COVERAGE IN SENSOR NETWORKS VIA PERSISTENT HOMOLOGY

VIN DE SILVA AND ROBERT GHRIST

ABSTRACT. We consider coverage problems in sensor networks with minimal sensing capabilities. In particular, we demonstrate that a stationary collection of sensor nodes with no localization can verify coverage in a bounded domain of unknown topological type, so long as the boundary is not too pinched. The only sensing capabilities required by the nodes are a binary form of distance estimation between nodes and a binary proximity sensor for the boundary. The methods we introduce come from persistent homology theory.

1. INTRODUCTION

Coverage problems arise in a variety of networks, communication and security being prominent examples. Given a collection of nodes in a fixed domain, each node having a neighborhood in which its sensors are active, one wants to know the extent of coverage by the nodes' sensor regions. In this paper we focus attention on the particular class of static, blanket coverage. By "static" is meant that the nodes are stationary; by "blanket" it is meant that one wants to determine if the entire domain is covered by sensor regions based at the nodes.

We give a homological criterion for certifying coverage. The criterion is centralized (as opposed to distributed) and conservative (failure of the criterion does not imply failure of coverage). This coverage represents a novel application of classical ideas in homology theory.

1.1. **Assumptions.** The methods we introduce are meant to work in settings where there are a large number of nodes with minimal and localized sensing capabilities. They have limited range and are devoid of localization and orientation capabilities, possessing merely a binary form of in-range distance measurement. More specifically, each node has a unique ID which it broadcasts. All other robots within range can "hear" its neighbor as either a strong or weak signal, depending on the distance to that node. We assume a small amount of information about the underlying domain $\mathcal{D} \subset \mathbb{R}^d$: one knows only the dimension and connectivity and that the domain is not too 'pinched' or 'wrinkled'. It is not necessary to assume knowledge of the topology of the domain, or of its large-scale geometry (e.g., volume).

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- **A1** Nodes have radially symmetric covering domains (for sensing or broadcasting) of *covering radius* r_c.
- **A2** Nodes broadcast their unique ID numbers. Each node can detect the identity of any node within radius r_s via a *strong* signal, or via a *weak* signal within a larger radius r_w.
- **A3** The radii of communication r_s, r_w and the covering radius r_c satisfy

$$r_c \geq r_s/\sqrt{2} \qquad ; \qquad r_w \geq r_s\sqrt{10}\,, \tag{1.1}$$

- **A4** Nodes lie in a compact domain $\mathcal{D} \subset \mathbb{R}^d$. Nodes can detect the presence (but not the location or direction) of the boundary $\partial\mathcal{D}$ within a *fence detection radius* r_f.
- **A5** The restricted domain $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ is connected, where

$$N_{\hat{r}}(\partial\mathcal{D}) = \{x \in \mathcal{D} : \|x - \partial\mathcal{D}\| \leq \hat{r}\}\,, \text{ where } \hat{r} = r_f + r_s/\sqrt{2}. \tag{1.2}$$

- **A6** The fence detection hypersurface $\Sigma = \{x \in \mathcal{D} : \|x - \partial\mathcal{D}\| = r_f\}$ has internal injectivity radius at least $r_s/\sqrt{2}$ and external injectivity radius at least r_s.

Assumptions **A1**-**A4** specify the communication capabilities of the nodes. Assumption **A5** is needed to prevent the domain from being too 'pinched'. This is clearly necessary since nodes with neither a map nor coordinates cannot distinguish between certain pinched domains and a disconnected domain. Assumption **A6** means that the outermost boundary cannot exhibit large-scale 'wrinkling'. This assumption is used in the details of the proof of Theorem 3.4 for eliminating pathological configurations. See Remark 4.5 for discussion on weakening this condition.

The last assumptions, **A5** and **A6**, are the only restrictions on the geometry of the domain. We emphasize that the number of boundary components is not assumed to be known: nodes have no information about the boundary other than whether they are within range r_f. This r_f is independent of the node-to-node communication radii r_s and r_w and the coverage radius r_c. The volume of the domain is not assumed to be known, and convexity is not at all required.

To summarize, the sensor data consists of three ingredients. Each node ascertains a primary list of node ID numbers associated to a 'strong' detection signal, as well as a secondary list of node ID numbers flagged as coming from the 'weak' signal. The third piece of data associated to each node is a fence-detection binary flag.

Remark 1.1. The numerical constants which appear in assumptions **A3**, **A5** and **A6** are independent of the ambient dimension d. There are tighter constants which depend on d, and we summarize those results later on. Our proofs will be structured so that we can read off the improved constants without additional work.

1.2. **Results.** This coordinate-free data can be sufficient to rigorously verify coverage of the domain (ignoring regions too close to the boundary) in certain cases.

Our strategy is as follows. We build a nested collection of graphs

$$\begin{array}{ccc} G_s & \xrightarrow{\subset} & G_w \\ \cup\uparrow & & \uparrow\cup \\ F_s & \xrightarrow{\subset} & F_w \end{array}$$

The graphs G_s and G_w are defined in the obvious manner via communication links: the vertices are the nodes of the network, and the edges are present between nodes which are within distance r_s and r_w respectively. These are *communication graphs* for the strong and weak signals respectively. The graphs F_s and F_w are the strong and weak *fence subgraphs* — the maximal subgraphs of G_s (respectively G_w) whose nodes all lie within the fence detection radius.

From these four graphs, we define a system of nested simplicial complexes,

$$\begin{array}{ccc} \mathcal{R}_s & \xrightarrow{\subset} & \mathcal{R}_w \\ \cup\uparrow & & \uparrow\cup \\ \mathcal{F}_s & \xrightarrow{\subset} & \mathcal{F}_w \end{array},$$

called Rips complexes, whose simplices are determined by 'filling in' the corresponding graphs: each is the largest simplicial complex with the corresponding graph as its 1-d skeleton.

The **sensor cover**, $\mathcal{U}$, is the union over $\mathcal{X}$ of discs of radius r_c. Our results link the topology of the cover $\mathcal{U}$ to the homology of the diagram of Rips complexes.

Main Theorem: *For a fixed set of nodes $\mathcal{X}$ in a domain $\mathcal{D} \subset \mathbb{R}^d$ satisfying assumptions* **A1**-**A6**, *the sensor cover $\mathcal{U}$ contains $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ if the homomorphism*

$$\iota_* : H_d(\mathcal{R}_s, \mathcal{F}_s) \to H_d(\mathcal{R}_w, \mathcal{F}_w) \tag{1.3}$$

induced by the inclusion $\iota : \mathcal{R}_s \hookrightarrow \mathcal{R}_w$ is nonzero.

Increasingly, homology is a practical tools in applications which require computation of global structure: see the texts [10, 12] for an introduction, the latter containing several current applications in science and engineering. Such applications include vision and recognition [1], hybrid systems and control theory [2], rigorous verification of dynamics from experimental data [17], and global analysis of large data sets [4]. In the last example, there is a growing literature on the importance of **persistent** homology — homology classes which persist as one changes a parameter in the system. It is this perspective that inspired the work in this paper.

1.3. **Related work.** The large literature on coverage problems for networks rests on two pillars of techniques. The first, the computational geometry approach, takes as its argument the geometric structure of the nodes — precise coordinates — and returns an auxiliary structure from which coverage or non-coverage can be deduced. Typical in this approach is the use of Delaunay triangulations (in 2-d), see, e.g., [16, 14, 20]. The precision with which coverage and lack of coverage can be determined is offset by the precision with which the coordinates of the nodes must be measured; such techniques are thus inapplicable in the context of 'minimal' coordinate-free sensors. The second approach uses probabilistic tools. Under assumptions of a uniform random distribution of points and a domain of known geometry, one can prove results about probability of coverage at a given density of nodes. Such methods are more appropriate in contexts for which the coordinates of nodes are unknown, see [13, 19, 15]; however, the assumptions on the uniformity of the distribution are crucial.

2. Factoring coverage through communication

Given a collection of nodes $\mathcal{X}$ in a domain, we wish to determine the global properties of $\mathcal{U}$, the union of coverage domains centered at these nodes. However, we are constrained to use only communication connectivity data between nodes. In this section, we outline the basic constructions used to form a coverage criterion in the subsequent section.

2.1. Simplicial complexes for covers.

The problem of computing the topological type of a union of sets is classical, and easily handled using the concept of a Čech complex (also called the **nerve** of the cover).

Definition 2.1. Given a collection of sets $\mathcal{U} = \{U_\alpha\}$, the **Čech complex** of $\mathcal{U}$, $\mathcal{C}(\mathcal{U})$, is the abstract simplicial complex whose k-simplices correspond to nonempty intersections of $k+1$ distinct elements of $\mathcal{U}$.

If the cover is **good** — that is, if the cover sets and all nonempty finite intersections of cover sets are contractible — then the Čech complex $\mathcal{C}$ captures the topology of the cover: (see, e.g., [10])

Theorem 2.2 (The Čech Theorem)**.** *The Čech complex of a good cover has the homotopy type of the union of the cover sets.*

Unfortunately, it is highly nontrivial to compute a Čech complex: one needs very precise data on pairwise distances between nodes. In the context of a sensor network with minimal range-sensing capabilities, the Čech complex is seemingly unattainable. Therefore, we consider the following related construction, which is more adapted to communication network constraints.

The Rips complex associated to a set of points is a notion originally developed by Vietoris in the earliest development of homology theory [18]. The concept was revived by Rips' work in geometric group theory and now generally goes by his name (see [9, 11]).

Definition 2.3. Given a set of points $\mathcal{X} = \{x_\alpha\} \subset \mathbb{R}^n$ in Euclidean n-space and a fixed radius ϵ, the **Rips complex** of $\mathcal{X}$, $\mathcal{R}_\epsilon(\mathcal{X})$, is the abstract simplicial complex whose k-simplices correspond to unordered $(k+1)$-tuples of points in $\mathcal{X}$ which are pairwise within Euclidean distance ϵ of each other.

Definition 2.4. In the setting of our main theorem, $\mathcal{R}_s$, $\mathcal{R}_w$ denote the Rips complexes on $\mathcal{X}$ with radii r_s, r_w respectively.

The Rips complex is ideally suited to communication networks, since the entire complex is determined by pairwise communication data. Unfortunately, the Rips complex does not necessarily capture the topology of the union of cover discs: we have traded computability for accuracy. Figure 1 gives a fundamental class of examples for which the Rips complex fails to capture the Čech complex.

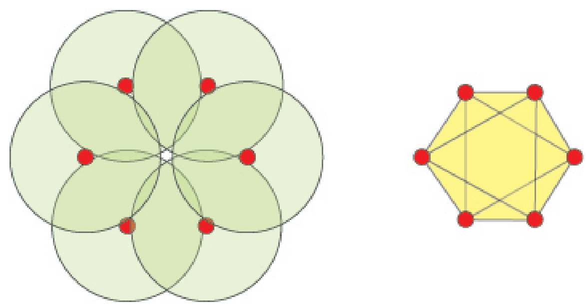

FIGURE 1. A class of examples for which the Rips complex $\mathcal{R}_s$ detects 'phantom' topological features. Take $2k+2$ points equidistributed on a circle of diameter $r_s + \lambda$ where $\lambda \ll 1$. The Čech complex (at the corresponding radius) is homotopy equivalent to a circle, as the Čech Theorem requires. The Rips complex however is isomorphic to the boundary of a *cross-polytope* in $k+1$ dimensions. This Rips complex is thus homeomorphic to the sphere S^k and accordingly is very different from the Čech complex for $k > 1$. Illustrated is the case $k = 2$, with $\mathcal{R}_s$ an octahedron.

2.2. **Optimal factorization of the Rips complex.** Since we assume that sensors can ascertain communication links, it follows that the 1-dimensional skeleton of the Čech complex *can* be determined directly from the sensor data. In the best of worlds, the Vietoris–Rips complex of the communication graph would suffice to capture the Čech complex: unfortunately, this is not true. However, one can "squeeze" the Čech complex (something hard to compute) in between $\mathcal{R}_s$ and $\mathcal{R}_w$ (something computable on the hardware level). In this subsection, we detail this nesting and prove optimality.

Theorem 2.5. *Let $\mathcal{X}$ be a set of points in $\mathbb{R}^d$ and $\mathcal{C}_\epsilon(\mathcal{X})$ the Čech complex of the cover of $\mathcal{X}$ by balls of radius $\epsilon/2$. Then there is chain of inclusions*

$$\mathcal{R}_{\epsilon'}(\mathcal{X}) \subset \mathcal{C}_\epsilon(\mathcal{X}) \subset \mathcal{R}_\epsilon(\mathcal{X}) \quad \textit{whenever} \quad \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{2d}{d+1}}.$$

Moreover, this ratio is the smallest for which the inclusions hold in general.

Proof: The second inclusion is trivial because the criterion for inclusion of a simplex in $\mathcal{R}_\epsilon$ is weaker than the criterion for inclusion of a simplex in $\mathcal{C}_\epsilon$ (if the balls of radius $\epsilon/2$ centered at the vertices have a common intersection then each pair of vertices is separated by distance at most ϵ).

The first inclusion is equivalent to the following assertion: if a collection of points in $\mathbb{R}^d$ is such that every pair is separated by a distance at most ϵ', then the balls of radius $\epsilon/2$ centered on these points have a common intersection. Proving this for a set of $k+1$ points implies that every k-simplex of $\mathcal{R}_{\epsilon'}$ belongs also to $\mathcal{C}_\epsilon$.

We will prove it first for a set of $d'+1$ points $\{x_0, x_1, \ldots, x_{d'}\}$, where $d' \leq d$. Consider the function $f : \mathbb{R}^d \to \mathbb{R}$ defined:

$$f(y) = \max_{0 \leq i \leq d'} \|x_i - y\|$$

This is continuous and moreover $f(y) \to +\infty$ as $\|y\| \to \infty$, so it follows that f has a global minimum $f(y_0)$, say. Define the *critical vertices* to be those points x_i for which $\|x_i - y_0\| = f(y_0)$.

Note that there is no vector v which *separates*, in the sense that $v \cdot (x_i - y_0) > 0$ for each critical vertex x_i. For such a vector we could calculate that

$$\begin{aligned} \|x_i - y_0\|^2 &= \|x_i - (y_0 + \lambda v)\|^2 + 2\lambda v \cdot (x_i - y_0) \\ &> \|x_i - (y_0 + \lambda v)\|^2 \end{aligned}$$

for all $\lambda > 0$ and therefore $f(y_0 + \lambda v) < f(y_0)$ for $0 < \lambda \ll 1$, contradicting minimality. Since no separating vector exists, y_0 must lie in the convex hull of the critical vertices.

It is convenient to make the translation $\hat{x}_i = x_i - y_0$. We can now find a convex combination $a_0\hat{x}_0 + a_1\hat{x}_1 + \cdots + a_{d''}\hat{x}_{d''} = 0$ for some $d'' \leq d'$, after relabeling so that $x_0, x_1, \ldots, x_{d''}$ are critical vertices, the coefficients a_i are strictly positive, and a_0 is the largest of the terms $a_0, a_1, \ldots, a_{d''}$. Then $-\hat{x}_0 = \sum_{i=1}^{d''}(a_i/a_0)\hat{x}_i$ and so

$$-f(y_0)^2 = -\|\hat{x}_0\|^2 = \sum_{i=1}^{d''}(a_i/a_0)\hat{x}_0 \cdot \hat{x}_i$$

At least one of the d'' terms on the right-hand side must satisfy $(a_i/a_0)\hat{x}_0 \cdot \hat{x}_i \leq -f(y_0)^2/d''$, which can be weakened and rearranged to $f(y_0)^2/d \leq -\hat{x}_0 \cdot \hat{x}_i$. We also know that $f(y_0)^2 = \|\hat{x}_0\|^2 = \|\hat{x}_i\|^2$. Putting this together

$$\begin{aligned} f(y_0)^2\,(1 + (2/d) + 1) &\leq \|\hat{x}_0\|^2 - 2\hat{x}_0 \cdot \hat{x}_i + \|\hat{x}_i\|^2 \\ &= \|\hat{x}_0 - \hat{x}_i\|^2 = \|x_0 - x_i\|^2 \leq (\epsilon')^2 \end{aligned}$$

and hence

$$f(y_0) \leq \frac{\epsilon'}{2}\sqrt{\frac{2d}{d+1}} \leq \frac{\epsilon}{2}$$

It follows that the balls of radius $\epsilon/2$ centered on the given $d+1$ points must meet at y_0.

For a set of greater than $d+1$ points, the result follows by applying Helly's theorem [7]. This asserts that a collection of $k \geq d+2$ convex sets in $\mathbb{R}^d$ has a nonempty common intersection provided only that the same is true for each subset of size $d+1$. If we have k points spanning a simplex in $\mathcal{R}_{\epsilon'}$, we have just established that each set of $d+1$ of the $\epsilon/2$-balls at these vertices must have a nonempty intersection. By Helly's theorem, the same is true for the entire set of k balls. Hence the vertices span a simplex in $\mathcal{C}_\epsilon$.

The lower bound on ϵ/ϵ' is tight in the case of a regular d-simplex. □

2.3. **Sensor complexes.** In order to determine coverage, it is necessary to know that there are no 'holes' in the interior of the cover; as well, one must check that the cover extends sufficiently far out to the boundary of the domain. This latter condition prompts the following:

Definition 2.6. Given a system with fence-detection as per assumption **A4**, define the strong and weak **fence subcomplexes**, $\mathcal{F}_s \subset \mathcal{R}_s$ and $\mathcal{F}_w \subset \mathcal{R}_w$ respectively, to be the maximal subcomplexes of $\mathcal{R}_s$ and $\mathcal{R}_w$ whose vertices lie within distance r_f of the $\partial\mathcal{D}$.

Lemma 2.7. *Under the assumptions of our main theorem (in particular $r_c \geq r_s/\sqrt{2}$), any collection of nodes in $\mathcal{D}$ which form a simplex of $\mathcal{R}_s$ has its convex hull entirely contained within $\mathcal{U}$.*

Remark 2.8. It follows from Lemma 2.7 that Theorem 3.4 is true in the trivial situation where $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ is entirely contained inside some d-simplex of $\mathcal{R}_s$.

The lemma can be read out of the following more precise result.

Lemma 2.9. *Let p belong to the convex hull of points $x_0, x_1, \ldots, x_k \in \mathbb{R}^d$ and suppose $\epsilon' \geq \epsilon\sqrt{2d/(d+1)}$. If $\|x_i - x_j\| \leq \epsilon$ for all i, j, then $\|p - x_i\| \leq \epsilon'/2$ for some i.*

Proof: By Theorem 2.5, the balls of radius $\epsilon'/2$ centered at the points x_i are guaranteed to meet at a common point y. Let $p = a_0x_0 + a_1x_1 + \cdots + a_kx_k$ be a convex combination of the nodes (so the a_i are nonnegative and sum to unity). We rearrange this as $0 = a_0\hat{x}_0 + a_1\hat{x}_1 + \cdots + a_k\hat{x}_k$ where $\hat{x}_i = x_i - p$. Taking the dot product with $\hat{y} = y - p$ we find that $0 = a_0\hat{x}_0 \cdot \hat{y} + a_1\hat{x}_1 \cdot \hat{y} + \cdots + a_k\hat{x}_k \cdot \hat{y}$ so for some i we must have $\hat{x}_i \cdot \hat{y} \leq 0$. In that case

$$\begin{aligned}(\epsilon'/2)^2 \geq \|x_i - y\|^2 \ = \ \|\hat{x}_i - \hat{y}\|^2 \ &= \ \|\hat{x}_i\|^2 - 2\hat{x}_i \cdot \hat{y} + \|\hat{y}\|^2 \\ &\geq \ \|\hat{x}_i\|^2 \ = \ \|x_i - p\|^2\end{aligned}$$

as required. □

Remark 2.10. If $k < d$ then the hypothesis in the preceding lemma can be weakened to $\epsilon' \geq \epsilon\sqrt{2k/(k+1)}$, since we can work in a k-dimensional affine subspace of $\mathbb{R}^d$.

3. A HOMOLOGICAL CRITERION FOR COVERAGE

Our criterion for coverage is based on the homology of the inclusion map $\iota : (\mathcal{R}_s, \mathcal{F}_s) \hookrightarrow (\mathcal{R}_w, \mathcal{F}_w)$. We claim that coverage is implied by having a nonzero generator for the top-dimensional relative homology group $H_d(\mathcal{R}_s, \mathcal{F}_s)$ which persists (remains nonzero) under the induced homomorphism ι_* to $H_d(\mathcal{R}_w, \mathcal{F}_w)$.

3.1. **Intuition and persistence.** With assumptions **A5** and **A6**, the top dimensional relative homology $H_d(\mathcal{D}, N_{\hat{r}}(\partial\mathcal{D}))$ has rank one. Furthermore, it is true that $H_d(\mathcal{U} \cup N_{\hat{r}}(\partial\mathcal{D}), N_{\hat{r}}(\partial\mathcal{D}))$ is nonzero if and only if $\mathcal{U}$ contains $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$. However, we cannot compute $\mathcal{U}$ directly. The simplicial complex which captures the topology of $\mathcal{U}$ — the Čech complex — is impossible to compute without coordinates or at least accurate distances between sensors. Rips complexes are, in contrast, very manageable merely with communication data (and hence computable on the hardware level). Thus, it would make sense to hope that if $H_d(\mathcal{R}_s, \mathcal{F}_s)$ is nonzero, then $\mathcal{U} \subset \mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$.

But this is not always the case. Consider the 2-d setting of Fig. 2, in which there is a cycle of points within $\mathcal{F}_s$ all of which are attached to a single vertex in $\mathcal{R}_s - \mathcal{F}_s$. This cycle is such that two of the edges are of length r_s, while the other two edges are of length $\epsilon \ll r_s$ As such, neither of the diagonals is of length r_s and is therefore not present in $\mathcal{F}_s$. This system has $H_2(\mathcal{R}_s, \mathcal{F}_s) \neq 0$: there exist "fake" relative 2-cycles which do not imply coverage of the entire domain. Other fake cycles can be generated from the examples of Fig. 1 in any dimension.

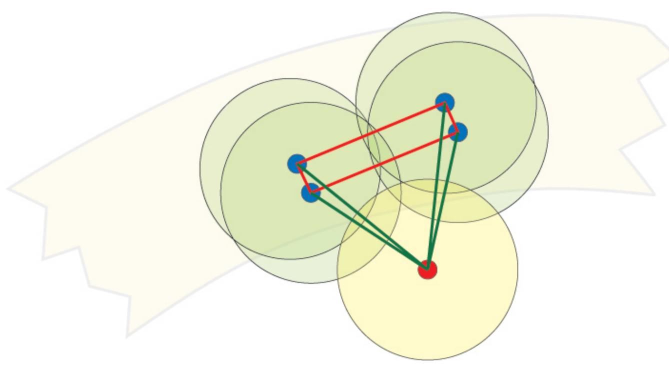

FIGURE 2. A fake generator for $H_2(\mathcal{R}_s, \mathcal{F}_s)$ which is annihilated by inclusion ι_* into $H_2(\mathcal{R}_w, \mathcal{F}_w)$. The strip illustrated is a collar of radius r_f.

Note, however, what happens to this relative 2-cycle under increasing the communication radius from r_s to r_w, then the loop in $\mathcal{F}_s$ is "filled in" by diagonals, and the image of this fake class under ι_* is the zero element of $H_2(\mathcal{R}_w, \mathcal{F}_w)$. Assuming that these points are a portion of a larger subset of nodes, it is *not necessarily* the case that $H_2(\mathcal{R}_w, \mathcal{F}_w) = 0$, since there may be a new fake 2-cycle which comes into existence at the longer communication lengths: but the original fake 2-cycle is annihilated by ι_*.

3.2. **Preliminary lemmas.** All of the difficulty in proving coverage comes from the analysis of the cover near the boundary $\partial\mathcal{D}$. For applications to sensor networks, we wish to minimize constraints on the number and types of boundary components. As a result, we can guarantee coverage only outside of a neighborhood of the boundary. We begin with some technical results concerning the geometry of how the fence nodes are situated.

Lemma 3.1. *Let $\overline{N_{\hat{r}}(\partial\mathcal{D})} = \mathbb{R}^d - (\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D}))$ denote the* **extended collar** *of $\mathcal{D}$. For any collection of nodes in $\mathcal{D}$ which form a simplex of $\mathcal{F}_s$, its convex hull lies within $\overline{N_{\hat{r}}(\partial\mathcal{D})}$, or else we are in the trivial case described in Remark 2.8.*

Proof: It suffices, by Carathéodory's Theorem [7], to show that the d-dimensional skeleton of $\mathcal{F}_s$ lies within $\overline{N_{\hat{r}}(\partial\mathcal{D})}$. First consider the $(d-1)$-skeleton. For $k \leq d-1$ let $x_0, x_1, \ldots, x_k \in \mathcal{X}_f$ define a k-simplex σ in $\mathcal{F}_s$. By Remark 2.10, for any $x \in \sigma$ there is an x_i such that $\|x_i - x\| \leq r_s\sqrt{(d-1)/2d}$. Combining this with the fact that $x_i \in \mathcal{X}_f$ using the triangle inequality yields $x \in \overline{N_{\hat{r}}(\partial\mathcal{D})}$.

Now suppose x lies in some d-simplex σ but is not in the $(d-1)$-skeleton. Then σ is not degenerate, and x must lie in its interior. Since $\mathcal{D} - \overline{N_{\hat{r}}(\partial\mathcal{D})} = \mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ is connected and does not meet $\partial\sigma$, it is either entirely contained in the interior of σ or it is disjoint from σ. Thus, either we are in the trivial situation of Remark 2.8, or $x \in \overline{N_{\hat{r}}(\partial\mathcal{D})}$. □

Remark 3.2. The proof is easier if we are not concerned with obtaining optimal dimension-dependent bounds. Then we can simply say that for any simplex σ (regardless of its dimension) and $x \in \sigma$ there is a vertex x_i such that $\|x_i - x\| \leq r_s/\sqrt{2}$.

The last and most technical lemma is a variant of Theorem 2.5 adapted to a $(d-1)$-cycle in a thickened hypersurface in $\mathbb{R}^d$ of thickness Δ. By this we mean a domain homeomorphic to a closed $(d-1)$-dimensional manifold cross an interval, which can be foliated by line segments of length no more than Δ.

Lemma 3.3. *Let $\mathcal{S} \subset \mathbb{R}^d$ be a thickened hypersurface of thickness Δ and let $\mathcal{X} \subset \mathcal{S}$ denote a collection of points which forms a $(d-1)$-cycle $[\gamma] \in H_{d-1}(\mathcal{R}_\epsilon(\mathcal{X}))$, for some $\epsilon > 0$ such that γ is contained entirely within $\mathcal{S}$. If $[\gamma] = 0$ in $H_{d-1}(\mathcal{S})$, then $[\gamma] = 0$ in the ϵ' Rips complex $\mathcal{R}_{\epsilon'}(\mathcal{X})$, where*

$$\epsilon' = \sqrt{\Delta^2 + 2\epsilon^2 \frac{d-1}{d}}. \tag{3.1}$$

Proof: Denote by γ the cycle as a geometric $(d-1)$-cycle in $\mathcal{S}$ and let $\mathcal{U}'$ denote the union over $\mathcal{X}$ of balls of radius $\epsilon'/2$. For our choice of ϵ' it follows from Remark 2.10

and Pythagoras' theorem that $\mathcal{U}'$ contains the set $\mathcal{U}$ obtained by covering every point of γ (simplices as well as vertices) with a ball of radius $\Delta/2$.

Assume by way of contradiction that $[\gamma] \neq 0$ in $H_{d-1}(\mathcal{R}_{\epsilon'}(\mathcal{X}))$ yet is trivial in $H_{d-1}(\mathcal{S})$. From Theorem 2.5, $\mathcal{R}_{\epsilon}(\mathcal{X}) \subseteq \mathcal{C}_{\epsilon'}(\mathcal{X}) \subseteq \mathcal{R}_{\epsilon'}(\mathcal{X})$. Thus, γ is a nontrivial cycle in $\mathcal{C}_{\epsilon'}(\mathcal{X})$. By the Čech Theorem and Alexander duality, there exists a point $p \in \mathcal{S} - \mathcal{U}'$ enclosed by γ.

Since $\mathcal{S}$ has thickness Δ, there is a line segment ℓ in $\mathcal{S}$ of length at most Δ passing through p and connecting the two boundary components of $\mathcal{S}$. As γ is trivial in $H_{d-1}(\mathcal{S})$, the two endpoints of ℓ are homologically not enclosed by γ, unlike p. For this reason ℓ must cross γ at least once on each side of p. Thus ℓ intersects $\mathcal{U}$ in at least two disjoint segments. Each such segment has length at least $\Delta/2$: contradiction. □

3.3. **The coverage criterion.** The following theorem is our principal coverage criterion:

Theorem 3.4. *For a fixed set of nodes $\mathcal{X}$ in a domain $\mathcal{D} \subset \mathbb{R}^d$ satisfying assumptions* **A1**-**A6**, *the sensor cover $\mathcal{U}$ contains $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ if the homomorphism*

$$\iota_* : H_d(\mathcal{R}_s, \mathcal{F}_s) \to H_d(\mathcal{R}_w, \mathcal{F}_w) \tag{3.2}$$

induced by the inclusion $\iota : \mathcal{R}_s \hookrightarrow \mathcal{R}_w$ is nonzero.

Proof: To start with, we can assume that we are not in the situation covered by Remark 2.8, where the theorem is trivially true.

Consider the simplicial realization map $\sigma : \mathcal{R}_s \to \mathcal{D}$ which sends vertices of $\mathcal{R}_s$ to the points $\mathcal{X} \subset \mathcal{D}$ and which sends a k-simplex of $\mathcal{R}_s$ to the (potentially degenerate) k-simplex given by the convex hull of the vertices implicated. Since the exceptional case is excluded, Lemma 3.1 implies that σ takes the pair $(\mathcal{R}_s, \mathcal{F}_s)$ to $(\mathbb{R}^d, \overline{N_{\hat{r}}(\partial\mathcal{D})})$; we therefore construct the following diagram from the long exact sequences of the pairs:

$$\begin{array}{ccc} H_d(\mathcal{R}_s, \mathcal{F}_s) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_s) \\ \downarrow{\sigma_*} & & \downarrow{\sigma_*} \\ H_d(\mathbb{R}^d, \overline{N_{\hat{r}}(\partial\mathcal{D})}) & \xrightarrow{\delta_*} & H_{d-1}(\overline{N_{\hat{r}}(\partial\mathcal{D})}) \end{array} \tag{3.3}$$

Here, δ_* acts on a class $[\alpha] \in H_d(\mathcal{R}_s, \mathcal{F}_s)$ by taking the boundary: $\delta_*[\alpha] = [\partial\alpha] \in H_{d-1}(\mathcal{F}_s)$. The diagram of Eqn. (3.3) is commutative: $\delta_*\sigma_* = \sigma_*\delta_*$. The homology class $\sigma_*\delta_*[\alpha]$ measures the degree of $\partial\alpha$, or how many times the boundary of α "wraps around" the extended collar $\overline{N_{\hat{r}}(\partial\mathcal{D})}$.

Now let $[\alpha] \in H_d(\mathcal{R}_s, \mathcal{F}_s)$ be a class for which $\iota_*[\alpha] \neq 0$.

Case 1: $\sigma_*\delta_*[\alpha] \neq 0$.

By commutativity of Eqn. (3.3), $\delta_*\sigma_*[\alpha] = \sigma_*\delta_*[\alpha] \neq 0$. Hence, $\sigma_*[\alpha] \neq 0$. Assume that $\mathcal{U}$ does not contain $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ and choose $p \in \mathcal{D} - (N_{\hat{r}}(\partial\mathcal{D}) \cup \mathcal{U})$. Since, by Lemma 2.7, every point in $\sigma(\mathcal{R}_s)$ lies within $\mathcal{U}$, this implies that $\sigma : (\mathcal{R}_s, \mathcal{F}_s) \to (\mathbb{R}^d, \overline{N_{\hat{r}}(\partial\mathcal{D})})$ factors through the pair $(\mathbb{R}^d - p, \overline{N_{\hat{r}}(\partial\mathcal{D})})$. However, $H_d(\mathbb{R}^d - p, \overline{N_{\hat{r}}(\partial\mathcal{D})}) = 0$ since, by Alexander duality, $H_d(\mathbb{R}^d - p, \overline{N_{\hat{r}}(\partial\mathcal{D})}) = H^0(\mathbb{R}^d - \overline{N_{\hat{r}}(\partial\mathcal{D})}, p)$, which vanishes since $\mathbb{R}^d - \overline{N_{\hat{r}}(\partial\mathcal{D})}$ is connected. This gives the contradiction $\sigma_*[\alpha] = 0$. Thus $\mathcal{U}$ contains $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ after all.

Case 2: $\sigma_*\delta_*[\alpha] = 0$.

We demonstrate that this case is impossible under the hypothesis $\iota_*[\alpha] \neq 0$. We construct the following commutative diagram with three rows, the top and bottom of which come from the long exact sequence of the pairs $(\mathcal{R}_s, \mathcal{F}_s)$ and $(\mathcal{R}_w, \mathcal{F}_w)$ respectively. The middle row comes from the pair $(\mathcal{R}_m, \mathcal{F}_m)$ — the Rips and Fence complexes computed at the "midrange" signal of radius

$$r_m = r_s\sqrt{\frac{7d - 5 + 2\sqrt{2d(d-1)}}{2d}}. \tag{3.4}$$

The inclusion map $\iota : (\mathcal{R}_s, \mathcal{F}_s) \hookrightarrow (\mathcal{R}_w, \mathcal{F}_w)$ factors through the pair $(\mathcal{R}_m, \mathcal{F}_m)$.

$$\begin{array}{ccccc}
H_d(\mathcal{R}_s) & \xrightarrow{j_*} & H_d(\mathcal{R}_s, \mathcal{F}_s) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_s) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
H_d(\mathcal{R}_m) & \xrightarrow{j_*} & H_d(\mathcal{R}_m, \mathcal{F}_m) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_m) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
H_d(\mathcal{R}_w) & \xrightarrow{j_*} & H_d(\mathcal{R}_w, \mathcal{F}_w) & \xrightarrow{\delta_*} & H_{d-1}(\mathcal{F}_w)
\end{array} \tag{3.5}$$

Represent the relative homology class $[\alpha]$ by an explicit cycle α, comprised of simplices in $\mathcal{R}_s - \mathcal{F}_s$. We claim that the geometric $(d-1)$-cycle $\sigma(\partial\alpha)$ is contained in a particular shell $\mathcal{S}$, defined as follows. Let Σ denote the hypersurface(s) of points at the precise fence detection radius:

$$\Sigma = \{x \in \mathcal{D} : \|x - \partial\mathcal{D}\| = r_f\}.$$

Let $\mathcal{S}$ denote the set of points in $\mathbb{R}^d$ within distance $r_s\sqrt{(d-1)/2d}$ of Σ on the interior side (i.e. the side of Σ corresponding to the interior of $\mathcal{D}$), and within distance r_s of Σ on the exterior side. It is helpful to define a signed distance function $|h(x)| = d(x, \Sigma)$ with $h(x)$ positive iff x is on the exterior side; then $\mathcal{S}$ is defined by the inequalities $-r_s\sqrt{(d-1)/2d} \leq h(x) \leq r_s$.

Let σ be a simplex in the geometric $(d-1)$-cycle $\partial\alpha$. Since $\sigma \in \mathcal{F}_s$, each vertex x_i satisfies $h(x_i) \geq 0$. On the other hand, σ is a face of some d-simplex $\tau \in \mathcal{R}_s - \mathcal{F}_s$ from the cycle α. Since $\tau \notin \mathcal{F}_s$ the other vertex y of τ must be on the interior side of Σ, so $h(y) < 0$.

For all $p \in \sigma$ we have

$$h(p) \leq h(y) + \|p - y\| \leq h(y) + \max_i(\|x_i - y\|) < 0 + r_s = r_s$$

Next, by Remark 2.10, we have $\|p - x_i\| \leq r_s\sqrt{(d-1)/2d}$ for some i. Therefore

$$h(p) \geq h(x_i) - \|p - x_i\| \geq 0 - r_s\sqrt{(d-1)/2d} = -r_s\sqrt{(d-1)/2d}$$

These inequalities prove that $\sigma \subset \mathcal{S}$; and in general this shows that the geometric realisation of $\partial\alpha$ is entirely contained in $\mathcal{S}$.

From **A6**, we know that $\mathcal{S}$ is a disjoint collection of thickened $(d-1)$-dimensional surfaces in $\mathbb{R}^d$ each of thickness at most

$$\Delta = r_s\left(1 + \sqrt{\frac{d-1}{2d}}\right). \tag{3.6}$$

Since $\sigma_*\delta_*[\alpha] = 0$, we know that the cycle $\partial\alpha'$ is nullhomologous within $\mathcal{S}$. Apply Lemma 3.3 with $\epsilon = r_s$, Δ as above, and

$$\epsilon' = \sqrt{\Delta^2 + 2\epsilon^2\frac{d-1}{d}} = r_s\sqrt{\frac{7d - 5 + 2\sqrt{2d(d-1)}}{2d}}, \tag{3.7}$$

to conclude that by increasing the radius from r_s to r_m, the cycle $\partial\alpha'$ becomes trivial: hence, $\iota_*\delta_*[\alpha] = 0 \in H_{d-1}(\mathcal{F}_m)$.

We may now rule out Case 2 as follows. By hypothesis, $[\alpha] \in H_d(\mathcal{R}_s, \mathcal{F}_s)$ is nonzero, as is $\iota_*\iota_*[\alpha] \in H_d(\mathcal{R}_w, \mathcal{F}_w)$. In the present case, $\iota_*\delta_*[\alpha] = 0$ in $H_{d-1}(\mathcal{F}_m)$. Commutativity of Eqn. (3.5) implies that $\delta_*\iota_*[\alpha] = 0$. By exactness of this row, $\iota_*[\alpha] = j_*[\zeta]$ for some $[\zeta] \in H_d(\mathcal{R}_m)$. An application of Theorem 2.5 implies that the map $\iota_* : H_d(\mathcal{R}_m) \to H_d(\mathcal{R}_w)$ factors through the homology of the Čech complex $\mathcal{C}_w = \mathcal{C}_w(\mathcal{X})$ of the cover of $\mathcal{X}$ with balls of radius $r_w/2$:

$$\iota_* : H_d(\mathcal{R}_m) \to H_d(\mathcal{C}_w) \to H_d(\mathcal{R}_w). \tag{3.8}$$

From the Čech Theorem, $\mathcal{C}_w$ has the homotopy type of a subset of $\mathbb{R}^d$. Any such subset has vanishing homology in dimension d; hence $H_d(\mathcal{C}_w) = 0$. We conclude that $\iota_*[\zeta] = 0$. It follows from commutativity of Eqn. (3.5) that

$$0 = j_*(\iota_*[\zeta]) = \iota_*(j_*[\zeta]) = \iota_*(\iota_*[\alpha]) \neq 0. \tag{3.9}$$

Contradiction. Case 2 is impossible under the assumption that

$$r_w \geq r_s\left(\sqrt{\frac{2d}{d+1}}\right)\left(\sqrt{\frac{7d-5+2\sqrt{2d(d-1)}}{2d}}\right) = r_s\sqrt{\frac{7d-5+2\sqrt{2d(d-1)}}{d+1}} \tag{3.10}$$

which is satisfied for any value of d when, as in **A3**, $r_w \geq r_s\sqrt{10}$. □

4. Remarks

Remark 4.1. This is by no means a sharp criterion. It is first of all clearly possible to have the criterion always fail for injudicious choices of r_f, r_c, or r_w. For example, if r_w is extremely large, then all nodes will be in (weak) communication and the complex $\mathcal{R}_w$ will be a single high-dimensional simplex with vanishing homology. Likewise, if r_c is much larger than the bound in Assumption **A3**, then there will be many instances of coverage without a homological forcing.

This being said, we note that even if one chooses the minimal acceptable bounds from Assumption **A3**, it is still not hard to arrange the points to cover $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ without the homological criterion detecting this. The companion paper [5] gives a detailed examination in the single-radius case which is generally applicable in this setting.

Remark 4.2. The complexes $(\mathcal{R}_m, \mathcal{F}_m)$ used in case 2 of the proof of Theorem 3.4 are purely auxiliary: there is no need to ever compute these objects. They are required to determine the degree of the boundary of the relative cycle in the collar of the domain. As can be seen from the convolutions of the midrange signal construction, this is a delicate task.

Remark 4.3. One can improve the constants of Assumptions **A3**, **A5**, and **A6** by using the expressions in the proof which depend on the dimension d. Specifically, we have

A3 The radii of communication r_s, r_w and the covering radius r_c satisfy

$$r_c \geq r_s\sqrt{\frac{d}{2(d+1)}} \qquad ; \qquad r_w \geq r_s\sqrt{\frac{7d-5+2\sqrt{2d(d-1)}}{d+1}}, \tag{4.1}$$

where d is the dimension of the domain.

A5 The restricted domain $\mathcal{D} - N_{\hat{r}}(\partial\mathcal{D})$ is connected, where

$$N_{\hat{r}}(\partial\mathcal{D}) = \{x \in \mathcal{D} : \|x - \partial\mathcal{D}\| \leq \hat{r}\}, \text{ where } \hat{r} = r_f + r_s\sqrt{\frac{d-1}{2d}}. \tag{4.2}$$

A6 The fence detection boundary $\{x \in \mathcal{D} : \|x - \partial\mathcal{D}\| = r_f\}$ has internal injectivity radius at least $r_s\sqrt{\frac{d-1}{2d}}$ and external injectivity radius at least r_s.

For example, in the case $d = 2$, the constants for **A3** become:

$$r_c \geq r_s\sqrt{\frac{1}{3}} \qquad ; \qquad r_w \geq r_s\sqrt{\frac{13}{3}}. \tag{4.3}$$

the latter of which is a significant improvement over the $\sqrt{10}$ bound for arbitrary d.

Remark 4.4. We note that if the homological criterion is satisfied with a class $[\alpha] \in H_d(\mathcal{R}_s, \mathcal{F}_s)$, then the cover is generated only by the vertices of the chain α. Thus, by minimizing the choice of generator α within its persistent homology class, we can relax the redundancy of the cover. This has clear implications to issues of power conservation in sensor networks: see [5] for details.

Remark 4.5. The precise statement of **A6** in terms of injectivity radii requires the curve to be smooth. From the proof of Theorem 3.4, it is clear that the crucial condition is to have the shell $\mathcal{S}$ represent annular domains of thickness bounded by $\frac{3}{2}r_s$. In practice, having $\mathcal{D}$ piecewise-linear is admissible: even though the injectivity radii degenerate to zero, the set $\mathcal{S}$ is still an annular region(s) of width bounded by some larger length, depending on the sharpness of the curves. For a piecewise-linear $\partial\mathcal{D}$, an increase in r_w based on the angle of the sharpest corner in the outermost boundary component makes the criterion rigorous.

Remark 4.6. The coverage criterion presented here is a very specific type of coverage: stationary blanket coverage. There are interesting questions involving, e.g., barrier coverage (in which one want the cover to separate a given domain) and sweeping coverage (in which the nodes move and 'sweep' a cover over time). The paper [5] gives homological criteria for these settings and more in the simpler case of $d = 2$ and controlled boundary nodes. We believe that the techniques of the current paper may be used to derive a persistent homology criterion applicable to these broader problems. The primary difficulty is in controlling what happens near the boundary of the domain.

Remark 4.7. We note that homology is computable, and that the coverage criterion of Theorem 3.4 can be checked in practice. We do not emphasize the computational issues. Is suffices to note that we have used the computational homology software package `Plex` [22]. Simulations have been written using `MATLAB` as the frontend (primarily for generating the simplicial complexes from various point-data sets, and for data formatting and visualization.) The current implementation of `Plex` computes the dimensions of persistent homology groups (using algorithms as in [8, 21]), which is enough to check whether the homomorphism ι_* in the criterion of Theorem 3.4 is nonvanishing.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, POMONA COLLEGE, CLAREMONT CA 91711, USA

E-mail address: vin.desilva@.pomona.edu

DEPARTMENT OF MATHEMATICS AND COORDINATED SCIENCES LABORATORY, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, USA

E-mail address: ghrist@math.uiuc.edu