GENERALIZED ROOK POLYNOMIALS

Jay Goldman and James Haglund

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Abstract. Generalizing the notion of placing rooks on a Ferrers board leads to a new class of combinatorial models and a new class of rook polynomials. Connections are established with absolute Stirling numbers and permutations, Bessel polynomials, matchings, multiset permutations, hypergeometric functions, Abel polynomials and forests, and polynomial sequences of binomial type. Factorization and reciprocity theorems are proved and a q-analogue is given.

Dedicated to the memory of Gian-Carlo Rota, friend and inspiring colleague

1. Introduction

Rook theory has a long history arising from problems of permutations with restricted positions [21], [28]. The rook theory of Ferrers boards was started by Foata and Schützenberger [10], who gave a full characterization of rook equivalence via bijective proofs. Goldman, Joichi, and White [14] then introduced a new version of the rook polynomial of a Ferrers board and gave it a combinatorial interpretation, which showed that it had all integer roots. Subsequent work with Ferrers boards and this rook polynomial have led to models for binomial type theorems [17], connections with chromatic polynomials [15], orthogonal polynomials [12], [16], hypergeometric series [19], q-analogues and permutation statistics [9], [11], [20], statistical problems in probability [6], algebraic geometry [7], and digraph polynomials [3], [5]. Today there is a very large literature about Ferrers boards.

Now we generalize the classic notion of placing non-taking rooks on a Ferrers board, where as rooks are placed in the columns of the board, moving from left to right, new rows are created. Together with a more general notion of rook polynomial, this leads to a large new class of combinatorial models with connections to polynomial sequences of binomial type and many other models, e.g. permutations of sets and multisets, forests and Abel polynomials, and Bessel polynomials and matchings. In this paper, we concentrate on these examples, constructing bijections and reasoning with the rook polynomials.

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In section 2, we review a few classic notions, define i-creation rook placements and their associated rook numbers and i-rook polynomials, prove a factorization theorem, discuss rook equivalence, and prove that every monic polynomial with non-positive integer roots is the rook polynomial of a Ferrers board with a 1-creation rule.

In section 3, we study i-creation placements on 1-jump boards, and, when $i = 1$, the connection with $c(n, k)$, the number of permutations of $\{1, 2, \dots, n\}$ with k cycles (absolute Stirling numbers of the first kind).

Section 4 deals with 2-creation placements on a 1-jump board. We construct a bijection between the rook placements and matchings in complete graphs graded by the number of edges. The corresponding rook numbers are the coefficients of the Bessel polynomials and we have a new polynomial relation involving these coefficients. There is another bijection between these rook numbers and permutations of multisets, which include the Stirling permutations of Gessel and Stanley.

In section 5 we see that the 1-creation rook polynomials of $n \times n - 1$ boards are Abel polynomials, whose coefficients count forests of rooted labeled trees. We construct a bijection to explain this.

In section 6, we further generalize the notion of rook placements, prove a factorization theorem for Ferrers boards and generalize Chow's reciprocity theorem.

Finally, in section 7 we derive q-analogues of some of our results.

Notation: LHS and RHS are abbreviations for "left-hand-side" and "right-hand-side", respectively. $\mathbb{N}$ denotes the nonnegative integers.

2. Basic Concepts

We first review some classical definitions. We think of a **board** intuitively as a finite array of squares or **cells** arranged in rows and columns, i.e., a subset of cells of some $n \times n$ chessboard. A **Ferrers board** is made up of adjacent solid columns of cells with a common lower edge such that the heights (number of cells) $h_1, h_2, \dots, h_n$ of the columns form a non-decreasing sequence reading from left to right. Fig. 2.1 gives examples of Ferrers boards with their height vectors $(h_1, h_2, \dots)$ and columns labeled from left to right. See [14] for more formal definitions. It is convenient to allow columns of zero height.

For the rest of this paper we use the term "board" to mean "Ferrers board" unless otherwise noted (as in section 6).

Classic rook theory studies $r_k(B)$, the number of ways of placing k markers on the cells of the board B so that no two are in the same row or column. In chess terminology, we place k non-taking rooks on the board.

The basis of this paper is the notion of an i-row creation rook placement or just an **i-creation rook placement** *(or a rook placement with an i-creation rule).* This means that first we choose the columns where we will place the rooks. Then, as we place non-taking rooks in these columns, from left to right, each time a rook is placed *i new rows are created* drawn to the right end and immediately above where we placed the rook. For $i > 0$, as we place a rook, the next rook to be placed has an increased number of possible positions ($i = 0$ corresponds to the classic rook placement). We give two examples in detail.

Example 1: Take the board of Fig. 2.1(a). We will place non-taking rooks in

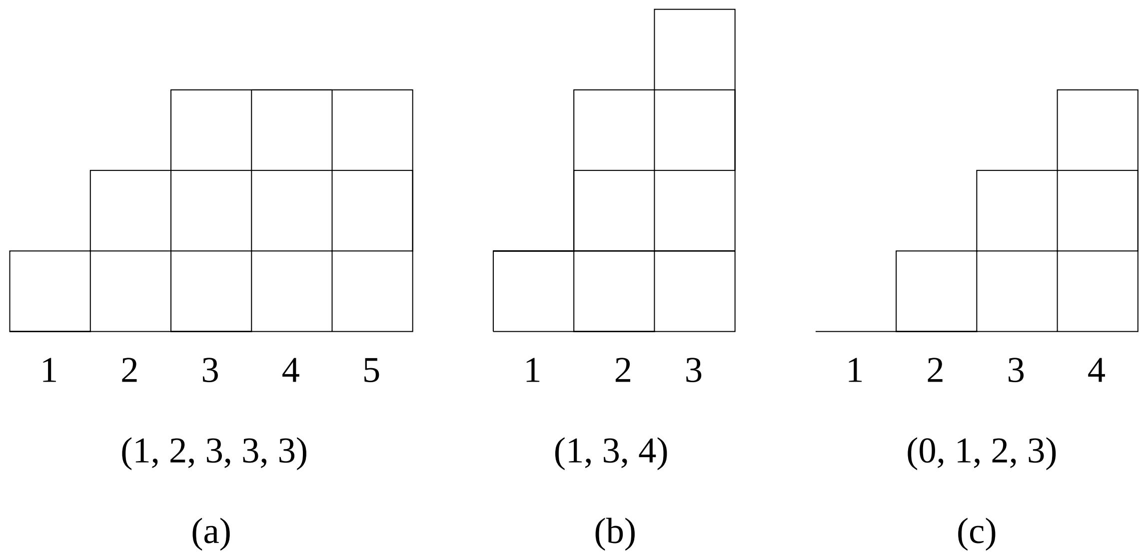

FIGURE 2.1

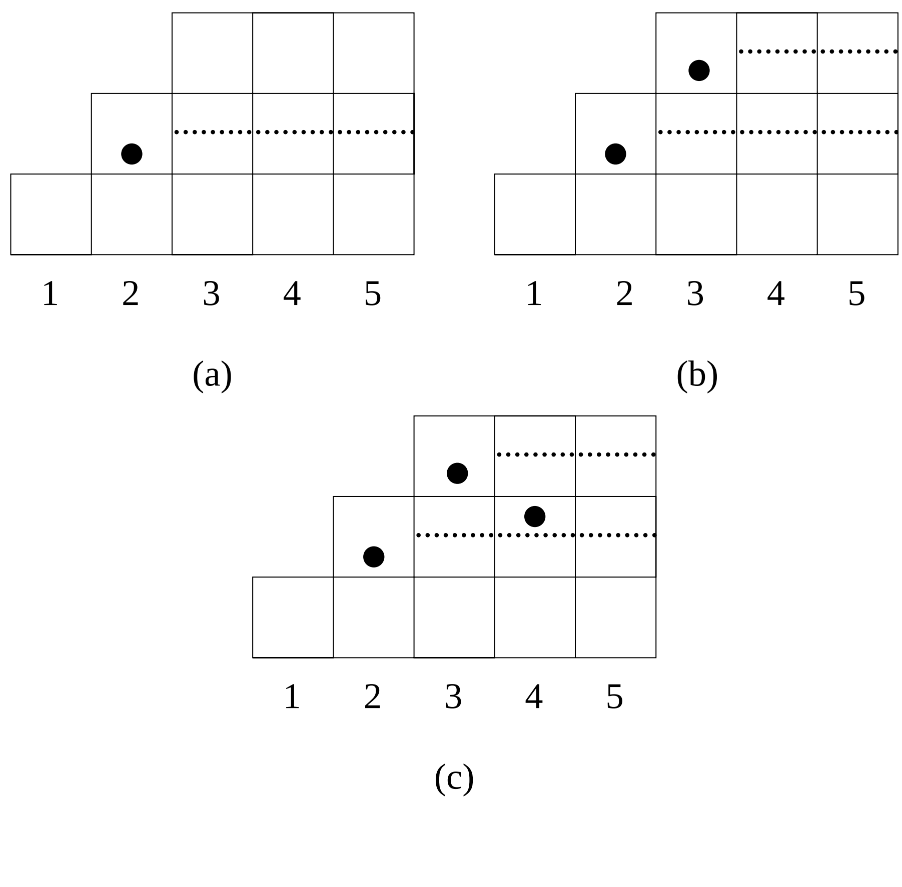

FIGURE 2.2

columns 2, 3, and 4, using a 1-creation rule. There are two choices for placing the first rook in column 2. We choose the second cell from the bottom and create one new row to the right and above this rook (Fig. 2.2(a)). Now there are 4 cells in column 3, but only 3 are available to place a rook in since row 2 has already been used in column 2. Place the second rook in the third available cell from the bottom and create a new row to the right and above it (Fig. 2.2(b)). Finally there are now 5 cells in column 4, 3 of which are allowed (rows 2 and 4 have been used) and we place the third rook in the second available space from the bottom (Fig. 2.2(c)).

Example 2: We use the same board as for example 1 (Fig. 2.1(a)) and again place a rook in columns 2, 3. and 4, but this time use a 2-*creation rule*. Place the first rook in column 2 in the second cell from the bottom and create two new rows as shown in Fig. 2.3(a). Now there are 5 cells in column 3, but only 4 are available (row 2 has been used). Place the second rook in the third available space from the bottom and create two new rows. Finally, there are now 7 cells in column 4, with 5 available ones (rows 2 and 4 have been used). Place the third rook in the second available space.

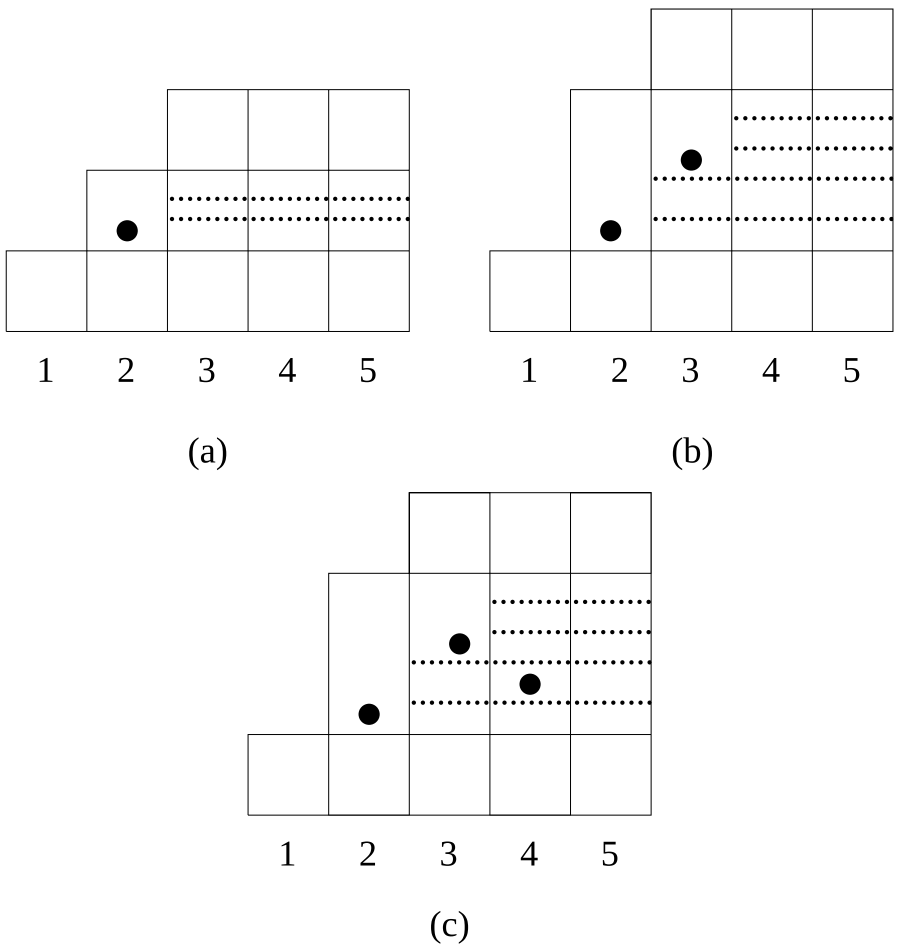

FIGURE 2.3

To describe the placement of cells more compactly, we introduce the coordinates of the rooks in a rook placement. Given a board and an i-creation rook placement, we say that a rook of the placement has **coordinates** (s,t) if the rook is in the column labeled s and was placed in the t^{th} available space from the bottom as the rooks are placed from left to right. In all of our examples, the columns of an n column board have been labeled with the label set $\{1,2,\ldots,n\}$ in the natural order. We shall see in section 5 that this is not the only useful labeling.

In example 1, the coordinates of the rooks are $(2,2)$, $(3,3)$, and $(4,2)$. In example 2, the rooks also have coordinates $(2,2)$, $(3,3)$, and $(4,2)$, but with a different creation rule. If we know the board B, the labeling of the columns, and the creation rule being used, then the set of coordinates of the rooks completely determines the rook placing.

The **i-rook number, $r_k^{(i)}(B)$,** $i=1,2,\ldots,n$, of an n column board B is the number of i-creation rook placements of k rooks on B, with $r_0^{(i)}=1$. We just use $r_k^{(i)}$ when no confusion can arise. The **i-rook vector of** B, $r^{(i)}(B)$, is the vector $(r_0^{(i)}, r_1^{(i)},\ldots)$, where clearly $r_k^{(i)}=0$ for $k>n$. The **i-rook polynomial of** B, $r^{(i)}(B,x)$, is defined as

$$r^{(i)}(B,x) := \sum_{k=0}^{n} r_k^{(i)}(B)x^{(n-k,i-1)},$$

where $x^{(n,m)} := x(x+m)(x+2m)\cdots(x+(n-1)m)$, for $n>0$, and $x^{(0,m)}=1$.

We see why $r^{(i)}(B,x)$ is a natural object of study in the following theorem.

Factorization Theorem. *Let B be a Ferrers board with column heights $h_1 \le h_2 \le \ldots \le h_n$. Then*

$$r^{(i)}(B,x) = \prod_{j=1}^{n}(x+h_j+(j-1)(i-1)). \tag{1}$$

Pf: Our proof is an immediate generalization of the proof in [14].

Let $x \in \mathbb{N}$ and B_x be the board B with an $x \times n$ rectangle adjoined (Fig. 2.4). This is a board with column heights $x+h_1 \le x+h_2 \le \ldots \le x+h_n$. Recall that some of the h_j's can be zero. The RHS of (1) counts the number of ways to place n rooks on B_x using an *i-creation rule*. But this can also be counted by cases, according to the number of rooks placed on B. There are $r_k^{(i)}(B)$ ways to place k rooks on B. This uses k of the columns of B_x and thus there are $x^{(n-k,i-1)}$ ways to place the remaining $n-k$ rooks in the remaining $n-k$ columns of the $x \times n$ rectangle. $\square$

Specializing (1) by taking a 1-creation rule yields $r^{(1)}(B,x)=\prod_{j=1}^{n}(x+h_j)$ and, since the h's can be any non-decreasing sequence of nonnegative integers, we have

Corollary 2.1. *Any monic polynomial with non-positive integer roots is the 1-rook polynomial of a Ferrers board.*

We call two boards B and B' *i-rook equivalent* if $r_k^{(i)}(B)=r_k^{(i)}(B')$, for all k. If $i=0$, the classical theory, there is an interesting and complete classification and enumeration of the equivalence classes for Ferrers boards [14]. If $i>0$, then it follows almost immediately from the factorization theorem that, except for the trivial case of boards differing by empty columns, no two boards are i-rook equivalent.

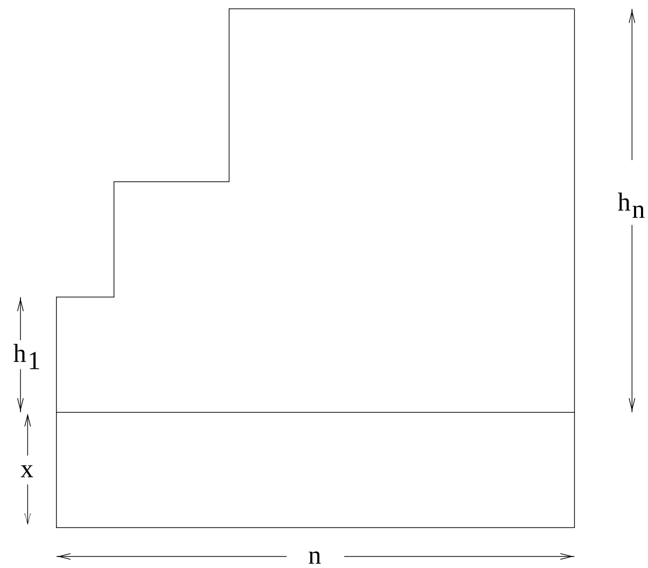

FIGURE 2.4: THE FERRERS BOARD B_x

Example 3: An interesting class of boards are the *m − jump boards* $\mathbf{J_{n,m}}$ with column heights $(0, m, 2m, \ldots, (n-1)m)$. Applying the factorization theorem we get

$$r^{(i)}(J_{n,m}, x) = x^{(n,m+i-1)} = \sum_{k=0}^{n} r_k^{(i)}(J_{n,m}) x^{(n-k,i-1)}. \tag{2}$$

Recall that for fixed b, the polynomial sequence $\{x^{(n,b)}, n = 0, 1, 2, \ldots\}$ is of binomial type and, since (2) connects two such binomial type sequences, the $r_k^{(i)}(J_{n,m})$ are also coefficients of binomial type sequences [23]. Hence we have a combinatorial interpretation, in terms of i-rook numbers, for the connection coefficients between these two rising factorial sequences. In particular, for $m = 1$, this relates $\{x^{(n,i)}\}$ and $\{x^{(n,i-1)}\}, n = 0, 1, 2, \ldots$. In the next two sections, we shall see that when $i = 1$ or 2, these are particularly interesting models.

We now derive a useful recurrence for the i-creation rook numbers. Let B be a board with heights $h_1 \leq h_2 \leq \cdots \leq h_n$, and let $\hat{B}$ be the $n-1$ column board obtained by removing the last column of B. The number of i-creation placements of k rooks on B with no rook in the last column is $r_k^{(i)}(\hat{B})$. On the other hand, for each placement of $k-1$ rooks on the first $n-1$ columns of B, there are $h_n + (k-1)(i-1)$ ways to place a rook in the last column. Thus

$$r_k^{(i)}(B) = r_k^{(i)}(\hat{B}) + (h_n + (k-1)(i-1)) r_{k-1}^{(i)}(\hat{B}). \tag{3}$$

If we let $m = 1, k = n-1$, we have by (3), or by a direct count, that $r_{n-1}^{(i)}(J_{n,1}) = 1^{(n-1,i)}$.

The factorization theorem, together with the notion of i-rook placements allows the construction of new combinatorial models for binomial type sequences, analogous to the development in [17]. We will not pursue that sequence of ideas here.

3. 1-Jump Board, 1-Row Creation

Specializing equation (2) from section 2 to the case $m = 1, i = 1$, we have $x^{(n,1)} = \sum r_k^{(1)}(J_{n,1})x^{(n-k,0)}$ or, in more familar notation, $x^{(n)} = \sum r_k^{(1)}(J_{n,1})x^{n-k}$, where $x^{(n)} = x^{(n,1)}$ is the ordinary rising factorial. But it is well known that $x^{(n)} = \sum_{k=0}^{n} c(n, n-k)x^{n-k}$, where $c(n,k)$, the absolute Stirling number of the first kind, counts the number of permutations of $\{1, 2, \ldots, n\}$ with k cycles [29]. Therefore we have

Theorem 3.1.

$$r_k^{(i)}(J_{n,1}) = c(n, n-k).$$

In this section we give two bijective proofs of this fact.

First Proof: To avoid cumbersome notation, we use an example to illustrate the idea. Let C be a 1-placement of 3 rooks on the board $J_{7,1}$, with coordinates $(2,1)$, $(4,2)$, and $(7,5)$.

Consider the sequence of permutations in Fig. 3.1. Start with the identity $I_7 \in S_7$. The first rook is $(2,1)$, so multiply the identity on the right by the transposition (12) to obtain π_1. This has the effect of erasing the cycle (2) and inserting 2 to the left of 1 in (1) (Fig. 3.1(i)). The second rook is $(4,2)$. Multiply π_1 by (24) on the right to obtain π_2 (equivalently, erase (4) in π, and insert 4 to the left of 2 in (21) - Fig. 3.1(ii)). The last rook is $(7,5)$ so multiply π_2 on the right by (75) to obtain π_3 (Fig. 3.1(iii)). Our correspondence is $C \longleftrightarrow \pi_3$. Since at each step we are merging a one cycle with another cycle, and there are three merges (rooks) we end up with a permutation, π_3, with $7-3$ cycles.

$(1)(2)(3)(4)(5)(6)(7) = I_7$

(i) $\downarrow$

$(21)(3)(4)(5)(6)(7) = I_7(12)$

(ii) $\downarrow$

$(421)(3)(5)(6)(7) = I_7\,(12)(24)$

(iii) $\downarrow$

$(421)(3)(75)(6) = I_7\,(12)(24)(57)$

Figure 3.1

In general if $(i_1, n_1), (i_2, n_2), \ldots, (i_k, n_k)$, $i_1 < i_2 < \cdots < i_k$ are the coordinates of a placement of rooks on $J_{n,1}$ and I_n is the identity in S_n, then the map

$$C \to \pi = I_n(n_1\, i_1)(n_2\, i_2)\cdots(n_k\, i_k)$$

is our required bijection. The reasoning of our example shows clearly why π has $n-k$ cycles.

The inverse map follows immediately. Let $\sigma \in S_n$ be a permutation with k cycles. If n is in a 1-cycle, erase the cycle. If n is immediately followed by j, (in cyclic order) in some cycle, then erase n from this cycle and add (n, j) as coordinates of a rook in our placement. In both cases we now have a permutation in S_{n-1} and we repeat this procedure. □

Second Proof: We will use the terminology of flags on flagpoles where the order in which the flags appear matters.

Let C be the placement of 4 rooks, with coordinates $(3,2),(4,3),(5,1),(7,1)$ on the board $J_{7,1}$ (Fig. 3.2). This defines a configuration of 4 flags on 3 flagpoles constructed as follows (Fig. 3.3). The first (empty) column has no rook, so we create flagpole 1. The second column has no rook; create flagpole 2. The first rook from the left has coordinates $(3,2)$, so place flag 3 in the second position (reading up the flagpoles from left to right). The second rook (in column 4) has coordinates $(4,3)$, so place flag 4 in position 3. The next rook is $(5,1)$ so flag 5 is placed in the first position. And finally rook $(7,1)$ tells us to place flag 7 in the first position.

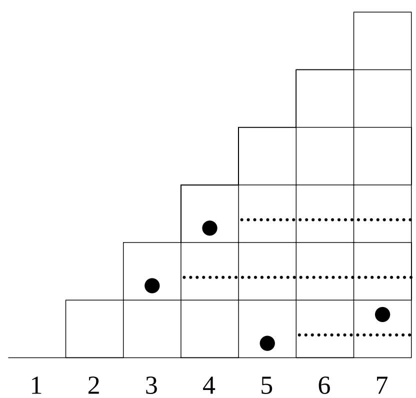

FIGURE 3.2

If we think of each flagpole with its flags as defining a cycle in the final configuration (reading from the bottom up), then we get the permutation (175)(234)(6). Since there are 3 columns with no rooks, we have 3 flagpoles and hence 3 cycles.

The inverse map (flags on flagpoles to rook placements) is straightforward. The flagpole labels $\{1,2,6\}$ tell us that columns $\{1,2,6\}$ of $J_{7,1}$ have no rooks. The largest numbered flag is 7 and it is in position 1; so $(7,1)$ is a rook. Erase flag 7, which leaves 5 as the largest flag, now in position 1; so $(5,1)$ is a rook. Erase 5 so flag 4 is now in position 3; so $(4,3)$ is a rook. Erase 3 and flag 3 is in position 2 and $(3,2)$ is a rook.

It is clear how to generalize to a bijection between 1-creation placements of k rooks and those $\sigma \in S_n$ with $n-k$ cycles. □

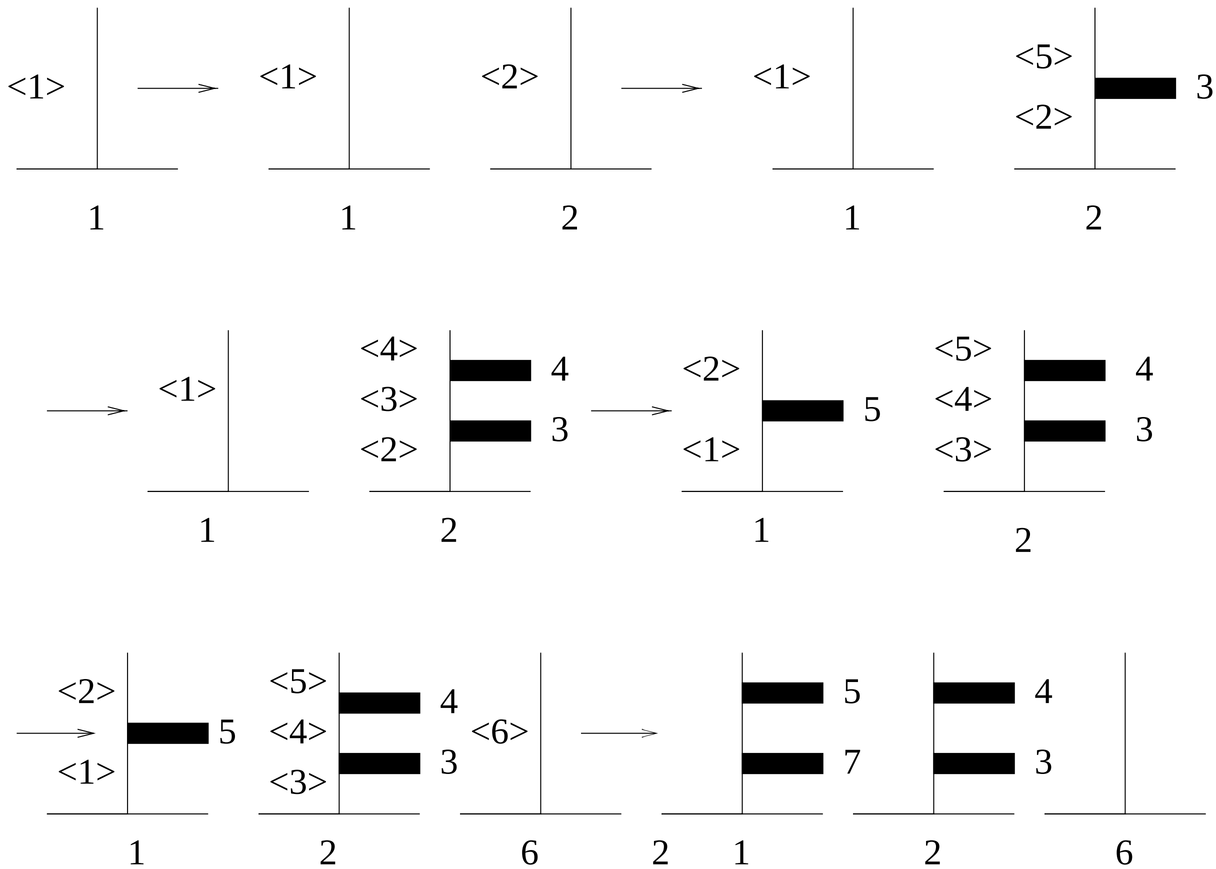

FIGURE 3.3

4. 1-JUMP BOARD, 2-ROW CREATION

A matching in a graph is a selection of disjoint edges. Dulucq and Favreau [8] have shown that the coefficients of the Bessel polynomial can be modeled by matchings, which together with Theorem 4.1 shows that $\sum r_k^{(2)}(J_{n+1,1})x^k$ is the Bessel polynomial of degree n (see [18] for background on Bessel polynomials). They also refer to a combinatorial proof of the orthogonality. Applying the factorization theorem to $J_{n+1,1}$ with $i = 2$ we get a new polynomial relation for these coefficients (see eq. (4) below), namely, as the connection coefficients between the two bases $x^{(n,1)}$ and $x^{(n,2)}$. Perhaps the orthogonality of the Bessel polynomials could also be proven using the methods of Gessel [12].

Theorem 4.1. *Let $m_k(K_n)$ denote the number of k-edge matchings in the complete graph K_n. Then*

$$r_k^{(2)}(J_{n,1}) = m_k(K_{n+k-1}).$$

Pf: Start with a placement of k rooks on $J_{n,1}$. If there is no rook in the last column, use induction to get a k-edge matching on $n-1+k-1$ vertices, then add the singleton vertex $n+k-1$ to this matching.

Now assume there is a rook in the last column, say with coordinates (n, j). Since each of the $k-1$ rooks in columns 1 through $n-1$ creates 2 rows, we have $1 \leq j \leq n+k-2$. Match vertex j with vertex $n+k-1$. This leaves $n+k-3$ vertices left unmatched in K_{n+k-1}. By induction, the $k-1$ rooks in columns 1 through $n-1$ determine a $k-1$ edge matching on $n-1+k-1-1 = n+k-3$ vertices. Use the edges from this matching, with vertices $j, j+1, \ldots, n+k-3$ relabeled as $j+1, j+2, \ldots, n+k-2$.

Now we describe the inverse of this correspondence. Say we have a matching M where vertex $n+k-1$ is a singleton. Then by induction we can associate a placement of k rooks on $J_{n-1,1}$ to $M - \{n+k-1\}$, to which we add an empty column of height $n+k-1$ on the right. On the other hand, if vertex $n+k-1$ is matched to vertex j, then again by induction we associate a placement of $k-1$ rooks on $J_{n-1,1}$ to $M - (j, n+k-1)$, to which we add a column of height $n+k-1$ with a rook in it with coordinates (n, j). □

Example 5*:* If C is the placement in the top left of Fig. 4.1(a), then $n+k-1 = 5$ and the above construction starts with vertices $\{v_1, v_2, v_3, v_4, v_5\}$. The rook in the last column with coordinates $(4, 2)$ matches v_2 and v_5, and we are then left with the rook placement in Fig. 4.1(b), applied to the set of vertices $\{v_1, v_3, v_4\}$. This placement matches v_1 and v_4, with the final matching displayed in Fig. 4.1(c).

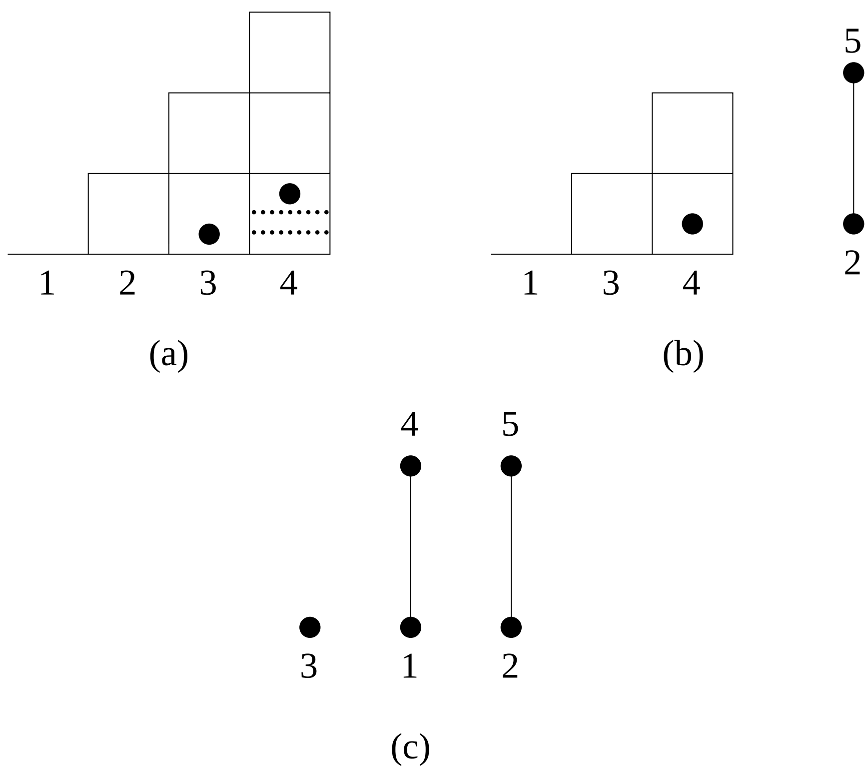

FIGURE 4.1

Pitman [24] has derived an alternative interpretation of the coefficients of the Bessel polynomials in terms of lattice paths, and it is an interesting question to try and connect the lattice paths and the rook placements directly.

By a simple combinatorial argument

$$m_k(K_n) = \binom{n}{2k} \frac{(2k)!}{k!2^k}.$$

Combining this with Theorem 4.1 yields

$$r_k^{(2)}(J_{n,1}) = \binom{n+k-1}{2k} \frac{(2k)!}{k!2^k},$$

and substituting this into the factorization theorem gives the identity

$$\sum_{k=0}^{n} \binom{n+k-1}{2k} \frac{(2k)!}{k!2^k}\, x^{(n-k,1)} = x^{(n,2)}. \tag{4}$$

When expressed in hypergeometric notation, this identity reduces to a special case of a well-known summation theorem of Kummer for a ${}_2F_1$ with argument $1/2$ (see [25, p. 69, ex. 3]).

In section 3, we constructed a bijection between 1-creation rook placements on a 1-jump board and flags on flagpoles (permutations by cycles). This generalizes immediately to a bijection between 2-creation rook placements on a 1-jump board and flags on flagpoles. The only difference is that when a flag, corresponding to a rook, is placed on a flagpole, two adjacent copies of the flag are placed, not one. Therefore two new positions are created on the flagpole, corresponding to 2-creation.

This is illustrated in Fig.'s 4.2 and 4.3. The first (empty) column has no rook; create flagpole 1. The second column has no rook; create flagpole 2. The first rook (from left to right) has coordinates $(3,1)$, so two copies of flag 3 are placed in the first position (reading up the flagpoles from left to right). The second rook has coordinates $(4,2)$, so two copies of flag 4 are placed in the second position. The third rook has coordinates $(5,6)$ and two copies of flag 5 are placed in the sixth position. Finally, column 6 has no rook so we create flagpole 6.

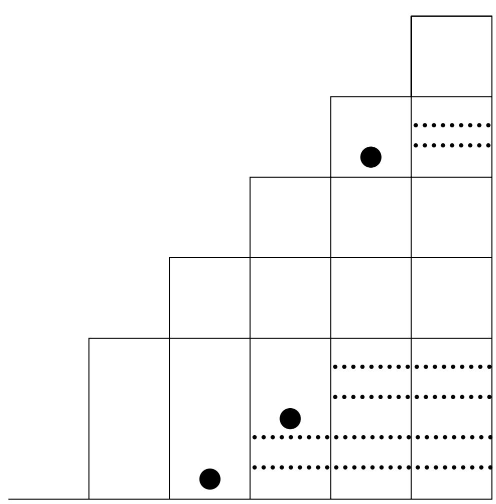

FIGURE 4.2

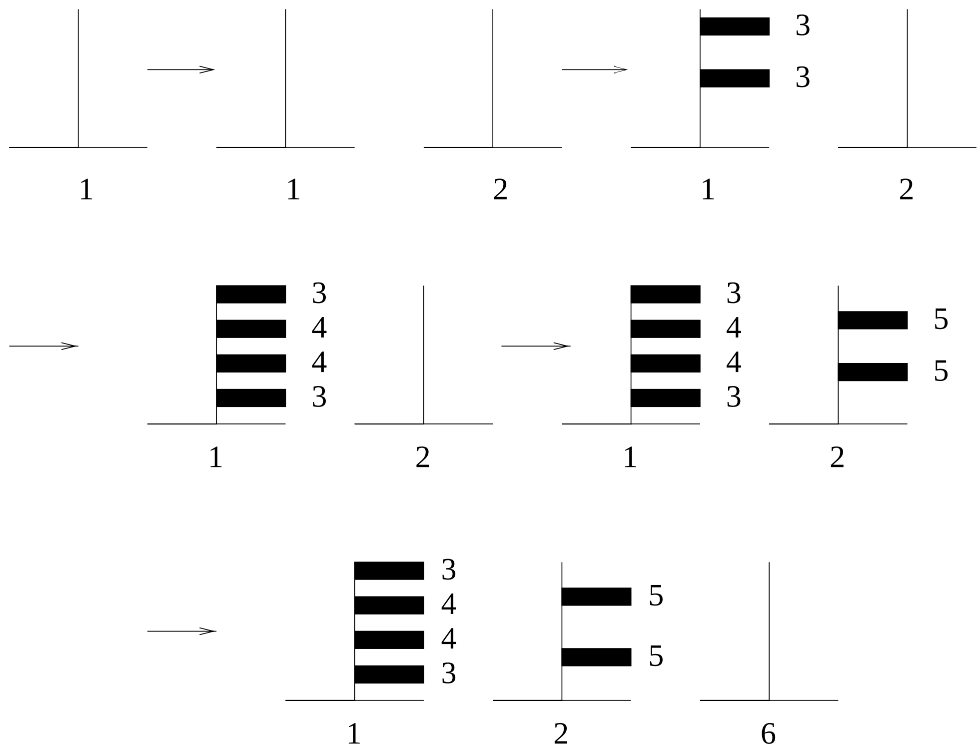

FIGURE 4.3

The inverse map is straightforward. The flagpole labels $\{1,2,6\}$ imply that columns $\{1,2,6\}$ have no rooks. The largest numbered pair of flags is labeled 5. They are adjacent (being there last flags placed), so erase one of them. The remaining flag 5 is in position 6 which gives $(5,6)$ as the coordinates of a rook. Now erase flag 5 and repeat with the next largest numbered flag and so on, yielding the coordinates of all the rooks.

Clearly for this example, and using the same procedure for any 2-creation placement on a 1-jump board, we have a bijection.

How can we interpret the flag-flagpole configurations as permutations?

One possibility is to consider each flagpole in Fig. 4.3 as determining a cycle. In our example above (Fig. 4.3), we then have (1 3 4 4 3), (2 5 5), (6), which could represent a permutation of the multiset $\{1,2,3,3,4,4,5,5,6\}$. Or possibly we should consider the elements of multiplicity one as defining an ordering or indexing of the cycles. See Knuth [22] for a discussion of multiset permutations.

We can generalize to a larger class of multisets. If we have i-creation on a j-jump board, then our bijection generalizes in an obvious way by using i copies of each flag (corresponding to a rook) and j copies of each flagpole (corresponding to columns with no rooks). Then we would be considering multisets with two types of elements, those of multiplicity i and those of multiplicity j. The best interpretations are not clear.

Consider 2-creation placements of $n-1$ rooks on $J_{n,1}$, the maximum possible.

There is one flagpole and each flag occurs twice. If we read the flag labels moving up the pole we get $a_1, \ldots, a_{2(n-1)}$, which is a rearrangement of the multiset $\{2, 2, 3, 3, ..., n-1, n-1\}$ with the property: if $i < j < k$ and $a_i = a_k$, then $a_j > a_i$. Gessel and Stanley [13] called these Stirling permutations and studied the number of them with a fixed number of descents. Allowing i-row creation with $n-1$ rooks on a 1-jump board, we have the multiset permutations defined in their open problem section. It might be possible to approach these open problems by defining a hit polynomial on a subset of the rook board.

5. Abel Boards and Forests

Let A_n denote the *Abel* board, the $n \times n$ board with column heights $(0, n, n, \ldots, n)$ (see Fig. 5.1).

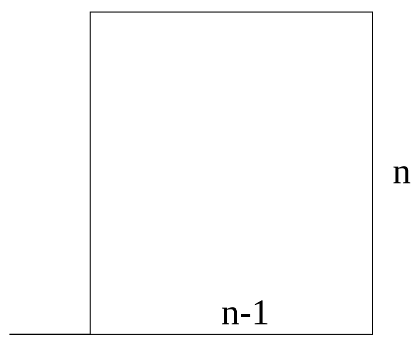

Figure 5.1

By the factorization theorem, the 1-rook polynomial of A_n is

$$r^{(1)}(A_n, x) = \sum_{k=1}^{n} r_k^{(1)} x^{n-k} = x(x+n)^{n-1}.$$

These polynomials are a special case of the general Abel polynomials $x(x+an)^{n-1}$ and the coefficient $r_k^{(1)} = t_{n,n-k}$, the number of labeled forests on n vertices composed of $n-k$ rooted trees [23]. In this section we explain this equality bijectively. In fact, we add some structure and construct a bijection between $R_{n,k} = \{(p, u), u \in \{1, 2, \ldots, n\}\}$, where p is a 1-creation placement of k rooks on A_n, and $F_{n,k} = \{$*marked rooted forests* of $n-k$ rooted trees on n vertices$\}$, where a marked rooted forest is a forest of rooted trees with one distinguished vertex in the forest (the mark). This yields $nr_k^{(1)} = nt_{n,n-k}$.

Recall that a partial endofunction is a function $f : V \to W$, where V is a subset of W. We construct a bijection between $R_{n,k}$ and $P_{n,k}$, a class of 'marked' partial endofunctions, and then specialize a bijection in [2] to a bijection between $F_{n,k}$ and the functional digraphs of $P_{n,k}$.

The definition of $P_{n,k}$ and the bijection between $R_{n,k}$ and $P_{n,k}$ is described with an example. On the board A_5, pick an integer from $\{1, 2, 3, 4, 5\}$, say 2, use 2 as the label of the first column, and label the other columns with the remaining labels

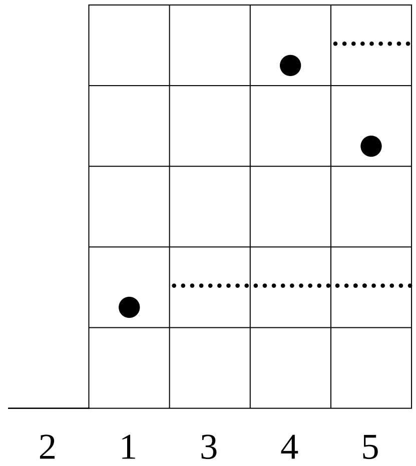

FIGURE 5.2

$\{1, 3, 4, 5\}$ in natural order. Then we use the 1-creation placement p of 3 rooks whose coordinates from left to right are $(1, 2)$, $(4, 5)$, and $(5, 4)$ (see Fig. 5.2).

The coordinates define the partial endofunction $f : \{1, 4, 5\} \to \{1, 2, 3, 4, 5\}$ given by $f(1) = 2$, $f(4) = 5$, and $f(5) = 4$. We call the pair $(f, 2)$, where 2 is the label chosen for the first column, a *marked partial endofunction*, with 2 as the mark.

Using the procedure just described, we see that any placement p of 3 rooks and a choice of label u for the first column defines, via its coordinates, a marked partial endofunction. Let $P_{n,k}$ be the set of marked partial endofunctions $\{(f, u)\}$, where A is a k-subset of $\{1, 2, \ldots, n\}$, $f : A \to \{1, 2, \ldots, n\}$ is a partial endofunction and $u \in \{1, 2, \ldots, n\} - A$. In our example we are considering $P_{5,3}$. The mapping $(p, u) \to (f_p, u)$ defines a bijection between $R_{5,3}$ and $P_{5,3}$, and the obvious generalization of our procedure leads to a bijection between $R_{n,k}$ and $P_{n,k}$.

The functional digraph $D_{f,2}$ of the marked partial endofunction in our example is the directed graph of Fig. 5.3. Every component of $D_{f,2}$ which is a tree has all its edges directed toward one vertex, which we designate the root. By our construction, these roots correspond to the labels of the empty columns of our rook placement. The label of the first column (2 in this case) is a specially marked root (vertex 2 with a square about it). We identify each marked partial endofunction with its functional digraph.

Now we specialize a bijection in [2, pp. 174-176] to a bijection between the functional digraphs of $P_{5,3}$ and $F_{5,3}$.

The set $D'_{f,2}$ of non-tree components of $D_{f,2}$ (which are cycles possibly with trees attached - none in our example), with vertex set S, is the functional digraph of the restriction of f to S, a regular function $f|_S : S \to S$. In our example $S = \{4, 5\}$, $f|_S(4) = 5$ and $f|_S(5) = 4$. We apply the Joyal bijection, between functional digraphs of regular functions and marked rooted trees [1], [2] to $D'_{f,2}$ in our example to obtain the marked rooted tree with marked vertex 4 as shown in Fig. 5.4. Vertex 4 (with the triangle) is the mark.

Now our forest is obtained by taking the trees in $D_{f,2}$ (Fig. 5.3) and the tree in Fig. 5.4, connecting the square vertex 2 to the mark 4 and erasing the triangle.

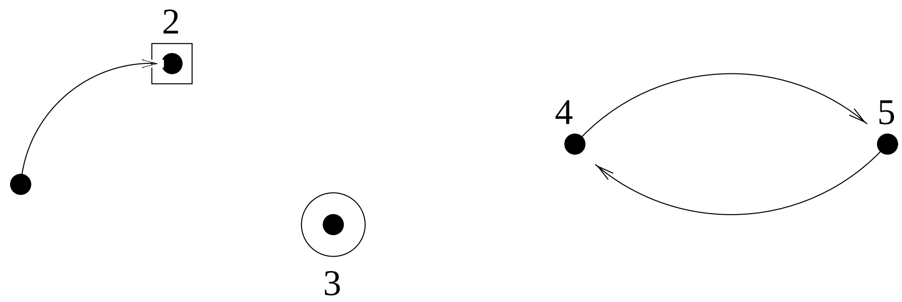

FIGURE 5.3

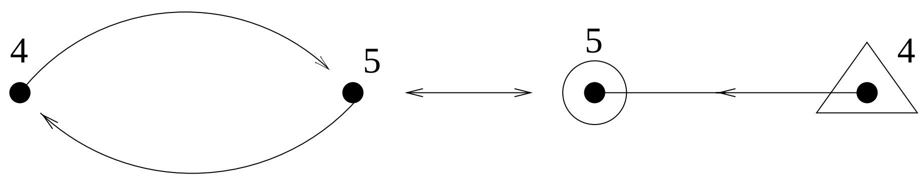

FIGURE 5.4

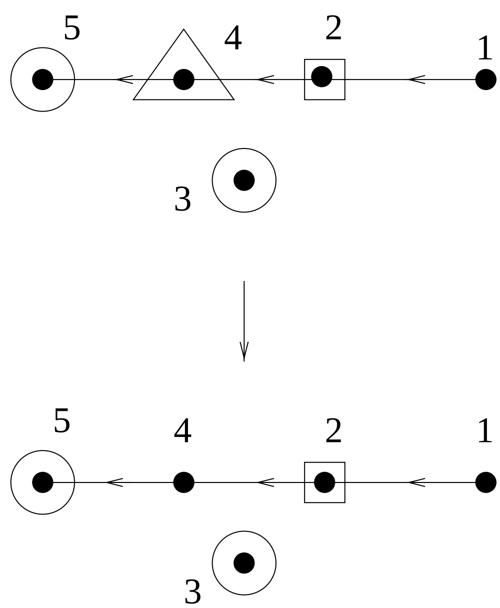

FIGURE 5.5

This yields a marked rooted forest with 2 as the marked vertex (Fig. 5.5).

This procedure, associating a marked rooted forest with the functional digraph of a marked partial endofunction, is easily seen to be a bijection between $P_{5,3}$ and $F_{5,3}$. It is also easy to see that this procedure generalizes to a bijection between $P_{n,k}$ and $F_{n,k}$.

Hence we have the two bijections $R_{n,k} \longleftrightarrow P_{n,k} \longleftrightarrow F_{n,k}$ and, by composing them, we are done.

6. The Alpha Parameter

In this section our placements can have at most one rook in any column but can have more than one rook in a given row. Rooks do not create new rows as in earlier sections, but instead are weighted according to the following scheme: if there are u rooks in a given row, that row has weight

$$\begin{cases} 1 & \text{if } 0 \le u \le 1 \\ \alpha(2\alpha-1)(3\alpha-2)\cdots((u-1)\alpha-(u-2)) & \text{if } u \ge 2. \end{cases}$$

The weight of a placement is the product of the weights of all the rows. Set

$$r_k^{(\alpha)}(B) = \sum_{\substack{C \\ k \text{ rooks on } B}} wt(C),$$

where the sum is over all placements of k rooks on B. If $\alpha = 0$, this definition reduces to the standard rook number. More generally, if α is a positive integer, $r_k^{(\alpha)}(B)$ equals the α-creation rook number from the earlier sections. (This follows from the factorization theorem from section 2 and the α-factorization theorem below). If α is a negative integer, $r_k^{(\alpha)}(B)$ equals the number of rook placements where each rook placed "deletes" $1-\alpha$ rows to the right of the rook, as in the theory of Remmel and Wachs [26]. If $\alpha = -1$, the rook numbers for a Ferrers board equal the matching numbers of a certain threshold graph corresponding to the board, as studied by Reiner and White [27].

α-Factorization Theorem. *Let B be a Ferrers board with column heights $h_1 \le h_2 \le \ldots \le h_n$. Then*

$$\sum_{k=0}^{n} r_k^{(\alpha)}(B) x^{(n-k,\alpha-1)} = \prod_{j=1}^{n} (x + h_j + (j-1)(\alpha-1)).$$

Pf: We proceed as in the proof of the factorization theorem. First assume $x \in \mathbb{N}$. Let B_x be the board of Fig. 2.4. We count the number of ways of placing n rooks on B_x, one in each column, with α-weighting, in two different ways. First we count the number of ways of placing one in the first column, then one in the second column, etc.. Assume that we have placed rooks in columns 1 through $j-1$, and that in these columns we have m_u rows with u rooks, $1 \le u$, $\sum_u u m_u = j-1$. Now we use the fact that if we have already placed j rooks in a row, then we can place another rook in this row, in the first available column, in $j\alpha - (j-1)$ ways. Thus the number of ways of placing a rook in column j in these rows is

$$\sum_u m_u(u\alpha - (u-1)).$$

In addition there are $x + h_j - \sum_u m_u$ open rows in column j, so the total number of placements in column j is

$$x + h_j + \sum_u -m_u + m_u(u\alpha - (u-1)) = x + h_j + (j-1)(\alpha-1).$$

It follows that the total number of placements is counted by the RHS of the α-factorization theorem. On the other hand we can begin by placing k rooks on B in $r_k^{(\alpha)}(B)$ ways. Each of these placements uses k of the columns of the $x \times k$ rectangle below B, and so we can place $n-k$ rooks on what's left in $x(x+\alpha-1)\cdots(x+(k-1)(\alpha-1))$ ways (by the above argument, applied to an empty board). This proves the theorem if $x \in \mathbb{N}$, and any two polynomials with infinitely many common values must be identical. $\square$

Theorem 6.1**.** *Let B be any subset of an $n \times n$ chessboard (not necessarily a Ferrers board). Let B^c denote the complementary board consisting of those squares $\notin B$. Then for any $\alpha \in \mathbb{C}$, as polynomials in x*

$$\sum_{k=0}^{n} r_k^{(\alpha)}(B)(-1)^k(x+k(\alpha-1)+n)(x+(k+1)(\alpha-1)+n)\cdots(x+(n-1)(\alpha-1)+n) =$$

$$\sum_{k=0}^{n} r_k^{(\alpha)}(B^c)x(x+\alpha-1)(x+2\alpha-2)\cdots(x+(n-k-1)(\alpha-1)).$$

Pf: Our proof is a straightforward generalization of Chow's proof of the $\alpha = 0$ case [4].

First assume $x \in \mathbb{N}$. Add x extra rows to $n \times n$. Then

$$r_k^{(\alpha)}(B)(x+k(\alpha-1)+n)\cdots(x+(n-1)(\alpha-1)+n)$$

is the number of ways of placing k rooks on B, with weights, and then placing $n-k$ more rooks anywhere (i.e. on B, B^c, or on the x extra rows). Here we again use the fact that if we have already placed j rooks in a row, then we can place another rook in this row, in the first available column, in $j\alpha-(j-1)$ ways. By inclusion-exclusion, we see that the resulting configuration in which the set S of rooks on B is nonempty cancel out of the above sum. The remaining details of the proof can be filled in by arguments contained in the proof of the α-factorization theorem. $\square$

7. A q-Analogue of $r_k^{(\alpha)}$

Let $[x] := (1-q^x)/(1-q)$ denote the standard q-analogue of the real number x. Let B be a Ferrers board and C a placement of rooks as in section 6, with no two rooks in the same column but (possibly) more than one rook in a given row. If γ is a square of B, let $v(\gamma)$ be the number of rooks strictly to the left of, and in the same row as, γ. Define the weight of γ to be

$$\mathrm{wt}(\gamma) := \begin{cases} 1 & \text{if there is a rook above and in the same column as } \gamma \\ [(\alpha-1)v(\gamma)+1] & \text{if } \gamma \text{ contains a rook} \\ q^{(\alpha-1)v(\gamma)+1} & \text{else} \end{cases}$$

Furthermore, define the weight of C to be

$$\mathrm{wt}(C) := \prod_{\alpha \in B} \mathrm{wt}(\alpha)$$

and the q-analogue of $r_k^{(\alpha)}$, $R_k^{(\alpha)}$, to be

$$R_k^{\alpha}(B) := \sum_{\substack{C \\ k \text{ rooks on } B}} \mathrm{wt}(C).$$

By a standard argument similar to those in [11] we can prove the following.

Theorem 7.1. *If B is a Ferrers board with column heights $h_1, h_2, \ldots, h_n$ then*

$$\sum_{k=0}^{n} R_k^{(\alpha)}(B)[x][x+\alpha-1]\cdots[x+(n-k-1)(\alpha-1)] = \prod_{j=1}^{n}[x+h_j+(j-1)(\alpha-1)].$$

Theorem 6.1 also has a q-analogue (at least for Ferrers boards), which is a simple corollary of Theorem 7.1.

Corollary 7.2. *Let B be a Ferrers board with column heights $h_1 \le h_2 \le \ldots \le h_n$. Set Area$(B) := h_1 + h_2 + \ldots h_n$ (the number of squares of B). Then*

$$\sum_{k=0}^{n} R_k^{(\alpha)}(B)(-1)^k q^{(n+x)k+(\alpha-1)\binom{k}{2}-\mathrm{Area}(B)}[x+k(\alpha-1)+n]\cdots[x+(n-1)(\alpha-1)+n] = \prod_{j=1}^{n}[x+n-h_{n-j+1}+(j-1)(\alpha-1)].$$

Pf: Note that for any w, $[w] = -q^w[-w]$. Making this substitution in every factor on both sides of Theorem 7.1 of the form $[x+p]$ for some p we get

$$\sum_{k=0}^{n} R_k^{(\alpha)}(B)(-1)^{n-k} q^x[-x]q^{x+\alpha-1}[-x-\alpha+1]\cdots q^{x+(n-k-1)(\alpha-1)}$$

$$\times[-x-(n-k-1)(\alpha-1)] = (-1)^n \prod_{j=1}^{n} q^{x+h_j+(j-1)(\alpha-1)}[-x-h_j-(j-1)(\alpha-1)].$$

Replacing x by $-x-n-(n-1)(\alpha-1)$ above and simplifying gives the Corollary. □

Corollary 7.2 is a q-analogue of Theorem 6.1 since, if B is an n-column Ferrers board, so is B^c, and thus in the q-case the RHS of Theorem 6.1 equals the RHS of Corollary 7.2 (using Theorem 7.1). One other identity we would like to mention is that

$$R_k^{(2)}(J_{n,1}) = q^{\binom{n-k}{2}} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix} \prod_{j=1}^{k}[2j-1].$$

This can be easily proven using recurrences and induction.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: goldman@math.umn.edu

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112
E-mail address: jhaglund@math.ucsd.edu