Statistics on Wreath Products, Perfect Matchings and Signed Words

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Abstract

We introduce a natural extension of Adin, Brenti, and Roichman's major-index statistic *nmaj* on signed permutations (*Adv. Appl. Math.* **27**, (2001), $210-244$) to wreath products of a cyclic group with the symmetric group. We derive "insertion lemmas" which allow us to give simple bijective proofs that our extension has the same distribution as another statistic on wreath products introduced by Adin and Roichman (*Europ. J. Combin.* **22**, (2001), $431-446$) called the *flag major index*. We also use our insertion lemmas to show that *nmaj*, the *flag major index*, and an inversion statistic have the same distribution on a subset of signed permutations in bijection with perfect matchings. We show that this inversion statistic has an interpretation in terms of q-counting rook placements on a shifted Ferrers board.

Many results on permutation statistics extend to results on multiset permutations (words). We derive a number of analogous results for signed words, and also words with higher order roots of unity attached to them.

Keywords: Major-Index Statistics, Wreath Products, Perfect Matchings, Signed Words, Rook Theory.

1 Introduction

A permutation statistic *stat* is a function $stat : S_n \to \mathbb{N}$, where S_n is the symmetric group. A statistic is called *Mahonian* if the distribution over S_n is the q-analogue of $n!$, i.e., if

$$\sum_{\sigma \in S_n} q^{stat(\sigma)} = [n]!_q,$$

where $[n]!_q = [1]_q[2]_q \cdots [n]_q$, with $[k]_q = 1 + q + \ldots + q^{k-1} = (1-q^k)/(1-q)$, and $0 < q < 1$. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be an element of S_n (in one-line notation) or more generally any sequence of nonnegative integers. The two most important examples of Mahonian statistics in combinatorics are the *inversion statistic*

$$inv(\sigma) = \sum_{1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)} 1$$

and the *major-index statistic*

$$maj(\sigma) = \sum_{1 \le i \le n-1 \text{ and } \sigma_i > \sigma_{i+1}} i.$$

Let $R(a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k})$ denote the set of words (multiset permutations) which have exactly n_i occurrences of the letter a_i. The statistic *maj* defined above was introduced by MacMahon [15], who showed that both *maj* and *inv* are *multiset* Mahonian, i.e. that

$$\sum_{\sigma \in R(a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k})} q^{inv(\sigma)} = \sum_{\sigma \in R(a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k})} q^{maj(\sigma)} = \begin{bmatrix} n_1 + \ldots + n_k \\ n_1, \ldots, n_k \end{bmatrix}_q, \tag{1}$$

where $\begin{bmatrix} n_1 + \ldots + n_k \\ n_1, \ldots, n_k \end{bmatrix}_q = \frac{[n_1 + \ldots + n_k]!_q}{[n_1]!_q \cdots [n_k]!_q}$ is the q-multinomial coefficient. Foata [5], [10] later found a classical involution on permutations which interchanges *maj* and *inv* and yields a bijective proof of the leftmost equality in (1). For more background information on these results see Chapter 1 of [21], Chapter 3 of [3] and exercise 5.1.2.18 of [14].

The first and third authors recently introduced a version of q-rook theory [11] which involves a number of inversion-based statistics on perfect matchings of the complete graph

K_n and which satisfy the following natural analog of the Mahonian property.

$$\sum_{\text{perfect matchings } P \text{ of } K_n} q^{stat(P)} = [1]_q[3]_q \cdots [2n-1]_q. \tag{2}$$

This led to the question of whether there exists a major-index statistic on perfect matchings with the same Mahonian distribution. A *signed permutation* is a permutation $\sigma \in S_n$ where each σ_i has a plus or minus sign attached to it. In sections 3 and 4 we first show how perfect matchings are in bijection with the set of signed permutations whose right-to-left minima have positive signs and then we define a major-index statistic on this subset of signed permutations which has the Mahonian property (2).

Statistics on the hyperoctahedral group B_n of signed permutations on n letters have been studied by many authors including Reiner [17], [18], [19], Steingrimsson [20], Clarke and Foata [6], [7], [8] and Foata and Krattenthaler [9]. It is known [12], that the natural inversion statistic $\ell(\sigma)$ (defined as the Coxeter group length) satisfies

$$\sum_{\sigma \in B_n} q^{\ell(\sigma)} = [2]_q[4]_q \cdots [2n]_q. \tag{3}$$

Reiner [17] obtained the distribution over B_n of the most obvious choice of a major-index statistic, but found it had a slightly different distribution than (3). On extending our major-index perfect matching statistic to all of B_n we found we had a statistic with the same distribution as (3). However, we later discovered that this result had already appeared in a recent article of Adin and Roichman [2]. Furthermore, Adin, Brenti and Roichman [1] have introduced another major-index statistic on signed permutations which also has the same distribution as (3). Our "insertion lemmas" from section 4 allow us to give new bijective proofs of these and other related results of theirs. In addition, we obtain the new result that their statistics satisfy the Mahonian property (2) when restricted to signed permutations whose right-to-left minima are positive. We show that many of our results apply to the wreath product of any cyclic group with S_n.

In section 5 we consider the distribution of statistics over signed words, and obtain many results similar in form to (1). We also consider major-index statistics corresponding to words with higher-order roots of unity attached to the elements which are multiset versions of wreath products of a cyclic group with S_n.

2 Statistics on $C_k \wr S_n$

The wreath product of the cyclic group C_k with S_n, $C_k \wr S_n$, reduces to the symmetric group S_n when $k = 1$ and the hyperoctahedral group B_n when $k = 2$. We can think of the group $C_k \wr S_n$ as the group of "signed" permutations where the signs are in the set of k^{th} roots of unity $\{1, \epsilon, \ldots, \epsilon^{k-1}\}$ where ϵ is defined by $\epsilon = e^{\frac{2\pi i}{k}}$. It is useful to describe the elements in two ways. First, we can think of $C_k \wr S_n$ as a group defined by generators and relations. There are n generators, $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau$, which satisfy the following relations:

$$\begin{aligned}
\sigma_i^2 &= 1, \quad i = 1, 2, \ldots, n-1, \\
\tau^k &= 1, \\
(\sigma_i \sigma_j)^2 &= 1, \quad |i - j| > 1, \\
(\sigma_i \sigma_{i+1})^3 &= 1, \quad i = 1, 2, \ldots, n-2, \\
(\tau \sigma_1)^{2k} &= 1.
\end{aligned}$$

In fact, one can realize the generators σ_i as the transpositions $(i, i+1)$ and the generator τ as $(\epsilon 1)$, that is, it maps 1 to ϵ times itself.

We can also write an element $\sigma \in C_k \wr S_n$ in two-line notation. For example, we could have

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & \epsilon^2 6 & \epsilon^2 7 & 10 & \epsilon 5 & \epsilon^2 2 & \epsilon 1 & 9 & \epsilon^2 8 & 4 \end{pmatrix} \in C_3 \wr S_n. \tag{4}$$

We can then write this in one-line form as

$$\sigma = \; 3 \quad \epsilon^2 6 \quad \epsilon^2 7 \quad 10 \quad \epsilon 5 \quad \epsilon^2 2 \quad \epsilon 1 \quad 9 \quad \epsilon^2 8 \quad 4 \; ,$$

or in cyclic notation as

$$\sigma = (\epsilon 1, 3, \epsilon^2 7)(\epsilon^2 2, \epsilon^2 6)(\epsilon 5)(\epsilon^2 8, 9). \tag{5}$$

Note that when using cyclic notation to determine the image of a number, one ignores the sign on that number and then considers only the sign on the next number in the cycle. Thus, in this example, we ignore the sign of ϵ^2 on the 7 and note that then 7 maps to $\epsilon 1$ since the sign on 1 is ϵ.

Building on work of Adin and Roichman [2], in [1] Adin, Brenti, and Roichman defined the following statistics on signed permutations in B_n. First given any sequence $\gamma = \gamma_1 \dots \gamma_n$ from an alphabet $\mathcal{A}$ which is totally ordered by $<$, we can define the following statistics.

$$\begin{aligned}
Des(\gamma) &= \{i : \gamma_i > \gamma_{i+1}\} && (6)\\
des(\gamma) &= |Des(\gamma)| && (7)\\
Neg(\gamma) &= \{i : \gamma_i < 0\} && (8)\\
neg(\gamma) &= |Neg(\gamma)| && (9)\\
maj(\gamma) &= \sum_{i \in Des(\gamma)} i && (10)\\
inv(\gamma) &= \sum_{1 \le i < j \le n} \chi(\gamma_i > \gamma_j) && (11)
\end{aligned}$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Then for any $\sigma = \sigma_1 \dots \sigma_n \in B_n$, Adin, Brenti and Roichman defined the following.

I. $NDes(\sigma) = Des(\sigma) \cup \{-\sigma_i : i \in Neg(\sigma)\}$ and $ndes(\sigma) = |NDes(\sigma)|$.
Here $NDes$ is a multiset. For example, if $\sigma = 3\ -1\ 4\ -5\ 2$, then $Des(\sigma) = \{1, 3\}$ and $\{-\sigma_i : i \in Neg(\sigma)\} = \{1, 5\}$ so that $NDes(\sigma) = \{1^2, 3, 5\}$ and $ndes(\sigma) = 4$.

II. $nmaj(\sigma) = \sum_{i \in NDes(\sigma)} i$.
For example, if $\sigma = 3\ -1\ 4\ -5\ 2$, then $nmaj(\sigma) = 1 + 1 + 3 + 5 = 10$.

III. $f\text{-}des(\sigma) = 2des(\sigma) + \chi(\sigma_1 < 0)$.
For example, if $\sigma = 3\ -1\ 4\ -5\ 2$, then $ndes(\sigma) = 2des(\sigma) + 0 = 4$.

IV. $f\text{-}maj(\sigma) = 2maj(\sigma) + neg(\sigma)$.
For example, if $\sigma = 3\ -1\ 4\ -5\ 2$, then $f\text{-}maj(\sigma) = 2maj(\sigma) + neg(\sigma) = 2(4) + 2 = 10$.

V. $\ell(\sigma) = inv(\sigma) - \sum_{i \in Neg(\sigma)} \sigma_i$.
For example, if $\sigma = 3\ -1\ 4\ -5\ 2$, then $\ell(\sigma) = inv(\sigma) - (-1 - 5) = 6 + 6 = 12$. We note ℓ is the usual length function for B_n considered as a Coxeter group, see [4], [13].

In [1], the authors proved that

$$\begin{aligned} [2]_q[4]_q \cdots [2n]_q &= \sum_{\sigma \in B_n} q^{\ell(\sigma)} \\ &= \sum_{\sigma \in B_n} q^{nmaj(\sigma)} \\ &= \sum_{\sigma \in B_n} q^{f\text{-}maj(\sigma)}. \end{aligned}$$

In addition, they proved that

$$\sum_{\sigma \in B_n} x^{ndes(\sigma)} q^{nmaj(\sigma)} = \sum_{\sigma \in B_n} x^{f\text{-}des(\sigma)} q^{f\text{-}maj(\sigma)}. \tag{12}$$

Adin and Roichman [2] defined a statistic they called the *flag major index* for $C_k \wr S_n$ in the case where $k \geq 2$. Their definition involved the following ordering on elements of the form $\epsilon^j m$ where $j \in \{0, \ldots, k-1\}$ and $m \in \{1, \ldots, n\}$,

$$\epsilon^{k-1}1 < \ldots < \epsilon^{k-1}n < \ldots < \epsilon^1 1 < \ldots < \epsilon^1 n < 1 < \ldots < n. \tag{13}$$

They defined the flag major index for $C_k \wr S_n$ by

$$flag\text{-}maj(\sigma) = k \cdot maj(\sigma) + \sum_{j=0}^{k-1} j \cdot sign_j(\sigma) \tag{14}$$

where $Sign_j(\sigma) = \{i : \frac{\sigma_i}{|\sigma_i|} = \epsilon^j\}$ and $sign_j(\sigma) = |Sign_j(\sigma)|$.

We note that the definitions of $f\text{-}maj$ and $flag\text{-}maj$ do not agree when we restrict ourselves to elements of B_n. That is, in the definition of $f\text{-}maj$, we use the order

$$\cdots > m > \cdots > 2 > 1 > -1 > -2 > \cdots > -m. \tag{15}$$

for the definition of the major index maj as opposed to the order

$$\cdots > m > \cdots > 2 > 1 > \cdots > -m > -(m-1) > \cdots > -1. \tag{16}$$

which we use to define maj in the definition of $flag\text{-}maj$. Thus in the case of B_n, we shall use $maj_{lex}(\sigma)$ for the major index of σ relative to the order given in (16) and use maj for the major index of σ relative to the order given in (15) if there is any chance of confusion. Thus it is not true that for all $\sigma \in B_n$, $2maj(\sigma) + neg(\sigma) = 2maj_{lex}(\sigma) + neg(\sigma)$. For example, if $\sigma = 1{-}3{-}2$, then $2maj(\sigma)+neg(\sigma) = 2(1)+2 = 4$ while $2maj_{lex}(\sigma)+neg(\sigma) = 2(3) + 2 = 8$. However, the results in section 4 will show that it is the case that

$$\sum_{\sigma\in B_n} q^{2maj(\sigma)+neg(\sigma)} = \sum_{\sigma\in B_n} q^{2maj_{lex}(\sigma)+neg(\sigma)} = \prod_{i=1}^{n}[2i]_q. \tag{17}$$

It turns out that there is also a natural extension of $nmaj$ to $C_k \wr S_n$ for $k > 2$ which we call *root-maj* that is defined as follows:

$$root\text{-}maj(\sigma) = maj(\sigma) + \sum_{j=0}^{k-1} \sum_{i\in Sign_j(\sigma)} j \cdot |\sigma_i|. \tag{18}$$

We shall show in section 3 that

$$\sum_{\sigma\in C_k\wr S_n} q^{flag\text{-}maj(\sigma)} = \sum_{\sigma\in C_k\wr S_n} q^{root\text{-}maj(\sigma)} = \prod_{j=1}^{n}[jk]_q. \tag{19}$$

3 Perfect Matching, Signed Permutations and Rook Theory

Let K_n denote the complete graph on n vertices. We shall assume that the vertex set of K_n is $[n] = \{1, \ldots, n\}$. Then it is well known that the number of perfect matchings of K_{2n} is equal to $\prod_{i=1}^{n}(2i-1)$.

Next we define an injection β from the set of perfect matchings $PM(K_{2n})$ of K_{2n} into B_n, in a manner which is is probably best explained with an example. Consider a perfect matching of K_{10},

$$P = (\{1, 3\}, \{2, 7\}, \{4, 9\}, \{5, 8\}, \{6, 10\}).$$

We start out with a graph consisting of two rows of vertices, the top row of vertices labeled $1, \ldots, n$ from left to right and the bottom row of vertices labeled $n + 1, \ldots, 2n$ from right to left. We start out with the edges $\{i, 2n + 1 - i\}$ for $i = 1, \ldots, n$. These are the dotted edges in Figure 1 which we shall call *non-matching edges*. Then if $\{i, j\} \in P$,

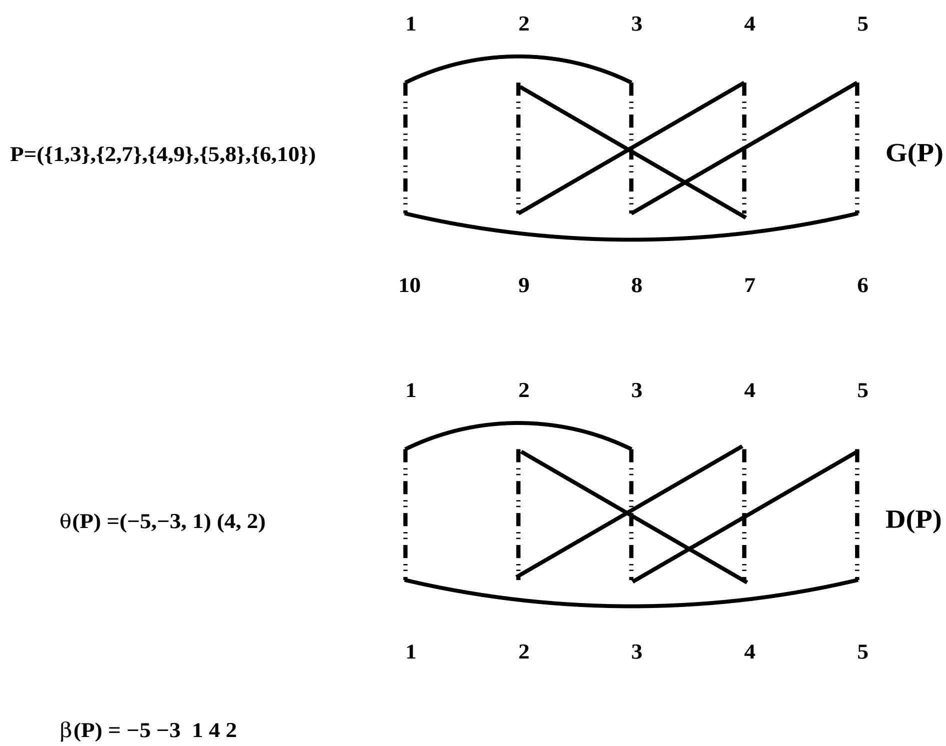

Figure 1: The β bijection

we add an edge from i to j. These are the solid edges in the Figure 1 which we call *matching edges.* In this way, we construct the graph of P, $G(P)$. Now we modify the graph of P by relabeling the vertex $2n+1-i$ by i for $i = 1, \ldots, n$. This has the effect of relabeling the bottom row of vertices of $G(P)$ by $1, \ldots, n$ from left to right to produce what we call the diagram of P, $D(P)$.

Next we use $D(P)$ to construct a permutation $\theta(P) \in B_n$. The idea is to use the diagram to construct the set of cycles of $\theta(P)$ in the following manner. First we start with vertex 1 in the top row of $D(P)$ and then follow the dotted edge to the 1 in the bottom row of $D(P)$ and then we follow a solid edge out of the 1 in the bottom row which in the case of Figure 1 leads to the 5 in the bottom row. In this case, we say that 1 is mapped to -5 since we ended up in a different row from where we started. Thus our cycle starts out $(1, -5, \ldots)$. Next we start with the 5 in the bottom row, follow the dotted edge to the 5 in the top row and then follow the matching edge out of the 5 in the top row to get to the 3 in the bottom row. In this case, since the 3 ended up in the same row as the 5 at which we started in the second step, we do not change signs. Thus the next element in the cycle is -3 and our cycle starts out $(1, -5, -3, \ldots)$. Since we ended up with the 3 in bottom row of $D(P)$, we follow the dotted edge out of the 3 in the bottom row to the 3 in the top row of $D(P)$ and then follow the matching edge out of the 3 in the top row to the 1 in the top row. Since the 1 is in a different row than the 3 in the bottom row, the next element changes sign so that the cycle would be $(1, -5, -3, 1)$ which obviously completes a cycle. The general procedure to construct cycles is then the following.

Step 1. Start with 1 in the top row. We follow a non-matching edge to the 1 in the bottom row and then follow a matching edge to some element i_2. If i_2 is in the same row as where we started, then the cycle starts out $(1, i_2, \ldots)$ and if the i_2 is in a different row than where we started then the cycle starts out $(1, -i_2, \ldots)$.

Step 2. Start with the i_2 that we ended up at the end of step 1. We follow a non-matching edge to the i_2 in the opposite row and then follow a matching edge to some element i_3. If i_3 is in the same row as the i_2 where we started, then the cycle starts out $(1, \pm i_2, \pm i_3, \ldots)$ where the signs on i_2 and i_3 are the same and if the i_3 is in a different row than the i_2 where we started, then the cycle starts out $(1, \pm i_2, \mp i_3, \ldots)$ where the signs on i_2 and i_3 are different.

Step $k+1$ Suppose that at the end of step k, we ended up at some vertex of $D(P)$ labeled

i_k. We follow a non-matching edge to the vertex labeled i_k in the opposite row and then follow a matching edge to some element i_{k+1}. If the resulting vertex i_{k+1} is in the same row as the vertex i_k where we started, then the cycle starts out $(1, \pm i_2, \pm i_3, \ldots, \pm i_k, \pm i_{k+1}, \ldots)$ where the signs on i_k and i_{k+1} are the same. If the resulting vertex is in a different row than the vertex labeled i_k where we started, then the cycle starts out $(1, \pm i_2, \pm i_3, \ldots, \pm i_k, \mp i_{k+1}, \ldots)$ where the signs on i_k and i_{k+1} are different.

Once we have completed the cycle, we then start the procedure over again starting with the smallest element in the top row that is not already in a cycle until we complete the next cycle. In general, having completed p cycles, we create the next cycle by following the same procedure starting with the smallest element in the top row which is not part of the previously constructed cycles. For example, if we return to the perfect matching P pictured in Figure 1, to create the next cycle, we start with the smallest element that is not in the previous cycle $(1, -5, -3)$ which in our example is 2. We then start with the 2 in the top row, follow the dotted edge to the 2 in the bottom row, and then follow the matching edge from the 2 in the bottom row to the 4 in the top row. Since the 4 we ended up with is in the same row that we started, we do not change signs so the second cycle starts out $(2, 4, \ldots)$. The next step is to take the 4 in the top row, follow the dotted edge to the 4 in the bottom row and then follow a matching edge to the 2 in the top row. Since this 2 is in the same row as the 4 that we started with in this step, we complete the cycle $(2, 4)$ and $\theta(P) = (1, -5, -3)(2, 4)$.

Next we cyclicly rearrange each cycle of $\theta(P)$ so that the smallest element of the cycle is on the right and then we order the cycles by increasing smallest elements. For the P pictured in Figure 1, this produces the list $(-3, -5, 1)(4, 2)$. Then to get $\beta(P)$, we simple erase the parenthesis and commas and get a permutation in one line notation. In our example, $\beta(P) = \ -3 - 5\ 1\ 4\ 2$.

There are several observations that we can make about this construction. First it is easy to see that the smallest element of each cycle of $\theta(P)$ is positive by construction. Next, by our conventions for ordering the cycles to obtain $\beta(P)$, it is easy to see that the end of each cycle is smaller in absolute value than all the elements of the cycles to its right. Thus it is easy to see that the smallest elements in the cycles in $\theta(P)$ are the right-to-left minima of $\beta(P)$ where we say that σ_i is *right-to-left minimum* of $\sigma = \sigma_1 \ldots \sigma_n \in B_n$ if $|\sigma_i| < |\sigma_j|$ for all $j > i$. Moreover, these right-to-left minima must be positive since the smallest elements in each cycle of $\theta(P)$ are positive. Thus we define $RLMin^+(B_n)$ to be the set of all $\sigma \in B_n$ such that all right-to-left minima in σ are positive. By the observations above, our construction ensures that $\beta(P) \in RLMin^+(B_n)$ for all $P \in PM(K_{2n})$.

Finally we observe that we can reconstruct P from $\beta(P)$. That is, we can reconstruct the cycles of $\theta(P)$ by simply cutting after the right-to-left minima of $\beta(P)$. Next it should be clear that we can use $\theta(P)$ to reconstruct $D(P)$ because if we reorder each cycle $c = (i_1, \ldots, i_k)$ of $\theta(P)$ so that its smallest element is on the left, then we know that the matching edges of $D(P)$ must connect a vertex labeled i_j to a vertex labeled i_{j+1} for $j = 1, \ldots, k-1$ and a vertex labeled i_k to a vertex labeled i_1. The only question is to determine in which rows do the various labeled vertices lie. However it is easy to see that this is completely determined by the fact that in the construction of each cycle, we always start with the i_1 in the top row and the signs in the cycle determine whether the matching edges stay in the same row or in opposite rows. That is, it is easy to see that if i_j and i_{j+1} have the same signs, then the matching edge must go from the top row to the bottom row or vice versa, and if i_j and i_{j+1} have the different signs, then the matching edge must stay in the same row. Thus we can reconstruct $D(P)$ from $\theta(P)$. Finally it is easy to see that we can construct $G(P)$ from $D(P)$ and P from $G(P)$. The following result now follows.

Theorem 3.1 *The map $\beta : PM(K_{2n}) \to RLMin^+(B_n)$ described above is a bijection.*

Haglund and Remmel [11] gave a rook theory interpretation for the set of perfect matchings involving a statistic u on rook placements such that if we q-count the rook placements that correspond to perfect matchings, then we obtain a q-analogue for the number of perfect matchings of B_n. Consider the board BD_n which consists of the cells $\{(i,j) : i < j\}$. For example, BD_{12} is pictured in Figure 2 where the row numbers i are labeled from top to bottom and the column numbers j are labeled from left to right.

We want to consider the set $RP_n(BD_{2n})$ of all placements of n rooks on BD_{2n} such that no two rooks share a common coordinate. Such rook placements naturally correspond to perfect matchings of K_{2n}. If a rook r is on square (i,j), then we will say that r cancels all cells (s,t) such that $s + t \leq i + j$ and $\{s,t\} \cap \{i,j\} \neq \emptyset$. For example in Figure 2, we have pictured the cells cancelled by the rook r in cell $(4,12)$ of BD_{12} that are not equal to $(4,12)$ by placing a dot in those cells. Given a placement $p \in RP_n(BD_{2n})$, we let $u(P)$

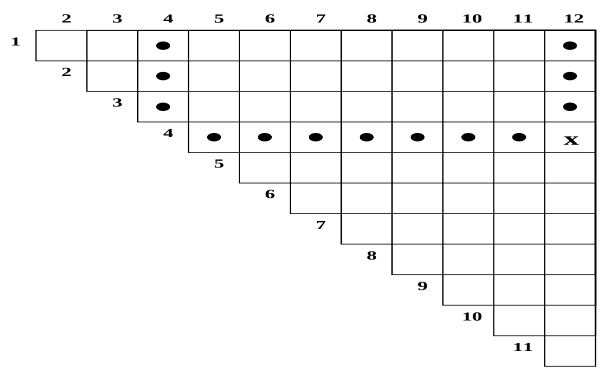

Figure 2: The Rook Board BD_{12}

denote the set of cells in BD_{2n} which are not cancelled by any rook in p. For example, for the placement $p \in RP_6(BD_{12})$ pictured in Figure 3, it is easy to check that $u(p) = 19$.

Using the standard technique of q-counting rooks first placed in the last column then moving to the left, one easily obtains that

$$\sum_{p \in RP_n(BD_{2n})} q^{u(p)} = \prod_{i=1}^{n} [2i-1]_q. \tag{20}$$

In light of (20) and our bijection β, it is natural to ask if there are statistics s such that

$$\sum_{\sigma \in RLMin^+(B_n)} q^{s(\sigma)} = \prod_{i=1}^{n} [2i-1]_q. \tag{21}$$

One of the main results of the next section is that any of the statistics ℓ, $nmaj$, or f-maj has this property.

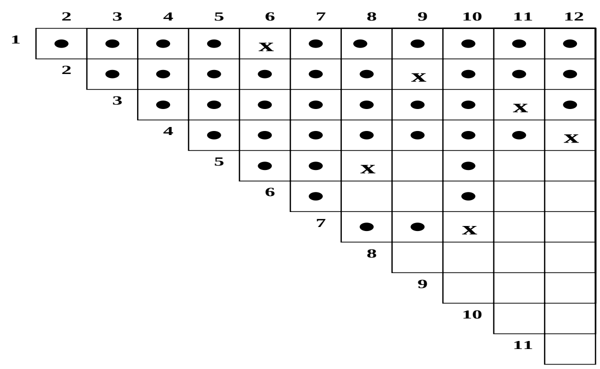

Figure 3: An element of $RP_6(BD_{12})$

4 Insertion Lemmas

Let $S_{\{t_1,\ldots,t_n\}}$ denote the set of permutations of some ordered set of elements $t_1 < \ldots < t_n$. Next fix some permutation $\sigma = \sigma_1 \ldots \sigma_n$ in $S_{\{t_1,\ldots,t_n\}}$ and let t be some element such that $t_{p-1} < t < t_p$. We want to see how the insertion of t in the sequence σ affects the major index and inversion statistics. There are clearly $n+1$ spaces where we can insert t into the sequence $\sigma_1 \ldots \sigma_n$. That is, for each $i = 1, \ldots, n$, there is the space immediately following σ_i which we call space i and there is the space immediately preceding σ_1 which we call space 0. We then let $(\sigma \downarrow j)$ be the sequence that results by inserting t into space j.

First we shall describe an insertion lemma for maj which will show that no matter what is the relative value of t with respect to the other elements of the sequence

$$\sum_{j=0}^{n+1} q^{maj((\sigma\downarrow j))} = q^{maj(\sigma)}[n+1]_q. \tag{22}$$

We shall classify the possible spaces where we can insert t into σ into two sets called the right-to-left spaces which we denote as RL-spaces and the left-to-right spaces which we denote as LR-spaces. That is, we say that a space i is a *RL-space of σ relative to t* if

1. $i = n$ and $\sigma_n < t$,

2. $i = 0$ and $t < \sigma_1$,

3. $0 < i < n$ and $\sigma_i > \sigma_{i+1} > t$,

4. $0 < i < n$ and $t > \sigma_i > \sigma_{i+1}$, or

5. $0 < i < n$ and $\sigma_i < t < \sigma_{i+1}$.

Then a space i is a *LR-space of σ relative to t* if it is not a RL-space of σ relative to t. Now suppose there are k RL-spaces for σ relative to t. Then we label the RL-spaces from right to left with $0, \ldots, k-1$ and we label the LR-spaces from left to right with $k, \ldots, n$ and call this labeling the canonical labeling for σ relative to t. For example suppose that $t = 5$ and $\sigma \in S_{1,\ldots,4,6,\ldots 10}$ is the permutation

$$\sigma = 10\ 1\ 9\ 8\ 2\ 7\ 4\ 3\ 6\ .$$

the RL-spaces of σ relative to 5 are $0, 2, 3, 5, 7$ and 8 and the LR-spaces of σ relative to 5 are $1, 4, 6$ and 9. The canonical labeling of σ relative to t is

$${}_5 10 {}_6 1 {}_4 9 {}_3 8 {}_7 2 {}_2 7 {}_8 4 {}_1 3 {}_0 6 {}_9.$$

This given we have the following.

Lemma 4.1 *Suppose that $\sigma = \sigma_1 \ldots \sigma_n$ is a permutation of the ordered set $t_1 < \cdots < t_n$ and t is such that $t_{p-1} < t < t_p$. Then if in the canonical labeling of σ relative to t space j receives the label k, then*

$$maj((\sigma \downarrow j)) = k + maj(\sigma). \tag{23}$$

For our example above $Des(\sigma) = \{1, 3, 4, 6, 7\}$ so that $maj(\sigma) = 1+3+4+6+7 = 21$. Note that in the canonical labeling space 4 receives the label 7 and $Des((\sigma \downarrow 4)) = Des(10\ 1\ 9\ 8\ 5\ 2\ 7\ 4\ 3\ 6\ ) = \{1, 3, 4, 5, 7, 8\}$ so that $maj((\sigma \downarrow 4)) = 28 = maj(\sigma) + 7$.

Proof: We proceed by induction on n, the case $n = 1$ being trivial. Consider any $\sigma = \sigma_1 \ldots \sigma_n \in S_{\{t_1,\ldots,t_n\}}$. The following facts are easy to establish from the definition of the major index.

1. If $\sigma_n < t$, then $maj((\sigma \downarrow n)) - maj(\sigma) = 0$.

2. If $\sigma_n > t$, then $maj((\sigma \downarrow n)) - maj(\sigma) = n$.

3. If $\sigma_1 < t$, then $maj((\sigma \downarrow 0)) - maj(\sigma) = 1 + des(\sigma_1 \ldots \sigma_n)$.

4. If $\sigma_1 > t$, then $maj((\sigma \downarrow 0)) - maj(\sigma) = des(\sigma_1 \ldots \sigma_n)$.

5. If $\sigma_i > \sigma_{i+1} > t$, then $maj((\sigma \downarrow i)) - maj(\sigma) = des(\sigma_{i+1} \ldots \sigma_n)$.

6. If $\sigma_i > t > \sigma_{i+1}$, then $maj((\sigma \downarrow i)) - maj(\sigma) = i + 1 + des(\sigma_{i+1} \ldots \sigma_n)$.

7. If $t > \sigma_i > \sigma_{i+1}$, then $maj((\sigma \downarrow i)) - maj(\sigma) = 1 + des(\sigma_{i+1} \ldots \sigma_n)$.

8. If $\sigma_i < \sigma_{i+1} < t$, then $maj((\sigma \downarrow i)) - maj(\sigma) = i + 1 + des(\sigma_{i+1} \ldots \sigma_n)$.

9. If $\sigma_i < t < \sigma_{i+1}$, then $maj((\sigma \downarrow i)) - maj(\sigma) = des(\sigma_{i+1} \ldots \sigma_n)$.

10. If $t < \sigma_i < \sigma_{i+1}$, then $maj((\sigma \downarrow i)) - maj(\sigma) = i + des(\sigma_{i+1} \ldots \sigma_n)$.

For example, consider case 8. Thus $\sigma_i < \sigma_{i+1} < t$ and and $(\sigma \downarrow i) = \sigma_1 \ldots \sigma_i \, t \, \sigma_{i+1} \ldots \sigma_n$. Clearly $Des(\sigma) = Des(\sigma_1 \ldots \sigma_i) \cup \{i + k : k \in Des(\sigma_{i+1} \ldots \sigma_n)\}$ while $Des((\sigma \downarrow i) = Des(\sigma_1 \ldots \sigma_i) \cup \{i + 1\} \cup \{1 + i + k : k \in Des(\sigma_{i+1} \ldots \sigma_n)\}$ so that $maj((\sigma \downarrow i)) = maj(\sigma) + i + 1 + des(\sigma_{i+1} \ldots \sigma_n)$.

Now assume Proposition 4.1 is true for all sequences of length n. Fix some permutation $\sigma^+ = \sigma_1 \ldots \sigma_{n+1}$ in $S_{\{t_1,\ldots,t_{n+1}\}}$ and let $\sigma = \sigma_1 \ldots \sigma_n$. By induction, we can assume that the canonical labeling of σ relative to t uses labels $0, \ldots, n$ and that if the insertion of t in space i increases the major index of σ by k, then space i is labeled with a k. We now consider the possibilities for σ_{n+1}. We will prove only one of these cases in detail, and merely list the other cases, as an aid to the reader who is interested in filling in all the details. For background on the process used the reader can consult [16].

Case I. $\sigma_n > \sigma_{n+1}$.

First it is easy to see from our equations for cases 3-10 above that whenever $\sigma_n > \sigma_{n+1}$ and $i < n$,

$$maj((\sigma^+ \downarrow i)) - maj(\sigma^+) = 1 + maj((\sigma \downarrow i)) - maj(\sigma). \tag{24}$$

That is, the only difference between the expression for $maj((\sigma^+ \downarrow i)) - maj(\sigma^+)$ versus the expression for $maj((\sigma \downarrow i)) - maj(\sigma)$ in each case is that the expression for $maj((\sigma^+ \downarrow i)) - maj(\sigma^+)$ involves $des(\sigma_{i+1} \ldots \sigma_n \sigma_{n+1})$ while the expression for $maj((\sigma \downarrow i)) - maj(\sigma)$ involves $des(\sigma_{i+1} \ldots \sigma_n)$. Thus since $des(\sigma_{i+1} \ldots \sigma_n \sigma_{n+1}) - des(\sigma_{i+1} \ldots \sigma_n \sigma_n) = 1$, (24) must hold for $i = 0, \ldots, n-1$. Hence in this case we must show that for each $0 \leq i \leq n-1$, if space i gets label k in the canonical labeling of σ with respect to t, then space i gets label $k+1$ in the canonical labeling of σ^+ with respect to t.

We now have three subcases.

Subcase I(a) $\sigma_n > \sigma_{n+1} > t$.

Note that in the canonical labeling of σ with respect to t, space n got label n since it was the rightmost LR-space for σ with respect to t. However in the canonical labeling of σ^+ with respect to t, space n gets label 0 since it is the rightmost RL-space for σ^+ with respect to t and space $n+1$ gets label $n+1$ since it is the right-most LR-space for σ^+ with respect to t. In pictures, we have the following.

In the canonical labeling of σ with respect to t	$\ldots \sigma_{n_{\overline{n}}}.$
In the canonical labeling of σ^+ with respect to t	$\ldots \sigma_{n_{\overline{0}}} \sigma_{n+1_{\overline{n+1}}}.$

It is now easy to check the following hold.

1. $maj(\sigma_1 \ldots \sigma_n\ t\ \sigma_{n+1}) = maj(\sigma_1 \ldots \sigma_{n+1})$ so that space n should be labeled 0 because the insertion of t into space n does not change the major index.

2. $maj(\sigma_1 \ldots \sigma_n\ \sigma_{n+1}\ t) = n+1+maj(\sigma_1 \ldots \sigma_{n+1})$ so that space $n+1$ should be labeled $n+1$ because the insertion of t into space $n+1$ adds $n+1$ to the major index.

3. Since space n was labeled n in the canonical labeling of σ with respect to t but is labeled with 0 in the canonical labeling of σ^+ with respect to t, our labeling algorithm ensures that for all $i \leq n-1$, if space i is labeled k in the canonical labeling of σ with respect to t, then space i is labeled with $k+1$ in the canonical labeling of σ^+ with respect to t as desired.

Subcase I(b) $\sigma_n > t > \sigma_{n+1}$.

Subcase I(c) $t > \sigma_n > \sigma_{n+1}$.

Case II. $\sigma_n < \sigma_{n+1}$.

Again, we have three subcases.

Subcase II(a) $t < \sigma_n < \sigma_{n+1}$.

Subcase II(b) $\sigma_n < t < \sigma_{n+1}$.

Subcase II(c) $\sigma_n < \sigma_{n+1} < t$.
□

We note that we immediately have the following corollary of Proposition 4.1.

Corollary 4.2 *Suppose that* $\sigma = \sigma_1 \ldots \sigma_n$ *is a permutation of the ordered set* $t_1 < \cdots < t_n$ *and* t *is such that* $t_{i-1} < t < t_{i+1}$. *Then*

$$\sum_{j=0}^{n} q^{maj((\sigma \downarrow j))} = [n+1]_q q^{maj(\sigma)}. \tag{25}$$

We note that the analogue of Corollary 4.2 fails for the inversion statistic. That is, suppose that we want to insert 2 into the sequence $\sigma = 1\ 3$. Then clearly $inv(2\ 1\ 3) = 1$, $inv(1\ 2\ 3) = 0$ and $inv(1\ 3\ 2) = 1$ so that $\sum_{j=0}^{2} q^{inv((\sigma\downarrow j))} = 1 + 2q \neq [3]_q q^{inv(\sigma)}$. However there clearly are insertion lemmas for inv in the special cases where either $t > t_n$ or $t < t_n$. That is, it is easy to see that the following lemma holds.

Lemma 4.3 *Suppose that* $\sigma = \sigma_1 \ldots \sigma_n$ *is a permutation of the ordered set* $t_1 < \cdots < t_n$.

1. *If* $t_n < t$, *then*
$$inv((\sigma \downarrow j)) = n - j + inv(\sigma). \tag{26}$$

2. *If* $t < t_1$, *then*
$$inv((\sigma \downarrow j)) = j + inv(\sigma). \tag{27}$$

This means that if $t_n < t$, the canonical labeling for *inv* of the spaces of any permutation $\sigma = \sigma_1 \ldots \sigma_n$ of the ordered set $t_1 < \cdots < t_n$ is to simply label the spaces from right to left with $0, \ldots, n$. In pictures, we have the following.

$$\text{The canonical labeling for } inv \text{ of } \sigma \text{ with respect to } t > t_n \; {}_{\overline{n}}\sigma_1{}_{\overline{n-1}} \cdots {}_{\overline{1}}\sigma_n{}_{\overline{0}}. \tag{28}$$

Similarly if $t < t_1$, the canonical labeling for *inv* of the spaces of any permutation $\sigma = \sigma_1 \ldots \sigma_n$ of the ordered set $t_1 < \cdots < t_n$ is to simply label the spaces from left to right with $0, \ldots, n$. In pictures, we have the following.

$$\text{The canonical labeling for } inv \text{ of } \sigma \text{ with respect to } t < t_1 \; {}_{\overline{0}}\sigma_1{}_{\overline{1}} \cdots {}_{\overline{n-1}}\sigma_n{}_{\overline{n}}. \tag{29}$$

Moreover the following corollary is immediate from Lemma 4.3.

Corollary 4.4 *Suppose that $\sigma = \sigma_1 \ldots \sigma_n$ is a permutation of the ordered set $t_1 < \cdots < t_n$.*

1. *If $t > t_n$, then*
$$\sum_{j=0}^{n} q^{inv((\sigma\downarrow j))} = [n+1]_q q^{inv(\sigma)}. \tag{30}$$
2. *If $t < t_1$, then*
$$\sum_{j=0}^{n} q^{inv((\sigma\downarrow j))} = [n+1]_q q^{inv(\sigma)}. \tag{31}$$

This given, we can now easily establish the following results.

Theorem 4.5

$$\begin{aligned} \prod_{i=1}^{n} [2i]_q &= \sum_{\sigma \in B_n} q^{\ell(\sigma)} \\ &= \sum_{\sigma \in B_n} q^{nmaj(\sigma)} \\ &= \sum_{\sigma \in B_n} q^{f\text{-}maj(\sigma)} \end{aligned} \tag{32}$$

$$\begin{aligned}\prod_{i=1}^{n}[2i-1]_q &= \sum_{\sigma\in RLMin^+(B_n)} q^{\ell(\sigma)} \\ &= \sum_{\sigma\in RLMin^+(B_n)} q^{nmaj(\sigma)} \\ &= \sum_{\sigma\in RLMin^+(B_n)} q^{f\text{-}maj(\sigma)}\end{aligned} \tag{33}$$

$$\begin{aligned}\prod_{i=1}^{n}[ki]_q &= \sum_{\sigma\in C_k\wr S_n} q^{root\text{-}maj(\sigma)} \\ &= \sum_{\sigma\in C_k\wr S_n} q^{flag\text{-}maj(\sigma)}\end{aligned} \tag{34}$$

Proof: Each part is straightforward to prove by induction. That is, consider (32). Assume that by induction that

$$\begin{aligned}\prod_{i=1}^{n-1}[2i]_q &= \sum_{\sigma\in B_{n-1}} q^{\ell(\sigma)} \\ &= \sum_{\sigma\in B_{n-1}} q^{nmaj(\sigma)} \\ &= \sum_{\sigma\in B_{n-1}} q^{f\text{-}maj(\sigma)}\end{aligned}$$

Let $\sigma = \sigma_1 \dots \sigma_{n-1} \in B_{n-1}$ and let $(\sigma \downarrow^n j)$ be the result of inserting n into the j-th space of σ and let $(\sigma \downarrow^{-n} j)$ be the result of inserting $-n$ into the j-th space of σ. Then it is easy to see from Lemma 4.3 and Corollary 4.4 since $n > \sigma_i$ for all i,

$$\sum_{j=0}^{n-1} q^{inv((\sigma\downarrow^n j))} = [n]_q q^{inv(\sigma)}. \tag{35}$$

Similarly since $-n < \sigma_i$ for all i,

$$\sum_{j=0}^{n-1} q^{inv((\sigma\downarrow^{-n} j))} = [n]_q q^{inv(\sigma)}. \tag{36}$$

Moreover for each j, if $\alpha = (\sigma \downarrow^n j) = \alpha_1 \dots \alpha_n$, then $-\sum_{i \in Neg(\alpha)} \alpha_i = -\sum_{i \in Neg(\sigma)} \sigma_i$ and if $\beta = (\sigma \downarrow^{-n} j) = \beta_1 \dots \beta_n$, then $-\sum_{i \in Neg(\beta)} \beta_i = n - \sum_{i \in Neg(\sigma)} \sigma_i$. Thus

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{\ell((\sigma\downarrow^n j))} &= \sum_{j=0}^{n-1} q^{inv((\sigma\downarrow^n j)) - \sum_{i\in Neg(\sigma\downarrow^n j)} (\sigma\downarrow^n j)_i} \\
&= q^{(-\sum_{i \in Neg(\sigma)} \sigma_i)} \sum_{j=0}^{n-1} q^{inv((\sigma\downarrow^n j))} \\
&= [n]_q q^{inv(\sigma) - \sum_{i\in Neg(\sigma)} \sigma_i} \\
&= [n]_q q^{\ell(\sigma)}. \qquad (37)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{\ell((\sigma\downarrow^{-n} j))} &= \sum_{j=0}^{n-1} q^{inv((\sigma\downarrow^{-n} j)) - \sum_{i\in Neg(\sigma\downarrow^{-n} j)} (\sigma\downarrow^{-n} j)_i} \\
&= q^{(n-\sum_{i \in Neg(\sigma)} \sigma_i)} \sum_{j=0}^{n-1} q^{inv((\sigma\downarrow^{-n} j))} \\
&= q^n [n]_q q^{inv(\sigma) - \sum_{i\in Neg(\sigma)} \sigma_i} \\
&= q^n [n]_q q^{\ell(\sigma)}. \qquad (38)
\end{aligned}$$

Hence it follows that for any $\sigma \in B_{n-1}$,

$$\sum_{j=0}^{n-1} q^{\ell((\sigma\downarrow^n j))} + q^{\ell((\sigma\downarrow^{-n} j))} = ([n]_q + q^n [n]_q) q^{\ell(\sigma)} = [2n]_q q^{\ell(\sigma)} \tag{39}$$

so that

$$\sum_{\tau \in B_n} q^{\ell(\tau)} = [2n]_q \sum_{\sigma \in B_{n-1}} q^{\ell(\sigma)} = \prod_{i=1}^{n} [2i]_q. \tag{40}$$

Next it is easy to see from Proposition 4.1 and Corollary 4.2 that

$$\sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^n j))} = \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^{-n} j))} = [n]_q q^{maj(\sigma)}. \tag{41}$$

Thus

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{nmaj((\sigma\downarrow^n j))} &= \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^n j))-\sum_{i\in Neg(\sigma\downarrow^n j)}(\sigma\downarrow^n j)_i} \\
&= q^{(-\sum_{i\in Neg(\sigma)}\sigma_i)} \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^n j))} \\
&= [n]_q q^{maj(\sigma)-\sum_{i\in Neg(\sigma)}\sigma_i} \\
&= [n]_q q^{nmaj(\sigma)}.
\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{nmaj((\sigma\downarrow^{-n} j))} &= \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^{-n} j))-\sum_{i\in Neg(\sigma\downarrow^{-n} j)}(\sigma\downarrow^{-n} j)_i} \\
&= q^{(n-\sum_{i\in Neg(\sigma)}\sigma_i)} \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^{-n} j))} \\
&= q^n [n]_q q^{maj(\sigma)-\sum_{i\in Neg(\sigma)}\sigma_i} \\
&= q^n [n]_q q^{nmaj(\sigma)}.
\end{aligned} \tag{43}$$

Hence as was the case for ℓ, it follows that for any $\sigma \in B_{n-1}$,

$$\sum_{j=0}^{n-1} q^{nmaj((\sigma\downarrow^n j))} + q^{nmaj((\sigma\downarrow^{-n} j))} = [2n]_q q^{nmaj(\sigma)} \tag{44}$$

so that

$$\sum_{\tau\in B_n} q^{nmaj(\tau)} = [2n]_q \sum_{\sigma\in B_{n-1}} q^{nmaj(\sigma)} = \prod_{i=1}^{n} [2i]_q. \tag{45}$$

Finally observe that for any j, $neg((\sigma\downarrow^n j)) = neg(\sigma)$ and $neg((\sigma\downarrow^{-n} j)) = 1 +$

$neg(\sigma)$. It thus follows that

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{f\text{-}maj((\sigma\downarrow^n j))} &= \sum_{j=0}^{n-1} q^{2maj((\sigma\downarrow^n j))+neg(\sigma\downarrow^n j)} \\
&= q^{neg(\sigma)} \sum_{j=0}^{n-1} q^{2maj((\sigma\downarrow^n j))} \\
&= [n]_{q^2} q^{2maj(\sigma)+neg(\sigma)} \\
&= [n]_{q^2} q^{f\text{-}maj(\sigma)} \qquad (46)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{f\text{-}maj((\sigma\downarrow^{-n} j))} &= \sum_{j=0}^{n-1} q^{2maj((\sigma\downarrow^{-n} j))+neg(\sigma\downarrow^{-n} j)} \\
&= q^{1+neg(\sigma)} \sum_{j=0}^{n-1} q^{2maj((\sigma\downarrow^{-n} j))} \\
&= q[n]_{q^2} q^{2maj(\sigma)+neg(\sigma)} \\
&= q[n]_{q^2} q^{f\text{-}maj(\sigma)}. \qquad (47)
\end{aligned}$$

Hence it follows that for any $\sigma \in B_{n-1}$,

$$\sum_{j=0}^{n-1} q^{f\text{-}maj((\sigma\downarrow^n j))} + q^{f\text{-}maj((\sigma\downarrow^{-n} j))} = ([n]_{q^2} + q[n]_{q^2}) q^{f\text{-}maj(\sigma)} = [2n]_q q^{f\text{-}maj(\sigma)} \qquad (48)$$

so that

$$\sum_{\tau \in B_n} q^{f\text{-}maj(\tau)} = [2n]_q \sum_{\sigma \in B_{n-1}} q^{f\text{-}maj(\sigma)} = \prod_{i=1}^{n} [2i]_q. \qquad (49)$$

In fact it is easy to see that our proof actually provides a bijective proof that

$$\sum_{\tau \in B_n} q^{\ell(\tau)} = \sum_{\tau \in B_n} q^{nmaj(\tau)} = \sum_{\tau \in B_n} q^{f\text{-}maj(\tau)}. \qquad (50)$$

That is, it not difficult to see from (35 - 45) that for any $\sigma = \sigma_1 \ldots \sigma_{n-1} \in B_{n-1}$ and for any $0 \leq i \leq n-1$ that there are j_1 and j_2 such that

$$\begin{aligned} \ell((\sigma \downarrow^n j_1)) &= i + \ell(\sigma) \text{ and} \\ nmaj((\sigma \downarrow^n j_2)) &= i + nmaj(\sigma). \end{aligned}$$

Similarly, for any $n \leq i \leq 2n-1$, there are j_3 and j_4 such that

$$\begin{aligned} \ell((\sigma \downarrow^{-n} j_3)) &= i + \ell(\sigma) \text{ and} \\ nmaj((\sigma \downarrow^{-n} j_4)) &= i + nmaj(\sigma). \end{aligned}$$

In the case of f-maj, it follows from (46- 49) that for any $0 \leq i \leq n-1$, there are j_5 and j_6 such that

$$\begin{aligned} f\text{-}maj((\sigma \downarrow^n j_5)) &= 2i + f\text{-}maj(\sigma) \text{ and} \\ f\text{-}maj((\sigma \downarrow^n j_6)) &= 2i + 1 + f\text{-}maj(\sigma). \end{aligned}$$

Thus it follows that for either ℓ, $nmaj$, or f-maj, we can increase the statistic by i for any $0 \leq i \leq 2n-1$ by inserting n or $-n$ in the appropriate space in σ.

We say that a function $f : \{1,\ldots,n\} \to \{0,\ldots,2n-1\}$ is an *inversion table* if $f(i) \leq 2i-1$ for all i. If $f : \{1,\ldots,n\} \to \{0,\ldots,2n-1\}$ is an inversion table, we let $|f| = \sum_{i=1}^n f(i)$. It should be clear that if $\mathbf{F}_n$ is the set of all inversion tables $f : \{1,\ldots,n\} \to \{0,\ldots,2n-1\}$, then

$$\sum_{f \in \mathbf{F}_n} q^{|f|} = \prod_{i=1}^n [2i]_q. \tag{51}$$

It follows from our discussion above that if s is any one of the three statistics ℓ, $nmaj$, or f-maj, then for any inversion table $f \in \mathbf{F}_n$, we can create a sequence of permutations $\sigma_f^1, \ldots, \sigma_f^n$ such that for all $1 \leq i \leq n$, (i) $\sigma_f^i \in B_i$, (ii) $s(\sigma_f^i) = f(1) + \cdots + f(i)$, and $\sigma_f^i = (\sigma \downarrow^\nu j_i)$ for some j_i and $\nu \in \{i, -i\}$. Vice versa, given any permutation $\sigma \in B_n$, let $\sigma^n = \sigma$ and for any $1 \leq i < n$, let σ^i be the permutation of B_i that results by removing all elements of σ whose absolute value is greater than i. Then we can define an inversion table f_σ by letting $f_\sigma(1) = 0$ and $f_\sigma(i) = s(\sigma^i) - s(\sigma^{i-1})$ for $1 < i \leq n$. This shows that there is a bijection $\theta_s : \mathbf{F}_n \to B_n$ such that $|f| = s(\theta_s(f))$ for all $f \in \mathbf{F}_n$. An example of

i	1	2	3	4	5	6
f(i)	0	1	3	2	6	7

l (σ)	*nmaj* (σ)	*f−maj* (σ)
1	1	1
2 1	2 1	−2 1
−3 2 1	2 −3 1	−2 −3 1
−3 4 2 1	4 2 −3 1	−2 4 −3 1
−3 −5 4 2 1	4 −5 2 −3 1	−2 5 4 −3 1
−3 −6 −5 4 2 1	4 −6 −5 2 −3 1	−1 −6 5 4 −3 1

Figure 4: Inversion Table to Permutation Statistics

these maps is given in Figure 4. One can then use the maps θ_s and θ_t and their inverses to construct a map $\theta_{s,t} : B_n \to B_n$ such that for all $\sigma \in B_n$,

$$s(\sigma) = t(\theta_{s,t}(\sigma)) \tag{52}$$

for any pair of statistics s and t from ℓ, $nmaj$, or $f\text{-}maj$.

We note that part (33) immediately follows from our proof of (32) once one observes that our labeling lemmas ensure that for any statistic s from ℓ, $nmaj$, or $f\text{-}maj$ and any $\sigma \in B_{n-1}$,

$$s(\sigma \downarrow^{-n} n) = 2n - 1 + s(\sigma). \tag{53}$$

That is, for all three statistics, placing $-n$ at the end of σ increases the statistics by $2n-1$. Since $RLMin^+(B_n)$ is constructed from $RLMin^+(B_{n-1})$ taking any $\sigma \in RLMin^+(B_{n-1})$ and inserting n into any space of σ and inserting $-n$ into any space of σ except space n, it follows that for any $\sigma \in RLMin^+(B_{n-1})$,

$$\sum_{i=0}^{n-1} q^{s((\sigma\downarrow^{n} i))} + \sum_{i=0}^{n-2} q^{s((\sigma\downarrow^{-n} i))} = [2n-1]_q q^{s(\sigma)}. \tag{54}$$

Hence it is easy to prove by induction that

$$\sum_{\sigma \in RLMin^+(B_n)} q^{s(\sigma)} = \prod_{i=1}^{n} [2i-1]_q. \tag{55}$$

Moreover if we let $RLMin^+(\mathbf{F}_n)$ be the set all inversion tables from $\mathbf{F}_n$ such that $f(1) = 0$ and $f(i) \leq 2i-2$ for $1 < i \leq n$, then the restriction of θ_s to $RLMin^+(\mathbf{F}_n)$ gives a bijection from $RLMin^+(\mathbf{F}_n)$ onto $RLMin^+(B_n)$ such that for all $f \in RLMin^+(\mathbf{F}_n)$, $|f| = s(\theta_s(f))$. Thus the bijections $\theta_{s,t}$ when restricted to $RLMin^+(B_n)$ provide bijections from $RLMin^+(B_n)$ to $RLMin^+(B_n)$ such that for all $\sigma \in RLMin^+(B_n)$, $s(\sigma) = t(\theta_{s,t}(\sigma))$.

For (34), first observe that it follows that it follows from Proposition 4.1 and Corollary 4.2 that if $\sigma \in C_k \wr S_{n-1}$ and $0 \leq p \leq k-1$, then

$$\sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^{(\epsilon^p n)} j))} = [n]_q q^{maj(\sigma)}. \tag{56}$$

Thus

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{flag\text{-}maj((\sigma\downarrow^{(\epsilon^p n)} j))} &= \sum_{j=0}^{n-1} q^{k\cdot maj((\sigma\downarrow^{(\epsilon^p n)} j)) + \sum_{r=0}^{k-1} r sign_r(\sigma\downarrow^{(\epsilon^p n)} j)} \\
&= q^{(p+\sum_{r=0}^{k-1} r\cdot sign_r(\sigma))} \sum_{j=0}^{n-1} q^{k\cdot maj((\sigma\downarrow^{(\epsilon^p n)} j))} \\
&= q^p [n]_{q^k} q^{k\cdot maj(\sigma) + \sum_{r=0}^{k-1} r\cdot sign_r(\sigma)} \\
&= q^p [n]_{q^k} q^{flag\text{-}maj(\sigma)}.
\end{aligned} \tag{57}$$

Hence it follows that for any $\sigma \in C_k \wr S_{n-1}$,

$$\sum_{j=0}^{n-1}\sum_{p=0}^{k-1} q^{flag\text{-}maj((\sigma\downarrow^{\epsilon^p n} j))} = \sum_{p=0}^{k-1} q^p [n]_{q^k} q^{flag\text{-}maj(\sigma)} = [kn]_q q^{flag\text{-}maj(\sigma)} \tag{58}$$

so that

$$\sum_{\tau \in C_k \wr S_n} q^{flag\text{-}maj(\tau)} = [kn]_q \sum_{\sigma \in C_k \wr S_{n-1}} q^{nmaj(\sigma)}. \tag{59}$$

Thus it is easy to prove by induction that

$$\sum_{\tau \in C_k \S_n} q^{flag\text{-}maj(\tau)} = \prod_{i=1}^{n} [ki]_q. \tag{60}$$

Similarly for any $\sigma \in C_k \wr S_{n-1}$ and $0 \le p \le k-1$,

$$\begin{aligned}
\sum_{j=0}^{n-1} q^{root\text{-}maj((\sigma\downarrow^{(\epsilon^p n)} j))} &= \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^{\epsilon^p n} j))+\sum_{r=0}^{k-1}\sum_{i \in Sign_r(\sigma\downarrow^{(\epsilon^p n)} j)} r\cdot|(\sigma\downarrow^{(\epsilon^p n)} j)_i|} \\
&= q^{(pn+\sum_{r=0}^{k-1}\sum_{i\in Sign_r(\sigma))} r\cdot|\sigma_i|)} \sum_{j=0}^{n-1} q^{maj((\sigma\downarrow^{(\epsilon^p n)} j))} \\
&= q^{pn}[n]_q q^{maj(\sigma)+\sum_{r=0}^{k-1}\sum_{i\in Sign_r(\sigma)} r\cdot|\sigma_i|} \\
&= q^{np}[n]_q q^{root\text{-}maj(\sigma)}.
\end{aligned} \tag{61}$$

Hence it follows that for any $\sigma \in C_k \wr S_{n-1}$,

$$\sum_{j=0}^{n-1}\sum_{p=0}^{k-1} q^{root\text{-}maj((\sigma\downarrow^{\epsilon^p n} j))} = \sum_{p=0}^{k-1} q^{pn}[n]_q q^{root\text{-}maj(\sigma)} = [kn]_q q^{root\text{-}maj(\sigma)} \tag{62}$$

so that

$$\sum_{\tau \in C_k \wr S_n} q^{root\text{-}maj(\tau)} = [kn]_q \sum_{\sigma \in C_k \wr S_{n-1}} q^{nmaj(\sigma)}. \tag{63}$$

Thus again it is easy to prove by induction that

$$\sum_{\tau \in C_k \wr S_n} q^{root\text{-}maj(\tau)} = \prod_{i=1}^{n} [ki]_q. \tag{64}$$

As in the bijective proof of (32), it is not difficult to see that we can use our labeling lemmas to show that there is a bijection $\Theta_n : C_k \wr S_n \rightarrow C_k \wr S_n$ such that $flag\text{-}maj(\sigma) = root\text{-}maj(\Theta_n(\sigma))$. $\Box$.

We should note that it is not the case that $ndes$ and $fdes$ have the same distribution over $RLMin^+(B_n)$ for $n > 1$. That is, it is easy to check that the maximum value of $ndes(\sigma)$ for $\sigma \in RLMin^+(B_n)$ is $2n-3$ which is realized when $\sigma = -2\ -3 \ldots\ -(n-1)1$ while the maximum value of $f\text{-}des(\sigma)$ for $\sigma \in RLMin^+(B_n)$ is $2n-2$ which is realized when $\sigma = n\ (n-1)\ \ldots\ 2\ 1$. Thus even though $(ndes, nmaj)$ and $(f\text{-}des, f\text{-}maj)$ have the same distribution over B_n, it is certainly not the case that $(ndes, nmaj)$ and $(f\text{-}des, f\text{-}maj)$ have the same distribution over $RLMin^+(B_n)$ for $n > 1$.

Finally we end this section by observing that one can also construct a weight-preserving bijection between inversion tables in $RLMin^+(\mathbf{F}_n)$ and rook placements in $RP_n(BD_{2n})$. That is, it is easy to see that we can place the rook in the last column so that the number of uncanceled squares in the last column is anything between 0 and $2n-1$. Thus given any $f \in RLMin^+(\mathbf{F}_n)$, we place the rook in the last column so that there are exactly $f(n)$ uncanceled squares in the last column. Then we can simply proceed recursively since we are reduced to finding a weight preserving map from $RLMin^+(\mathbf{F}_{n-1})$ onto $RP_{n-1}(BD_{n-2})$. For example, it is easy to check that following this procedure for the inversion table given in Figure 4 results in the rook placement given in Figure 3.

5 Signed Words

In this section we consider statistics on signed words. Let $A = \{a_1 < a_2 < \ldots < a_k\}$ be a k-letter alphabet with a total ordering $<$. (Frequently, we let $A = \{1, 2, \ldots, k\}$ with the usual ordering of positive integers.) Let $v = v_1 v_2 \ldots v_N$ be a word where each v_i is a letter in A and we allow repeated letters.

A *signed word* is defined to be a pair $\alpha = (v, \epsilon)$, where $v = v_1 v_2 \ldots v_N$ is a list of N positive integers and $\epsilon = \epsilon_1 \epsilon_2 \ldots \epsilon_N$ is a list of N "signs" where each ϵ_i is $+1$ or -1. Sometimes, we write $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ where α_i is the "biletter" (v_i, ϵ_i), which we also identify with the integer $\epsilon_i \cdot v_i$. Given α, we define the following statistics.

1. The *signed major index* of α is

$$smaj(\alpha) = \left[\sum_{i=1}^{N-1} 2i \cdot \chi(\epsilon_i = \epsilon_{i+1} \text{ and } v_i > v_{i+1})\right] + \left[\sum_{i=1}^{N-1} i \cdot \chi(\epsilon_i \neq \epsilon_{i+1})\right] + N\chi(\epsilon_N = -1). \tag{65}$$

2. The *lexical major index* of α is

$$maj_{lex}(\alpha) = \sum_{i=1}^{N-1} i\chi\left(\epsilon_i > \epsilon_{i+1} \text{ or } (\epsilon_i = \epsilon_{i+1} \text{ and } v_i > v_{i+1})\right). \tag{66}$$

   This is just the ordinary major index of α relative to the following total ordering of the alphabet:

$$\cdots > m > \cdots > 2 > 1 > \cdots > -m > \cdots > -2 > -1. \tag{67}$$

3. The *negative count* of α is

$$neg(\alpha) = \sum_{i=1}^{N} \chi(\epsilon_i = -1),$$

   which is just the number of negative signs in α.

4. The *flag major index* of α is

$$flag\text{-}maj(\alpha) = 2maj_{lex}(\alpha) + neg(\alpha) \tag{68}$$

   It is easy to see that if $\alpha \in B_n$, then this definition reduces to the definition of flag major index given in section 2.

5. The *length* of α is

$$\ell(\alpha) = \left[\sum_{1 \le i < j \le N} \chi\left((v_i > v_j \text{ and } \epsilon_j = +1) \text{ or } (v_i < v_j \text{ and } \epsilon_j = -1)\right)\right] + \sum_{i=1}^{N} i\chi(\epsilon_i = -1). \tag{69}$$

The length statistic is one analogue of the inversion statistic for signed words. In contrast, we refer to N (the number of biletters in α) as the "size" of α.

In this case, the definition of $\ell(\sigma)$ given by (69) agrees with the definition of $\ell(\sigma)$ given in section 2 when we restrict ourselves to either S_n or B_n. That is, it is easy to see that if $\sigma \in S_n$, then by (69), $\ell(\sigma) = inv(\sigma)$. To see that the definition of $\ell(\sigma)$ given in section 2 agrees with the definition of $\ell(\sigma)$ given by (69) for elements of B_n, we can proceed by induction. That is, if $\sigma \in B_{n-1}$, then it is easy to see from our insertion lemmas in section 4 that for the definition of ℓ given in section 2, we have

$$\ell(\sigma \downarrow^n j) = n - j + \ell(\sigma) \tag{70}$$

and

$$\ell(\sigma \downarrow^{-n} j) = j + n + \ell(\sigma) \tag{71}$$

It is also easy to see that (70) and (71) hold for the definition of ℓ given by (69). That is, the insertion of n into the j-th space of σ causes ℓ to increase by 1 for each σ_k with $k > j$ for which σ_k is positive since it contributes to the sum $\left[\sum_{1\le i<j\le N} \chi\left((v_i > v_j \text{ and } \epsilon_j = +1) \text{ or } (v_i < v_j \text{ and } \epsilon_j = -1)\right)\right]$. However,the insertion of n into the j-th space of σ causes ℓ to increase by 1 for each σ_k with $k > j$ for which σ_k is negative since it contributes an extra 1 to the sum $\sum_{i=1}^N i\chi(\epsilon_i = -1)$. Thus (70) holds. Similarly it is easy to see that the insertion of $-n$ into the j-th space of σ causes the sum $\left[\sum_{1\le i<j\le N} \chi\left((v_i > v_j \text{ and } \epsilon_j = +1) \text{ or } (v_i < v_j \text{ and } \epsilon_j = -1)\right)\right]$ to increase by j plus the number of σ_k with $k > j$ and σ_k is positive and causes the sum $\sum_{i=1}^N i\chi(\epsilon_i = -1)$ to increase by j plus the number of σ_k with $k > j$ and σ_k is negative. Thus (71) holds.

6. The number of *cross-inversions* of α is

$$crinv(\alpha) = \sum_{i<j} \chi\left(\epsilon_i = -1 \text{ and } \epsilon_j = +1 \text{ and } v_i > v_j\right). \tag{72}$$

Proposition 5.1 *$smaj(\alpha) = flag\text{-}maj(\alpha)$ for every signed word α.*

Proof: Write $\alpha = \alpha_1 \cdots \alpha_n$, where $\alpha_i = (v_i, \epsilon_i)$. Proceed by induction on n. If $n = 0$, so that α is the empty word, we adopt the definition $smaj(\alpha) = 0 = flag\text{-}maj(\alpha)$. If $n = 1$, we have $smaj(\alpha) = \chi(\epsilon_1 = -1) = flag\text{-}maj(\alpha)$. Assume that $n > 1$, and that the result holds for all words β having size less than n. Let α have size n, and write $\alpha = \beta\alpha_n$, where $\beta = \alpha_1 \cdots \alpha_{n-1}$ has size $n-1$. Let P denote the logical proposition:

$$(\epsilon_{n-1} > \epsilon_n, \text{ or } (\epsilon_{n-1} = \epsilon_n \text{ and } v_{n-1} > v_n)).$$

We will show that

$$smaj(\alpha) = smaj(\beta) + \chi(\epsilon_n = -1) + 2(n-1)\chi(P); \tag{73}$$

$$flag\text{-}maj(\alpha) = flag\text{-}maj(\beta) + \chi(\epsilon_n = -1) + 2(n-1)\chi(P). \tag{74}$$

Since $smaj(\beta) = flag\text{-}maj(\beta)$ by induction, this will imply that $smaj(\alpha) = flag\text{-}maj(\alpha)$, completing the proof.

From the defining formula for $smaj$, we have

$$\begin{aligned} smaj(\alpha) - smaj(\beta) &= 2(n-1)\chi(\epsilon_{n-1} = \epsilon_n \text{ and } v_{n-1} > v_n) + (n-1)\chi(\epsilon_{n-1} \neq \epsilon_n) \\ &\quad + n\chi(\epsilon_n = -1) - (n-1)\chi(\epsilon_{n-1} = -1). \end{aligned}$$

To prove (73), we show that this expression always equals $\chi(\epsilon_n = -1) + 2(n-1)\chi(P)$, by considering the six possible cases that can occur.

- Case 1: $\epsilon_{n-1} = \epsilon_n = +1$ and $v_{n-1} > v_n$. Then P is true, and $smaj(\alpha) - smaj(\beta) = 2(n-1) = \chi(\epsilon_n = -1) + 2(n-1)\chi(P)$.
- Case 2: $\epsilon_{n-1} = \epsilon_n = -1$ and $v_{n-1} > v_n$. Then P is true, and $smaj(\alpha) - smaj(\beta) = 2(n-1) + 1 = \chi(\epsilon_n = -1) + 2(n-1)\chi(P)$.
- Case 3: $\epsilon_{n-1} = \epsilon_n = +1$ and $v_{n-1} \leq v_n$. Then P is false, and $smaj(\alpha) - smaj(\beta) = 0 = \chi(\epsilon_n = -1) + 2(n-1)\chi(P)$.
- Case 4: $\epsilon_{n-1} = \epsilon_n = -1$ and $v_{n-1} \leq v_n$. Then P is false, and $smaj(\alpha) - smaj(\beta) = 1 = \chi(\epsilon_n = -1) + 2(n-1)\chi(P)$.
- Case 5: $\epsilon_{n-1} = -1$ and $\epsilon_n = +1$. Then P is false, and $smaj(\alpha) - smaj(\beta) = (n-1) - (n-1) = 0 = \chi(\epsilon_n = -1) + 2(n-1)\chi(P)$.

- Case 6: $\epsilon_{n-1} = +1$ and $\epsilon_n = -1$. Then P is true, and $smaj(\alpha) - smaj(\beta) = (n-1) + n = 2n - 1 = \chi(\epsilon_n = -1) + 2(n-1)\chi(P)$.

Proving (74) is much easier. First, note that $majlex(\alpha) = majlex(\beta) + (n-1)\chi(P)$ by definition of the major index. Second, note that $neg(\alpha) = neg(\beta) + \chi(\epsilon_n = -1)$. Since $flag\text{-}maj = 2majlex + neg$, equation (74) follows immediately. $\square$

Theorem 5.2 *Let R denote the set of all rearrangements of the signed word $\beta = (-k)^{m_k} \cdots (-1)^{m_1} 1^{n_1} \cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k + m_1 + \cdots + m_k$. Then*

$$\sum_{\alpha \in R} q^{smaj(\alpha)} = \sum_{\alpha \in R} q^{flag\text{-}maj(\alpha)} = q^{m_1 + \cdots + m_k} \begin{bmatrix} N \\ n_1, \ldots, n_k, m_1, \ldots, m_k \end{bmatrix}_{q^2}.$$

Proof: The first equality is immediate from 5.1. By (1) we have

$$\sum_{\alpha \in R} u^{maj_{lex}(\alpha)} = \begin{bmatrix} N \\ n_1, \ldots, n_k, m_1, \ldots, m_k \end{bmatrix}_u$$

for any expression u. Observing that $neg(\alpha) = neg(\beta) = m_1 + \cdots + m_k$, we therefore have

$$\sum_{\alpha \in R} q^{flag\text{-}maj(\alpha)} = \sum_{\alpha \in R} q^{neg(\alpha)} (q^2)^{maj_{lex}(\alpha)} = q^{m_1 + \cdots + m_k} \begin{bmatrix} N \\ n_1, \ldots, n_k, m_1, \ldots, m_k \end{bmatrix}_{q^2}. \qquad \square$$

Proposition 5.3 *Suppose that $\alpha = \alpha_1 \alpha_2 \cdots \alpha_N$ is a rearrangement of the signed word $\beta = (-k)^{m_k} \cdots (-1)^{m_1} 1^{n_1} \cdots k^{n_k}$. Then*

$$\ell(\alpha) = \tbinom{neg(\beta)+1}{2} + crinv(\beta) + inv(\alpha), \tag{75}$$

where $inv(\alpha)$ is the usual (unsigned) inversion statistic computed using the standard total ordering of the integers.

Proof: For each rearrangement α of β, there is a sequence $\beta = \alpha_0, \alpha_1, \ldots, \alpha_n = \alpha$ such that for $i \geq 1$, α_i is obtained from α_{i-1} by interchanging two adjacent symbols. We let $n(\alpha)$ denote the smallest such n. We shall prove the lemma by induction on $n(\alpha)$.

If $n(\alpha) = 0$, then $\alpha = \beta$. Clearly, $\text{inv}(\beta) = 0$ since the symbols of β (viewed as integers) are in increasing order. We must therefore show that $\ell(\beta) = \binom{neg(\beta)+1}{2} + crinv(\beta)$. Since all the negative symbols in β occur at the beginning, the term $\sum_{i=1}^{N} i\chi(\epsilon_i = -1)$ in the definition of $\ell(\beta)$ evaluates to $1+2+\cdots+(m_1+\cdots+m_k) = \binom{neg(\beta)+1}{2}$. We claim that the other term in (69) evaluates to $crinv(\beta)$ for the word β. First, suppose that $i<j$ and $\epsilon_j = -1$. Then $\epsilon_i = -1$ also, since all negative symbols occur first in β. These negative symbols are ordered from largest to smallest (in absolute value) in the word β. Hence, $v_i < v_j$ does not hold when $\epsilon_j = -1$. Second, suppose that $i<j$ and $\epsilon_j = +1$. If $\epsilon_i = +1$, then $v_i > v_j$ cannot hold because of the ordering of the positive symbols in β. We conclude that, for the special word β, the bracketed sum appearing in (69) simplifies to

$$\sum_{i<j} \chi(\epsilon_i = -1 \text{ and } \epsilon_j = +1 \text{ and } v_i > v_j) = crinv(\beta).$$

Thus, formula (75) is true when $n=0$.

Next, assume that formula (75) holds for any word α that $n(\alpha) \le n$. Let $n(\gamma) = n+1$ and assume that γ is obtained from α by interchanging two adjacent symbols where $n(\alpha) = n$. To prove that the formula still holds for γ, we just compute the change in the left side and the change in the right side in all possible cases. On the right side, the terms $\binom{neg(\beta)+1}{2}$ and $crinv(\beta)$ are unaffected by the interchange. Thus, we need only compare $\ell(\gamma)-\ell(\alpha)$ to $inv(\gamma)-inv(\alpha)$ in all possible cases. Let the symbols being interchanged have absolute values a and b. There are twelve cases.

Assumption	Interchange	$\ell(\gamma)-\ell(\alpha)$	$inv(\gamma)-inv(\alpha)$
$a<b$	$+a,+b \to +b,+a$	$+1$	$+1$
$a<b$	$+a,-b \to -b,+a$	-1	-1
$a<b$	$-a,+b \to +b,-a$	$+1$	$+1$
$a<b$	$-a,-b \to -b,-a$	-1	-1
$a=b$	$+a,+b \to +b,+a$	0	0
$a=b$	$+a,-b \to -b,+a$	-1	-1
$a=b$	$-a,+b \to +b,-a$	$+1$	$+1$
$a=b$	$-a,-b \to -b,-a$	0	0
$a>b$	$+a,+b \to +b,+a$	-1	-1
$a>b$	$+a,-b \to -b,+a$	-1	-1
$a>b$	$-a,+b \to +b,-a$	$+1$	$+1$
$a>b$	$-a,-b \to -b,-a$	$+1$	$+1$

It is trivial to verify the entries for $inv(\gamma) - inv(\alpha)$, since the symbols being interchanged are adjacent. Let us verify the entry for $\ell(\gamma) - \ell(\alpha)$ in the second row. When $+a, -b$ is replaced by $-b, +a$, a negative sign moved one position to the left, decrementing the length by one. Observe that the two strings $+a, -b$ and $-b, +a$ both contribute 1 to the bracketed sum in (69): $+a, -b$ contributes because $a < b$ and the sign of b is negative, while $-b, +a$ contributes because $b > a$ and the sign of a is positive. Thus, the total change in the length is -1 as claimed. The other entries are verified similarly. Since the increments in the last two columns agree in all cases, the proof of (75) is complete. □

Theorem 5.4 *Let R denote the set of all rearrangements of the signed word $\beta = (-k)^{m_k} \cdots (-1)^{m_1} 1^{n_1} \cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k + m_1 + \cdots + m_k$. Then*

$$\sum_{\alpha \in R} q^{\ell(\alpha)} = q^{\binom{neg(\beta)+1}{2} + crinv(\beta)} \begin{bmatrix} N \\ n_1, \ldots, n_k, m_1, \ldots, m_k \end{bmatrix}_q .$$

Proof: This is immediate from 5.3 and the result quoted in the introduction for the ordinary inversion statistic on words relative to any total order. □

Note that Theorems 5.2 and 5.4 show that it is not the case that l and $flag\text{-}maj$ have the same distribution on the set of all rearrangements of the the signed word $\beta = (-k)^{m_k} \cdots (-1)^{m_1} 1^{n_1} \cdots k^{n_k}$ unless $m_1 + \cdots + m_k = \binom{neg(\beta)+1}{2} + crinv(\beta)$. However $m_1 + \cdots + m_k = neg(\beta)$ so that l and $flag\text{-}maj$ have the same distribution on the set of all rearrangements of the signed word $\beta = (-k)^{m_k} \cdots (-1)^{m_1} 1^{n_1} \cdots k^{n_k}$ only if

$$neg(\beta) = \binom{neg(\beta)+1}{2} + crinv(\beta). \tag{76}$$

However it is easy to see that (76) holds only if $neg(\beta) = 0$ or $neg(\beta) = 1$ and the negative element of β has absolute value which is less than or equal to all the positive letters occurring in β.

Proposition 5.5 *Let v_0 be a fixed rearrangement of the unsigned word $1^{n_1} \cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k$. Then*

$$\sum_{\epsilon \in \{\pm 1\}^N} q^{flag\text{-}maj(v_0,\epsilon)} = \sum_{\epsilon \in \{\pm 1\}^N} q^{smaj(v_0,\epsilon)} = q^{maj(v_0)} \prod_{i=1}^{N} (1+q^i) = q^{maj(v_0)} \prod_{i=1}^{N} [2]_{q^i},$$

where $maj(v_0)$ is the usual (unsigned) major index of v_0.

Proof: Let G be the multiplicative group $\{\pm 1\}^N$ and let S be the set of all signed words (v_0, ϵ) where $\epsilon \in G$. We will describe a procedure that uniquely constructs each object in S from a sequence of N binary choices $c_1, \ldots, c_N \in \{0, 1\}$ such that the word α constructed from these choices satisfies $smaj(\alpha) = maj(v_0) + \sum_{i=1}^{N} ic_i$. Then

$$q^{smaj(\alpha)} = q^{maj(v_0)} \prod_{i=1}^{N} (q^i)^{c_i}.$$

If we add this formula over all sequences $c_1, \ldots, c_N$ and use the distributive law, we obtain the formula in the proposition. [Specifically, choosing $c_i = 0$ corresponds to choosing the term 1 from the i'th factor $(1 + q^i)$. Choosing $c_i = 1$ corresponds to choosing the term q^i from this factor.]

We now describe the procedure for constructing the word $\alpha = (v, \epsilon)$ from the choices c_i. Since $v = v_0$ is fixed, we need only determine the sign vector ϵ. Define $g = (g_1, \ldots, g_N) \in G$ as follows. Set $g_N = +1$. For $i = N-1, \ldots, 2$, set $g_{i-1} = g_i$ if $(v_0)_{i-1} \leq (v_0)_i$, and set $g_{i-1} = -g_i$ if $(v_0)_{i-1} > (v_0)_i$. Next, for $1 \leq i \leq N$, define $h_i \in G$ to be a sequence of i (-1)'s followed by $N - i$ $(+1)$'s. Finally, given the choices $c_1, \ldots, c_N$, set

$$\epsilon = \epsilon(c_1, \ldots, c_N) = g \prod_{i=1}^{N} (h_i)^{c_i}, \tag{77}$$

where the product is taken in the group G. [In combinatorial terms, we start with the sign vector given by g. Then, for $1 \leq i \leq N$, we flip the first i signs in the current sign vector if $c_i = 1$, but we do nothing if $c_i = 0$.]

Every element $k \in G$ has a unique expression of the form (77). To prove this, switch from multiplicative notation to additive notation for $G = \{+1, -1\}^N$ via the isomorphism sending -1 to 1 and $+1$ to 0. Then we want to prove that every element $k \in \{0, 1\}^N$ has a unique expression of the form

$$k = g + \sum_{i=1}^{N} c_i h_i, \quad (c_i \in \{0, 1\}).$$

This says that $\{h_1, \ldots, h_N\}$ is a basis for G, viewed as an N-dimensional vector space over the field $\{0, 1\}$. But this is clear, since the N vectors $h_i = (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0)$ are obviously linearly independent.

We claim that

$$smaj(v_0, g) = maj(v_0) \text{ and } smaj(v_0, g\prod_{j=1}^{i-1}(h_j)^{c_j}h_i) = smaj(v_0, g\prod_{j=1}^{i-1}(h_j)^{c_j}) + i.$$

Assuming these claims are true, note that the second claim can be written

$$smaj(v_0, g\prod_{j=1}^{i}(h_j)^{c_j}) = smaj(v_0, g\prod_{j=1}^{i-1}(h_j)^{c_j}) + ic_i,$$

by considering the cases $c_i = 0$ and $c_i = 1$. Iterating this relation and using $smaj(v_0, g) = maj(v_0)$, we obtain

$$smaj(v_0, \epsilon(c_1, \ldots, c_N)) = maj(v_0) + \sum_{i=1}^{N} ic_i,$$

as desired.

To prove that $smaj(v_0, g) = maj(v_0)$, recall from (65) that

$$smaj(v_0, g) = \left[\sum_{i=1}^{N-1} 2i \cdot \chi(g_i = g_{i+1} \text{ and } (v_0)_i > (v_0)_{i+1})\right] + \left[\sum_{i=1}^{N-1} i \cdot \chi(g_i \neq g_{i+1})\right] + N\chi(g_N = -1). \tag{78}$$

By definition of g, we have $g_N = +1$ and for $i < N$, $g_i = g_{i+1}$ iff $(v_0)_i \leq (v_0)_{i+1}$. Thus, the first and third terms in the formula for $smaj$ contribute nothing to $smaj(v_0, g)$. On the other hand, the condition $\chi(g_i \neq g_{i+1})$ in the second term is true iff $(v_0)_i > (v_0)_{i+1}$ iff the unsigned word v_0 has a descent at position i. It follows that

$$smaj(v_0, g) = 0 + \sum_{i=1}^{N-1} i\chi((v_0)_i > (v_0)_{i+1}) + 0 = maj(v_0).$$

Finally, we must prove that $smaj(v_0, g\prod_{j=1}^{i-1}(h_j)^{c_j}h_i) = smaj(v_0, g\prod_{j=1}^{i-1}(h_j)^{c_j}) + i$ for all i. Let $s = g\prod_{j=1}^{i-1}(h_j)^{c_j} \in G$, and let $t = sh_i$. We must show that $smaj(v_0, t) -$

$smaj(v_0, s) = i$. We have

$$smaj(v_0, s) = \left[\sum_{k=1}^{N-1} 2k \cdot \chi(s_k = s_{k+1} \text{ and } (v_0)_k > (v_0)_{k+1})\right] + \left[\sum_{k=1}^{N-1} k \cdot \chi(s_k \neq s_{k+1})\right] + N\chi(s_N = -1); \tag{79}$$

$$smaj(v_0, t) = \left[\sum_{k=1}^{N-1} 2k \cdot \chi(t_k = t_{k+1} \text{ and } (v_0)_k > (v_0)_{k+1})\right] + \left[\sum_{k=1}^{N-1} k \cdot \chi(t_k \neq t_{k+1})\right] + N\chi(t_N = -1). \tag{80}$$

Note that multiplying g by h_j, where $j < i$, does not change the sign of any g_ℓ with $\ell \geq i$. In particular, $s_i = g_i$ and (when $i < N$) $s_{i+1} = g_{i+1}$.

We first consider the cases where $i < N$. By definition of h_i, we have ($s_k = s_{k+1}$ iff $t_k = t_{k+1}$) for all $k \neq i$ ($k < N$) and ($s_k \neq s_{k+1}$ iff $t_k \neq t_{k+1}$) for all $k \neq i$ ($k < N$). On the other hand, note that ($s_i = s_{i+1}$ iff $t_i \neq t_{i+1}$). Since $i < N$, we have $t_N = s_N = g_N = +1$. Using these facts in the formulas above, we find that

$$\begin{aligned} smaj(v_0, t) - smaj(v_0, s) &= 2i\chi(t_i = t_{i+1} \text{ and } (v_0)_i > (v_0)_{i+1}) + i\chi(t_i \neq t_{i+1}) \\ &\quad -2i\chi(s_i = s_{i+1} \text{ and } (v_0)_i > (v_0)_{i+1}) - i\chi(s_i \neq s_{i+1}). \end{aligned}$$

First, suppose $(v_0)_i > (v_0)_{i+1}$. Then $s_i = g_i \neq g_{i+1} = s_{i+1}$ by definition of g, and so $t_i = t_{i+1}$. Therefore

$$smaj(v_0, t) - smaj(v_0, s) = 2i - i = i,$$

Second, suppose $(v_0)_i \leq (v_0)_{i+1}$. Then $s_i = g_i = g_{i+1} = s_{i+1}$, and hence $t_i \neq t_{i+1}$. Therefore

$$smaj(v_0, t) - smaj(v_0, s) = i - 0 = i.$$

Finally consider the case where $i = N$. By definition of h_N, we have ($s_k = s_{k+1}$ iff $t_k = t_{k+1}$) for all $k < N$, and ($s_k \neq s_{k+1}$ iff $t_k \neq t_{k+1}$) for all $k < N$. However, $s_N = g_N = +1$ while $t_N = -1$. Using these facts in the formulas above, we get

$$smaj(v_0, t) - smaj(v_0, s) = N - 0 = N = i.$$

This completes the proof of 5.5. $\square$

Example: Let $v_0 = 2132212$. Then $g = (-1, +1, +1, -1, -1, +1, +1)$, and

$$smaj(v_0, g) = smaj(-2, 1, 3, -2, -2, 1, 2) = 9 = maj(v_0).$$

Suppose we use the choice sequence $(0, 0, 1, 1, 0, 1, 1)$, which corresponds to choosing the terms $1, 1, q^3, q^4, 1, q^6, q^7$ when expanding

$$\prod_{i=1}^{N}(1 + q^i) = (1 + q)(1 + q^2)(1 + q^3)(1 + q^4)(1 + q^5)(1 + q^6)(1 + q^7).$$

Here, we must multiply g by $h_3 = (-1, -1, -1, +1, +1, +1, +1)$, then by h_4, h_6, and h_7. We calculate

$$\begin{aligned}
gh_3 &= (+1, -1, -1, -1, -1, +1, +1);\\
smaj(v_0, gh_3) &= smaj(2, -1, -3, -2, -2, 1, 2) = 12 = 9 + 3;\\
gh_3h_4 &= (-1, +1, +1, +1, -1, +1, +1);\\
smaj(v_0, gh_3h_4) &= smaj(-2, 1, 3, 2, -2, 1, 2) = 16 = 9 + 3 + 4;\\
gh_3h_4h_6 &= (+1, -1, -1, -1, +1, -1, +1);\\
smaj(v_0, gh_3h_4h_6) &= smaj(2, -1, -3, -2, 2, -1, 2) = 22 = 9 + 3 + 4 + 6;\\
gh_3h_4h_6h_7 &= (-1, +1, +1, +1, -1, +1, -1);\\
smaj(v_0, gh_3h_4h_6h_7) &= smaj(-2, 1, 3, 2, -2, 1, -2) = 29 = 9 + 3 + 4 + 6 + 7.
\end{aligned}$$

Thus, for this choice sequence, $\epsilon = (-1, +1, +1, +1, -1, +1, -1)$.

Theorem 5.6 *Let R be the set of rearrangements of the unsigned word $1^{n_1} \cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k$. Then*

$$\sum_{v \in R, \epsilon \in \{\pm 1\}^N} q^{flag\text{-}maj(v,\epsilon)} = \sum_{v \in R, \epsilon \in \{\pm 1\}^N} q^{smaj(v,\epsilon)} = \begin{bmatrix} N \\ n_1, \ldots, n_k \end{bmatrix}_q \prod_{i=1}^{N} [2]_{q^i}.$$

Proof: Add up the formulas in 5.5 over all choices of v_0, and use MacMahon's result (1) to obtain the multinomial coefficient $\begin{bmatrix} N \\ n_1, \ldots, n_k \end{bmatrix}_q$. $\square$

Proposition 5.7 *Let R be the set of rearrangements of the unsigned word $1^{n_1}\cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k$. Let $\epsilon_0 \in \{\pm 1\}^N$ be a fixed choice of N signs. Then*

$$\sum_{v\in R} q^{\ell(v,\epsilon_0)} = q^{\sum_{i=1}^N i\chi((\epsilon_0)_i=-1)} \begin{bmatrix} N \\ n_1,\ldots,n_k \end{bmatrix}_q.$$

Proof: We will define a bijection $f : R \to R$ such that

$$\ell(f(w),\epsilon_0) = inv(w) + \sum_{i=1}^N i\chi((\epsilon_0)_i = -1).$$

The desired formula will follow, since

$$\sum_{w\in R} q^{\ell(f(w),\epsilon_0)} = \sum_{v\in R} q^{inv(w)+\sum_{i=1}^N i\chi((\epsilon_0)_i=-1)} \text{ and } \sum_{w\in R} q^{inv(w)} = \begin{bmatrix} N \\ n_1,\ldots,n_k \end{bmatrix}_q.$$

We define f as follows. Fix $w = w_1 \ldots w_N \in R$. To obtain the word $f(w)$, write down N blanks underneath the symbols of the word ϵ_0. Put the successive letters of w underneath the -1's in ϵ_0 from right to left, and then put the remaining letters of w underneath the 1's in ϵ_0 from left to right. For example, if $w = 15241523536$ and

$$\begin{array}{rcl} \epsilon_0 &=& (+1,+1,-1,-1,-1,+1,-1,+1,-1,-1,+1), \text{ then} \\ f(w) &=& \;\;2 \quad\; 3 \quad\; 5 \quad\; 1 \quad 4 \quad\; 5 \quad\; 2 \quad 3 \quad\; 5 \quad\; 1 \quad 6. \end{array}$$

Fix $w \in R$, and let $f(w) = v_1 v_2 \ldots v_N$. We must prove that

$$\ell(f(w),\epsilon_0) = inv(w) + \sum_{i=1}^N i\chi((\epsilon_0)_i = -1).$$

From the definition of length (see (69)), this is equivalent to

$$inv(w) = \sum_{1\le m<n\le N} \chi\left((v_m > v_n \text{ and } \epsilon_n = +1) \text{ or } (v_m < v_n \text{ and } \epsilon_n = -1)\right). \tag{81}$$

To prove this, suppose ϵ_0 has k minus signs and $N-k$ plus signs. Note that $inv(w)$ is the sum of the sizes of the three sets

$$S_1 = \{(i,j) : i < j \le k \text{ and } w_i > w_j\};$$

$$S_2 = \{(i,j) : k < i < j \text{ and } w_i > w_j\};$$

$$S_3 = \{(i,j) : i \le k < j \text{ and } w_i > w_j\}.$$

Similarly, the right side of (81) is the sum of the sizes of the three sets

$$T_1 = \{(i',j') : i' > j', (\epsilon_0)_{i'} = (\epsilon_0)_{j'} = -1, \text{ and } v_{j'} < v_{i'}\};$$

$$T_2 = \{(i',j') : i' < j', (\epsilon_0)_{i'} = (\epsilon_0)_{j'} = +1, \text{ and } v_{i'} > v_{j'}\};$$

$$T_3 = \{(i',j') : (\epsilon_0)_{i'} = -1, \ (\epsilon_0)_{j'} = +1, \text{ and } v_{i'} > v_{j'}\}.$$

Note that T_3 allows the possibility that $i' < j'$ or $i' > j'$.

Define a permutation $g : \{1,2,\ldots,N\} \to \{1,2,\ldots,N\}$ by letting $g(i)$ be the position of the letter w_i in $f(w)$. In the example above, we have

$$(g(1),\ldots,g(N)) = (10,9,7,5,4,3,1,2,6,8,11).$$

Observe that $v_{g(k)} = w_k$ for all k. We claim that the correspondence $(i,j) \mapsto (g(i),g(j))$ gives a bijection of S_1 onto T_1, S_2 onto T_2, and S_3 onto T_3. The proof is simple. Given that $i < j \le k$, we have $(\epsilon_0)_{g(i)} = (\epsilon_0)_{g(j)} = -1$ and $g(i) > g(j)$ since the first k symbols of w are placed underneath the minus signs from right to left. We have $w_i > w_j$ iff $v_{g(j)} < v_{g(i)}$, since $v_{g(k)} = w_k$ for all k. Furthermore, all pairs of indices $i' > j'$ with $(\epsilon_0)_{i'} = (\epsilon_0)_{j'} = -1$ arise from pairs of indices (i,j) with $i < j \le k$ via the map g. This proves that $|S_1| = |T_1|$. Similarly, given that $k < i < j$, we have $(\epsilon_0)_{g(i)} = (\epsilon_0)_{g(j)} = +1$ and $g(i) < g(j)$ since the last $N-k$ symbols of w are placed underneath the plus signs from left to right. We have $w_i > w_j$ iff $v_{g(i)} > v_{g(j)}$, proving that $|S_2| = |T_2|$. Similarly, $|S_3| = |T_3|$, since $i \le k < j$ iff $((\epsilon_0)_{g(i)} = -1$ and $(\epsilon_0)_{g(j)} = +1)$, whereas $w_i > w_j$ iff $v_{g(i)} > v_{g(j)}$. This completes the proof of 5.7. □

Theorem 5.8 *Let R be the set of rearrangements of the unsigned word $1^{n_1}\cdots k^{n_k}$, and* *let $N = n_1 + \cdots + n_k$. Then*

$$\sum_{\epsilon \in \{\pm 1\}^n, v \in R} q^{\ell(v,\epsilon)} = \begin{bmatrix} N \\ n_1, \ldots, n_k \end{bmatrix}_q \prod_{i=1}^{N} [2]_{q^i}.$$

Proof: Add up the formulas in 5.7 over all choices of ϵ_0. There is a common factor of $\left[\begin{smallmatrix} N \\ n_1,\ldots,n_k \end{smallmatrix}\right]_q$, which is multiplied by

$$\sum_{\epsilon_0} \prod_{i=1}^{N} (q^i)^{\chi((\epsilon_0)_i=-1)} = \left(\prod_{i=1}^{N} \sum_{(\epsilon_0)_i \in \{-1,1\}} (q^i)^{\chi((\epsilon_0)_i=-1)} \right) = \prod_{i=1}^{N} (1+q^i) = \prod_{i=1}^{N} [2]_{q^i}. \qquad \square$$

Proposition 5.9 *Length, signed major index, and flag major index have the same distribution on the set of signed words (v,ϵ) where ϵ is arbitrary and v is a rearrangement of a fixed word $1^{n_1} \cdots k^{n_k}$. There is an explicit bijection sending the length statistic to the flag major index.*

Proof Sketch: The first statement follows by combining 5.6 and 5.8. The second statement follows by looking at the proofs of 5.5, 5.6, 5.7 and 5.8. Every equality appearing has a bijective proof (for Foata's bijection sending unsigned maj to unsigned inv, see [10]), so we can combine all these bijections to get a map on the given set of words sending length to flag major index, or vice versa. $\square$

Example: Let $\alpha = (-2, 1, 3, 2, -2, 1, -2)$, which has $smaj(\alpha) = 29$. Using the example after 5.5, we see that α was constructed from the unsigned word $v_0 = 2132212$ and the choice sequence $(0, 0, 1, 1, 0, 1, 1)$. Here, $maj(v_0) = 9$. Using Foata's bijection on v_0 produces $w_0 = 2231212$, which has $inv(w_0) = 9 = maj(v_0)$. Next, regard the choice sequence as a sequence of signs $\epsilon_0 = (+1, +1, -1, -1, +1, -1, -1)$. Place the letters of w_0 underneath these signs as described in 5.7 to obtain $\beta = (2, 1, -1, -3, 2, -2, -2)$. We have $\ell(\beta) = 29 = smaj(\alpha)$.

Major Index for Words with Higher Roots of Unity. Fix an integer $m \geq 2$. We now consider words $\alpha = \alpha_1 \ldots \alpha_N$ such that $\alpha_j = (v_j, r_j)$, where v_j is a positive integer and $r_j \in \{0, 1, \ldots, m-1\}$. Sometimes we identify r_j with the m'th root of unity $e^{2\pi r_j i/m}$, and we may write $\alpha_j = e^{2\pi r_j i/m} v_j$.

We can consider various major index statistics on these new words.

1. Define a total ordering $>_{lex}$ on the alphabet of biletters (v, r) by setting $(v_1, r_1) >_{lex} (v_2, r_2)$ iff $(r_1 < r_2$, or $(r_1 = r_2$ and $v_1 > v_2))$. Then define the *lexical major index* by

$$maj_{lex}(\alpha) = \sum_{i=1}^{N-1} i\chi(\alpha_i >_{lex} \alpha_{i+1}),$$

which is the usual major index relative to the total order $>_{lex}$.

2. Define the *log sum* of α by

$$logsum(\alpha) = \sum_{i=1}^{N} r_i.$$

3. Define the *flag major index* of α by

$$flag\text{-}maj(\alpha) = maj_{lex}(\alpha)m + logsum(\alpha).$$

4. Define the *special major index* of α by

$$\begin{aligned} smaj(\alpha) \quad = \quad & \sum_{i=1}^{N-1} mi \cdot \chi(r_i = r_{i+1} \text{ and } v_i > v_{i+1}) \\ & + \sum_{k=1}^{m-1} \sum_{i=1}^{N-1} ki \cdot \chi(r_i - r_{i+1} = k) \\ & + \sum_{k=1}^{m-1} \sum_{i=1}^{N-1} (m-k)i \cdot \chi(r_i - r_{i+1} = -k) \\ & + Nr_N \end{aligned}$$

(This clearly reduces to the previous definition of *smaj* when $m = 2$.) In the last formula, we regard all numbers appearing as integers. If, instead, we view the r_i's as elements of a cyclic additive group $C_m = \{0, 1, \ldots, m-1\}$, we can rewrite the formula as

$$\begin{aligned} smaj(\alpha) &= \sum_{i=1}^{N-1} mi \cdot \chi(r_i = r_{i+1} \text{ and } v_i > v_{i+1}) \\ &+ \sum_{k=1}^{m-1} \sum_{i=1}^{N-1} \mathrm{int}(k) i \cdot \chi(r_i - r_{i+1} = k) \\ &+ N r_N. \end{aligned}$$

In this formula, the subtraction $r_i - r_{i+1}$ is performed in C_m, and $\mathrm{int}(k)$ denotes the unique integer in the range $\{0, 1, \ldots, m-1\}$ that represents the group element k. This version of the formula makes it clear that the value of *smaj* depends only on the letters v_i, the last letter r_N, and the *differences* of consecutive letters $r_i - r_{i+1}$ in the group C_m. This fact will be used in the proof of 5.12.

Proposition 5.10 *For every signed word α, $flag\text{-}maj(\alpha) = smaj(\alpha)$.*

Proof Sketch: The proof is like that of 5.1. By induction, it is enough to show that

$$smaj(\alpha) = smaj(\beta) + r_N + m(N-1)\chi(\alpha_{N-1} >_{lex} \alpha_N) \tag{82}$$

$$flag\text{-}maj(\alpha) = flag\text{-}maj(\beta) + r_N + m(N-1)\chi(\alpha_{N-1} >_{lex} \alpha_N) \tag{83}$$

where $\beta = (\alpha_1, \ldots, \alpha_{N-1})$. The second recursion is obvious from the definition of $flag\text{-}maj$. The first recursion is proved by calculating $smaj(\alpha) - smaj(\beta)$ in various cases.

- Case 1: $r_{N-1} = r_N$ and $v_{N-1} \leq v_N$. When adding α_N to β to get α, we lose $(N-1)r_{N-1}$ since r_{N-1} is no longer last, but we gain Nr_N since r_N is now last. There is no other change to the *smaj* statistic. Thus the net gain is r_N, since $r_{N-1} = r_N$, and this change matches the formula (82).

- Case 2: $r_{N-1} = r_N$ and $v_{N-1} > v_N$. When going from β to α, we gain r_N as in Case 1, and we also gain $m(N-1)$ since $v_{N-1} > v_N$. This change in *smaj* also matches (82).

- Case 3: $r_{N-1} - r_N = k > 0$ (so that $\alpha_{N-1} \not>_{lex} \alpha_N$). Going from β to α, we lose $(N-1)r_{N-1}$ and gain Nr_N due to the new last letter. We also gain $k(N-1)$, for a net change of

$$Nr_N - (N-1)r_{N-1} + (N-1)(r_{N-1} - r_N) = r_N$$

in the *smaj* statistic, which matches (82).

- Case 4: $r_{N-1} - r_N = -k < 0$ (so that $\alpha_{N-1} >_{lex} \alpha_N$). Going from β to α, we lose $(N-1)r_{N-1}$ and gain Nr_N due to the new last letter. We also gain $(m-k)(N-1)$, for a net change of

$$Nr_N - (N-1)r_{N-1} + (m + r_{N-1} - r_N)(N-1) = r_N + m(N-1),$$

in the *smaj* statistic, which matches (82). □

Theorem 5.11 *Let $n_{i,j} \geq 0$ be given integers, for $1 \leq i \leq k$ and $0 \leq j \leq m-1$. Let R denote the set of words α that can be formed by rearranging $n_{i,j}$ copies of the biletter (i,j), for all i and j. Let $N = \sum_{i,j} n_{i,j}$. Then*

$$\sum_{\alpha \in R} q^{smaj(\alpha)} = \sum_{\alpha \in R} q^{flag\text{-}maj(\alpha)} = \begin{bmatrix} N \\ \ldots, n_{i,j}, \ldots \end{bmatrix}_{q^m} \cdot q^{\sum_{j=0}^{m-1} j \sum_{i=1}^{k} n_{i,j}}.$$

Proof: This is immediate from 5.10 and the definition of $flag$-maj, together with MacMahon's result (1) for the distribution of major index on a totally ordered alphabet. (Compare to 5.2.) □

Proposition 5.12 *Let v_0 be a fixed rearrangement of the unsigned word $1^{n_1} \cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k$. Then*

$$\sum_{r \in \{0,1,\ldots,m-1\}^N} q^{smaj(v_0,r)} = q^{maj(v_0)} \prod_{i=1}^{N} [m]_{q^i},$$

where $maj(v_0)$ is the usual (unsigned) major index of v_0.

Proof Sketch: The proof is like that of 5.5. Regard $C_m = \{0, 1, \ldots, m-1\}$ as the cyclic additive group of order m, and put $G = C_m^N$. We define a bijection $p : G \to G$ such that

$$smaj(v_0, p(c_1, \ldots, c_N)) = maj(v_0) + \sum_{i=1}^{N} ic_i \text{ for } (c_1, \ldots, c_N) \in G.$$

The stated formula will then follow from the distributive law, just as in 5.5. [Combinatorially, if $c_i = j_0$, then we choose the summand $(q^i)^{j_0}$ from the i'th factor $[m]_{q^i} = \sum_{j=0}^{m-1} (q^i)^j$.]

To define p, we first define special elements $g, h_1, \ldots, h_N \in G$. Set $g_N = 0$. For $i = N-1, \ldots, 1$, set $g_i = g_{i+1}$ if $v_i \le v_{i+1}$; set $g_i = g_{i+1} + 1 \in C_m$ (addition mod m) if $v_i > v_{i+1}$. As in 5.5, this definition of $g \in G$ implies that $smaj(v_0, g) = maj(v_0)$, since consecutive entries of g agree except at descents in v_0, where the entries of g differ by 1. Next, let h_i be the element of G consisting of i ones followed by $N-i$ zeroes. For $c \in C_m$, let ch_i be the element of G consisting of i elements c followed by $N-i$ zeroes. (This is the usual action of C_m on the C_m-module G.) Suppose we have two elements $r, s \in G$ such that $s = r + ch_i$ for some $c \in C_m$. If $i < N$, then $s_k - s_{k+1} = r_k - r_{k+1}$ for $k \ne i$, while $s_i - s_{i+1} = (r_i - r_{i+1}) + c$. Also, $s_N = r_N$. On the other hand, for $i = N$, we have $s_k - s_{k+1} = r_k - r_{k+1}$ for all $k < N$, while $s_N = r_N + c$.

Now, define the map $p : G \to G$ by

$$p(c_1, \ldots, c_N) = g + \sum_{i=1}^{N} c_i h_i.$$

This p is a bijection, since $(h_1, \ldots, h_N)$ is a basis for the N-dimensional free C_m-module G. To complete the proof, let $r = g + \sum_{j=1}^{i-1} c_j h_j$, and let $s = r + c_i h_i$. It is enough to check that $smaj(v_0, s) = smaj(v_0, r) + c_i i$. This equation follows from the observations at the end of the last paragraph. Specifically, if v_0 has no descent at position $i < N$, then $r_i = r_{i+1}$ (since $g_i = g_{i+1}$). Adding $c_i h_i$ will cause an increase in $smaj$ of precisely $c_i i$, by definition of $smaj$. If v_0 does have a descent at position $i < N$, then $r_i = r_{i+1} + 1$ in C_m by definition of g. Using the definition of $smaj$, it is easy to check that adding $c_i h_i$ causes an increase of $c_i i$ (consider the cases $c_i < m-1$ and $c_i = m-1$ separately). Finally, if $i = N$, it is clear that adding $c_i h_i$ increases $smaj$ by $c_i N$. $\square$

Example: Suppose $m = 4$, $v_0 = 21132122432$, and $(c_1, \ldots, c_N) = 01000303032$. (We write elements of G as words of length N for brevity.)

We calculate $g = 10003222210$ from the descents of v_0.
We compute $p(c_1, \ldots, c_N) = g + \sum_{i=1}^{N} c_i h_i$ in several stages, as follows:

$$\begin{array}{rcl} r_1 = g + 1h_2 &=& 21003222210, \\ r_2 = r_1 + 3h_6 &=& 10332122210, \\ r_3 = r_2 + 3h_8 &=& 03221011210, \\ r_4 = r_3 + 3h_{10} &=& 32110300100, \\ r_5 = r_3 + 2h_{11} &=& 10332122322. \end{array}$$

Here, $p(c_1, \ldots, c_N) = r_5 \in G$. We have:

$$\begin{array}{rcl} smaj(v_0, g) = 29 &=& maj(v_0) \\ smaj(v_0, r_1) = 31 &=& maj(v_0) + 1 \cdot 2 \\ smaj(v_0, r_2) = 49 &=& maj(v_0) + 1 \cdot 2 + 3 \cdot 6 \\ smaj(v_0, r_3) = 73 &=& maj(v_0) + 1 \cdot 2 + 3 \cdot 6 + 3 \cdot 8 \\ smaj(v_0, r_4) = 103 &=& maj(v_0) + 1 \cdot 2 + 3 \cdot 6 + 3 \cdot 8 + 3 \cdot 10 \\ smaj(v_0, r_5) = 125 &=& maj(v_0) + 1 \cdot 2 + 3 \cdot 6 + 3 \cdot 8 + 3 \cdot 10 + 2 \cdot 11. \end{array}$$

Hence, $smaj(v_0, p(c_1, \ldots, c_N)) = maj(v_0) + \sum_{i=1}^{N} ic_i$, as required.

Theorem 5.13 *Let R be the set of rearrangements of the unsigned word $1^{n_1} \cdots k^{n_k}$, and let $N = n_1 + \cdots + n_k$. Then*

$$\sum_{v \in R, r \in \{0,1,\ldots,m-1\}^N} q^{flag\text{-}maj(v,r)} = \sum_{v \in R, r \in \{0,1,\ldots,m-1\}^N} q^{smaj(v,r)} = \begin{bmatrix} N \\ n_1, \ldots, n_k \end{bmatrix}_q \prod_{i=1}^{N} [m]_{q^i}.$$

Proof: Add up the formulas in 5.12 over all choices of v_0, and invoke MacMahon's result (1) to obtain the multinomial coefficient $\begin{bmatrix} N \\ n_1, \ldots, n_k \end{bmatrix}_q$. $\square$

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