)$ is connected. Let Q'_n denote the set of prime, $n \times n$ Tesler matrices. We remark that prime Tesler matrices first occur in [AGH+12], while Tesler matrices were first explicitly defined by Glenn Tesler in unpublished work of Tesler's from the late 1990's on plethystic expressions for Macdonald's $D_{n,r}$ operators. Tesler then entered them into Sloane's online encyclopedia of integer sequences, as sequence $A008608$.

Our main result has the following equivalent expression in terms of prime Tesler matrices.

Theorem 3 *For any positive integer n,*

$$\nabla e_n(X) = \sum_{C \in Q'_{n+1}} \prod_{i : c_{ii} > 0} \left(e_{c_{ii}}(X) + w \binom{n}{i-1} (-1)^{n-i+1} e_{c_{ii}-1}(X) \right) (-M)^{pos'(C)-n} \prod_{\substack{i<j \\ c_{ij}>0}} [c_{ij}]_{q,t} \Big|_w. \tag{44}$$

Proof. View a given term B in the RHS of (23) as the product of three parts; a contribution P coming from $\tilde{G}(Z_n)$, a middle factor $\binom{n}{k}(-1)^{n-k} z_k$, and a monomial Q coming from $e_n[XZ'_n]$. To construct the corresponding term from the RHS of (44), first re-index the Z variables, replacing z_i by z_{i+1} in B, so the new alphabet is Z'_{n+1}. For each domino $-M[p]_{q,t}(z_i/z_j)^p$ in B, set $c_{ij} = p$ in our corresponding Tesler matrix. The diagonal elements c_{ii} are now uniquely defined by (40), and the value of c_{ii} is exactly the coefficient of z_i in Q if $i \neq k+1$, and is one more than the value of z_i in Q if $i = k+1$ (since after re-indexing the middle term is now $\binom{n}{k}(-1)^{n-k}z_{k+1}$, where $0 \le k \le n$). Let λ be the partition obtained by rearranging the set $\{c_{11}, c_{22}, \ldots, c_{k+1,k+1} - 1, \ldots c_{n+1,n+1}\}$ into partition order. Then the coefficient of $m_\lambda(Z'_{n+1})$ in $e_n\left[XZ'_{n+1}\right]$ is $\prod_i e_{\lambda_i}(X)$ (by the dual Cauchy identity) and the equivalence of (23) and (44) follows. □

Example 2 *The last three matrices on the RHS of (43) form the set Q'_3. The weights associated to these elements from the $n=2$ case of (44), in the same left-to-right order as in (43), are*

$$\left(e_1(X) - w\binom{2}{1}\right)\left(e_2(X) + w\binom{2}{2}e_1(X)\right)(-M)^0[1]^2_{q,t}\Big|_w = -2e_2 + e_1e_1 \tag{45}$$
$$\left(e_3(X) + w\binom{2}{2}e_2(X)\right)(-M)^0[1]_{q,t}[2]_{q,t}\Big|_w = (q+t)e_2$$
$$\left(e_3(X) + w\binom{2}{2}e_2(X)\right)(-M)^0[1]^2_{q,t}\Big|_w = e_2.$$

Adding these weights together gives

$$\begin{aligned}\nabla e_2(X) &= -2e_2 + e_1e_1 + (q+t)e_2 + e_2\\ &= s_2 + s_{1^2}(-2+1+q+t+1)\\ &= s_2 + s_{1^2}(q+t).\end{aligned}$$

By setting $e_k = 1$ for all $0 \le k \le n$ in (44) you get a formula for the q,t-Catalan $\langle \nabla e_n, s_{1^n}\rangle$. It is interesting to compare this to the following formula of Gorsky-Negut mentioned in the introduction (set $u=0, m=n+1$ in equation (62) from [GN13]):

$$\langle \nabla e_n, s_{1^n}\rangle = \sum_{C\in Q_n} \prod_{\substack{i\\ c_{i,i+1}>0}} ([c_{i,i+1}+1]_{q,t} - [c_{i,i+1}]_{q,t}) \prod_{\substack{j>i+1\\ c_{i,j}>0}} (-M)[c_{i,j}]_{q,t}. \tag{46}$$

Gorsky and Negut also [GN13, equation (60)] have a generalization of (46), where in addition to the other weights they weight each diagonal entry by $e_{c_{i,i}}$. The resulting expression equals ∇e_n. In fact, their formula applies to the generalized Frobenius characteristics corresponding to any coprime pair of integers (m,n), occurring in recent work of Armstrong, Gorsky and Mazin, Hikita, and others [ALW14], [Arm12], [GM13], [GM14], [Hik14]. In the case $m=n+1$ these reduce to ∇e_n. Although in the $m=n+1$ case their formula expresses ∇e_n as a sum over $Q(n)$, while ours is over the somewhat more complicated set Q'_{n+1}, one advantage our formula may have is that in ours the total power of $-M$ is the number of positive off-diagonal entries minus n, while in theirs they essentially get a factor of $-M$ for each positive off-diagonal entry. Hence terms in our formula will in general have fewer factors of $-M$, and could be viewed as being "closer" to a positive formula.

References

[AGH+12] D. Armstrong, A. Garsia, J. Haglund, B. Sagan, and B. Rhoades. Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics. *Journal of Combinatorics*, 3:451–494, 2012.

[ALW14] D. Armstrong, N. Loehr, and G. Warrington. Rational parking functions and Catalan numbers. *Math arXiv:1403.1845*, 2014.

[Arm12] D. Armstrong. Rational Catalan combinatorics 3. *Slides from a talk at the AIM December* 2012 *workshop in Palo Alto, CA, www.math.miami.edu/ armstrong/activity.html*, 2012.

[BGHT99] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler. Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions. *Methods Appl. Anal.*, 6:363–420, 1999.

[GH96] A. M. Garsia and M. Haiman. A remarkable q,t-Catalan sequence and q-Lagrange inversion. *J. Algebraic Combin.*, 5(3):191–244, 1996.

[GM13] E. Gorsky and M. Mazin. Compactified Jacobians and q,t-Catalan numbers, I. *Jour. of Combin. Theory, Ser. A*, 120:49–63, 2013.

[GM14] Evgeny Gorsky and Mikhail Mazin. Compactified Jacobians and q,t-Catalan numbers, II. *J. Algebraic Combin.*, 39(1):153–186, 2014.

[GN13] E. Gorsky and A. Negut. Refined knot invariants and Hilbert schemes. *Preprint on the math arXiv: http://arxiv.org/pdf/1304.3328.pdf*, 2013.

[Hag08] James Haglund. *The q,t-Catalan numbers and the space of diagonal harmonics*, volume 41 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. With an appendix on the combinatorics of Macdonald polynomials.

[Hag11] J. Haglund. A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants. *Adv. Math.*, 227:2092–2106, 2011.

[Hai02] M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, 149:371–407, 2002.

[HHL+05] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke Math. J.*, 126:195–232, 2005.

[Hik14] Tatsuyuki Hikita. Affine Springer fibers of type A and combinatorics of diagonal coinvariants. *Adv. Math.*, 263:88–122, 2014.