STRATIFIED INTEGRALS AND UNKNOTS IN INVISCID FLOWS

John B. Etnyre and Robert W. Ghrist

ABSTRACT. We prove that any steady solution to the C^ω Euler equations on a Riemannian S^3 must possess a periodic orbit bounding an embedded disc. One key ingredient is an extension of Fomenko's work on the topology of integrable Hamiltonian systems to a degenerate case involving stratified integrals. The result on the Euler equations follows from this when combined with some contact-topological perspectives and a recent result of Hofer, Wyzsocki, and Zehnder.

1. Introduction

The mathematical approach to knot theory initiated by Lord Kelvin began as a problem in fluid dynamics: to understand the manner in which closed flowlines in the æther are partitioned into various knot types, with the goal of recovering the periodic table [**Tho69**]. Unfortunately, the two subjects quickly diverged and have not since come into such close companionship. There are several key exceptions of which we mention two. The continual work of Moffatt [**Mof85, Mof86, Mof94**] from the engineering side to recognize the role that topology plays in physical fluid dynamics has been largely responsible for the acceptance of the definition of *helicity* (an important topological invariant) within the applied community. As an example of reversing this scenario, we mention the work of Freedman and He [**FH91b, FH91a**], who use a physical notion (hydrodynamical energy) to define a topological invariant for knots and links.

There are several unavoidable problems in the attempt to reconcile knot theory and fluid dynamics, not the least of which is that the fundamental starting point, the global existence of solutions to the Euler and Navier-Stokes equations on $\mathbb{R}^3$, is unknown and perhaps not true. Coupled with this difficulty is the fact that viscosity, unusual boundary conditions, and poorly-understood phenomenon of turbulence conspire to make it nearly impossible to rigorously analyze the solutions to the relevant equations of motion, even with powerful analytical techniques currently available.

However, since so little is known about the rigorous behavior of fluid flows, any methods which can be brought to bear to prove theorems about their behavior are of interest and of potential use in further understanding these difficult problems. We propose a view of the relevant equations of motion for inviscid (without viscosity)

1

fluid flows which sets up the possibility of a topological approach. We do so by not only restricting the class of flows considered (steady, nonsingular) but also by expanding the class via "forgetting" all information about the metric structure.

1.1. The Euler equations. For a recent topological treatment of the equations of motion for a fluid, see the excellent monograph of Arnold and Khesin [**AK98**]. Any discussion of fluid dynamics must begin with the relevant equations of motion, the most general of which is the Navier-Stokes equation. Let u denote a time-dependent vector field on $\mathbb{R}^3$ (the *velocity* of the fluid), p denote a time-dependent real-valued function on $\mathbb{R}^3$ (the *pressure*), and $\nu \geq 0$ denote a constant (the *viscosity*). Then, the Navier-Stokes equations are

$$(1.1) \qquad \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p + f \;\; ; \;\; \nabla \cdot u = 0,$$

where f is a time-dependent function on $\mathbb{R}^3$ representing body forces such as gravity, etc. The Euler equations (the form that we will concern ourselves with) are precisely the Navier-Stokes equations in the absence of viscosity and body forces:

$$(1.2) \qquad \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p \;\; ; \;\; \nabla \cdot u = 0.$$

We will for the remainder of this paper work under the following three simplifying assumptions:

1. All flows considered will be inviscid: $\nu = 0$.
2. All flows considered will be steady: $\frac{\partial u}{\partial t} = 0$.
3. All flows considered will be nonsingular: $u \neq 0$ anywhere.

The first step in embedding this problem into a topological setting is to expand the class of fluid flows that we consider. First, instead of restricting the flows to $\mathbb{R}^3$, we will allow for fluid flows on any three-manifold M. In order to make sense of the various operations (grad, curl) in Equation (1.2), we must choose a Riemannian metric, g, with respect to which these operations are taken. Finally, in order to work with the volume-preserving condition of the Euler equations, we must choose an appropriate volume form, μ. One can of course choose the precise volume form μ_g induced by the metric; however, for the sake of generality, we allow for arbitrary μ. This has physical significance, as noted in [**AK92**].

The form which the Euler equations now take is the following:

$$(1.3) \qquad \frac{\partial u}{\partial t} + \nabla_u u = -\nabla p \;\; ; \;\; \mathcal{L}_u \mu = 0,$$

where ∇_u is the covariant derivative along u defined by the metric g, and $\mathcal{L}_u$ is the Lie derivative along u. By a suitable identity for the covariant derivative (see [**AMR88**, p. 588]) one can transform the previous equation into an exterior differential system:

$$(1.4) \qquad \frac{\partial(\iota_u g)}{\partial t} + \iota_u d\iota_u g = -dP \;\; ; \;\; \mathcal{L}_u \mu = 0.$$

Here, $\iota_u g$ denotes the one-form obtained from u via contraction into the first slot of the metric, and P is a modified pressure function from M to the reals. It is this form of the Euler equation with which we will be concerned for the remainder of this paper.

1.2. Statement of results. In an earlier paper [**EG98**], the authors initiated the use of contact-topological ideas in the study of the Euler equations. There, it was shown that all steady solutions of sufficiently high regularity on S^3 possess a closed flowline: *i.e.*, the Seifert Conjecture is true in the hydrodynamical context. This opens the possibility for asking questions about knotting and linking phenomena common to all fluid flows on S^3.

The main result we prove in this note is the following:

THEOREM 1.1. *Any steady solution to the C^ω Euler equations on a Riemannian S^3 must possess a closed flowline which bounds an embedded disc.*

The proof of this theorem relies upon deep results due to several authors, most especially the work of Hofer et al. [**HWZ96a**] on unknotted orbits in Reeb fields, as well as the theorem of Wada [**Wad89**] on nonsingular Morse-Smale flows. A key ingredient is a generalization in §4 of a theorem of Fomenko and Nguyen [**FN91**] and of Casasayas et al [**CAN93**] to a degenerate case:

THEOREM 1.2. *Any nonsingular vector field on S^3 having a C^ω integral of motion must possess a pair of unknotted closed orbits.*

This theorem has certain peripheral implications in the Fomenko-style approach to two degree-of-freedom integrable Hamiltonian systems. We elaborate upon these themes in §5.

We note that Theorem 1.1 is but one small piece of data concerning knot theory within hydrodynamics (note in particular the work of Moffatt et al. [**Mof94**]). In a future paper [**EG99**], we will consider the other end of the spectrum: namely, what is possible as opposed to what is inevitable. There, we will construct steady nonsingular solutions to the Euler equation possessing knotted orbits of all possible knot types simultaneously.

2. Contact / integrable structures for steady Euler flows

After providing a brief background on contact geometry/topology, we consider the class of steady Euler flows on the three-sphere in the C^ω category. We demonstrate that there is a dichotomy between integrable solutions, and solutions which are related to contact forms.

2.1. Contact structures on three-manifolds. A more thorough introduction to the field of contact topology can be found in the texts [**Aeb94, MS95**]. A CONTACT STRUCTURE on an odd-dimensional manifold is a completely nonintegrable hyperplane distribution. We will restrict to the case of a three-manifold. In this case, a contact structure is a completely nonintegrable smoothly varying field of 2-dimensional subspaces of the tangent spaces. Unlike vector fields, such plane fields do not necessarily integrate to form a foliation. This integrability of a plane field ξ is measured by the Frobenius condition on a defining (local) 1-form α. If $\xi = \ker \alpha$, then ξ is a contact structure if and only if $\alpha \wedge d\alpha$ vanishes nowhere. Such a form α is a CONTACT FORM for ξ.

The topology of the structures ξ and the geometry of the associated forms α has of late been a highly active and exciting field. As contact structures are in a strong sense the odd-dimensional analogue of a symplectic structure [**MS95, Arn80**], many of the interesting phenomena of that discipline carry over. Of particular importance is the existence of a certain class of contact structures – the TIGHT structures – which possess topological restrictions not otherwise present. Such structures

are fairly mysterious: basic questions concerning existence and uniqueness of such structures are as yet unanswered. See [**Eli89, ET98**] for more information.

Contact geometry has found applications in a number of disciplines. We note in particular the utility of contact geometry in executing a form of reduction in dynamics with special symmetry properties [**HM97**]. In a recent work [**EG98**], the authors initiated the use of modern contact-topological methods in hydrodynamics. Most of these dynamical applications revolve around the notion of a REEB FIELD for a contact form. The Reeb field associated to a contact form α is the unique vector field X satisfying the equations:

$$(2.1) \qquad\qquad \iota_X \alpha = 1 \quad ; \quad \iota_X d\alpha = 0.$$

The Reeb field forms a canonical section of the CHARACTERISTIC LINE FIELD $\ker d\alpha$ on M. Reeb fields are by definition nonsingular and preserve the volume form $\alpha \wedge d\alpha$. In §2.3, we review a result on the topology of Reeb fields due to Hofer et al.

2.2. The dichotomy for steady Euler flows. The following theorem is a specialized version of the general correspondence between solutions to the Euler equation and Reeb fields in contact geometry derived in [**EG98**]. We include the [simple] proof for completeness.

THEOREM 2.1. *Let u denote a steady nonsingular solution to the Euler equations of class C^ω on a Riemannian S^3. Then at least one of the following is true:*

1. *There exists a nontrivial integral for u; or*
2. *u is a nonzero section of the characteristic line field of a contact form α.*

Proof: If u is a steady solution then

$$(2.2) \qquad\qquad \iota_u d\iota_u g = -dP.$$

As all the data in the equation is assumed real-analytic, the differential dP must be C^ω and hence vanishes identically if and only if it vanishes on an open subset on S^3. Note that

$$(2.3) \qquad\qquad \mathcal{L}_u P = \iota_u dP = \iota_u \iota_u(d\iota_u g) \equiv 0,$$

and thus a nonconstant P yields a nontrivial integral for the vector field u.[1]

In the case where $dP \equiv 0$, we have that $\iota_u d\iota_u g \equiv 0$. Consider the nondegenerate 1-form $\alpha := \iota_u g$ dual to the vector field u via the metric. In addition, denote by $\beta := \iota_u \mu$ the 2-form obtained by pairing u with the volume form μ. Since u is μ-preserving, it is the case that $d\beta \equiv 0$. Also, by definition, β has a one-dimensional kernel spanned by u: $\iota_u \beta = 0$. As $\iota_u d\iota_u g = 0$, it follows that $d(\iota_u g) = h\iota_u \mu$ for some function $h : S^3 \to \mathbb{R}$.

It is a classical fact that h is an integral of u: observe that

$$(2.4) \qquad\qquad 0 = d^2\alpha = d(h\beta) = dh \wedge \beta + h d\beta = dh \wedge \beta,$$

which implies that $\iota_u dh = 0$.

Thus, the only instance in which u is not integrable is when h is constant. If $h \equiv 0$, then by the Frobenius condition, the 1-form $\iota_u g$ defines a C^ω codimension-one foliation of S^3 which is transverse to u. This is impossible due to the presence of a T^2 leaf in the foliation (guaranteed by Novikov's Theorem [**Nov67**]) which is transverse to the volume-preserving flow of u. Or, alternatively, the C^ω codimension-one foliation of S^3 violates the Haefliger Theorem [**Hae56**].

---

[1]This is a "modern" reformulation of the classical Bernoulli Theorem for fluids.

Thus, if u is not integrable via h, then h is a nonzero constant. So, the 1-form $\iota_u g$ dual to u is a contact form since $\alpha \wedge d\alpha = h\alpha \wedge \beta \neq 0$. Note that in this case u is a section of the characteristic line field of this contact form: $\iota_u(d\iota_u g) = 0$. $\qquad\square$

2.3. The results of Hofer et al. The recent deep work of Hofer [**Hof93**] and of Hofer, Wyszocki, and Zehnder [**HWZ96b, HWZ96a**] utilizes analytical properties of pseudoholomorphic curves in products of a contact manifold with $\mathbb{R}$ to elucidate the dynamics and topology of Reeb fields. As such, these results are directly applicable to the understanding of steady nonsingular Euler flows. The specific result we employ for this paper is the following:

THEOREM 2.2 (Hofer et al. [**HWZ96a**]). *Let α be a contact form on a homology 3-sphere M. Then the Reeb field associated to α possesses a periodic orbit which bounds an embedded disc.*

This theorem, combined with Theorem 2.1, yields a proof of the main result [Theorem 1.1] in the difficult case where the velocity field is a section of a characteristic line field for a contact form. The integrable case must yet be considered.

3. Knotted orbits in integrable systems

Having dispensed with the nonintegrable cases, we turn in this section to consider the knot types associated to periodic orbits in integrable Hamiltonian systems. Several fundamental results about such flows are presented in the work by Casasayas et al. [**CAN93**] and by Fomenko and Nguyen [**FN91**]: we will review these results and extend them to the degenerate framework we require in order to complete the proof of the main theorem.

Recall that a two degree-of-freedom Hamiltonian system on a symplectic four-manifold (W, ω) with Hamiltonian H splits into invariant codimension-one submanifolds $Q_c = H^{-1}(c)$ (on the regular values of H). The system is said to be INTEGRABLE on Q_c if there exists a function $F : Q_c \to \mathbb{R}$ such that F is independent of H and $\{F, H\} = 0$: in other words, F is invariant on orbits of the Hamiltonian flow.

A topological classification of such two degree-of-freedom integrable systems exists when the integrals are nondegenerate. An integral F is said to be BOTT or BOTT-MORSE if the critical point set $cp(F)$ consists of a finite collection of submanifolds Σ_i of Q_c which are transversally nondegenerate: the restriction of the Hessian d^2F to the normal bundle $\nu\Sigma_i$ of $cp(F)$ is nondegenerate. Such integrals are generic among integrable systems, but given a particular system, it is by no means easy to verify whether an integral is Bott-Morse.

The classification of knotted orbits in Bott-integrable Hamiltonian systems on S^3 is best accomplished via the classification of round-handle decompositions of S^3. Recall that a ROUND HANDLE in dimension three is a solid torus $H = D^2 \times S^1$ with a specified index and exit set $E \subset T^2 = \partial(D^2 \times S^1)$ as follows:

index 0: $E = \emptyset$.
index 1: E is either (1) a pair of disjoint annuli on the boundary torus, each of which wraps once longitudinally; or (2) a single annulus which wraps twice longitudinally.
index 2: $E = T^2$.

A ROUND HANDLE DECOMPOSITION (or RHD) for a manifold M is a finite sequence of submanifolds

$$(3.1) \qquad \emptyset = M_0 \subset M_1 \subset \cdots M_n = M,$$

where M_{i+1} is formed by adjoining a round handle to ∂M_i along the exit set E_{i+1} of the round handle. The handles are added in order of increasing index. Asimov [**Asi75**] and Morgan [**Mor78**] used round handles to classify nonsingular MORSE-SMALE vector fields: that is, vector fields whose recurrent sets consist entirely of a finite number of hyperbolic closed orbits with transversally intersecting invariant manifolds. In short, the cores of an RHD, labelled by the index, correspond to the periodic orbits of a nonsingular Morse-Smale vector field, labelled by the Morse index [**Asi75**]. The classification of RHD's (in the context of nonsingular Morse-Smale flows) on S^3 was achieved by Wada [**Wad89**] following work of Morgan [**Mor78**].

THEOREM 3.1 (Wada [**Wad89**]). *Let $\mathcal{W}$ be the collection of indexed links determined by the following eight axioms:*

O *The Hopf link indexed by 0 and 2 is in $\mathcal{W}$.*

I *If $L_1, L_2 \in \mathcal{W}$ then $L_1 \circ L_2 \circ u \in \mathcal{W}$, where u (here and below) is an unknot in S^3 indexed by 1, and $\circ$ denotes the SPLIT SUM of knots (i.e., separable by means of an embedded 2-sphere).*

II *If $L_1, L_2 \in \mathcal{W}$ and K_2 is a component of L_2 indexed by 0 or 2, then $L_1 \circ (L_2 - K_2) \circ u \in \mathcal{W}$.*

III *If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 with indices 0 and 2 (resp.), then $(L_1 - K_1) \circ (L_2 - K_2) \circ u \in \mathcal{W}$.*

IV *If $L_1, L_2 \in \mathcal{W}$ and K_1, K_2 are components of L_1, L_2 (resp.) each with index 0 or 2, then*

$$((L_1, K_1) \# (L_2, K_2)) \cup m \in \mathcal{W},$$

where $K_1 \# K_2$ shares the index of either K_1 or K_2 and m is a meridian of $K_1 \# K_2$ indexed by 1.

V *If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L replacing a tubular neighborhood of K with a solid torus with three closed orbits, K_1, K_2, and K_3. K_1 is the core and so has the same knot type as K. K_2 and K_3 are parallel (p, q) cables of K_1. The index of K_2 is 1. The indices of K_1 and K_3 may be either 0 or 2, but at least one of them must be equal to the index of K.*

VI *If $L \in \mathcal{W}$ and K is a component of L indexed by $i = 0$ or 2, then $L' \in \mathcal{W}$, where L' is obtained from L by changing the index of K to 1 and placing a $(2, q)$-cable of K in a tubular neighborhood of K, indexed by i.*

VII *$\mathcal{W}$ is minimal. That is, $\mathcal{W} \subset \mathcal{W}'$ for any collection, $\mathcal{W}'$, satisfying O-VI.*

Then the class of indexed periodic orbit links arising within nonsingular Morse-Smale flows on S^3 is precisely $\mathcal{W}$.

COROLLARY 3.2 (Wada [**Wad89**]). *Every smooth nonsingular Morse-Smale vector field on S^3 possesses a pair of unknotted closed orbits.*

Proof: The base Hopf link is such a pair. It is clear that the Wada moves I-VI leave this property invariant. □

The relationship between Wada's Theorem and nonsingular integrable Hamiltonian systems was developed by Casasayas et al. [**CAN93**]. The idea is straightforward: given a Bott-integrable nonsingular Hamiltonian system with integral P, the vector field $-\nabla P$ is a field with curves of Bott-Morse type critical points. A small perturbation tangent to the critical curves yields a nonsingular Morse-Smale flow.

COROLLARY 3.3 (Casasayas et al. [**CAN93**]). *Every Bott-integrable C^∞ Hamiltonian flow on a symplectic 4-manifold having a nonsingular S^3 energy surface possesses a pair of unknotted invariant critical curves.*

Similar results were obtained by Fomenko and Nguyen [**FN91**].

4. Proof of Theorem 1.1

As is clear from the previous section, we may obtain information about the knot data of integrable Euler fields if we can ensure that the critical sets are all of Bott-Morse type. Nowhere in the literature is there a discussion of the non-Bott case with respect to knotting and linking phenomena. This is very difficult if not impossible to control in the general C^∞ case; however, in the real-analytic case, we may still analyze the degenerate critical point sets.

LEMMA 4.1. *Any critical set of a nontrivial C^ω integral P for a nonsingular vector field X on S^3 is a [Whitney] stratified set of (topological) dimension at most two.*

Proof: Denote by $cp(P)$ the critical points of P and by Σ a connected component of the inverse image of the critical values of P. It follows from the standard theorems concerning real-analytic varieties [**Whi57, GM88**] that the set Σ is a (Whitney) stratified set. That is, although Σ is not a manifold, it is composed of manifolds — or STRATA — glued together along their boundaries in a controlled manner. It follows from analyticity that Σ has topological dimension less than or equal to two; otherwise, P would be a constant. □

LEMMA 4.2. *The critical set Σ is either an embedded closed curve in S^3, or else is a (non-smoothly) branched 2-manifold, where the non-manifold set of Σ is a finite invariant link in S^3. The complement of this set in Σ [the 2-strata] consists of critical tori, as well as annuli and Möbius bands glued to the singular link along their boundaries.*

Proof: As $X|_\Sigma$ is a nonsingular vector field, Σ must have a stratification devoid of 0-strata: only 1- and 2-strata are permitted. Furthermore, the topology of Σ must be transversally homogeneous with respect to the flow: a neighborhood of any point x in Σ is homeomorphic to a product of a 1-dimensional stratified space with $\mathbb{R}$ (the local orbits of the nonsingular vector field). Compactness and finiteness of the stratification imply that Σ is everywhere locally homeomorphic to the product of a K-pronged radial tree with $\mathbb{R}$.

If Σ is one-dimensional, then by transverse homogeneity and compactness of Σ, it is a compact one-manifold — a circle.

If Σ is two-dimensional, then every point of Σ is locally homeomorphic to a K-pronged radial tree cross $\mathbb{R}$, where K may vary but is always nonzero. The non-manifold points of Σ are precisely those points where $K \neq 2$. This set must be invariant under the flow, otherwise the uniqueness theorem for the vector field

is violated. Hence, by compactness of the stratification, the non-manifold set is a finite link L.

Consider the space Σ' obtained by removing from Σ a small open tubular neighborhood of L. Since L is an invariant set for the flow, the vector field may be perturbed in such a way as to leave Σ' invariant. As all of the non-manifold points of Σ have been removed, Σ' is a 2-manifold with boundary. The perturbed vector field on Σ' is nonsingular; thus, Σ' consists of annuli and Möbius bands (plus perhaps tori which do not encounter the singular link). □

We conclude the proof of Theorem 1.1 with the following theorem.

THEOREM 4.3. *Any nonsingular vector field on S^3 having a C^ω integral of motion must possess a pair of unknotted closed orbits.*

Proof: We will prove this theorem for the slightly larger class of *stratified* integrals which are not necessarily C^ω but whose critical sets are finite, [Whitney] stratified, and of positive codimension (see §5 for details and extensions). Induct upon κ the number of non-Bott connected components in the inverse images of the critical value set of the integral. If there are no such sets, then the system is Bott and the theorem follows from Corollary 3.3.

Let c denote a (transversally) degenerate critical value of P and Σ a connected component of $P^{-1}(c)$. Denote by $N(\Sigma)$ the connected component of $P^{-1}([c-\epsilon, c+\epsilon])$ containing Σ. For ϵ sufficiently small N is well-defined up to isotopy. The boundary components, T_k, of N are all in the inverse image of regular values of the integral: as such, each T_k is an embedded closed surface in S^3 supporting a nonsingular vector field – a 2-torus.

Each boundary torus T_k bounds a solid torus in S^3 on at least one side [**Rol77**, p.107]. Denote by $\mathcal{S}$ the set of boundary components of N which bound a solid torus containing Σ. Denote by $\mathcal{S}_0 \subset \mathcal{S}$ the subset of bounding neighborhood tori which are unknotted in S^3.

Case 1: $\mathcal{S} - \mathcal{S}_0 \neq \emptyset$

Denote by V the nontrivially-knotted solid torus containing Σ. Redefine the integral on V in a C^∞ manner so that there is a single (Bott) critical set on the nontrivially knotted core of V, reducing κ. By the induction hypothesis, there is a pair of unknotted closed curves, neither of which can be the core of V. Note that although the new integral is not necessarily C^ω, this stratified Bott integral is sufficient to apply Corollary 3.3 and the induction hypothesis.

Case 2: $\mathcal{S} = \emptyset$

In this case, the non-Bott component Σ must have a neighborhood N such $S^3 - N$ consists of a disjoint collection of solid tori. We may then place a round-handle decomposition (RHD) on S^3 as follows.

By Lemma 4.2, one can decompose Σ into a finite number of critical circles (1-strata) to which are attached annuli and Möbius bands (2-strata) in a way which satisfies the Whitney condition. Place an RHD on N by thickening up each 1-stratum to a round 0-handle. The annular and Möbius 2-strata then thicken up in N to round 1-handles of orientable and nonorientable type respectively. Since all of the boundary components of N bound solid tori on the exterior of N, we can glue in round 2-handles, completing the RHD of S^3.

According to the previously cited results of Asimov and Morgan, there is a nonsingular Morse-Smale flow on S^3 which realizes the indexed cores of this RHD as the periodic orbit link. Hence, by Theorem 3.1 there is a pair of unknots among

the cores of this RHD. The index-2 cores are all nontrivially knotted by assumption; hence, the unknots have index zero or one. If zero, then these cores are the invariant 1-strata for the original flow. If an index-1 core is unknotted, then there exists an invariant 2-stratum (annulus or Möbius band) which is unknotted. If the core of an embedded annulus is unknotted, then both boundary components (invariant 1-strata in the flow) are unknotted. In the case where the core of the invariant Möbius 2-stratum is unknotted, we show that there exists an unknotted flowline as well (see the proof of Lemma 4.4 below). Hence, both unknotted RHD cores are realized by isotopic invariant curves of the original flow.

Case 3: $\mathcal{S} = \mathcal{S}_0 \neq \emptyset$

Construct a round-handle decomposition of S^3 as in Case 2 — this is possible since all the exterior regions can be made into round 2-handles. As before, there must exist a pair of unknotted cores to the RHD. Any which are of index zero or index one correspond to unknotted invariant curves in the original flow, by the arguments of Case 2. Assume that V corresponds to an unknotted round 2-handle: a component of $S^3 - N$. Replace the integral on $S^3 - V$ (which is an unknotted solid torus as well) to have a single unknotted core critical set. Then by the induction hypothesis, there must have been an unknotted invariant curve within V. Hence, each unknotted round 2-handle corresponds to an unknotted invariant curve in the original flow. $\qquad\square$

LEMMA 4.4. *Any nonsingular vector field on a Möbius band has a periodic orbit isotopic to the core.*

Proof: The Poincaré-Bendixson Theorem holds for the Klein bottle (and thus for the Möbius band which is a subset) by the theorem of Markley [**Mar69**]. The boundary curve of the Möbius band is invariant, and either this curve has nontrivial holonomy or it does not. If the holonomy is nontrivial, then index theory and the Poincaré-Bendixson Theorem imply the existence of another closed orbit which is either twice-rounding (in which case is separates a smaller invariant Möbius band — repeat the analysis) or is once-rounding, in which case it is isotopic to the core. For the case of trivial holonomy, there is a 1-parameter family of twice-rounding invariant curves, which either limits onto a closed curve with nontrivial holonomy, or else limits onto a once-rounding invariant core curve. $\qquad\square$

We note that it is not necessarily the case that a stratified integral on S^3 must have an unknotted curve of critical points (as is the case for a Bott integral). The centers of the Möbius 2-strata may be the only unknotted orbits in the flow: one may construct an example in a manner reminiscent of a Seifert-fibred structure on S^3 in which the critical sets of the integral are a pair of $(2, 2n + 1)$ torus knots whose (unknotted) cores are arranged in a Hopf link.

5. Stratified integrable systems

The argument repeatedly stated in the Fomenko programme for restricting attention to Bott-integrable Hamiltonian systems is that this condition appears to be ubiquitous in physical integrable systems (*i.e.*, ones in which the integrals can be written out explicitly) [**Fom91, Fom88**]. However, the examples cited as evidence for this hypothesis are often real-analytic integrals. Hence, it would appear sensible to recast the Fomenko program of topological classification of integrable Hamiltonian systems in the analytic case. Or, better still, one could allow for

integrals which are less smooth yet satisfy the following more general conditions: one might call such integrals STRATIFIED.

1. The critical values of the integral are isolated;
2. The inverse images of critical values are [Whitney] stratified sets of codimension greater than zero.

These assumptions allow for a controlled degeneracies in the integral, yet are by no means unnatural: codimension-one bifurcations of "physical" integrable Hamiltonian systems can and do exhibit such degeneracies. Since the topological results of the Fomenko program can be obtained by using pre-existing RHD-theory [**CAN93**], and since the case of stratified non-Bott singularities also reduces to RHD's (see the proof of Theorem 4.3 above), there is seemingly no reason to exclude stratified integrals.

Many, if not all, of the key results of Fomenko's programme hold for this larger class of integrals. Unlike the Bott-Morse condition, it is often trivial to check whether the above criteria are met in the case of an explicit integral (as these are almost always analytic functions). We summarize a few results pertaining to the topology of flowlines which hold for stratified integrable systems. First, however, we recall that a GRAPH-MANIFOLD is a 3-manifold obtained by gluing together Seifert-fibred manifolds with boundary along mutually incompressible tori [**Wal67a, Wal67b**].

THEOREM 5.1. *Given a nonsingular flow on a closed three-manifold M possessing a stratified integral P, then M is a graph-manifold. Furthermore, if $M = S^3$, then the following statements hold:*

1. *There exists a pair of unknotted flowlines;*
2. *Every closed orbit of the flow is a knot which belongs to the family of zero-entropy knots described in Theorem 3.1.*
3. *The critical point set is* NONSPLITTABLE: *there does not exist an embedded S^2 which separates distinct components of $cp(P)$.*

Proof: The fact that M is a graph-manifold follows trivially from the proof of Lemma 4.2 upon noting that a neighborhood of a two-dimensional degenerate set has the structure of a round-handle decomposition. The results of Morgan [**Mor78**] then imply that M is a graph-manifold. Or, equivalently, one may perturb the integral on neighborhoods of singular sets to be Bott-Morse without changing the topology of the underlying manifold. This result was stated in [**Fom88**, p. 325].

Proof of Item (1): Item 1 follows from the proof of Theorem 4.3 above. $\square_1$

Proof of Item (2): The following classification of knotted periodic orbits in stratified integrable dynamics is an extension of the theorems of Casasayas et al. [**CAN93**] and Fomenko and Nguyen [**FN91**] in the Bott case. Choose any periodic orbit γ whose P-value is not critical. Then, there exists a suitably small neighborhood of the non-Bott singular sets not containing γ. Perturb the integral on this small neighborhood to be a Bott integral. Although this changes the vector field near the singular sets, it does not alter the knot type of γ, which must be a zero-entropy knot by the aforementioned results.

If γ is lying on a 2-stratum of the singular set then one may push γ or a 2-cable of γ off into a regular torus T. This regular torus is also a regular torus in a Bott integral. We may alter this integral to another Bott integral for which the solid torus that T bounds contains a single critical level at the core. Since the core of

this torus is a zero entropy knot we know that γ or a 2-cable of γ is also a zero entropy knot. Thus γ is a zero entropy knot (see below).

The only case left to consider is a periodic orbit γ lying on the 1-stratum of the singular set. Some cable of γ is a knot on the 2-stratum of the singular set and we have merely to show, then, that if κ is a zero-entropy knot which is a cable of γ, then γ is also zero-entropy.

Recall that zero-entropy knots are the closure of the unknot under connected sum and cabling. If κ is also a cable of a zero-entropy knot γ', then we claim that γ and γ' are isotopic. Let T and T' denote the cabling tori for γ and γ' respectively. By transversality, $T \cap T'$ consists of disjoint circles having κ as a component. Any nullhomotopic circles can be inductively removed, leaving a finite collection of intersection curves isotopic to κ. These slice T and T' into pairs of annuli attached along their boundaries pairwise to form tori. One then uses the solid tori these bound to inductively cancel intersection curve pairs. Hence the cores γ and γ' are isotopic.

In the other possibility, where κ is the connected sum of two nontrivial knots, one has a contradiction upon showing that the nontrivial cable of a knot is always prime. A proof of this fact may be obtained by a similar geometric argument as the previous step, or by an algebraic argument in [**BZ85**, p.93]. Hence, any periodic orbit is always a zero-entropy knot. $\square_2$

Proof of Item (3): Item (3) is seen to be true for Bott-integrable systems without critical tori by analyzing the operations of Theorem 3.1 (see [**CAN93**]). In the presence of critical tori, one can perturb the integral to have critical curves on the torus which renders the [now smaller] critical set unsplittable; hence the full critical set was unsplittable as well. In the stratified case, assume that S is an embedded 2-sphere which separates the critical point set $cp(P)$. Then there exists a bound such that all sufficiently small smooth perturbations to the integral do not create critical points along S. Applying such a perturbation to a neighborhood of the inverse image of the critical values yields a Bott system with S as a splitting sphere for the critical points set: contradiction. $\square_3$

This completes the proof of the Theorem. $\blacksquare$

These results are noteworthy in that the existence of a single hyperbolic knot (*e.g.*, a figure-eight knot) in a nonsingular vector field on S^3 implies the nonexistence of an integral.

References

[Aeb94]   B. Aebisher, et al. *Symplectic Geometry*. Number 124 in Progress in Math. Birkhaüser, Berlin, 1994.

[AK92]   V. I. Arnold and B. Khesin. Topological methods in hydrodynamics. *Annu. Rev. Fuid Mech.*, 24:145–166, 1992.

[AK98]   V. I. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*. Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[AMR88]   R. Abraham, J. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Number 75 in Applied Math. Sci. Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[Arn80]   V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[Asi75]   D. Asimov. Round handles and non-singular Morse-Smale flows. *Ann. Math.*, 102:41–54, 1975.

[BZ85]   G. Burde and H. Zieschang. *Knots*. De Gruyter, Berlin, 1985.

[CAN93]	J. Casasayas, J. Martinez Alfaro, and A. Nunes. Knotted periodic orbits and integrability. In *Hamiltonian Systems and Celestial Mechanics (Guanajuato 1991)*, number 4 in Adv. Ser. Nonlinear Dynam., pages 35–44. World Sci. Pub., River Edge, NJ, 1993.

[EG98]	J. Etnyre and R. Ghrist. Contact topology and hydrodynamics I: existence and the Seifert conjecture. Preprint, 1998.

[EG99]	J. Etnyre and R. Ghrist. Contact topology and hydrodynamics III: knotted flowlines. In preparation, 1999.

[Eli89]	Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98:623–637, 1989.

[ET98]	Y. Eliashberg and W. Thurston. *Confoliations*, volume 13 of *University Lecture Series*. American Mathematical Society, 1998.

[FH91a]	M. Freedman and Z.-X. He. Divergence-free fields: energy and asymptotic crossing number. *Annals of Math.*, 134(1):189–229, 1991.

[FH91b]	M. Freedman and Z.-X. He. Links of tori and the energy of incompressible flow. *Topology*, 30(2):283–287, 1991.

[FN91]	A. T. Fomenko and T. Z. Nguyen. Topological classification of integrable nondegenerate Hamiltonians on isoenergy three-dimensional spheres. In A. T. Fomenko, editor, *Advances in Soviet Mathematics, Vol. 6*, pages 267–296. American Mathematical Society, 1991.

[Fom88]	A. T. Fomenko. *Integrability and nonintegrability in geometry and mechanics*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian by M. V. Tsaplina.

[Fom91]	A. T. Fomenko. *Advances in Soviet Mathematics, Vol. 6*. American Mathematical Society, Providence, RI, 1991.

[GM88]	M. Goresky and R. MacPherson. *Stratified Morse Theory*. Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[Hae56]	André Haefliger. Sur les feuilletages analytiques. *C. R. Acad. Sci. Paris*, 242:2908–2910, 1956.

[HM97]	G. Haller and I. Mezić. Reduction of three-dimensional volume-preserving flows with symmetry. Preprint, 1997.

[Hof93]	H. Hofer. Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. *Invent. Math.*, 114:515–563, 1993.

[HWZ96a]	H. Hofer, K. Wysocki, and E. Zehnder. Unknotted periodic orbits for Reeb flows on the three-sphere. *Topol. Methods Nonlinear Anal.*, 7(2):219–244, 1996.

[HWZ96b]	H. Hofer, K. Wysocki, and E. Zender. A characterisation of the tight three-sphere. *Duke J. Math.*, 81(1):159–226, 1996.

[Kup96]	G. Kuperberg. A volume-preserving counterexample to the Seifert conjecture. *Comment. Math. Helv.*, 71(1):70–97, 1996.

[Mar69]	N. G. Markley. The Poincaré-Bendixson theorem for the Klein bottle. *Trans. Amer. Math. Soc.*, 135:159–165, 1969.

[Mof85]	H. Moffatt. Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology: part I. *J. Fluid Mech.*, 159:359–378, 1985.

[Mof86]	H. Moffatt. Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology: part II. *J. Fluid Mech.*, 166:359–378, 1986.

[Mof94]	H. Moffatt, editor. *Topological aspects of the dynamics of fluids and plasmas. Proceedings of the Program of the Institute for Theoretical Physics on Topological Fluid Dynamics held August–December 1991 and the NATO Advanced Research Workshop held at the University of California, Santa Barbara, California, November 1–5, 1991*, number 218 in NATO Advanced Science Institutes Series E: Applied Sciences. Kluwer Academic Press, 1994.

[Mor78]	J. Morgan. Nonsingular Morse-Smale flows on 3-dimensional manifolds. *Topology*, 18:41–54, 1978.

[MS95]	D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Oxford University Press, New York, 1995.

[Nov67]	S. Novikov. Topology of foliations. *Trans. Moscow Math. Soc.*, 14:268–304, 1967.

[Rol77]	D. Rolfsen. *Knots and Links*. Publish or Perish, Berkely, CA, 1977.

[Tho69]	W. Thomson. On vortex motion. *Trans. R. Soc. Edin.*, 25:217–260, 1869.

[Wad89]   M. Wada. Closed orbits of nonsingular Morse-Smale flows on S^3. *J. Math. Soc. Japan*, 41(3):405–413, 1989.

[Wal67a]   F. Waldhausen. Eine Klasse von 3-dimensionalen Mannigfaltigkeiten I. *Invent. Math.*, 2:308–333, 1967.

[Wal67b]   F. Waldhausen. Eine Klasse von 3-dimensionalen Mannigfaltigkeiten II. *Invent. Math.*, 4:87–117, 1967.

[Whi57]   H. Whitney. Elementary structure of real analytic varieties. *Annals of Math.*, 66(3):545–556, 1957.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, PALO ALTO, CA 94305
E-mail address: `etnyre@math.stanford.edu`

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332-0160
E-mail address: `ghrist@math.gatech.edu`