AN INDEX FOR CLOSED ORBITS IN BELTRAMI FIELDS

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Abstract. We consider the class of Beltrami fields (eigenfields of the curl operator) on three-dimensional Riemannian solid tori: such vector fields arise as steady incompressible inviscid fluids and plasmas. Using techniques from contact geometry, we construct an integer-valued index for detecting closed orbits in the flow which are topologically inessential (they have winding number zero with respect to the solid torus). The most important feature of the index is its computability: it can be rigorously determines from a C^1-approximation to the vector field on any meridional disc of the solid torus. As a result, we obtain a test for detecting the non-existence of a global cross-section to the 3-d flow via purely 2-d information.

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1. Introduction and summary

Consider the class of *Beltrami fields* — the volume-preserving eigenfields of the curl operator. Such vector fields are the source of numerous interesting phenomena in inviscid fluids and plasmas. For example, Beltrami fields are the only steady three-dimensional Euler flows which admit chaotic Lagrangian dynamics. Beltrami fields are also common approximations to the magnetic field lines in large-scale structures within the solar corona.

Despite their importance and inherent intricacy, very little is known about the dynamics of Beltrami fields apart from numerical simulation [25, 8] and Melnikov analyses of near-integrable Beltrami fields [19, 38, 23, 6, 29] – an important but extremely small class of solutions. We consider the subtle problem of understanding how much and what kinds of dynamics Beltrami fields are forced to possess given the underlying topological features of the fluid domain.

In a series of papers [14, 15, 16], the authors develop techniques for determining forced behaviors in steady inviscid fluids via the topology of *contact structures*, the odd-dimensional analogues of symplectic structures (see, *e.g.*, [1, 12] for an introduction to contact geometry). In this paper, we give an

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application of these techniques to Beltrami fields on solid tori. One of the features of our topological approach is that it is independent of the Riemannian metric and is furthermore robust with respect to perturbations of the vector field, without resorting to any hyperbolicity or nondegeneracy assumptions usually required to preserve closed orbits.

We restrict attention to Beltrami fields on Riemannian solid tori, such as would occur in the case of a force-free plasma in a containment device. In [15], it is shown using techniques from contact topology and pseudo-holomorphic curves that steady Euler fields on an invariant solid torus *always* possess a closed orbit, independent of the Riemannian structure and volume form:

Theorem: ([15]) *Any C^2 Beltrami field on any invariant Riemannian solid torus possesses a closed orbit.*

The actual result of [15] is stated for general steady solutions to (2.2) for a more restrictive smoothness condition. However, in the setting of pure Beltrami fields, the techniques are valid up to smoothness class C^2 (following the analytic details of [26]).

Initially, the problem of detecting closed orbits seems trivial. In the simplest examples of nonsingular vector fields on a solid torus, one has a global cross-section: a meridional disc D and a well-defined Poincaré return map on D induced by the flow. For such systems, the Brouwer Fixed Point Theorem guarantees a closed orbit. In cases where no section exists, it is very difficult to determine the existence of closed orbits. Indeed, the recent examples of fixed-point-free vector fields on a solid torus without any periodic orbits (constructible via [32] in the real-analytic case, and by [31] in the C^1 volume-preserving case), demonstrate the delicacy of the problem.

In this paper, we give a computable test for determining when a Beltrami field does *not* possess a global cross section by detecting the presence of *contractible* closed orbits — those closed orbits which can be shrunk to a point within the solid torus. We do so by means of an integer-valued index.

The index we construct is a type of linking number with respect to a contact structure — the so-called *self-linking number* of a transverse knot, well-known to contact topologists. The data required to compute the index is minimal: one needs information about the Beltrami field along some (arbitrary) meridional disc in the solid torus. The principal contribution of this note is to retool the contact-topological index in a form which requires no knowledge of the contact structure *per se*. Since contact structures are very stable with respect to perturbation, the vector field need only be known approximately (C^1) along the disc. Hence, this index can be computed numerically with full rigor.

A related scenario to which our results apply is that of a Beltrami field on a long tube $D^2 \times \mathbb{R}$ which is periodic in the third variable. The problem of finding contractible closed orbits in the solid torus obtained by quotienting out the periodicity is precisely the problem of finding an orbit which is closed in the long tube.

Sections 2 through 4 assemble the relevant ingredients. Section 5 presents the technical result on contact structures used to define the index and prove its major properties. The section §6 gives a very simple method for computing this index from a minimal amount of data: one need simply know what the vector field X approximately looks like near some finite number of points on a meridional disc of the solid torus. We do not include analytic examples of Beltrami fields in non-Euclidean solid tori as these are relatively difficult to write out explicitly: indeed, attention seems to have been focused on systems with global cross-sections precisely because most explicit analytic models are so simple as to have this property. It is our hope that this computability may allow for utilization of this index in the analysis of experimental data, which may well yield highly complex Beltrami fields.

2. BELTRAMI FIELDS ON RIEMANNIAN MANIFOLDS

Let M be an arbitrary 3-manifold with Riemannian metric g and (arbitrary) volume form μ. Given a vector field X on M, one can consider the dual 1-form $g(X, \cdot)$ to X that pairs with a vector Y to give the inner product $g(X, Y)$. In this general setting, the *curl* of a vector field X on M is the unique vector field $\nabla \times X$ satisfying

$$\mu(\nabla \times X, \cdot, \cdot) = d(g(X, \cdot)), \tag{2.1}$$

where d denotes the exterior derivative on forms. The curl operator is linear and its μ-preserving eigenfields are known as the *Beltrami fields.* In other words, X is Beltrami if and only if it is volume-preserving and $\nabla \times X = \lambda X$ for some constant λ. One can also consider the class of eigenfields with scalar fields as eigenvalues: $\nabla \times X = fX$ for $f : M \to \mathbb{R}$. Our techniques are adaptable to this more general format, but we restrict to pure eigenfields here for simplicity.

Beltrami fields arise in several contexts:

1. Beltrami fields are always steady solutions to the Euler equations for an inviscid incompressible fluid

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p \quad ; \quad \mathcal{L}_u \mu = 0, \tag{2.2}$$

   where $\nabla_u u$ is the covariant derivative of the velocity field u along itself, and $p : M \to \mathbb{R}$, the pressure function, can be chosen to be

$\frac{1}{2}\|u\|^2$. The Lie derivative $\mathcal{L}_u\mu$ of the volume form along u vanishing is equivalent to u being divergence-free.

(2) Beltrami fields also yield steady solutions to the ideal MHD equations

$$\begin{gathered}\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p + (\nabla \times B) \times B \\ \frac{\partial B}{\partial t} - \nabla \times (u \times B) = 0 \\ \mathcal{L}_u\mu = \mathcal{L}_B\mu = 0,\end{gathered} \tag{2.3}$$

where B denotes the magnetic field. In this context, Beltrami fields are known as force-free fields.

(3) Beltrami fields are all extrema of the L^2 energy functional

$$\|u\|_2 := \frac{1}{2}\int_M \|u\|^2 \, d\mu \tag{2.4}$$

under the action of the volume-preserving diffeomorphism group of M. Eigenfields of curl having the smallest nonzero eigenvalue globally minimize the energy [2, 3].

Beltrami fields also play a role in the analysis of the stability of matter [33] and in the formation of dynamos [7].

The topology and dynamics of Beltrami fields are subtle: witness the complex dynamics of the well-known *ABC fields* on the Euclidean 3-torus [8]. The existence of fixed-point-free Beltrami fields on general Riemannian 3-manifolds is highly nontrivial [14], as is the presence of closed orbits within such fields [14, 15].

3. CONTACT STRUCTURES AND TOPOLOGY

Contact structures are the natural complements to Beltrami fields. Loosely put, a contact structure on an odd-dimensional manifold M is a hyperplane field which is maximally nonintegrable. More specifically, on a three-dimensional manifold, a contact structure is a smoothly-varying plane field (a choice of a two-dimensional subspace ξ_p in each tangent space T_pM) which cannot be stitched together into leaves of a foliation, not even at a point. Locally, every contact structure ξ is the kernel of a differential 1-form α satisfying the contact condition:

$$\alpha \wedge d\alpha \neq 0. \tag{3.1}$$

Otherwise said, $\alpha \wedge d\alpha$ is locally a volume form on M. Any 1-form satisfying (3.1) is called a contact form. If $\alpha \wedge d\alpha$ is a globally defined volume form on M, then the contact structure is said to be *cooriented*: all contact structures which arise in connection with Beltrami fields are of necessity cooriented, and we will restrict entirely to this category.

A contact form α on an oriented three-manifold M is said to be *positive* if the sign of $\alpha \wedge d\alpha$ is positive with respect to the orientation of M. Otherwise, α is said to be *negative*. The sign is a property of the contact structure and is independent of the defining 1-form.

Canonical examples of positive contact forms on $\mathbb{R}^3$ include $dz + x\,dy$, $dz + r^2 \sin r\,d\theta$, and $\cos r\,dz + r \sin r\,d\theta$, the latter two being given in cylindrical coordinates.

Much of the current interest in contact structures arises from fairly recent elucidations of their topological and dynamical properties (see, *e.g.*, [1, 12]). Studying contact structures by means of *characteristic foliations* is most fruitful. Given a two-dimensional surface S embedded in M, the characteristic foliation of S, S_ξ, is the [singular] one-dimensional foliation generated by the intersections of the tangent planes of T_pS with the contact planes ξ_p in T_pM. For all intents and purposes, S_ξ may be thought of as a vector field on S generated by ξ (by orienting the foliation). The singularities which arise on a characteristic foliation are generically saddles or spiral sources/sinks: pure centers cannot ever appear from a contact structure — the plane field twists too much for this.

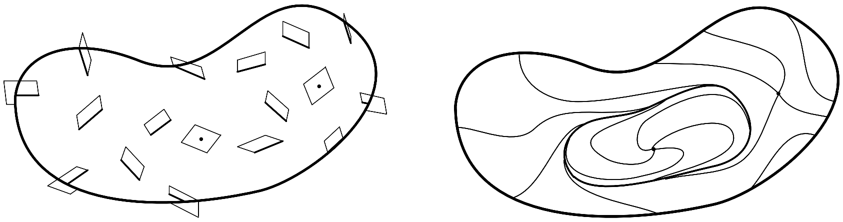

FIGURE 1. A contact structure is overtwisted if the induced characteristic foliation on some embedded disc possesses a limit cycle.

The dynamical properties of S_ξ are closely related to the topological classification of contact structures. A contact structure ξ is said to be *overtwisted* if there exists an embedded disc $D \subset M$ such that D_ξ possesses a limit cycle — a closed orbit along which nearby orbits accumulate [see Figure 1]. A contact structure is said to be *tight* if there are no such overtwisted discs anywhere in M. The contact structures for $dz + x\,dy$ and $dz + r^2 d\theta$ are tight [4], while that of $\cos r\,dz + r \sin r\,d\theta$ is overtwisted (*e.g.*, at the disc $\{r \leq 1, z = r^2\}$).

It is by no means apparent that the above definition is at all helpful: but in fact, the entire topological theory hangs on this dichotomy. Many major questions about contact structures are solved in the overtwisted category and unknown in the tight category (or, if known, then only recently and then by great skill and effort). For example, while overtwisted contact structures have been completely classified up to isotopy [9], the classification of tight structures appears [from what is presently known] to be delicate at best, intractable at worst [4, 21, 30, 13, 27, 28, 17, 18].

4. Contact dynamics

The connection between contact structures and Beltrami fields is, in the present context, quite straightforward. Several authors have noted the presence of a contact structure within the Beltrami condition (*e.g.,* [5, 20, 24]). We are unaware of any prior applications of this underlying structure to the dynamics of the flow.

Given any fixed-point-free Beltrami field X on M, the Beltrami condition states that

$$\mu(\lambda X, \cdot, \cdot) = d(g(X, \cdot)). \tag{4.1}$$

From this one can derive the crucial observation that the plane field orthogonal to any nonvanishing Beltrami field is indeed a contact structure as follows. Witness the 1-form $\alpha := g(X, \cdot)$ dual to X via g. The kernel of this 1-form represents the orthogonal plane field ξ to X. This form α is a contact form on M since

$$\alpha \wedge d\alpha := \lambda g(X, \cdot) \wedge \mu(X, \cdot, \cdot), \tag{4.2}$$

which, for $\lambda \neq 0$, is nowhere vanishing, as one can easily check by evaluating on local orthogonal coordinate bases of the form $(e_1 := X/\left\|X\right\|, e_2, e_3)$.

A little more is in fact true: a Beltrami field annihilates the exterior derivative of the associated contact form, since

$$(d\alpha)(X, \cdot) = \mu(X, X, \cdot) = 0. \tag{4.3}$$

Such vector fields are classical objects known as *Reeb fields.* The Reeb field of a contact form α is the unique vector field Z such that $d\alpha(Z, \cdot) = 0$ and $\alpha(Z) = 1$. We have thus observed that any Beltrami field X (nonsingular with nonzero eigenvalue) is a Reeb field for a contact form, after a possible rescaling to force $\alpha(X) = 1$. In [14], a broader version of this result was demonstrated: namely, that the class of nonsingular Beltrami fields on M (up to scaling, for any Riemannian structure and volume form) is identical to the class of Reeb fields (up to scaling, for any contact form).

This theorem allows one to build "custom" solutions to the steady Euler equations of very high regularity. For example, [16] builds a single Beltrami field on a Riemannian $\mathbb{R}^3$ which possesses closed orbits of all possible knot and link types simultaneously. This can be viewed as a rigorous manifestation of the delightful results of Moffatt on knotting in the Euler equations [37].

In the present context, we will use this simple correspondence between Beltrami and Reeb fields to import technology from contact dynamics. Most specifically, we are interested in the utility of the tight/overtwisted dichotomy in describing the dynamics of Beltrami fields. One extremely important result in contact dynamics is the following theorem of Hofer [26]: *Let ξ be an overtwisted contact structure on M a compact 3-manifold without boundary. Then the Reeb field of any contact form associated to ξ possesses a closed orbit, some multiple of which bounds a disc in M.*

Loosely speaking, a closed orbit (limit cycle) in the characteristic foliation of a disc in M implies the existence of a closed Reeb orbit which bounds a disc in M. The proof relies on the delicate techniques of pseudo-holomorphic curves in symplectic manifolds. One indication of the implicit nature of the proof is that the location of the overtwisted disc has little to no correlation with the location of the implicated Reeb orbit.

It follows from well-known properties of pseudo-holomorphic curves that the proof of Hofer's theorem remains valid for a three-manifold with invariant boundary (see [15] for details). Thus, for the solid torus, it follows that an overtwisted Reeb field must possess a contractible periodic orbit (since the fundamental group of the solid torus contains no elements of finite order except the identity).

5. Definition of $\mathbb{I}$

From Hofer's theorem, then, one way to force a contractible orbit in a Beltrami field is by finding an overtwisted disc in the orthogonal contact structure. This is far from trivial, since it requires searching for overtwisted discs among all possible embedded discs in M: not a computationally feasible task, even if the vector field were known analytically (which, in the context of an experimentally generated flow is not generally the case).

The classification of contact structures has been successfully completed only on a selected class of three-manifolds. The classification of contact structures on the solid torus is quite recent and subtle [22, 27]. In particular, it is known that on the solid torus there are tight contact structures which are *virtually overtwisted* — taking some finite covering space of the solid torus and lifting the tight contact structure yields an overtwisted contact structure on the cover [35, 15] (see Theorem 5.2). While this is a complication for contact topologists, it is a benefit to dynamicists.

Lemma 5.1. *Any Beltrami field orthogonal to a virtually overtwisted contact structure must possess a contractible closed orbit.*

Proof: Assume that ξ is a contact structure, some finite cover $\tilde{\xi}$ of which is overtwisted. Then, given any Beltrami field X orthogonal to ξ, lift this to a Beltrami field $\tilde{X}$ on the overtwisted cover. Applying Hofer's theorem to the cover implies the existence of a contractible periodic orbit for $\tilde{X}$; however, since a covering space projection takes orbits to orbits, the closed orbit upstairs (along with the disc that it bounds) *must* project to a contractible closed orbit of the original Beltrami field X. □

Thus, our goal is to effectively determine the existence of an overtwisted or virtually (with respect to coverings) overtwisted structure on a solid torus given the least amount of information about a Beltrami field transverse to it.

To do so, we recall a common index used in contact topology (see [1, 10] for an introduction). Given a contact structure ξ on a three-manifolds M, a simple closed curve (knot) is called *transverse* if its tangents are everywhere transverse to the contact planes. Assume that γ is an oriented simple closed curve in M which bounds a compact oriented surface Σ in M. Then the *self-linking number* of γ with respect to ξ and Σ is defined as follows. Choose any vector field Z on a neighborhood of Σ which has no fixed points and which is always tangent to ξ (that this is possible is a simple argument involving the classification of plane bundles). Then, evolve γ for a small amount of time under the flow of Z to obtain a "push-off" curve γ_Z. The self-linking number of γ, $s\ell k(\gamma)$, is then defined as the intersection number of γ_Z with Σ — *i.e.*, the number of transverse intersections, counted algebraically using the orientations. This integer, which can be shown to be independent of the vector field Z chosen, is an invariant of transverse curves up to isotopy through transverse curves [4]. On S^3, the self-linking number is also independent of the surface Σ chosen so long as it is bounded by γ.

The following recent result allows for an application of this index to Beltrami fields.

Theorem 5.2 ([15]). *Assume α is a positive contact form on a solid torus V whose Reeb field is tangent to the boundary ∂V. Choose any transverse curve γ on the boundary torus ∂V which bounds a meridional disc in V. If the self-linking number $s\ell k(\gamma)$ of this meridian is not equal to -1, then the pullback of α under some finite cover is an overtwisted contact form.*

It known from the inequality of [4] that if the initial contact structure is tight, the self linking number must satisfy $s\ell k \leq -\chi(D) = -1$. Thus, any self-linking number greater than -1 automatically implies an overtwisted structure (which is of course preserved under covers). The novel result of this theorem is that for a tight structure, a self-linking number less than -1 implies an overtwisted cover. The techniques used in the proof of this theorem are a combination of perturbing characteristic foliations, manipulating singularities of characteristic foliations, and using dynamical properties of the characteristic foliation on ∂V.

Remark 5.3. It is necessary to distinguish between positive and negative contact structures. In the case of a negative contact form (one for which $\alpha \wedge d\alpha < 0$) on an invariant solid torus, the structure possesses an overtwisted cover if and only if the self-linking number of a transverse meridian is not equal to $+1$. It is an easy exercise to show that the sign of the contact form dual to a curl eigenfield is precisely the sign of the eigenvalue.

From these ingredients the following index may be defined:

Definition 5.4. Given a nonsingular Beltrami field X on an invariant Riemannian solid torus V, define the index $\mathbb{I}$ as follows.

(1) If the eigenvalue λ of X with respect to the curl operator is zero, define $\mathbb{I} := 0$.
(2) Otherwise, consider the characteristic foliation $(\partial V)_\xi$ of the contact structure ξ orthogonal to X on ∂V. If possible, choose γ any meridional curve on ∂V transverse to ∂V_ξ and define

$$\mathbb{I} := \mathrm{Sign}(\lambda)\,(s\ell k(\gamma)) + 1. \tag{5.1}$$

(3) If no transverse curve γ exists, define $\mathbb{I} := \mathrm{Sign}(\lambda)$.

Proofs concerning the well-definedness and effectiveness of the index will rely on the following crucial lemma:

Lemma 5.5. *Assume that X is a Beltrami field and ξ is the orthogonal contact structure on an invariant solid torus V. If the characteristic foliation $(\partial V)_\xi$ on the boundary possesses a closed curve, then ∂V is completely foliated by closed curves.*

Proof: Consider the Reeb field Z associated to the contact form α dual to X. From the definition of the Reeb field and the Cartan formula for the Lie derivative, one computes that the Lie derivative of α along Z vanishes:

$$\mathcal{L}_Z\alpha = \iota_Z d\alpha + d(\iota_Z\alpha) = 0. \tag{5.2}$$

Thus, the flow of Z preserves the contact form α, and hence the contact structure ξ and the characteristic foliation $(\partial V)_\xi$. If $(\partial V)_\xi$ contains a closed curve, then, since Z is everywhere transverse to $(\partial V)_\xi$, the entire boundary torus is swept out by forward images of this curve under the flow of Z, and $(\partial V)_\xi$ is a foliation by closed curves. □

Proposition 5.6. *The index $\mathbb{I}$ is well-defined.*

Proof: It suffices to show that the index is independent of the choice of transverse meridional curve γ. The self-linking number is an invariant of a closed curve up to isotopy through closed curves which are transverse to the contact structure [4]. If the characteristic foliation $(\partial V)_\xi$ on the boundary possesses a closed leaf, then the boundary torus is foliated by closed leaves via Lemma 5.5, and any closed meridian transverse to $(\partial V)_\xi$ can slide around the longitude to every other transverse meridian through transverse isotopy. If $(\partial V)_\xi$ does not possess a closed leaf, then, by the Poincaré-Bendixson Theorem, the foliation $(\partial V)_\xi$ must be conjugate to an irrational linear foliation, and once again all transverse meridians are connected by a transverse isotopy which preserves the index. □

Theorem 5.7. *Any C^2 or smoother nonvanishing Beltrami field on an invariant Riemannian solid torus V having nonzero index $\mathbb{I}$ possesses a contractible closed orbit.*

Proof: Assume that $\lambda > 0$ as the negative case follows similarly. Since $\mathbb{I} \neq 0$, we are either in the case where there is a transverse meridian on the boundary of V with self-linking number not equal to -1, or there is no transverse meridian. In the case where the transverse meridian exists, some finite cover of the Beltrami field has a contractible closed orbit which is preserved by the covering projection.

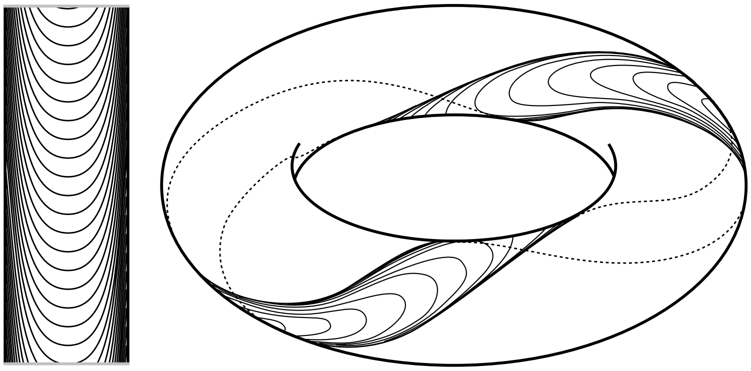

Figure 2. A Reeb component in a two-dimensional foliation is a foliation of an annulus by curves which limit onto the boundary components as illustrated [left, identify top and bottom]. On the boundary torus ∂V, if the characteristic foliation $(\partial V)_\xi$ possesses a Reeb component [right], then there does not exist a transverse meridional curve.

If there does not exist a closed transversal, then a basic result in foliation theory implies that either (1) the characteristic foliation $(\partial V)_\xi$ is entirely by meridional curves; or (2) the characteristic foliation $(\partial V)_\xi$ possesses a *Reeb component*, illustrated in Figure 2. In the former case, the contact structure is clearly overtwisted: any meridional disc in general position which spans one of these meridional curves has this boundary curve as a limit cycle in its characteristic foliation. The existence of a periodic orbit then follows as earlier. The latter case, where a Reeb component exists in $(\partial V)_\xi$, is impossible thanks to Lemma 5.5, since this characteristic foliation has isolated closed leaves. $\square$

The following corollary is obvious and would not be worth mentioning except for the fact that we present a method for computing $\mathbb{I}$ which requires information about the vector field on any generic 2-d meridional slice of the solid torus.

Corollary 5.8. *Any Beltrami field on an invariant solid torus with* $\mathbb{I} \neq 0$ *does not possess a global cross-section to the flow.*

Note that determining the non-existence of a global section is a difficult procedure in general.

6. Computation of $\mathbb{I}$

Given a contact structure ξ on a 3-manifold M, the determination of whether it is a tight or overtwisted structure is a difficult question in general. By the Darboux Theorem (see, *e.g.*, [1, 36]), every contact structure in dimension three is locally equivalent to the kernel of $dz + x\,dy$, which is a tight structure [4]. Thus, on the one hand, the property of being overtwisted is a decidedly global feature. However, the process of *Lutz twisting* [34] allows one to change a tight structure into an overtwisted structure by means of an alteration on an arbitrarily small open set in M. Thus, given a Beltrami field X on M, determining whether the contact structure orthogonal to X is overtwisted is computationally intractable. Determining whether the universal cover is overtwisted is no less difficult.

However, to compute the index $\mathbb{I}$, one does not need information about the vector field on the entire three-dimensional regime, but rather on some (arbitrary) two-dimensional meridional disc. We outline a method for easily computing $\mathbb{I}$ from a C^1 approximation to X along a two-dimensional slice of the solid torus.

Choose a meridional disc D with boundary curve γ. Orient γ so that the contact form $\alpha := g(X, \cdot)$ evaluates to a positive number on the tangents to γ: *i.e.*, γ points in roughly the same direction as X. This orientation on γ induces in the usual way an orientation on the disc D. Denote by X_D the projection of the vector field X on to D by orthogonal projection onto the tangent planes (orthogonal with respect to the metric g).

Proposition 6.1. *If the vector field* X_D *is generic (possesses a finite number of nondegenerate rest points), the index* $\mathbb{I}$ *of the Beltrami field* X *can be computed by*

$$\mathbb{I} = \mathrm{Sign}(\lambda)\left(1 + \sum_{p : X_D(p)=0} \sigma(p)\mathrm{Ind}(X_D; p)\right) \tag{6.1}$$

where for every rest point p *of* X_D, $\sigma(p)$ *is defined to be the sign* $(+/-)$ *of the dot product of* $X(p)$ *with the positive normal vector to* D *at* p, *and the term* $\mathrm{Ind}(X_D; p)$ *denotes the standard Euler-Poincaré index of the planar vector field* X_D *at* p.

Proof: Assume that the characteristic foliation D_ξ is generic in the above sense: there are a finite number of singular points p at which the contact structure ξ is tangent to D, and the characteristic foliation about these points appears locally as a source/sink or a saddle. Following [4, 10], there is a standard formula for computing the self-linking number of the transverse curve $\gamma = \partial D$:

$$s\ell k(\gamma) = \sum_{p:D_\xi(p)=0} \operatorname{sign}(T_pD, \xi_p)\operatorname{Ind}(D_\xi; p) \tag{6.2}$$

Here, the sign of (T_pD, ξ_p) is $+1$ when the orientations on the contact plane ξ and the orientations on the tangent plane to disc D at p agree. Otherwise the sign is -1. However, we only know the Beltrami field X and the disc D — not the characteristic foliation. Determining the indices of the rest points of the characteristic foliation D_ξ is accomplished via the projected field X_D as follows. Since the contact structure for X is the plane field ξ orthogonal to X, the characteristic foliation at every point $q \in D$ is given by

$$D_\xi(q) := \xi_q \cap T_qD = (X_D(q))^\perp,$$

the line field orthogonal to X_D at q. The proposition is proved by noting (1) fixed points of X_D occur exactly at fixed points of D_ξ; (2) the Euler-Poincaré index of D_ξ at a fixed point p is unchanged by looking at the orthogonal vector field, as illustrated in Figure 3. □

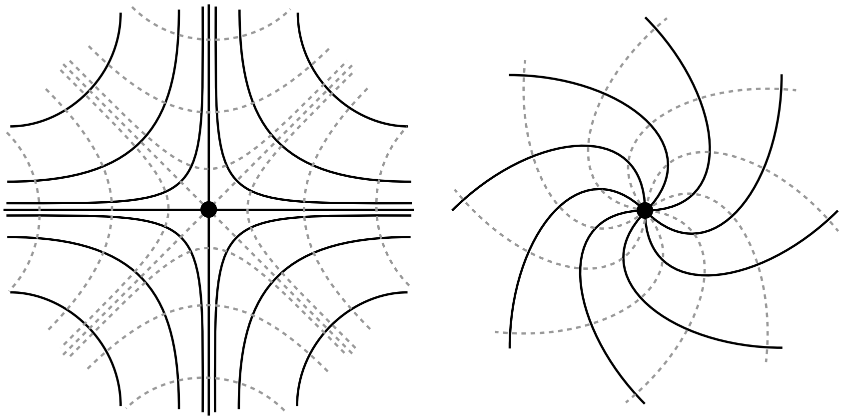

FIGURE 3. Taking the orthogonal line field X_D (dashed) to the characteristic foliation D_ξ (solid) leaves the index invariant.

Computationally, this is extremely simple as D can be chosen almost arbitrarily (one presumably chooses a D which is "nice" in coordinates) and information about X is required only on D itself. The local index calculation is the most delicate portion of the computation: the location of the orthogonal point p is easy and the sign $\sigma(p)$ merely measures whether X agrees with the oriented normal to D, which is trivial to determine.

An example of a characteristic foliation D_ξ and the resulting self-linking number is given in Figure 4.

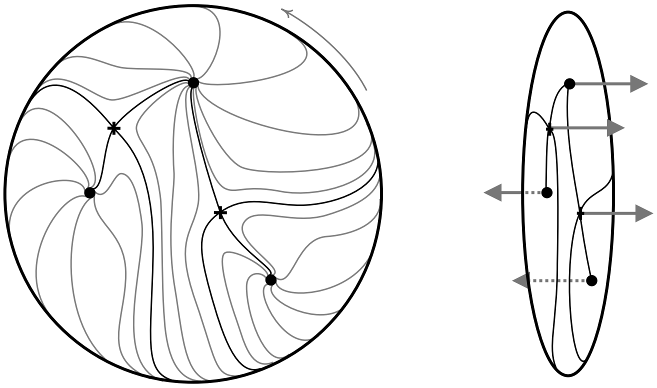

Figure 4. An example of an $s\ell k$ computation given the characteristic foliation on an oriented spanning disc D. [left] D_ξ; [right] the directions of the field X, positive to the left and negative to the right. The self-linking number is $s\ell k = -3$. Thus the Beltrami field X has index $\mathbb{I} = -2$.

Note that, with reagrds to using the index to determine the non-existence of a cross-section to the flow, a nonzero index indicates the presence of either a contractible periodic orbit or a fixed point in the Beltrami field.

7. Conclusions

The index $\mathbb{I}$ is an unusual object in that one inputs information about the vector field which is strictly *two*-dimensional, yet one obtains data about the dynamics which is fully *three*-dimensional. In Figure 4, the *only* information about the Beltrami field known is that (1) it is orthogonal to the disc D at five points; (2) locally near those five points the projected field is either source/sink or saddle type; and (3) at those five points the field points out or in as illustrated. From this data, it is inevitable that *somewhere* in the flow there exists a contractible closed flowline. This is a corollary of the contact-topological methods used — the Beltrami condition hides within it certain constraints on the dynamics which couple the dynamics of the vector field to the topology of the orthogonal plane field.

A deficiency of our theory is that it is not sharp. It is certainly possible for a Beltrami field to have the value $\mathbb{I} = 0$ and yet still have a contractible periodic orbit: indeed, any universally tight (*i.e.*, not virtually overtwisted) contact structure which is then Lutz twisted along a contractible closed curve necessarily has trivial index as well as a contractible orbit. It appears certain (due to this mechanism of arbitrarily small Lutz twists) that no completely sharp computable index can be defined. What $\mathbb{I}$ does, however, is detect if there are contractible orbits forced by the presence of non-localized overtwisted discs, and in this regime it is efficacious.

It would be interesting to find a sharp lower bound on the number of periodic orbits present in the case of $\mathbb{I} \neq 0$ (cf. the recent body of theory surrounding the *contact homology* of Eliashberg, Givental, and Hofer [11]).

In addition, it would be worthwhile exploring whether or not the non-existence of a global cross-section to a Beltrami field automatically forces the presence of a contractible periodic orbit.

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