Radon-Brascamp-Lieb Inequalities and Model Operators

Fourier Analysis @ 200, ICMS

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1. Geometric Averaging Operators

The Setup

Consider a map $x \to {}^x\Sigma$ from points in $\mathbb{R}^n$ to submanifolds ${}^x\Sigma$ of $\mathbb{R}^{n'}$ and construct an operator which averages functions over the submanifolds:

$$Tf(x) := \int_{{}^x\Sigma} f(y) w(x,y) d\sigma(y).$$

More precisely, we say (Ω, π, Σ) is a smooth incidence relation on $\mathbb{R}^n \times \mathbb{R}^{n'}$ of codimension k when

- **DOMAIN:** $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{n'}$ is open
- **DEFINING FUNCTION:** $\pi : \Omega \to \mathbb{R}^k$ is smooth

- **JACOBIAN:** $||d_x\pi(x,y)||_\omega$ for any n-tuple $\omega := \{\omega_i\}_{i=1}^n$ of vectors in $\mathbb{R}^n$ is given by

$$\left[\frac{1}{k!}\sum_{i_1,\ldots,i_k=1}^{n}\left|\det\left[(\omega_{i_1}\cdot\nabla_x)\pi \quad \cdots \quad (\omega_{i_k}\cdot\nabla_x)\pi\right]\right|^2\right]^{\frac{1}{2}}$$

  When ω is omitted, use default coordinates.
- **INCIDENCE RELATION:**

$$\Sigma := \left\{(x,y) \;\middle|\; \pi(x,y)=0, ||d_x\pi(x,y)||, ||d_y\pi(x,y)|| > 0\right\}.$$

- **SLICES:** ${}^x\Sigma$ and Σ^y are slices with fixed x and y, resp.
- **NATURAL MEASURE:** On each ${}^x\Sigma$ and Σ^y, σ denotes what will be called the coarea measure (or the Leray or microcanonical measure).

- Similar operators appear in many contexts and are beyond Calderón-Zygmund theory.
- There are (at least) two main types of estimates: L^p-improving estimates and L^p-Sobolev.
  - The relationship between the two types is quite complicated.
  - L^p-improving properties are implied by Fourier restriction.
- A long-term goal is to identify structural properties that allow one to read off boundedness properties. Tao and Wright (2003) embodies this idea.

The Big Problem: There's essentially no idea what the right type of **quantitative** nondegeneracy criterion is.

Agenda for Today

- A new, non-local testing condition for a family of Radon-Brascamp-Lieb inequalities.
- Exploration of the implications for “model operators” whose properties are governed by the order 2 Taylor jets of submanifolds.
- Initial steps towards understanding the new sort of uniform sublevel set inequalities that arise; development of local criteria.
- ~~Extensive details of proofs.~~
- ~~Passage from algebraic to smooth.~~
- ~~In-depth study of uniform sublevel set inequalities.~~

2. Testing Conditions

Theorem: Testing Conditions

- **ENSEMBLE OF RADON-LIKE OPERATORS:** For each $j = 1, \ldots, m$, let T_j equal

$$T_j f(x) := \int_{{}^x\Sigma_j} f_j(y_j) w_j(x, y_j) d\sigma_j(y_j)$$

for all nonnegative Borel-measurable f_j on $\mathbb{R}^{n_j}$ associated to an **algebraic** $\pi_j : \mathbb{R}^n \times \mathbb{R}^{n_j} \to \mathbb{R}^{k_j}$.

- **CRITICAL SCALING LINE:** Let $p_1, \ldots, p_m \in [1, \infty)$ and $q_1, \ldots, q_m \in (0, \infty)$ satisfy

$$n = \sum_{j=1}^{m} \frac{k_j q_j}{p_j}.$$

Then

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} |T_j f_j(x)|^{q_j} dx \leq C \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\mathbb{R}^{n_j})}^{q_j} \quad \forall f_1, \ldots, f_m$$

if and only if

$$\prod_{j\,:\,p_j=1} \sup_{y_j \in {}^x\Sigma_j} \frac{|w_j(x, y_j)|^{q_j}}{||d_x \pi_j(x, y_j)||_{\omega}^{q_j}} \prod_{j\,:\,p_j>1} \left[\int_{{}^x\Sigma_j} \frac{|w_j(x, y_j)|^{p'_j} d\sigma_j(y_j)}{||d_x \pi_j(x, y_j)||_{\omega}^{p'_j - 1}} \right]^{\frac{q_j}{p'_j}}$$

is uniformly bounded over all x and all $\{\omega_i\}_{i=1}^{n}$ of determinant 1. Here p_j and p'_j are Hölder dual exponents.

Lemma (Visibility Lemma, Continuous Version)

For any Borel measurable, nonnegative integrable function ψ on the box $B_R := [-R, R)^n$, there exist Borel measurable $\mathbb{R}^n$-valued functions $\omega_1^x, \ldots, \omega_n^x$ on B_R such that $|\det\{\omega_i^x\}_{i=1}^n| = 1$ at all points and a nonnegative Borel-measurable function $\widetilde{\psi}$ on B_R equal to ψ a.e. such that every polynomial map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ with $1 \leq k \leq n$ satisfies

$$\int_{\Sigma_\pi \cap B_R} \left[\widetilde{\psi}(x)\right]^{\frac{n-k}{n}} ||d\pi(x)||_{\omega^x} d\sigma(x)$$
$$\leq C_n(\deg \pi) \left[\int_{B_R} \psi(x) dx\right]^{\frac{n-k}{n}}.$$

What does this lemma mean?

Let $f_1(x), \ldots, f_n(x)$ be (arbitrary) coordinate functions that map B_R to some box B' and let $\psi(x) := \left|\det \frac{\partial f}{\partial x}\right|$.

- Regard $\psi^{-1/n}(x)\nabla f_1(x), \ldots, \psi^{-1/n}(x)\nabla f_n(x)$ as "unit-size" covectors with respect to some norm. The normalization makes the basis unit volume.
- Now take $\omega_x^1, \ldots, \omega_x^n$ to be the dual basis of vectors.
- The change of variables formula implies that

$$\int_{\Sigma_\pi \cap B_R} [\psi(x)]^{\frac{n-k}{n}} ||d\pi(x)||_{\omega^x} d\sigma(x) \leq C_n K \left[\int_{B_R} \psi(x) dx\right]^{\frac{n-k}{n}}$$

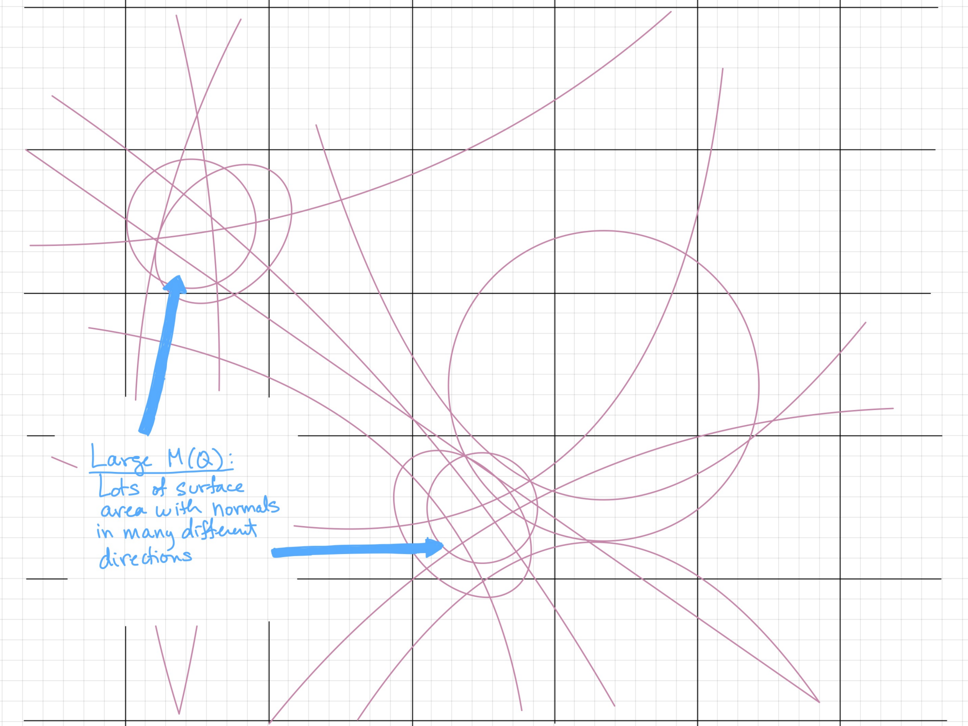
where $K =$ max no. of transverse intersections of a k-dim'l affine coordinate subspace and Σ_π.

Visibility Lemma   Guth (2010), Carbery-Valdimarsson (2013)

For any nonnegative integer-valued function $M(Q)$ defined on the lattice of unit cubes $\Lambda_1 \subset \mathbb{R}^n$, $\exists$ an algebraic hypersurface Z of degree at most $C_n(\sum_Q M(Q))^{1/n}$ such that $\overline{\text{Vis}}[Z \cap Q] \geq M(Q)$ for all $Q \in \Lambda_1$, where $\overline{\text{Vis}}[Z \cap Q]$ is the mollified visibility, i.e., the reciprocal of the Euclidean volume of the convex set of vectors u for which $||u|| \leq 1$ and

$$\frac{1}{|B(Z,\epsilon)|} \int_{B(Z,\epsilon)} \int_{Z' \cap Q} |u \cdot \widehat{n}(z')| d\mathcal{H}^{n-1}(z') dZ' \leq 1.$$

Here $\widehat{n}(z)$ is the unit normal to Z' at the point z'.


Large M(Q):
Lots of surface area with normals in many different directions

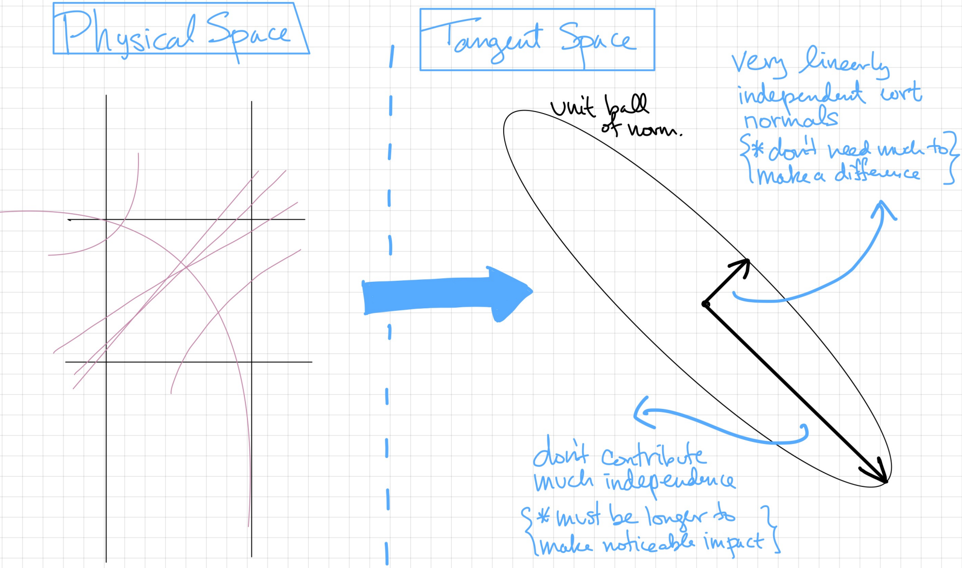
Point

Combining with a change of variables formula of Zhang (2018) shows how to measure the size of $d\pi$ in some **unnormalized**, pointwise-varying system $\{\omega_1^x, \ldots, \omega_n^x\}$ with volume like $\overline{\text{Vis}}[Z \cap Q]$ on Q. This system doesn't depend on π and

$$\int_{\Sigma_\pi} ||d\pi(x)||_{\omega^x} d\sigma(x) \lesssim (\deg \pi) \left(\sum_Q M(Q) \right)^{\frac{n-k}{n}}$$

Normalizing the ω_i^x above replaces $||d\pi(x)||_{\omega^x}$ here by $(\overline{\text{Vis}}[Z \cap Q])^{\frac{n-k}{n}} ||d\pi(x)||_{\omega^x}$.

We now do a typical combo of rescaling and approx. to get the continuous Visibility Lemma.



In the end, the norm matters in the proof because of its quantitative properties, but the variety generating it does not.

There is a coarea/Fubini-type identity:

$$\int_{B_R} [\widetilde{\psi}(x)]^{\frac{n-k}{n}} \int_{{}^x\Sigma} |f(y)|^p ||d_x\pi(x,y)||_{\omega^x} d\sigma(y)\, dx = \int_{\mathbb{R}^{n'}} |f(y)|^p \int_{\Sigma^y \cap B_R} [\widetilde{\psi}(x)]^{\frac{n-k}{n}} ||d_x\pi(x,y)||_{\omega^x} d\sigma(x)\, dy$$

- Bound RHS with continuous Visibility Lemma.
- Estimate LHS from below via Hölder:

$$\int_{{}^x\Sigma} f(y) w(x,y) d\sigma(y) \leq \left[\int_{{}^x\Sigma} |f(y)|^p ||d_x\pi(x,y)||_\omega d\sigma(y) \right]^{\frac{1}{p}} \cdot \left[\int_{{}^x\Sigma} \frac{|w(x,y)|^{p'} d\sigma(y)}{||d_x\pi(x,y)||_\omega^{p'-1}} \right]^{\frac{1}{p'}}.$$

Argument structure is reminiscent of Christ (1998).

3. Model Operators

The New Geometric Integral Game

1. You have some decomposable k-form $\mu(t) := \mu_1(t) \wedge \cdots \wedge \mu_k(t)$ in $\Lambda^k(\mathbb{R}^n)$, $t \in U \subset \mathbb{R}^d$.
2. You choose some basis $\{\omega_i\}_{i=1}^n$ of $\mathbb{R}^n$ which is volume-normalized; let $\{\nu_i\}_{i=1}^n$ be its dual basis.
3. You define $\mu_{i_1 \cdots i_k}(t)$ to be the coefficient of $\nu_{i_1} \wedge \cdots \wedge \nu_{i_k}$ when $\mu(t)$ is expressed in this basis $(i_1 < i_2 < \cdots < i_k)$.
4. You let $||\mu(t)||_\omega := \left(\sum_{i_1,\ldots,i_k} |\mu_{i_1 \cdots i_k}(t)|^2\right)^{1/2}$.
5. You want a uniform (in ω) estimate for the integral

$$\int_U \frac{dt}{||\mu(t)||_\omega^\tau}.$$

Moves of the Game

- For restricted strong type, you only need uniform sublevel set estimates.
- If $\omega' = M\omega$, then $||\mu(t)||_{\omega'} \leq C_{k,n}(\max_{ij} |M_{ij}|^k)||\mu(t)||_{\omega}$. Thus you can replace ω by ω' as long as M has bounded entries.
- Row reducing and permuting, if $V_1 \supset V_2 \supset \cdots \supset V_n$ and $\dim V_j = \mathbb{R}^{n+1-j}$, wlog $\omega_j \in V_j$ for each j.
- We can write as determinants:

$$||\mu(t)||_{\omega} \approx \sum_{i_1,\ldots,i_k} \left| \det \begin{bmatrix} \mu_1(t)\cdot\omega_{i_1} & \cdots & \mu_1(t)\cdot\omega_{i_k} \\ \vdots & \ddots & \vdots \\ \mu_k(t)\cdot\omega_{i_1} & \cdots & \mu_k(t)\cdot\omega_{i_k} \end{bmatrix} \right|.$$

Simplifications

- At any particular point t_0, wlog $\mu_j(t) \cdot \omega_i = 0$ for $i > k$.
- The best-case degeneracy scenario would be that $\det(\mu_j(t_0) \cdot \omega_i)_{i,j=1,\dots,k}$ is not zero and that $\mu_j(t) \cdot \omega_i$ vanishes at most to first order at t_0 for $i > k$.
- In this best scenario, there is the following structure:

$$\mu(t) = \overbrace{\mu_{1\cdots k}(t)}^{\neq 0 \text{ at } t=t_0} \nu_1 \wedge \cdots \wedge \nu_k + \sum_{\substack{i_1 < \cdots < \cdots < i_k \\ \neq (1,\dots,k)}} \underbrace{\mu_{i_1 \cdots i_k}(t)}_{*} \nu_{i_1} \wedge \cdots \wedge \nu_{i_k},$$

$*$ vanishes to ord. s & can be written as $s \times s$ det.

Setup

- **Recall Setup:** Incidence relation of codimension k inside $\mathbb{R}^n \times \mathbb{R}^{n_1}$. Let $d := n - k$ and $d_1 := n_1 - k$.
- **Target Exponents:** $L^{p_b} \to L^{q_b}$ indicated by scaling and Knapp-type examples:

$$p_b = \frac{kd}{nd_1} + 1 \text{ and } q_b = \frac{n_1 d}{kd_1} + 1.$$

- **Graph Structure:** Assume that the submanifold associated to $x \in \mathbb{R}^n$ is the graph $(t, \phi(x, t))$ for $t \in \mathbb{R}^{d_1}$. Suppose the Jacobian matrix $D_x\phi$ (rows are coordinates of ϕ and columns are coordinates of x) is rank k at (x, t).

Curvature Trilinear Form

- **Curvature Form:** Let $w_1, \dots, w_d$ be orthonormal in $\mathbb{R}^n$ spanning the kernel of $D_x\phi$ at the point (x, t). For $i \in \{1, \dots, d_1\}$, $i' \in \{1, \dots, k\}$, and $i'' \in \{1, \dots, d\}$, let

$$Q_{ii'i''} := \sum_{\ell=1}^{n} w_{i''}^{\ell} \frac{\partial^2 \phi^{i'}}{\partial t^i \partial x^\ell}(x, t).$$

- **Notation:** Given a multiindex $\beta \in \mathbb{Z}^k_{\geq 0}$ and sequence $\mathcal{I} := \{i_1, \dots, i_s\} \subset \{1, \dots, k\}$, say that β **counts** $\mathcal{I}$ when the ℓ-th entry of β equals the number of times that ℓ appears in $\mathcal{I}$.

Generalized Newton Polytope

Let $N(Q)$ be the convex hull in $[0,\infty)^{d_1+k+d}$ of the triples $(\alpha,\beta,\gamma)\in\mathbb{Z}^{d_1}_{\geq 0}\times\mathbb{Z}^{k}_{\geq 0}\times\mathbb{Z}^{d}_{\geq 0}$, $|\alpha|=|\beta|=|\gamma|\leq\min\{d,k\}$, $(\alpha,\beta,\gamma)=(0,0,0)$ or $\exists\,\mathcal{I}:=\{i_1,\ldots,i_s\}\subset\{1,\ldots,k\}$ and $\mathcal{J}:=\{j_1,\ldots,j_s\}\subset\{1,\ldots,d\}$ such that β counts $\mathcal{I}$, γ counts $\mathcal{J}$, and

$$\partial^{\alpha}_{\tau}\big|_{\tau=0}\det\begin{bmatrix}Q(\tau,e_{i_1},e_{j_1}) & \cdots & Q(\tau,e_{i_1},e_{j_s})\\ \vdots & \ddots & \vdots\\ Q(\tau,e_{i_s},e_{j_1}) & \cdots & Q(\tau,e_{i_s},e_{j_s})\end{bmatrix}\neq 0,$$

where $\{e_i\}_{i=1}^{k}$ is the standard basis of $\mathbb{R}^k$, $\{e_j\}_{j=1}^{d}$ is the standard basis of $\mathbb{R}^d$, and $\tau\in\mathbb{R}^{d_1}$.

Generalized Newton Distance

$$\mathcal{N}_{\mathcal{R}}(Q) := \bigcap \Big\{ N(Q') \mid Q'(x,y,z) = Q(O_1 x, O_2 y, O_3 z) \text{ for orthogonal matrices } O_1, O_2, O_3 \Big\}.$$

The functional Q will be called nondegenerate when the point

$$\Big(\overbrace{\frac{dk}{d_1 n}, \dots, \frac{dk}{d_1 n}}^{d_1 \text{ copies}}, \overbrace{\frac{d}{n}, \dots, \frac{d}{n}}^{k \text{ copies}}, \overbrace{\frac{k}{n}, \dots, \frac{k}{n}}^{d \text{ copies}} \Big)$$

belongs to $\mathcal{N}_{\mathcal{R}}(Q)$.

Codim 1 Example:

$$Q(x, y, z) = y(x \cdot z)$$

$x \to \mathbb{R}^{n-1}$, $y \to \mathbb{R}$, $z \to \mathbb{R}^{n-1}$

S = 1 terms only

$$(e_i \cdot \nabla_i)\, Q(\tau, e, e_i) = 1 \quad \forall i$$

→ always valid in any orthogonal coordinate system.

$$(1, 0, \dots, 0, 1, \underbrace{1, 0, \dots, 0}_{\text{Counts a single 1.}}) \cdot \tfrac{1}{n}$$

$$(0, 1, 0 \dots 0, 1, 0, 1, 0, \dots 0) \cdot \tfrac{1}{n}$$

$$\vdots$$

$$(0, \dots 0, 1, 1, \underbrace{0, \dots 0, 1}_{\text{Counts a single } n-1}) \cdot \tfrac{1}{n}$$

$$(0, \dots, 0, 0, 0, \dots, 0) \cdot \tfrac{1}{n}$$

$\left(\tfrac{1}{n}, \dots \tfrac{1}{n}, \tfrac{n-1}{n}, \tfrac{1}{n}, \dots, \tfrac{1}{n}\right)$ belongs to convex hull.

Codim 2 Example:

$x, y, z \in \mathbb{R}^2$.

$$Q(x,y,z) = y_1 (x \cdot z) + y_2 \det(x, z)$$

$s = 2$ terms

$$\det \begin{bmatrix} Q(\tau, e_1, e_1) & Q(\tau, e_1, e_2) \\ Q(\tau, e_2, e_1) & Q(\tau, e_2, e_2) \end{bmatrix} = \det \begin{bmatrix} \tau_1 & \tau_2 \\ -\tau_2 & \tau_1 \end{bmatrix} = \tau_1^2 + \tau_2^2$$

det symmetry in rows leaves invariant under rotations in second position

det symmetry in columns leaves invariant under rotations in position 3

$(e_1 \cdot \nabla_\tau)^2$ & $(e_2 \cdot \nabla_\tau)^2$ are nonzero.

$(2, 0, 1, 1, 1, 1) \cdot \frac{1}{4}$ (counts 1&2, counts 1&2.)

$(0, 2, 1, 1, 1, 1) \cdot \frac{1}{4}$

$(0, 0, 0, 0, 0, 0) \cdot \frac{1}{2}$

$\rightarrow (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$\frac{k}{n} = \frac{d}{n} = \frac{1}{2}$ in this example.

Local Characterization

Local Characterization of Model Operators

Suppose ϕ is polynomial. Let $\Delta \subset [0,1]^2$ be the closed triangle with vertices $(0,0)$, $(1,1)$ and $(1/p_b, 1/q_b)$. There exists a smooth cutoff function η nonvanishing at (x,t) such that the cutoff Radon-like operator $T_\eta : L^{p,1} \to L^q$ for all pairs $(p^{-1}, q^{-1}) \in \Delta$ if and only if Q is nondegenerate at (x,t).

Necessity is a Dressed-up Knapp Example

- We compute $\int \chi_F(x) T_\eta \chi_G(x) dx$. Let $F \subset \mathbb{R}^n$ be a product of two ellipsoids: one tangential and one transverse. Let $G \subset \mathbb{R}^{n_1}$ be points (t, y) where $t \in$ third ellipsoid and $y \in$ image of F under $x \mapsto \phi(t, x)$.
- To leading order, for each t, the slice G_t is also an ellipsoid. Its volume is comparable to

$$\sum_s \sum_{i,j} \left| \det \begin{bmatrix} Q(t, v_{i_1}, w_{j_1}) & \cdots & Q(t, v_{i_1}, w_{j_s}) \\ \vdots & \ddots & \vdots \\ Q(t, v_{i_s}, w_{j_1}) & \cdots & Q(t, v_{i_s}, w_{j_s}) \end{bmatrix} \right|$$

for some bases $\{v_i\}$ and $\{w_i\}$ determined by F.
- ∂_t^α used to quantify how often $|G_t|$ is large/small.

Next Steps

- Known
  - Details of the proof
  - Passage from algebraic to smooth for restricted weak type
  - Dealing with more degenerate objects
- Some Ideas
  - Equivalent algebraic characterizations of the nondegeneracy condition
- Unknown
  - Upgrading sublevel set inequalities to integrability exponent inequalities
  - Moving off the critical scaling line

Thank You