# Combinatorial Formulas for (Type A) Macdonald Polynomials

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Abstract. We give an expository overview of the combinatorial formula of Haglund, Haiman and Loehr for the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$, as well as versions of the formula which correspond to other variants of Macdonald polynomials. These include Macdonald's original $P_\mu(X;q,t)$ and the nonsymmetric Macdonald polynomials $E_\alpha(X;q,t)$, as well as their integral forms $J_\mu(X;q,t)$ and $\mathcal{E}_\alpha(X;q,t)$. We also discuss more compact versions of these formulas, as well as a quasisymmetric Macdonald polynomial, which grew out of work connecting Macdonald polynomials to the asymmetric simple exclusion process (ASEP).

## 1. Introduction

In 1988 Macdonald introduced an important family of symmetric functions $P_\mu(X;q,t)$, which depend on a countable set of variables $\{x_1, x_2, \ldots\}$, a partition $\mu$, and two parameters $q,t$ [**27**], [**28**][Ch. 6]. Special cases of the $P_\mu$ include the famous Hall Littlewood polynomials (i.e. $P_\mu(X;0,t)$), the Jack polynomials $J_\mu^\beta(X)$ (defined as $\lim_{t\to 1} P_\mu(X;t^\beta,t)$), as well as the classical bases of monomial symmetric functions $m_\lambda$, elementary symmetric functions $e_\lambda$, and Schur functions $s_\lambda$. We refer the reader to [**34**][Ch. 7], [**28**][Ch. 1] for general background on these bases for the ring $\mathrm{Sym}(X)$ of symmetric functions. In order to maintain a certain amount of brevity we will often use notation common to these sources without giving explicit definitions.

In general the coefficient of a given monomial in $P_\mu(X;q,t)$ is a rational function in $q,t$, but Macdonald found that if he multiplied $P_\mu(X;q,t)$ by a certain product of linear terms in $q,t$ to form the *integral form* $J_\mu(X;q,t)$, the monomial coefficients seemed to always be in $\mathbb{Z}[q,t]$. (We let $\mathbb{Z}, \mathbb{N}, \mathbb{Z}_+$ denote the integers, nonnegative integers, and positive integers, respectively.) More importantly, Macdonald conjectured the much stronger fact that when the $J_\mu$ are expanded in a certain variant of the Schur basis, the coefficients are in $\mathbb{N}[q,t]$. Garsia and Haiman found a refinement of this, their famous $n!$ Conjecture, which says that if a modified version of $J_\mu$ they denoted $\tilde{H}_\mu(X;q,t)$ is expanded in the Schur basis, the coefficients count the multiplicities of irreducible $S_n$ modules in the decomposition of a certain $S_n$ module $V_\mu$. The $n!$ Conjecture was finally proved by Haiman in 2001

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using algebraic geometry. It is still an open problem though to find a combinatorial interpretation for the Schur coefficients of $\tilde{H}_\mu(X;q,t)$.

In 1996 several groups of researchers independently found proofs of "integrality", i.e. that the monomial coefficients in $J_\mu$ are in $\mathbb{Z}[q,t]$. This is equivalent to the same result holding for $\tilde{H}_\mu(X;q,t)$. In 2004 the author found a (conjectural) combinatorial formula for $\tilde{H}_\mu(X;q,t)$ which showed these coefficients were also nonnegative, i.e. in $\mathbb{N}[q,t]$, and this was proved shortly after in joint work with Haiman and Loehr, giving the so-called HHL formula for $\tilde{H}_\mu(X;q,t)$. This eventually led to a second proof of the stronger result of Schur positivity, as well as other nice applications.

This paper is an expository account of the HHL formula and its variants. Section 2 contains a description of the original formula, and outlines some of its main applications. In Section 3 we show how to obtain a corresponding formula for $J_\mu(X;q,t)$. Section 4 describes an extension of this to the nonsymmetric Macdonald polynomial $E_\alpha(X;q,t)$ and its integral form $\mathcal{E}_\alpha(X;q,t)$. We also discuss how this formula for $E_\alpha$ inspires equivalent versions of the HHL formulas for $\tilde{H}_\mu(X;q,t)$ and $J_\mu(X;q,t)$ which involve fillings of any "skyline" diagram obtained by permuting the columns of a partition $\mu$. Section 5 contains somewhat more compact versions of the above formulas, as well as a new quasisymmetric Macdonald polynomial, all of which grew out of the study of the ASEP.

## 2. The Formula for $\tilde{H}_\mu(X;q,t)$

By the notation $\mu \vdash n$, we mean $\mu$ is a partition of $n$. We let $\mu$ refer to both a partition and its Ferrers diagram. We let $\mathrm{dg}(\mu)$ denote the "augmented" diagram of $\mu$, consisting of $\mu$ together with a row of squares below $\mu$, referred to as the *basement*, with $(x,y)$ coordinates $(j,0)$, $1 \le j \le \mu_1$. Define a *filling* $\sigma$ of $\mu$ to be an assignment of a positive integer to each square of $\mu$. For $s \in \mu$, we let $\sigma(s)$ denote the integer assigned to $s$, i.e the integer occupying $s$. If $s$ has $(x,y)$ coordinates $(i,j)$, we let $\sigma(i,j)$ denote $\sigma(s)$. By the bottom row of $\mu$, also referred to as the first row, or row 1, we mean the set of squares $\{(k,1), 1 \le k \le \mu_1\}$. Define the *reading word* of $\sigma$, denoted $\mathrm{read}(\sigma)$, to be the word $\sigma_1\sigma_2\cdots\sigma_n$, obtained be reading the occupants of $\mu$ across rows left to right, starting with the top row and working downwards. We say $\sigma$ is a *standard filling* if $\mathrm{read}(\sigma)$ is a permutation. See Figure 1.

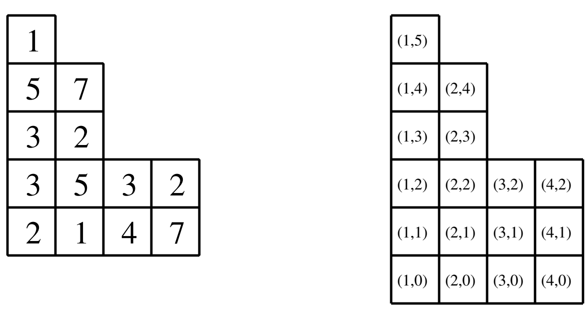


FIGURE 1. On the left, a filling of $(4,4,2,2,1)$ with reading word 1573235322147, and on the right dg(44221), with the coordinates of each square. Here $\sigma_3 = \sigma(2,4) = 7$.

For each filling $\sigma$ of $\mu$ we associate $x$, $q$ and $t$ weights. The $x$ weight is defined in a similar fashion to the usual semi-standard Young tableaux (SSYT), namely

$$x^\sigma = \prod_{s\in\mu} x_{\sigma(s)}. \tag{2.1}$$

For $s \in \mu$, let North$(s)$ denote the square of $\mu$ right above $s$ (if it exists) in the same column, and South$(s)$ the square of dg$(\mu)$ directly below $s$, in the same column. Let the descent set of $\sigma$, denoted Des$(\sigma, \mu)$, be the set of squares $s$ where $\sigma(s) > \sigma(\mathrm{South}(s))$. (We regard the basement as containing virtual infinity symbols, so no square in the bottom row of $\sigma$ can be in Des$(\sigma, \mu)$.) For $s \in \mu$, let leg$(s)$ denote the number of squares strictly above $s$ in its column, and arm$(s)$ the number of squares strictly to the right of $s$ in its row. Finally set

$$\mathrm{maj}(\sigma,\mu) = \sum_{s\in \mathrm{Des}(\sigma,\mu)} 1 + \mathrm{leg}(s). \tag{2.2}$$

Note that $\mathrm{maj}(\sigma, 1^n) = \mathrm{maj}(\sigma)$, where $\sigma$ is viewed as a word, and maj is the usual major index statistic (i.e., the sum of the elements of the descent set of the word). For the filling $\sigma$ on the left in Figure 1, we have $\mathrm{maj}(\sigma) = 2 + 4 + 1 + 3 = 10$.

We say a square $u \in \mu$ attacks all other squares $v \in \mu$ in its row and strictly to its right, and all other squares $v \in \mu$ in the row below and strictly to its left. We say $u, v$ attack each other if $u$ attacks $v$ or $v$ attacks $u$. An *inversion pair* of $\sigma$ is a pair of squares $u, v$ where $u$ attacks $v$ and $\sigma(u) > \sigma(v)$. Let Invset$(\sigma, \mu)$ denote the set of inversion pairs of $\sigma$, $\mathrm{Inv}(\sigma, \mu) = |\mathrm{Invset}(\sigma, \mu)|$ its cardinality and set

$$\mathrm{inv}(\sigma,\mu) = \mathrm{Inv}(\sigma,\mu) - \sum_{s\in \mathrm{Des}(\sigma,\mu)} \mathrm{arm}(s). \tag{2.3}$$

For example, if $\sigma$ is the filling on the left in Figure 1 then

$$\begin{aligned}
\mathrm{Des}(\sigma) &= \{(1,2),(1,4),(2,2),(2,4)\},\\
\mathrm{maj}(\sigma) &= 4+2+3+1 = 10,\\
\mathrm{Invset}(\sigma) &= \{[(2,4),(1,3)],[(1,3),(2,3)],[(1,2),(4,2)],[(2,2),(3,2)],[(2,2),(4,2)],\\
&[(2,2),(1,1)],[(3,2),(4,2)],[(3,2),(1,1)],[(3,2),(2,1)],[(4,2),(2,1)],[(1,1),(2,1)]\}\\
\mathrm{inv}(\sigma) &= |\mathrm{Invset}(\sigma)| - (3+1+2+0) = 11-6 = 5.
\end{aligned}$$

Note that when $\mu$ is a single row, $\mathrm{inv}(\sigma, (n))$ is the usual inversion statistic on words, and when $\mu$ is a single column, $\mathrm{maj}(\sigma, (1^n))$ is the usual major index statistic.

Macdonald originally defined $P_\mu(X; q, t)$ as the unique polynomial satisfying certain orthogonality conditions. Later in this section we will describe a translation of these conditions which can be used to define the $\tilde{H}_\mu(X; q, t)$. It is nontrivial to show that there exists an element of Sym$(X)$ satisfying these conditions. We now introduce a polynomial defined combinatorially which one can show satisfies the orthogonality conditions, and hence equals $\tilde{H}_\mu(X; q, t)$.

Definition 2.1. For $\mu \vdash n$, let

$$C_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} x^\sigma t^{\mathrm{maj}(\sigma,\mu)} q^{\mathrm{inv}(\sigma,\mu)}. \tag{2.4}$$

Given a word $\sigma \in \mathbb{Z}_+^n$, we Stan$(\sigma)$ denote the *standardization* of $\sigma$, which is the unique permutation in the symmetric group $S_n$ satisfying $\mathrm{Stan}(\sigma)_i < \mathrm{Stan}(\sigma)_j$ if and only if $\sigma_i \le \sigma_j$ for all $1 \le i < j \le n$. For example, Stan$(24254) = 13254$.

Furthermore we define the standardization of a filling $\sigma$, again denoted $\mathrm{Stan}(\sigma)$, as the standard filling whose reading word is the standardization of $\mathrm{read}(\sigma)$. Figure 2 gives an example of this. Note that the statistics $\mathrm{inv}(\sigma,\mu)$ and $\mathrm{maj}(\sigma,\mu)$ are invariant under standardization. It thus follows easily from Definition 2.1 that

$$C_\mu(X;q,t) = \sum_{\beta\in S_n} t^{\mathrm{maj}(\beta,\mu)} q^{\mathrm{inv}(\beta,\mu)} F_{\mathrm{Des}(\beta^{-1})}(X), \tag{2.5}$$

where we identify a permutation $\beta$ with the standard filling whose reading word is $\beta$. Here for $n$ fixed and any subset $\gamma$ of $\{1,2,\ldots,n-1\}$, $F_\gamma(X)$ is the Gessel Fundamental Quasisymmetric Function, which can be defined as

$$F_\gamma(X) = \sum_{\substack{1\le a_1\le a_2\le\cdots\le a_n\\ a_i=a_{i+1}\implies i\notin\gamma}} x_{a_1}x_{a_2}\cdots x_{a_n}, \tag{2.6}$$

or equivalently as

$$F_\gamma(X) = \sum_{\substack{\sigma\in\mathbb{Z}_+^n\\ \mathrm{Des}(\mathrm{Stan}(\sigma)^{-1})=\gamma}} x_{\sigma_1}x_{\sigma_2}\cdots x_{\sigma_n}. \tag{2.7}$$

| | | | |
|---|---|---|---|
| 6 | 4 | | |
| 2 | 4 | | |
| 3 | 3 | 2 | |
| 1 | 4 | 1 | 1 |

| | | | |
|---|---|---|---|
| 11 | 8 | | |
| 4 | 9 | | |
| 6 | 7 | 5 | |
| 1 | 10 | 2 | 3 |

FIGURE 2. On the left, a filling of $(4,3,2,2)$ with reading word 64243321411 and on the right, its standardization.

REMARK 2.2. There is another way to view $\mathrm{inv}(\sigma,\mu)$ which is makes it clear it is nonnegative, and will be important in the sequel. Call three squares $u,v,w$, with $u,v\in\mu$, $w=\mathrm{South}(u)$, and with $v$ in the same row as $u$ and strictly to the right of $\mu$, a *triple*. Given a standard filling $\sigma$, we define an orientation on such a triple by starting at the square, either $u,v$ or $w$, with the smallest element of $\sigma$ in it, and going in a circular motion, towards the next largest element, and ending at the largest element. We say the triple is an inversion triple or a coinversion triple depending on whether this circular motion is counterclockwise or clockwise, respectively. Note that since $\sigma(j,0)=\infty$, if $u,v$ are in the bottom row of $\sigma$, they are part of a counterclockwise triple if and only if $\sigma(u)>\sigma(v)$.

Extend this definition to (possibly non-standard) fillings by defining the orientation of a triple to be the orientation of the corresponding triple for the standardized filling $\mathrm{Stan}(\sigma)$. (So for two equal numbers, the one which occurs first in the reading word is regarded as being smaller). It is an easy exercise to show that $\mathrm{inv}(\sigma,\mu)$ is the number of counterclockwise triples. For example, for the filling on the left in Figure 1, the inversion triples are

$$\begin{gathered}[(2,3),(1,3),(1,2)],[(2,1),(3,2),(2,2)],\\ [(2,1),(4,2),(2,2)],[(4,2),(3,2),(3,1)],[(2,1),(1,1)].\end{gathered} \tag{2.8}$$

It is useful to note that if $\sigma(u) = \sigma(v)$ for $u, v$ two entries in the same row, then any triple containing them is always a clockwise triple. Also, if $\sigma(u) = \sigma(w) \neq \sigma(v)$, then $u, v, w$ is always form a counterclockwise triple.

We now introduce the operation of *plethysm*, which is central to the study of Macdonald polynomials. Let $p_k(X) = \sum_i x_i^k$ be the $k$th *power sum* symmetric function. Given an alphabet of parameters $t_1, t_2, \ldots$ and a function $E(t_1, t_2, \ldots)$ on this alphabet, we define the plethystic substitution of $E$ into $p_k$, denoted $p_k[E]$, by

$$p_k[E(t_1, t_2, \ldots)] = E(t_1^k, t_2^k, \ldots). \tag{2.9}$$

More generally, for any $G \in \mathrm{Sym}$, we define the plethystic substitution of $E$ into $G$, denoted $G[E]$, as the result obtained by first expressing $G$ as a polynomial in the power sums (which can be done in a unique way since the power sums generate the ring Sym) and then replacing each $p_k$ by $E(t_1^k, t_2^k, \ldots)$.

Remark 2.3. Note $p_k[-X] = -p_k(X)$ (since $-1$ is not a parameter, the plethystic operation doesn't replace it by its $k$th power.) We define a special symbol $\epsilon$ satisfying $p_k[\epsilon X] = (-1)^k p_k(X)$ which gives us a way of distinguishing a negative sign inside plethystic brackets by the operation of negating a set of variables $X$. If $\omega$ is the standard involution on Sym which sends the Schur function $s_\lambda$ to $s_{\lambda'}$, where $\lambda'$ is the conjugate partition, then $\omega p_k(X) = (-1)^{k-1} p_k = p_k[-\epsilon X]$, so for any $G \in \mathrm{Sym}$, $\omega G(X) = G[-\epsilon X]$. For a more detailed discussion of plethysm see [**18**][Chap. 1].

Macdonald's Existence Theorem, when translated into a statement about the $\tilde{H}_\mu$, gives the following [**19**], [**14**].

Theorem 2.4 (Existence Theorem for the modified Macdonald polynomial). *The following three conditions uniquely determine a family $\tilde{H}_\mu(X; q, t)$ which form a basis for the ring of symmetric functions with coefficients in $\mathbb{Q}(q, t)$.*

$$\tilde{H}_\mu[X(q-1); q, t] = \sum_{\rho \leq \mu'} c_{\rho,\mu}(q, t) m_\rho(X), \tag{2.10}$$

$$\tilde{H}_\mu[X(t-1); q, t] = \sum_{\rho \leq \mu} d_{\rho,\mu}(q, t) m_\rho(X), \tag{2.11}$$

(2.12) *The coefficient of $x_1^n$ in $\tilde{H}_\mu(X; q, t) = 1$,*

*where in the sums in (2.10) and (2.11) we use the usual dominance partial order, i.e.*

$$\rho \leq \mu \iff \rho_1 + \rho_2 + \ldots + \rho_k \leq \mu_1 + \mu_2 + \ldots + \mu_k \quad \textit{for} \quad 1 \leq k. \tag{2.13}$$

The following result was conjectured in [**13**] and proved shortly after in [**14**].

Theorem 2.5 (HHL Formula). *For any partition $\mu$,*

$$C_\mu[X; q, t] = \tilde{H}_\mu[X; q, t]. \tag{2.14}$$

Open Problem 2.6. The $n!$ Theorem implies the well-known symmetry relation

$$\tilde{H}_\mu[Z; q, t] = \tilde{H}_{\mu'}[Z; t, q]. \tag{2.15}$$

(This can also be derived fairly easily from the three axioms in Theorem 2.4 above). Prove that (2.4) satisfies this symmetry bijectively. Note that in the case $\mu = (n)$

this question is equivalent to asking for a bijective proof that maj and inv have the same distribution on arbitrary multisets, which is exactly what Foata's $\phi$ map [**10**] gives (see also [**18**][Chap.1]).

Before outlining the proof of Theorem 2.5, we discuss a few of its important corollaries.

• The proof of Theorem 2.5 gives a new proof of Macdonald's Existence Theorem.

• Theorem 2.5 not only gives a new proof that the monomial coefficients of $\tilde{H}_\mu$ are in $\mathbb{Z}[q,t]$, but shows they are actually in $\mathbb{N}[q,t]$. Haiman's Schur positivity result mentioned earlier implies this immediately, but (2.14) gave the first proof of monomial positivity that didn't use that.

• With a little work one can show that the HHL formula implies a famous result of Lascoux-Schützenberger [**24**] which says that

$$\langle \tilde{H}_\mu(X;0,t), s_\lambda(X)\rangle = \sum_{T\in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{cocharge}(T)}. \tag{2.16}$$

Here $\langle , \rangle$ is the usual Hall scalar product with respect to which the Schur functions are orthonormal, and so (2.16) gives the Schur expansion of the $q = 0$ case of $\tilde{H}_\mu$. The statistic cocharge$(T)$ on SSYT of shape $\lambda$ and weight $\mu$ will be given implicitly by the algorithm outlined below; see [**18**][Ch. 1 and Appendix A] for an explicit definition of cocharge, background on the Hall scalar product, and a full description of how to obtain (2.16) from (2.14).

Say we are given multisets of positive integers $R_i$ of cardinality $\mu_i$ for $i \geq 1$, and we want to create a filling $\sigma$ of $\mu$ such that for each $i$ the entries in the $i$th row of $\mu$ are the elements of $R_i$, and furthermore that there are no inversion triples (so $\mathrm{inv}(\sigma,\mu) = 0$). This can be done in a unique way as follows. Form the bottom row of $\sigma$ by placing the entries of $R_1$ in the bottom row in nondecreasing order, read left-to-right. Let $w$ denote the smallest of these entries (i.e. the leftmost). Next, for the first entry in the second row of $\sigma$, choose the smallest element of $R_2$ which is larger than $w$, or choose the smallest entry in $R_2$ if none are larger then $w$. Remove this entry from $R_2$ to form $R_2'$, and iterate; let $w'$ denote the entry in the second column of the first row; choose the smallest element of $R_2'$ which is larger than $w'$, or choose the smallest entry in $R_2'$ if there are not any larger then $w'$, and let this be the entry in the second column of the second row, and continue filling out the second row in this way by moving one column to the right at a time. To define the entry in the first column, third row of $\sigma$, let $w$ denote the entry in the first column, second row of $\sigma$, and choose the smallest element of $R_3$ which is larger than $w$, or choose the smallest entry in $R_3$ if there are none larger then $w$. Remove this entry from $R_3$ to form $R_3'$, and so on. See Figure 3 for a filling with no inversion triples.

We construct a word cword$(\sigma)$ by initializing cword to the empty string, then scanning through read$(\sigma)$, from the beginning to the end, and each time we encounter a 1, adjoin the number of the row containing this 1 to the beginning of cword. After recording the row numbers of all the 1's in this fashion, we go back to the beginning of read$(\sigma)$, and adjoin the row numbers of squares containing 2's to the beginning of cword. For example, if $\sigma$ is the filling in Figure 3, then cword$(\sigma) = 11222132341123$.

We now translate the statistic $\mathrm{maj}(\sigma,\mu)$ into a statistic on cword$(\sigma)$. Note that $\sigma(1,1)$ corresponds to the rightmost 1 in cword$(\sigma)$ - denote this 1 by $w_{11}$.

| | | | | |
|---|---|---|---|---|
| 2 | | | | |
| 3 | 1 | 2 | | |
| 2 | 4 | 4 | 1 | 5 |
| 1 | 1 | 3 | 6 | 7 |

FIGURE 3. A filling with no inversion triples

If $\sigma(1,2) > \sigma(1,1)$, $\sigma(1,2)$ corresponds to the rightmost 2 which is left of $w_{11}$, otherwise it corresponds to the rightmost 2 (in $\mathrm{cword}(\sigma)$). In any case denote this 2 by $w_{12}$. More generally, the element in $\mathrm{cword}(\sigma)$ corresponding to $\sigma(1,i)$ is the first $i$ encountered when travelling left from $w_{1,i-1}$, looping around and starting at the right end of $\mathrm{cword}(\sigma)$ if necessary. To find the subword $w_{21}w_{22}\cdots w_{2\mu'_2}$ corresponding to the second column of $\sigma$, we do the same algorithm on the word $\mathrm{cword}(\sigma)'$ obtained by removing the elements $w_{11}w_{12}\cdots w_{1\mu'_1}$ from $\mathrm{cword}(\sigma)$, then remove $w_{21}w_{22}\cdots w_{2\mu'_2}$ and apply the same process to find $w_{31}w_{32}\cdots w_{3\mu'_3}$ etc..

Clearly $\sigma(i,j) \in \mathrm{Des}(\sigma,\mu)$ if and only if $w_{ij}$ occurs to the left of $w_{i,j-1}$ in $\mathrm{cword}(\sigma)$. The statistic $\mathrm{maj}(\sigma,\mu)$ now becomes transparently equal to the statistic $\mathrm{cocharge}(\mathrm{cword}(\sigma))$ introduced by Lascoux and Schützenberger [**24**]. See also [**18**][pp. 16-17].

Next we associate a two-line array $A(\sigma)$ to a filling $\sigma$ with no inversions by letting the upper row $A_1(\sigma)$ be nonincreasing with the same weight as $\sigma$, and the lower row $A_2(\sigma)$ be $\mathrm{cword}(\sigma)$. For example, to the filling in Figure 3 we associate the two-line array

$$\begin{matrix} 7\,6\,5\,4\,4\,3\,3\,2\,2\,2\,1\,1\,1\,1 \\ 1\,1\,2\,2\,2\,1\,3\,2\,3\,4\,1\,1\,2\,3 \end{matrix} \tag{2.17}$$

By construction, below equal entries in the upper row the entries in the lower row are nondecreasing. Later in this section we will show $C_\mu(X;0,t)$ is a symmetric function, so we can reverse the variables, replacing $x_i$ by $x_{n-i+1}$ for $1 \le i \le n$, without changing the sum. This has the effect of reversing $A_1(\sigma)$, and we end up with an ordered two-line array, of the type occurring in the RSK algorithm. This algorithm associates such an array with a pair of SSYT. We can invert this correspondence since from the two-line array we get the multiset of elements in each row of $\sigma$, which uniquely determines $\sigma$. Thus

$$C_\mu(x_1,x_2,\ldots,x_n;0,t) = \sum_{\sigma} x^{\mathrm{weight}(A_1(\sigma))} t^{\mathrm{cocharge}(A_2(\sigma))} \tag{2.18}$$

$$= \sum_{(A_1,A_2)} x^{\mathrm{weight}(A_1)} t^{\mathrm{cocharge}(A_2)}, \tag{2.19}$$

where the sum is over ordered two-line arrays satisfying $\mathrm{weight}(A_2) = \mu$.

It is well known that for any word $w$ of partition weight, cocharge $(w)$ = cocharge $(\mathrm{read}(P_w))$, where $\mathrm{read}(P_w)$ is the reading word of the insertion tableau $P_w$ under the RSK algorithm [**30**, pp.48-49], [**34**, p.417]. Hence applying the RSK algorithm to (2.19),

$$C_\mu(x_1, x_2, \ldots, x_n; 0, t) = \sum_{(P,Q)} x^{\text{weight}(Q)} t^{\text{cocharge}(\text{read}(P))}, \tag{2.20}$$

where the sum is over all pairs $(P, Q)$ of SSYT of the same shape with weight$(P) = \mu$. Since the number of different $Q$ tableau of weight $\nu$ matched to a given $P$ tableau of shape $\lambda$ is the Kostka number $K_{\lambda,\nu}$,

$$\begin{aligned}
C_\mu[X; 0, t] &= \sum_\nu m_\nu \sum_\lambda \sum_{\substack{P \in SSYT(\lambda,\mu) \\ Q \in SSYT(\lambda,\nu)}} t^{\text{cocharge}(\text{read}(P))} \\
&= \sum_\lambda \sum_{P \in SSYT(\lambda,\mu)} t^{\text{cocharge}(\text{read}(P))} \sum_\nu m_\nu K_{\lambda,\nu} \\
&= \sum_\lambda s_\lambda \sum_{P \in SSYT(\lambda,\mu)} t^{\text{cocharge}(\text{read}(P))}. \qquad (2.21)
\end{aligned}$$

This completes the outline of the proof of (2.16).

• A final segment of a word is the last $k$ letters of the word, for some $k$. We say a word $\sigma$ is a *Yamanouchi word* if in any final segment of $\sigma$, there are at least as many $i$'s as $i+1$'s, for all $i \geq 1$. In [**14**] the following result is derived from Theorem 2.5.

THEOREM 2.7. *For any partition $\mu$ with at most two columns,*

$$\langle \tilde{H}_\mu(X; q, t), s_\lambda(X) \rangle = \sum_{Y \in Yam(\lambda)} q^{inv(Y,\mu)} t^{maj(Y,\mu)}. \tag{2.22}$$

*Here the sum is over all Yamanouchi words of weight $\lambda$, i.e. with $\lambda_i$ copies of $i$ for $i \geq 1$, and in the summand we identify a word $Y$ with the filling whose reading word is $Y$.*

Other combinatorial formulas for the left-hand-side of (2.22) are known, but the one above is arguably the most elegant. The proof of Theorem 2.7 involves a combinatorial construction which groups together fillings which have the same maj and inv statistics. This is carried out with the aid of *crystal graphs*, which occur in the representation theory of Lie algebras.

• Below we show how $C_\mu$ is a positive sum of symmetric functions introduced by Lascoux, Leclerc and Thibon, commonly called LLT polynomials [**25**]. These functions depend on a set of variables $X$, a tuple of skew shapes, and a parameter $q$. In an unpublished preprint on Haiman's website titled "Affine Hecke algebras and positivity of LLT and Macdonald polynomials" which is generally accepted by experts to be true, Grojnowski and Haiman prove that LLT polynomials are Schur positive. Using this we have a new proof of the Schur positivity of $\tilde{H}_\mu$.

We now give an overview of the main ideas in the proof of the HHL formula, some of which will be used to derive a corresponding formula for $J_\mu(X; q, t)$ described in the next section. To begin with, we prove $C_\mu$ is in Sym (which is not at all obvious) by showing it can be written as a sum of LLT polynomials as described in [**16**], [**18**][Ch. 6].

Fix a descent set $D$, and let

$$R_D(X; q) = \sum_{\substack{\sigma:\mu \to \mathbb{Z}_{>0} \\ \text{Des}(\sigma,\mu)=D}} q^{\text{inv}(\sigma,\mu)} x^\sigma. \tag{2.23}$$

If $\mu$ has one column, then $R_D$ is a ribbon Schur function. More generally, $R_D(X;q)$ is an LLT product of ribbons. We illustrate how to transform a filling $\sigma$ into a term $\gamma(\sigma)$ in the corresponding LLT product in Figure 4. Note that inversion pairs in $\sigma$ are in direct correspondence with LLT inversion pairs in $\gamma(\sigma)$. Since the shape of the ribbons in $\gamma(\sigma)$ depends only on $\mathrm{Des}(\sigma,\mu)$ we have

$$C_\mu(X;q,t) = \sum_D t^L q^{-A} R_D(X;q), \tag{2.24}$$

where the sum is over all possible descent sets $D$ of fillings of $\mu$, with

$$L = \sum_{s\in D} 1 + \mathrm{leg}(s), \quad A = \sum_{s\in D} \mathrm{arm}(s).$$

Since LLT polynomials are symmetric functions, we have

THEOREM 2.8. *For all $\mu$, $C_\mu(X;q,t)$ is a symmetric function.*

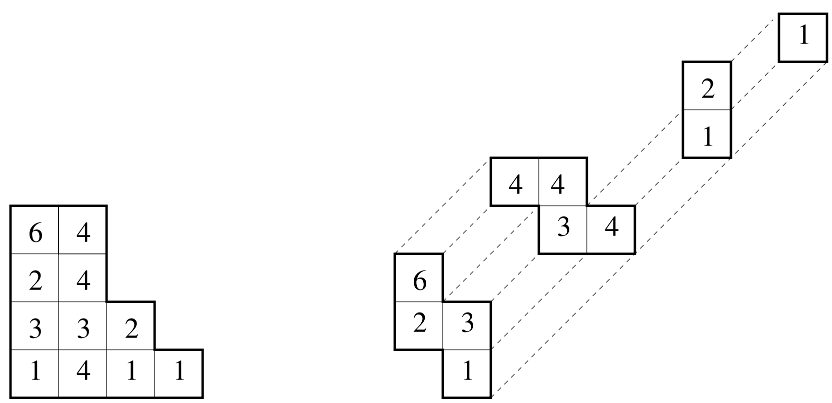


FIGURE 4. On the left, a filling, and on the right, the term in the corresponding LLT product of ribbons

Now that we know $C_\mu \in \mathrm{Sym}$, we proceed to show that $C_\mu$ satisfies the three defining conditions of $\tilde{H}_\mu$. We mention that any symmetric function $f(X)$ of homogeneous degree $n$ is uniquely determined by its value on $X_n = \{x_1,\ldots,x_n\}$ so in Definition 2.1 we may as well restrict ourselves to fillings from the set $\{1,2,\ldots,n\}$. First of all, if we consider the filling of $\mu$ where all entries equal 1, then there are no descents and no inversion pairs, and the $x$-weight is $x_1^n$, so $C_\mu$ satisfies (2.12).

The two other conditions are more involved, and depend on a process called the *superization* of an element of $\mathrm{Sym}(X)$. This involves two alphabets, a positive alphabet $\mathcal{A}_+ = \{1,2,\ldots,n\}$ and a negative alphabet $\mathcal{A}_- = \{\bar{1},\bar{2},\ldots,\bar{n}\}$; we denote their union by $\mathcal{A}_\pm$. We define a *super word* $\tilde{\sigma}$ to be a word in the alphabet $\mathcal{A}_\pm$, and let $|\tilde{\sigma}|$ be the word obtained by replacing each negative letter $\bar{i}$ in $\tilde{\sigma}$ by its "absolute value" $i$. Set

$$(xy)^{\tilde{\sigma}} = \prod_{\substack{i\\ \tilde{\sigma}_i\in\mathcal{A}_+}} x_{\tilde{\sigma}_i} \prod_{\substack{i\\ \tilde{\sigma}_i\in\mathcal{A}_-}} y_{|\tilde{\sigma}_i|}. \tag{2.25}$$

So, for example, if $\tilde{\sigma} = 21\bar{1}3\bar{2}$, $(xy)^{\tilde{\sigma}} = x_2x_1y_1x_3y_2$, and $|\tilde{\sigma}| = 21132$.

We extend the concept of standardization to super words as follows. Fix an arbitrary ordering of the elements of $\mathcal{A}_{\pm}$. Two examples of orderings we will be working with are

$$1 < \bar{1} < 2 < \bar{2} \cdots < n < \bar{n} \tag{2.26}$$

$$1 < 2 < \cdots < n < \bar{n} \cdots < \bar{2} < \bar{1}. \tag{2.27}$$

To standardize a word according to a specific ordering, order two equal positive letters by assuming the leftmost is the smaller of the two, and order two equal negative letters by assuming the leftmost is the larger of the two. For unequal letters order them by the given total ordering on $\mathcal{A}_{\pm}$. For example, if we use (2.26), then $\mathrm{Stan}(31\bar{1}1\bar{1}\bar{2}\bar{2}2) = 81423765$, while if we use (2.27), then $\mathrm{Stan}(31\bar{1}1\bar{1}\bar{2}\bar{2}2) = 41827653$. See Figure 5.

$$\begin{array}{|c|c|cc}
\hline
3 & \bar{3} & & \\
\hline
4 & \bar{2} & & \\
\hline
4 & 5 & \multicolumn{1}{c|}{3} & \\
\hline
\bar{1} & 3 & \multicolumn{1}{c|}{\bar{1}} & \multicolumn{1}{c|}{1} \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|cc}
\hline
5 & 8 & & \\
\hline
9 & 4 & & \\
\hline
10 & 11 & \multicolumn{1}{c|}{6} & \\
\hline
3 & 7 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|cc}
\hline
2 & 8 & & \\
\hline
5 & 9 & & \\
\hline
6 & 7 & \multicolumn{1}{c|}{3} & \\
\hline
11 & 4 & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{1} \\
\hline
\end{array}$$

FIGURE 5. On the left, a super filling. In the middle, its standardization assuming the ordering (2.26) and on the right, its standardization assuming the ordering (2.27).

The following lemma, true for any fixed ordering of $\mathcal{A}_{\pm}$, is proved in [**16**]; see also [**18**][pp. 99-100].

LEMMA 2.9. *Say $G \in Sym(X)$ satisfies*

$$G(X) = \sum_{\beta \in S_n} c(\beta) F_{Des(\beta^{-1})}(X), \tag{2.28}$$

*for some constants $c(\beta)$ independent of $X$. Then if $Y = \{y_1, y_2, \ldots\}$ is another set of variables,*

$$\omega^Y G[X+Y] = \sum_{\beta \in S_n} c(\beta) \tilde{F}_{Des(\beta^{-1})}(X,Y), \tag{2.29}$$

*where $\omega^Y$ is the usual involution, acting only on the $Y$ variables, and*

$$\tilde{F}_{\gamma}(X,Y) = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \in \mathcal{A}_+ \implies i \notin \gamma \\ a_i = a_{i+1} \in \mathcal{A}_- \implies i \in \gamma}} \prod_{a_i \in \mathcal{A}_+} x_{a_i} \prod_{a_i \in \mathcal{A}_-} y_{|a_i|} \tag{2.30}$$

*is a superized version of the Gessel Fundamental $F_{\gamma}(X)$.*

We note that for any fixed $\beta \in S_n$ with $\mathrm{Des}(\beta^{-1}) = \gamma$,

$$\tilde{F}_{\gamma}(X,Y) = \sum_{\substack{\tilde{\sigma} \in \mathcal{A}_{\pm}^n \\ \mathrm{Des}(\mathrm{Stan}(\tilde{\sigma})) = \beta}} (xy)^{\tilde{\sigma}}. \tag{2.31}$$

By Lemma 2.9 and (2.5) we have

$$\omega^Y C_{\mu}[X+Y;q,t] = \sum_{\beta \in S_n} t^{\mathrm{maj}(\beta,\mu)} q^{\mathrm{inv}(\beta,\mu)} \tilde{F}_{\mathrm{Des}(\beta^{-1})}(X,Y). \tag{2.32}$$

Now for a real parameter $\alpha$, $C_\mu[X(\alpha-1);q,t] = C_\mu[X\alpha - X;q,t]$, and by Remark 2.3 can be obtained from $\omega^Y C_\mu[X+Y;q,t] = C_\mu[X-\epsilon Y;q,t]$ by replacing $x_i$ by $\alpha x_i$ and $y_i$ by $\epsilon x_i$, for $1 \le i \le n$ (since $\omega f(X) = (-1)^n f[-X]$). Hence (2.32) implies

$$C_\mu[X\alpha - X;q,t] = \sum_{\tilde\sigma:\mu\to\mathcal{A}_\pm} x^{|\tilde\sigma|} t^{\mathrm{maj}(\tilde\sigma,\mu)} q^{\mathrm{inv}(\tilde\sigma,\mu)} \alpha^{\mathrm{pos}(\tilde\sigma)} (-1)^{\mathrm{neg}(\tilde\sigma)}. \tag{2.33}$$

Here $\mathrm{pos}(\tilde\sigma)$ and $\mathrm{neg}(\tilde\sigma)$ denote the number of positive letters and negative letters in $\tilde\sigma$, respectively.

Define the *critical pair* of squares of a filling $\tilde\sigma$ as the last pair (in the reading word order) of attacking squares $u, v$ which both contain elements of the set $\{1, \bar 1\}$, with $u$ say denoting the "critical square", i.e. the earliest of these two squares in the reading order. If $\tilde\sigma$ has no such attacking pair, then we define the critical pair as the last pair of attacking squares which both contain elements of the set $\{2, \bar 2\}$, and if no such pair exists, as the last pair of attacking squares which both contain elements of the set $\{3, \bar 3\}$, etc. Let $\mathrm{IH}(\tilde\sigma)$ be the sign-reversing involution obtained by starting with $\tilde\sigma$ and switching the sign of the element in the critical square. For example, for the filling on the left in Figure 5, the critical pair is $(3,1),(4,1)$ and IH changes the $\bar 1$ in $(3,1)$ to a 1.

Assume our alphabets satisfy (2.26), and let $\alpha = q$ in (2.33). Clearly $\mathrm{IH}(\tilde\sigma)$ fixes the $x$ weight. Let $u, v$ be critical with $|\tilde\sigma(u)| = a$. If $|\tilde\sigma(\mathrm{North}(u))| = a$ we have $u \in \mathrm{Des}(\tilde\sigma,\mu)$ if and only if $\tilde\sigma(\mathrm{North}(u)) = \bar a$. Furthermore if $u$ is not in the bottom row of $\mu$, then $|\tilde\sigma(\mathrm{South}(u))| \ne a$, since otherwise $v$ and $\mathrm{South}(u)$ are an attacking pair later in the reading order, and $u, v$ would not be the critical pair. It follows that $\mathrm{IH}(\tilde\sigma)$ fixes the descent set and hence also the $t$ weight.

Since $\mathrm{IH}(\tilde\sigma)$ fixes the descent set, it will fix the $q$-weight if and only if changing $a$ to $\bar a$ increases the number of inversion pairs Inv by 1 (since this change decreases pos by 1). Changing $a$ to $\bar a$ or vice-versa has no affect on inversion pairs involving numbers not equal to $a$ or $\bar a$. The definition of critical implies that changing $a$ to $\bar a$ creates a new inversion pair between $u$ and $v$, but does not affect any other inversion pairs. Similarly, changing $\bar a$ to $a$ decreases the number of inversion pairs by 1, hence if $\alpha = q$ in (2.33), then $\mathrm{IH}(\tilde\sigma)$ fixes the $q$-weight.

Call a super filling nonattacking if there are no critical pairs of squares. (These are the fixed points of IH). The above reasoning shows

(2.34)
$$C_\mu[X(q-1);q,t] = \sum_{\tilde\sigma:\mu\to\mathcal{A}_\pm,\ \text{nonattacking}} x^{|\tilde\sigma|} t^{\mathrm{maj}(\tilde\sigma,\mu)} q^{\mathrm{inv}(\tilde\sigma,\mu)} q^{\mathrm{pos}(\tilde\sigma)} (-1)^{\mathrm{neg}(\tilde\sigma)},$$

where the sum is over all nonattacking superfillings of $\mu$. Since nonattacking implies there is at most one occurrence of a number from the set $\{a, \bar a\}$ in any row, the exponent of $x_1$ in $x^{|\tilde\sigma|}$ is at most $\mu'_1$, the sum of the exponents of $x_1$ and $x_2$ is at most $\mu'_1 + \mu'_2$, etc., which shows $C_\mu$ satisfies axiom (2.10).

Call the first square $w$ (in the reading word order) such that $|\tilde\sigma(w)| = 1$, with $w$ not in the bottom row, the *pivotal square*. If there is no such square, let $w$ denote the first square, not in the bottom *two* rows of $\mu$, with $|\tilde\sigma(w)| = 2$, and again if there is no such square, search for the first square not in the bottom three rows satisfying $|\tilde\sigma(w)| = 3$, etc. Let $\mathrm{IL}(\tilde\sigma)$ denote the sign-reversing involution obtained by switching the sign on $\tilde\sigma(w)$. For the filling on the left in Figure 5, the pivotal square is $(2,3)$, and IL switches this $\bar 2$ to a 2.

Let our alphabets satisfy (2.27), with $\alpha = t$ in (2.33). As before, IL$(\tilde\sigma)$ trivially fixes the $x$-weight. Note that by construction, if the pivotal square is in row $k$, then after standardization it contains either the smallest element occurring anywhere in rows $k-1$ and higher or the largest element occurring in rows $k-1$ and higher. Thus in any triple containing the pivotal square, IL either switches an element smaller than the other two to an element larger than the other two, or vice-versa, and this preserves the orientation of the triple. By Remark 2.2, IL thus fixes the $q$-weight. We leave it as an exercise for the reader to show that IL also fixes the $t$-weight.

Call a super filling *primary* if there are no pivotal squares. (These are the fixed points of IL). We have

$$C_\mu[X(t-1);q,t] = \sum_{\tilde\sigma:\mu\to\mathcal{A}_\pm,\text{primary}} x^{|\tilde\sigma|} t^{\mathrm{maj}(\tilde\sigma,\mu)} q^{\mathrm{inv}(\tilde\sigma,\mu)} t^{\mathrm{pos}(\tilde\sigma)} (-1)^{\mathrm{neg}(\tilde\sigma)}, \tag{2.35}$$

where the sum is over all primary superfillings of $\mu$. Now primary implies that all 1 or $\bar 1$'s are in the bottom row, all 2 or $\bar 2$'s are in the bottom two rows, etc., so in $x^{|\tilde\sigma|}$ the power of $x_1$ is at most $\mu_1$, the sum of the powers of $x_1$ and $x_2$ is at most $\mu_1+\mu_2$, etc. Thus $C_\mu$ satisfies axiom (2.11), which completes the proof of Theorem 2.5.

Remark 2.10. In a private communication to the author in 2007, Haiman noted that the involution IH can be easily extended to any LLT polynomial. This implies that any LLT polynomial, plethystically evaluated at $X(q-1)$, can be expanded as a sum over nonattacking superfillings of the tuple of skew shapes.

## 3. Formulas for $J_\mu$

By definition, Macdonald's original integral form $J_\mu$ is related to Garsia and Haiman's modified Macdonald as follows, where $n(\mu) = \sum_i (i-1)\mu_i$.

$$J_\mu(X;q,t) = t^{n(\mu)} \tilde H_\mu[X(1-t);q,1/t] \tag{3.1}$$
$$= t^{n(\mu)} \tilde H_\mu[Xt(1/t-1);q,1/t]$$
$$= t^{n(\mu)+n} \tilde H_{\mu'}[X(1/t-1);1/t,q] \tag{3.2}$$

using (2.15). Formula (2.34), with $q,t$ interchanged thus implies

$$J_\mu(X;q,t) = \sum_{\text{nonattacking super fillings } \tilde\sigma \text{ of } \mu'} x^{|\tilde\sigma|} q^{\mathrm{maj}(\tilde\sigma,\mu')} t^{\mathrm{coinv}(\tilde\sigma,\mu')} (-t)^{\mathrm{neg}(\tilde\sigma)}, \tag{3.3}$$

where coinv $= n(\mu) -$ inv is the number of coinversion triples (i.e. triples with a clockwise orientation), and we use the ordering (2.26).

We can derive a more compact form of (3.3) by grouping together all the $2^n$ super fillings $\tilde\sigma$ of $\mu'$ whose absolute value equals a fixed positive filling $\sigma$ of $\mu'$. The basic idea is that if you consider an entry $\tilde\sigma(s)$, where $|\tilde\sigma(\mathrm{South}(s))| \neq |\tilde\sigma(s)|$, then switching the sign of $\tilde\sigma(s)$ will not change the descent set, nor will it affect the orientation of any triples. The only thing that changes is the number of negative entries. This observation uses the fact that the filling is nonattacking. On the other hand, if $|\tilde\sigma(\mathrm{South}(s))| = |\tilde\sigma(s)|$, then switching the sign of $\tilde\sigma(s)$ from $a$ to $\bar a$ will create a new descent, while switching it from $\bar a$ to $a$ will remove a descent there. Putting everything together, from (3.3) we get a nice combinatorial formula for $J_\mu$.

Corollary 3.1. [**14**]

$$J_\mu(X;q,t) = \sum_{\text{nonattacking fillings } \sigma \text{ of } \mu'} x^\sigma q^{\text{maj}(\sigma,\mu')} t^{\text{coinv}(\sigma,\mu')} \times \prod_{\substack{u\in\mu' \\ \sigma(u)=\sigma(\text{South}(u))}} (1-q^{\text{leg}(u)+1}t^{\text{arm}(u)+1}) \prod_{\substack{u\in\mu' \\ \sigma(u)\neq\sigma(\text{South}(u))}} (1-t), \tag{3.4}$$

where each square in the bottom row is included in the last product, and nonattacking means no two squares which attack each other contain the same number.

Example 3.2. Let $\mu = (3,3,1)$. Then for the nonattacking filling $\sigma$ of $\mu'$ in Figure 6, maj $= 3$, coinv $= 3$, squares $(1,1),(1,2),(2,1),(2,3)$ and $(3,1)$ each contribute a $(1-t)$, square $(1,3)$ contributes a $(1-qt^2)$, and $(2,2)$ contributes a $(1-q^2t)$. Thus the term in (3.4) corresponding to $\sigma$ is

$$x_1x_2^3x_3^2x_4q^3t^3(1-qt^2)(1-q^2t)(1-t)^5. \tag{3.5}$$

$$\begin{array}{|c|c|c}
\hline
2 & 4 & \\
\hline
2 & 3 & \\
\hline
1 & 3 & \multicolumn{1}{c|}{2} \\
\hline
\end{array}$$

Figure 6. A nonattacking filling of $(3,3,1)'$

Let $\alpha$ denote a real parameter. The (integral form) Jack polynomial $J_\mu^{(\alpha)}(X)$ can be obtained from $J_\mu(X;q,t)$ by

$$J_\mu^{(\alpha)}(X) = \lim_{t\to 1} \frac{J_\mu(X;t^\alpha,t)}{(1-t)^{|\mu|}}. \tag{3.6}$$

If we set $q = t^\alpha$ in (3.4) and then divide by $(1-t)^{|\mu|}$ and take the limit as $t \to 1$ we get the following result of Knop and Sahi [**23**].

$$J_\mu^{(\alpha)}(X) = \sum_{\text{nonattacking fillings } \sigma \text{ of } \mu'} x^\sigma \prod_{\substack{u\in\mu' \\ \sigma(u)=\sigma(\text{South}(u))}} (\alpha(\text{leg}(u)+1)+\text{arm}(u)+1). \tag{3.7}$$

There is another formula for $J_\mu$ corresponding to involution IL and (2.27). First note that by applying the $t \to 1/t$, $X \to tX$ case of (2.35) to the right-hand side of (3.1) we get

$$J_\mu(X;q,t) = \sum_{\tilde\sigma:\mu\to\mathcal{A}_\pm,\ \text{primary}} x^{|\tilde\sigma|} t^{\text{nondes}(\tilde\sigma,\mu)} q^{\text{inv}(\tilde\sigma,\mu)} (-t)^{\text{neg}(\tilde\sigma)}, \tag{3.8}$$

where the sum is over all primary super fillings of $\mu$, and nondes $= n(\mu) - \text{maj}$ is the sum of $\text{leg}+1$ over all squares of $\mu$ which are *not* in $\text{Des}(\tilde\sigma,\mu)$.

For a positive filling $\sigma$ and $s \in \mu$, let

$$\operatorname{maj}_s(\sigma,\mu) = \begin{cases} \operatorname{leg}(s) \text{ if } \operatorname{North}(s) \in \operatorname{Des}(\sigma,\mu) \\ 0 \text{ otherwise} \end{cases} \tag{3.9}$$

$$\operatorname{nondes}_s(\sigma,\mu) = \begin{cases} \operatorname{leg}(s)+1 \text{ if } s \notin \operatorname{Des}(\sigma,\mu) \\ 0 \text{ otherwise,} \end{cases} \tag{3.10}$$

so $\operatorname{maj} = \sum_s \operatorname{maj}_s$ and $\operatorname{nondes} = \sum_s \operatorname{nondes}_s$. Note that a square $s$ makes a nonzero contribution to $\operatorname{nondes}(\tilde{\sigma},\mu)$ if it contains a positive letter and $\operatorname{nondes}_s(|\tilde{\sigma}|,\mu) > 0$, or it contains a negative letter and $\operatorname{maj}_s(|\tilde{\sigma}|,\mu) > 0$. Thus by letting $q = 1$ in (3.8) we get

$$J_\mu(X;1,t) = \sum_{\sigma:\mu\to\mathcal{A}_+,\text{primary}} x^\sigma \prod_{s\in\mu} (t^{\operatorname{nondes}_s(\sigma,\mu)} - t^{1+\operatorname{maj}_s(\sigma,\mu)}). \tag{3.11}$$

By a similar analysis we can incorporate the $q$-parameter into (3.11). Given a triple $u, v, w$ of $\mu$ with $w \in \mu$, we define the "middle square" of $u, v, w$, with respect to $\tilde{\sigma}$, to be the square containing the middle of the three numbers of the set $\{\operatorname{Stan}(|\tilde{\sigma}|)(u), \operatorname{Stan}(|\tilde{\sigma}|)(v), \operatorname{Stan}(|\tilde{\sigma}|)(w)\}$. In other words, we replace all letters by their absolute value, then standardize, then take the square containing the number which is neither the largest, nor the smallest, of the three. For squares $(1,1), (1,2), (3,2)$ for the super filling on the left in Figure 5, square (3,2) is the middle square. For squares $(1,4), (1,3), (2,4)$ square $(2,4)$ is the middle square. Extend this definition to triples $u, v, w$ of $\mu$ with $u, v$ in the bottom row by letting the middle square be $u$ if $|\tilde{\sigma}|(u) \le |\tilde{\sigma}|(v)$, otherwise let it be $v$ (here, as usual, $v$ is to the right of $u$). By checking the eight possibilities for choices of signs of the letters in $u, v, w$ (or four possibilities if $u, v$ are in the bottom row), one finds that if $u, v, w$ form a coinversion triple in $|\tilde{\sigma}|$, then they form a coinversion triple in $\tilde{\sigma}$ if the middle square contains a positive letter, otherwise they form an inversion triple in $\tilde{\sigma}$. Similarly, if $u, v, w$ form an inversion triple in $|\tilde{\sigma}|$, they form a coinversion triple in $\tilde{\sigma}$ if the middle square contains a negative letter, otherwise they form an inversion triple. Thus by defining $\operatorname{coinv}_s(\sigma,\mu)$ to be the number of coinversion triples for which $s$ is the middle square, and $\operatorname{inv}_s(\sigma,\mu)$ to be the number of inversion triples for which $s$ is the middle square, we get the following result of the author [**18**][p.133].

COROLLARY 3.3.

$$J_\mu(X;q,t) = \sum_{\substack{\sigma:\mu\to\mathcal{A}_+ \\ \text{primary}}} x^\sigma \prod_{s\in\mu} (q^{inv_s(\sigma,\mu)} t^{\operatorname{nondes}_s(\sigma,\mu)} - q^{coinv_s(\sigma,\mu)} t^{1+\operatorname{maj}_s(\sigma,\mu)}). \tag{3.12}$$

Although the products in (3.12) are more complicated than those in (3.4), one advantage this formula has is that the $q$ and $t$-weights are invariant under standardization, hence we can express it as

$$J_\mu(X;q,t) = \sum_{\substack{\beta\in S_n \\ \text{primary}}} F_{\operatorname{Des}(\beta^{-1})}(X) \prod_{s\in\mu} (q^{\operatorname{inv}_s(\beta,\mu)} t^{\operatorname{nondes}_s(\beta,\mu)} - q^{\operatorname{coinv}_s(\beta,\mu)} t^{1+\operatorname{maj}_s(\beta,\mu)}), \tag{3.13}$$

which gives an expansion of $J_\mu$ into Gessel's fundamental quasisymmetric functions. The sum is over all standard fillings (permutations) which are also primary fillings, although the equation is still true if we extend the sum to all permutations, since if $\beta$ is a standard filling which is not primary, with pivotal square $s$, then the term $(q^{\mathrm{inv}_s(\sigma,\mu)}t^{\mathrm{nondes}_s(\sigma,\mu)} - q^{\mathrm{coinv}_s(\sigma,\mu)}t^{1+\mathrm{maj}_s(\sigma,\mu)})$ is zero. M. Yoo has used (3.12) to obtain the expansion of $J_\mu(X;q,t)$ into Schur functions for certain shapes $\mu$ [**35**], [**36**].

## 4. Nonsymmetric Macdonald Polynomials

In 1995 Macdonald [**29**] introduced polynomials $E_\alpha(X_n;q,t)$ which form a basis for the polynomial ring $\mathbb{Q}(q,t)[X_n]$, and in many ways are analogous to the $P_\lambda(X;q,t)$. Here $X_n = \{x_1,\ldots,x_n\}$ as in Section 2. Further development of the theory was made by Cherednik [**5**], Sahi [**33**], Knop [**22**], Ion [**20**], and others. Both the $E_\alpha$ and the $P_\lambda$ have versions for any affine root system, and are both orthogonal families of polynomials with respect to a certain scalar product.

We will be working with the type $A_{n-1}$ case, where there is a special structure which allows us to assume $\alpha$ is a composition, i.e. $\alpha \in \mathbb{N}^n$. In [**15**] Haiman, Loehr and the author showed how the $E_\alpha(X_n;q,t)$ and their integral forms $\mathcal{E}_\alpha(X_n;q,t)$ also satisfy a combinatorial formula, and that this formula implies formula (3.4) for $J_\mu(X;q,t)$. The proof of the formula for $\mathcal{E}_\alpha(X_n;q,t)$ involves operators $T_i$ typically used to define the type $A$ Hecke algebra. Briefly, the $T_i$ are linear operators on the polynomial ring, which act on monomials $x^\lambda$ by

$$T_i x^\lambda = t x^{s_i(\lambda)} + (t-1)\frac{x^\lambda - x^{s_i(\lambda)}}{1-x^{\alpha_i}} \quad \text{for all } i = 0,1,\ldots,n-1. \tag{4.1}$$

Here for $1 \le i \le n-1$, $s_i$ permutes $x_i$ and $x_{i+1}$, while for any polynomial $f(x_1,\ldots,x_n)$,

$$s_0 f(x_1,\ldots,x_n) = f(qx_n, x_2,\ldots,x_{n-1}, x_1/q).$$

See [**15**][p. 3] for background on the $T_i$.

The HHL formula for $E_\alpha(X_n;q,t)$ from [**15**] involves nonattacking fillings of $\mathrm{dg}(\alpha')$, as in the formula (3.4) for $J_\mu$, but with the added feature of having the number $n-i+1$ in the "basement" square with coordinates $(i,0)$, for $1 \le i \le n$. Shortly after the publication of this formula, in a private communication to the author Haiman conjectured that for any $\sigma \in S_n$, the polynomial $T_\sigma E_\alpha(X_n;q,t)$ satisfies the same HHL formula, with the basement replaced by $\sigma$. Here $T_\sigma$ is a product of the $T_i$ corresponding to a reduced word for $\sigma$. Utilizing some unpublished notes of the author which outlined a proof of the conjecture, Per Alexandersson [**1**] completed the proof and obtained many related results about this more general function. Below we describe Alexandersson's formula for the *permuted basement nonsymmetric Macdonald polynomial*.

We first extend some of the definitions from earlier sections to more general "skyline" diagrams which can be obtained by permuting the columns of a partition. Given $\alpha \in \mathbb{N}^n$, let $\alpha'$ denote the transpose graph of $\alpha$, consisting of the squares

$$\alpha' = \{(i,j), \quad 1 \le i \le n, \quad 1 \le j \le \alpha_i)\} \tag{4.2}$$

Furthermore let $\mathrm{dg}(\alpha')$ denote the augmented diagram obtained by adjoining the "basement" row of $n$ squares with coordinates $\{(j,0), 1 \le j \le n\}$ below $\alpha'$. Given $s \in \alpha'$, we let $\mathrm{leg}(s)$ be the number of squares of $\alpha'$ above $s$ and in the same column of $s$. Define $\mathrm{Arm}(s)$ to be the set of squares of $\mathrm{dg}(\alpha')$ which are either to the right and in the same row as $s$, and also in a column not taller than the column containing $s$, or to the left and in the row below the row containing $s$, and in a column strictly shorter than the column containing $s$. Then set $\mathrm{arm}(s) = |\mathrm{Arm}(s)|$. For example, for $\alpha = (1,0,3,2,3,0,0,0,0)$, the leg lengths of the squares of $(1,0,3,2,3,0,0,0,0)'$ are listed on the left in Figure 7 and the arm lengths on the right. Note that if $\alpha$ is a partition $\mu$, the leg and arm definitions agree with those previously given for $\mu'$.

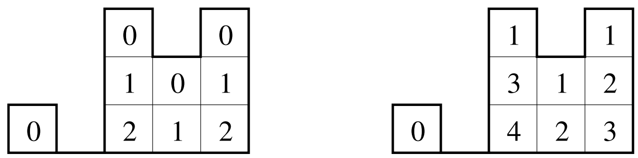

FIGURE 7. The leg lengths (on the left) and the arm lengths (on the right) for $(1,0,3,2,3)'$.

A filling $\sigma$ of $\alpha'$ is an assignment of positive integers to the squares of $\mathrm{dg}(\alpha')$. The reading word of such a $\sigma$ is obtained by reading the entries iin $\alpha'$ along rows, right-to-left, top-to-bottom, as in the case of partition diagrams. We standardize as before; given two equal letters in $\sigma$, the one occurring first in the reading word is viewed as being smaller. We say two squares $u, v \in \mathrm{dg}(\alpha')$ attack each other if they are in the same row, or if $u \in \alpha'$ and $v \in \mathrm{dg}(\alpha')$, $v$ is in the row below $u$, and in a column strictly to the left of $u$. A filling (from a positive alphabet) of $\mathrm{dg}(\alpha')$ is nonattacking if no two squares which attack each other contain the same number.

A triple of $\mathrm{dg}(\alpha')$ is three squares with $u \in \alpha'$, $v \in \mathrm{Arm}(u)$, and $w = \mathrm{South}(u)$. Note that both $v$ and $w$ could be in the basement. Given a filling $\sigma$ of $\alpha'$, we say a triple $u, v, w$ is a coinversion triple if either $v$ is in a column to the right of $u$, and $u, v, w$ has a clockwise orientation, or $v$ is in a column to the left of $u$, and $u, v, w$ has a counterclockwise orientation. A triple of the first kind, with $v$ to the right of $u$, is called a Type A triple, while a triple of the second kind, where $v$ is to the left of $u$ and one row below, is called a Type B triple. Let $\mathrm{coinv}(\sigma, \alpha')$ denote the number of coinversion triples of $\sigma$. For example, the filling in Figure 8 has 7 coinversion triples:

(4.3)
$$\{[(3,2),(3,1),(4,2)],[(3,2),(3,1),(5,2)],[(3,2),(3,1),(1,1)],[(4,2),(4,1),(1,1)]$$

(4.4)
$$[(5,1),(5,0),(1,0)],[(5,1),(5,0),(2,0)],[(5,1),(5,0),(4,0)]\},$$

where the first two are Type A triples and the rest are Type B.

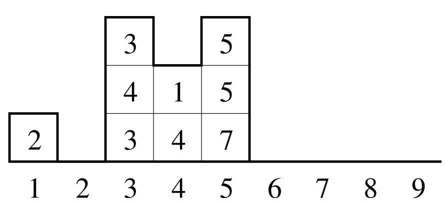

FIGURE 8. A nonattacking filling of $(1,0,3,2,3,0,0,0,0)'$.

Given $\beta \in S_n$ and $\alpha \in \mathbb{N}^n$, Alexandersson's formulas for the nonsymmetric Macdonald polynomial corresponding to $\alpha$ with basement $\beta$, denoted $E_\alpha^\beta(X_n; q, t)$, and its integral form $\mathcal{E}_\alpha^\beta(X_n;q,t)$ are [**1**];

$$E_\alpha^\beta(X_n;q,t) = \sum_{\substack{\sigma \text{ nonattacking} \\ 1\le\sigma_i\le n}} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \prod_{\substack{s\in\alpha' \\ \sigma(s)\neq\sigma(\mathrm{South}(s))}} \frac{1-t}{1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1}} \tag{4.5}$$

and

$$\mathcal{E}_\alpha^\beta(X_n;q,t) = \sum_{\substack{\sigma \\ \text{nonattacking}}} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \tag{4.6}$$

$$\times \prod_{\substack{s\in\alpha' \\ \sigma(s)\neq\sigma(\mathrm{South}(s))}} (1-t) \prod_{\substack{s\in\alpha' \\ \sigma(s)=\sigma(\mathrm{South}(s))}} (1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1}). \tag{4.7}$$

Here the sums are over all nonattacking fillings $\sigma$ of the diagram $\mathrm{dg}(\alpha')$ with basement $\beta$, i.e. $\sigma(j,0)=\beta_j$ for $1\le j\le n$. By definition,

$$\mathcal{E}_\alpha^\beta(X_n;q,t) = E_\alpha^\beta(X_n;q,t)\prod_{s\in\alpha'}(1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1}), \tag{4.8}$$

and it is easy to see formulas (4.5) and (4.6) are equivalent.

Example 4.1. By (4.3) the nonattacking filling in Figure 8 has coinv $=7$. There are descents at squares $(1,1)$, $(3,2)$ and $(5,1)$, with maj-values 1, 2 and 3, respectively. The squares $(3,1)$, $(4,1)$ and $(5,3)$ satisfy the condition $\sigma(u)=\sigma(\mathrm{South}(u))$ and contribute factors $(1-q^3t^5)$, $(1-q^2t^3)$ and $(1-qt^2)$, respectively. Hence the total weight associated to this filling in (4.5) would be

$$x_1x_2x_3^2x_4^2x_5^2x_7\, q^6t^7(1-q^3t^5)(1-q^2t^3)(1-qt^2)(1-t)^6. \tag{4.9}$$

If $\mathrm{rev}(\alpha)=(\alpha_n,\dots,\alpha_1)$, then $E_{\mathrm{rev}(\alpha)}^{(n,n-1,\dots,1)}(X_n;q,t)$ is Macdonald's original $E_\alpha(X_n;q,t)$, and $E_\alpha^{(1,2,\dots,n)}(X_n;q,t)$ is the variant $E_{\mathrm{rev}(\alpha)}(x_n,\dots,x_2,x_1;1/q,1/t)$ of Macdonald's $E$ studied by Marshall [**31**]. Hence (4.5) gives combinatorial formulae for both these special cases, which are equivalent to ones that first appeared in [**15**].

## 5. A formula for $P_\mu$ and a Quasisymmetric Macdonald Polynomial

Given a vector $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ of nonnegative integers, we let $\mathrm{inc}(\alpha)$ be the vector $\alpha$, sorted so the parts are in nondecreasing order, and $\mathrm{dec}(\alpha)$ the vector $\alpha$, sorted so the parts are in nonincreasing order. Furthermore let $\beta(\alpha)$ be the permutation in $S_n$ of maximal length with the property that $\beta$ applied to the vector $\alpha$ yields $\mathrm{inc}(\alpha)$, let $\alpha^+$ be the (strong) composition consisting of $\alpha$ with the zeros removed, and $\ell(\alpha)$ the number of parts of $\alpha^+$. For example, if $\alpha=(0,2,0,2,1,3)$ then $\mathrm{inc}(\alpha)=(0,0,1,2,2,3)$, $\mathrm{dec}(\alpha)=(3,2,2,1,0,0)$, $\beta(\alpha)=(3,1,5,4,2,6)$, $\alpha^+=(2,2,1,3)$, and $\ell(\alpha)=4$.

In [**4**][Lemma 3], Cantini, de Gier and Wheeler showed that

$$P_\mu(X_n;q,t) = \sum_{\substack{\alpha \\ \mathrm{dec}(\alpha^+)=\mu}} f_\alpha(X_n;q,t), \tag{5.1}$$

for certain polynomials $f_\alpha\in\mathbb{Q}(q,t)[X_n]$. In [**8**], Corteel, Mandelshtam and Williams proved that

$$f_\alpha(X_n;q,t) = E_{\mathrm{inc}(\alpha)}^{\beta(\alpha)}(X_n;q,t), \tag{5.2}$$

and using this in (5.1) we get

$$P_\mu(X_n;q,t) = \sum_{\substack{\alpha \\ \mathrm{dec}(\alpha^+)=\mu}} E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X_n;q,t) \tag{5.3}$$

Note that for all the terms in the right-hand-side of (5.2), the shape $\mathrm{inc}(\alpha)$ being filled is the same, namely the shape $0^{n-\ell(\mu)}\mathrm{inc}(\mu)$. Also, for this shape the definition of $\beta(\alpha)$ implies the entries in the basement below columns of $0^{n-\ell(\mu)}\mathrm{inc}(\mu)$ of equal height are decreasing left-to-right. Say a basement with this property is *ordered*. There is a bijection between ordered basements and permutations of the form $\beta(\alpha)$, where $\alpha \in \mathbb{N}^n$ and $\mathrm{dec}(\alpha^+) = \mu$. See Figure 9 for an example of a filling contributing to (5.1).

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | 3 | 2 |
| | | | | | | | 7 | 6 |
| | | | | 1 | 5 | 2 | 7 | 9 |
| | | | | 1 | 6 | 3 | 5 | 8 |
| | | | 9 | 7 | 4 | 1 | 5 | 3 |
| 8 | 6 | 2 | 9 | 7 | 4 | 1 | 5 | 3 |

FIGURE 9. A nonattacking filling of $(0,0,0,1,3,3,3,5,5)'$ with an ordered basement.

It is a general fact that in any nonattacking filling of a shape $\alpha'$ with a given basement, any entry in the first row of $\alpha'$ must equal the number just below it in the basement. Thus the basement $\beta(\alpha)$ in a filling $\sigma$ in a given term contributing to (5.3) is uniquely determined by $\sigma$. Furthermore, if $\sigma(u) = \sigma(w) \neq \sigma(v)$, then (as in the type A case) $u, v, w$ always from an inversion triple. Thus none of the entries in the basement columns of height 0 in (5.3) are involved in any coinversion triples, so contribute nothing to the power of $t$. They also contribute nothing to the $x$ or $q$-weights. Since this part of the basement is determined by the part below columns of positive height, we can erase it. But the rest of the basement is just a copy of the first row, so we can just sum over fillings of $\mathrm{inc}(\mu)$ with the property that in the first row, entries in columns of the same height are decreasing left-to-right; call such a filling ordered. We will keep the concept of the basement here, consisting of a copy of the first row, so our definition of coinv still applies. We get the following final formula for $P_\mu$, which is implicit in [**6**].

$$P_\mu(X;q,t) = \sum_{\substack{\text{fillings } \sigma \text{ of } \mathrm{inc}(\mu)' \\ \sigma \text{ nonattacking, ordered}}} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \prod_{\substack{s \in \mathrm{inc}(\mu)' \\ s \text{ not in bottom row} \\ \sigma(s) \neq \sigma(\mathrm{South}(s))}} \frac{1-t}{1-q^{\mathrm{leg}+1}t^{\mathrm{arm}+1}}. \tag{5.4}$$

This formula for $P_\mu(X;q,t)$ is more elegant than previous ones from ([**15**]). For example, one formula there starts with a decomposition of $P_\mu(X;q,t)$ as a sum of the Marshall $E_\alpha$ 's in a way similar to (5.3), but the coefficient of a given $E_\alpha$ there is a ratio of complicated products.

By iterating the intertwiner recurrence for nonsymmetric Macdonald polynomials, Ram and Yip [**32**] obtained a very general formula, parameterized in terms of alcove walks, for symmetric and nonsymmetric Macdonald polynomials associated to an arbitrary affine root system. In the type A (specifically $\mathrm{GL}_n$) case, these formulas have a lot more terms than the HHL formulas, but for a partition $\mu$ with distinct parts, Lenart [**26**] has shown how to "compress" the Ram-Yip formula for $P_\mu(X;q,t)$, and the resulting identity is equivalent to (5.4). More recently, Guo and Ram [**12**] have shown how type $\mathrm{GL}_n$ permuted basement nonsymmetric Macdonald polynomials (which they have suggested be renamed relative Macdonald polynomials - perhaps a good idea) satisfy a different "box-by-box" recurrence, which upon iteration leads directly to the nonattacking formulas (4.5) and (5.4). Hopefully their method will lead to compression of the Ram-Yip formula for other root systems.

For $\mu \vdash n$ and $\alpha \in \mathbb{N}^n$, let

$$\mathrm{PR1}(\mu) = \prod_{s\in\mu}(1-q^{\mathrm{arm}(s)}t^{\mathrm{leg}(s)+1}) = \prod_{s\in\mu'}(1-q^{\mathrm{leg}(s)}t^{\mathrm{arm}(s)+1}) \tag{5.5}$$

$$\mathrm{PR2}(\alpha) = \prod_{i\geq 1}(t;t)_{m_i} \prod_{\substack{s\in\mathrm{inc}(\alpha')\\ s \text{ not in the bottom row}}} (1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)+1}), \tag{5.6}$$

where for $i \geq 1$, $m_i$ is the number of times $i$ occurs in $\alpha$. In [**6**] it is noted that $\mathrm{PR1}(\mu) = \mathrm{PR2}(\mathrm{inc}(\mu))$. Now Macdonald's definition of $J_\mu$ in terms of $P_\mu$ is

$$J_\mu(X;q,t) = \mathrm{PR1}(\mu)P_\mu(X;q,t), \tag{5.7}$$

and so by multiplying both sides of (5.4) by $\mathrm{PR1}(\mu)$ we get an equivalent formula for $J_\mu$;

Corollary 5.1.

(5.8)

$$J_\mu(X;q,t) = \prod_i (t;t)_{m_i} \sum_{\text{ordered, nonattacking fillings } \sigma \text{ of } \mathrm{inc}(\mu)'} x^\sigma q^{\mathrm{maj}(\sigma)}t^{\mathrm{coinv}(\sigma)}$$
$$\times \prod_{\substack{s\in\mu',\ s \text{ not in row } 1\\ \sigma(s)=\sigma(\mathrm{South}(s))}} (1-q^{\mathrm{leg}(s)+1}t^{\mathrm{arm}(s)+1}) \prod_{\substack{s\in\mu',\ s \text{ not in row } 1\\ \sigma(s)\neq\sigma(\mathrm{South}(s))}} (1-t),$$

Note that (5.8) implies $J_\mu(X_n;q,t)$ is $\prod_i (t;t)_{m_i}$ times an element of $\mathbb{Z}[X_n,q,t]$. By taking limits as in (3.6) we recover a result of Knop and Sahi [**23**], namely that the coefficient of a given monomial in the integral form Jack polynomial $J_\mu^\alpha(X)$ is $\prod_{i\geq 1} m_i(\mu)!$ times an element of $\mathbb{N}[\alpha]$.

Let $\gamma \in \mathbb{Z}_+^k$, where $k \in \mathbb{Z}_+$, be a (strong) composition. Haglund, Luoto, Mason and van Willigenburg [**17**] showed that if we let

$$\mathrm{QS}_\gamma(X_n) = \sum_{\substack{\alpha\in\mathbb{N}^n\\ \alpha^+=\gamma}} E_\alpha(X_n;0,0) \tag{5.9}$$

then the $\{\mathrm{QS}_\gamma\}$ form a basis for the ring QSym of quasisymmetric functions they called the quasisymmetric Schur basis. They noted that

$$\sum_{\substack{\gamma\in\mathbb{Z}_+^{\ell(\mu)}\\ \mathrm{dec}(\gamma)=\mu}} \mathrm{QS}_\gamma(X_n) = s_\mu(X_n), \tag{5.10}$$

so the QS basis decomposes the Schur basis into positive components. Since $P_\mu(X_n;0,0) = s_\mu(X_n)$, a natural question to ask is whether or not there is a way to incorporate $q,t$ parameters into $\mathrm{QS}_\gamma(X_n)$ to form a quasisymmetric basis for $\mathrm{QSym}(q,t)$, the ring of quasisymmetric functions with coefficients in $\mathbb{Q}(q,t)$. Identity (5.2) has led to a solution to this question. In [**6**] it is shown if we define

$$\mathrm{G}_\gamma(X_n;q,t) = \sum_{\substack{\alpha\in\mathbb{N}^n\\ \alpha^+=\gamma}} E^{\beta(\alpha)}_{\mathrm{inc}(\alpha)}(X_n;q,t), \tag{5.11}$$

then the $G_\gamma$ form a basis for $\mathrm{QSym}(q,t)$, $G_\gamma(X_n;0,0) = \mathrm{QS}_\gamma(X_n)$, and (by (5.2) ),

$$P_\mu(X_n;q,t) = \sum_{\substack{\gamma\\ \mathrm{dec}(\gamma)=\mu}} G_\gamma(X_n;q,t). \tag{5.12}$$

We mention that Corteel, Mandelshtam and Roberts [**7**] have found the expansion of $G_\gamma$ into Gessel fundamental quasisymmetric functions.

## 6. Refinements of the HHL formula for $\tilde{H}_\mu(X;q,t)$

In this section we list several identities for $\tilde{H}_\mu(X;q,t)$ which refine and extend the HHL formula (2.4). First of all, in [**15**] it is noted that if in formula (3.3) we replace the sum over nonattacking fillings of $\mu'$ by a sum over nonattacking fillings of $\gamma'$, where $\gamma$ is any permutation of the parts of $\mu$ (i.e. $\gamma^+ = \mu$), and use the same $q,t,x$-weights, with type A and type B triples to define coinv in terms of triples (or skyline diagram leg and arm lengths to define coinv via inversion pairs and descents) then the result still equals $J_\mu(X;q,t)$. Furthermore one can show that if we apply the same type of idea to (2.1), the obvious extensions of the involutions IH and IL fix the $q,t,x$-weights and switch the sign. This implies that for any shape $\tau$ obtained by permuting the columns of $\mu$,

$$\tilde{H}_\mu(X;q,t) = \sum_{\sigma:\tau\to\mathbb{Z}_+} x^\sigma t^{\mathrm{maj}(\sigma,\tau)} q^{\mathrm{inv}(\sigma,\tau)}. \tag{6.1}$$

There is also [**6**] a more compact version of Theorem 2.5. We say a filling $\sigma$ of $\mu$ is *sorted* if entries in the first row, in columns of the same height, are weakly increasing left-to-right. Letting $\mathrm{Col}_j$ denotes the set of columns of $\mu$ of height $j$ for $1 \le j \le n$, for a sorted filling $\sigma$ of $\mu$ set

$$\mathrm{perm}_q(\sigma,\mu) = \prod_{j=1}^{n} \binom{m_1+m_2+\dots}{m_1,m_2,\dots}_q, \tag{6.2}$$

where inside the sum $m_j$ is the number of times $j$ occurs in $\sigma$ in the first row and in columns $\mathrm{Col}_j$, i.e. in squares $\{(p,1) : p \in \mathrm{Col}_j\}$. For the filling in Figure 10,

$$\begin{aligned}\mathrm{perm}_q &= \binom{3}{1\,,2}_q \binom{4}{2\,,2}_q \\ &= (1+q+q^2)(1+q+q^2)(1+q^2).\end{aligned} \tag{6.3}$$

Theorem 6.1 ([**6**]).

$$\tilde{H}_\mu(X;q,t) = \sum_{\substack{\sigma:\mu\to\mathbb{Z}_+\\ \sigma\ sorted}} x^\sigma t^{maj(\sigma,\mu)} q^{inv(\sigma,\mu)} perm_q(\sigma,\mu). \tag{6.4}$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | | | | | | | |
| 6 | 2 | 6 | 2 | | | | |
| 7 | 2 | 5 | 7 | | | | |
| 1 | 2 | 3 | 5 | | | | |
| 3 | 3 | 5 | 5 | 1 | 1 | 2 | 2 |

FIGURE 10. A sorted filling of 84441.

The proof of (6.4) is fairly long; it involves a detailed analysis of the structure of the statistic inv on fillings.

We now describe another recent result which grew out of the study of the ASEP. Given a filling $\sigma$ of $\mu$, we call a set of squares $u, v, w$, where $w = \text{South}(u)$, and $v$ is in the same row as $w$ and to the right of $w$, a quinv triple of $\sigma$. Also, we call two squares $w, v$ satisfying the second of these conditions, with both $w, v$ at the top of their respective columns, a (degenerate) quinv triple. Given a quinv triple with $\sigma(u)$, $\sigma(v)$ and $\sigma(w)$ all distinct, we say they form an inversion triple if they have a counterclockwise orientation; otherwise they form a coinversion triple. (Or, if $w, v$ form a degenerate triple, then they from an inversion triple if $\sigma(w) < \sigma(v)$, otherwise they form a coinversion triple.) If the three entries of $\sigma(u), \sigma(v), \sigma(w)$ are not all distinct, we standardize them, using the reading order that goes across rows from right-to-left, starting with the top row and going downwards one row at a time. For example, the quinv reading word for the filling in Figure 11 is 322431352332143215445751, some of the inversion triples are

$$\{[(2,2),(2,1),(3,1)],[(3,3),(4,3)],[(6,2),(6,1),(7,1)], \\ [(6,2),(6,1),(8,1)][(1,5),(2,5)]\}, \tag{6.5}$$

while some coinversion triples are

$$\{[(1,2),(1,1),(2,1)],[(6,2),(7,2)],[(6,2),(8,2)]\}. \tag{6.6}$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 3 | | | | | | |
| 4 | 2 | | | | | | |
| 5 | 3 | 1 | 3 | | | | |
| 2 | 3 | 4 | 1 | 2 | 3 | 3 | 2 |
| 1 | 5 | 7 | 5 | 4 | 4 | 5 | 1 |

FIGURE 11. A filling of 88422

THEOREM 6.2 ([**2**]).

$$\tilde{H}_\mu(X;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} x^\sigma t^{maj(\sigma,\mu)} q^{quinv(\sigma,\mu)}, \tag{6.7}$$

*where $quinv(\sigma,\mu)$ is the number of quinv inversion triples.*

There have been a number of other exciting combinatorial formulas for Macdonald polynomials that have appeared in recent years, but due to space and time considerations we will have to content ourselves with merely listing a few of them. Several of these formulas have emerged from statistical mechanics and representation theory. For example, Garbali and Wheeler [**11**] have a formula for the monomial expansion of $\tilde{H}_\mu(X;q,t)$ where, when $q=0$, the coefficients reduce to the *rigged configurations* occurring in the description Kirillov and Reshetikhin gave of the Schur expansion of modified Hall-Littlewood polynomials [**21**]. Also, Borodin and Wheeler [**3**] give a new expansion for the nonsymmetric Macdonald polynomial, and then show how their formula can be bijectively mapped to terms in the HHL formula. In addition, Diaconis and Ram [**9**] have a probabilistic model for the coefficients in the expansion of $P_\mu(X;q,t)$ into the power-sum basis, with impressive applications to the study of statistics and Markov chains.

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