Inversion of Euler integral transforms with applications to sensor data

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Abstract. Following the pioneering work of Schapira, we consider topological Radon-type integral transforms on constructible $\mathbb{Z}$-valued functions using the Euler characteristic as a measure. Contributions include: (1) application of the Schapira inversion formula to target localization and classification problems in sensor networks; (2) extension and application of the inversion formula to weighted Radon transforms; and (3) pseudo-inversion formulae for inverting annuli (sets of Euler measure zero).

1. Introduction

This paper considers problems of extracting information about a collection $\mathcal{U} = \{U_\alpha\}$ of subsets of $\mathbb{R}^n$ conglomerated and compressed as a sum of indicator functions $h = \sum_\alpha \mathbf{1}_{U_\alpha}$. Given $h : \mathbb{R}^n \to \mathbb{N}$, what about the decomposition can be recovered? We consider problems of enumeration (determine the cardinality $\#\alpha$ of $\mathcal{U}$), localization (determine the locations of the components U_α), and identification (determine the shapes of the components U_α).

This is motivated by problems in sensor networks. Consider a large, dense network of sensors which fill a domain. Each sensor (using whatever sensing modality is applicable) detects the number of nearby targets, but cannot determine location, distance, bearing, or even identity of the sensed targets. In a continuum limit, a dense network returns such a function $h : \mathcal{D} \to \mathbb{N}$ of the form $h = \sum_\alpha \mathbf{1}_{U_\alpha}$, where each U_α is the domain of a target's impact, or TARGET SUPPORT.

We address these problems via EULER CALCULUS — an ingenious integration theory first explored by Schapira [22, 23] and Viro [26], both of these being based on results in sheaf theory, particularly [18, 15]. There are historical antecedents in the theory of valuations for polynomials, as in the work of Blaschke [6], Groemer [12], and Hadwiger [14]. These methods are currently of great interest in algebraic geometry, forming a simple example of a MOTIVIC INTEGRATION, a topic of great impact in algebraic geometry [10]. These methods have projected shadows in the literature on combinatorics [8, 19, 21], probability [1, 2, 11], computational geometry [20, 13], and integral geometry and geometric probability [17].

The Euler calculus is recalled in §2, with integral transforms outlined in §3. It is there noted that the act of anonymous sensing — of converting targets into a counting function $h : \mathcal{D} \to \mathbb{N}$ — is an Euler-calculus integral transform of Radon-type, where the kernel is a relation encoding sensor-target visibility. In order to reconstruct target locations and geometries, we consider the problem of inverting the integral transform. This, also, is foreshadowed in the prescient paper of Schapira [23], which contains a general inversion formula: see §4. The contributions of this paper are:

(i) generation of new examples of sensor-target systems to which the Schapira inversion formula applies (§5);

(ii) an inversion formula for weighted Radon transforms (§6); and

(iii) new pseudo-inversion methods built specifically to detect sets of Euler measure zero, such as annuli (§8).

In this paper, we focus on the inversion aspects of the problem: we do not here address the many issues surrounding implementation in discrete sensor networks, with the attendant challenges in signal protocols, communication complexity, and numerical implementation.

2. Euler calculus

The Euler calculus is an integral calculus based on Euler characteristic as a scale-invariant measure.

2.1. Euler integration

Given a set $A \subset \mathbb{R}^n$ and a decomposition of A into a finite number of CELLS $A = \coprod_\alpha \sigma_\alpha$, where each k-cell σ_α is homeomorphic to $\mathbb{R}^k$, the (GEOMETRIC) EULER CHARACTERISTIC of A is defined as

$$\chi(A) = \sum_\alpha (-1)^{\dim \sigma_\alpha}. \tag{1}$$

This quantity is well-defined for finite cellular objects and is both independent of the decomposition of A into cells and of the homeomorphism type of A, since $\chi(A)$ has a (co)homological formulation. The geometric Euler characteristic is not a homotopy invariant, as, *e.g.*, it distinguishes $\chi((0,1)) = -1$ from $\chi([0,1]) = 1$. Nevertheless, it is ideally suited for an integration theory, since

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B), \tag{2}$$

for subsets of $\mathbb{R}^n$ on which all the above quantities make sense. One focuses attention on "tame" or DEFINABLE subsets of $\mathbb{R}^n$, for which finite cell decompositions hold, and for which the operations of union and intersection do not cause mischief. The axiomatization of these properties leads to the notion of an O-MINIMAL STRUCTURE. For brevity, we fix an o-minimal structure — a system of definable subsets and mappings — of globally subanalytic sets and analytic mappings. Narrower classes, such as semialgebraic or piecewise-linear, are possible, as are broader classes [25].

We briefly recall the theory of Euler integration, established as an integration theory in the constructible setting in [18, 22, 23, 26] and anticipated by a combinatorial version in [6, 12, 14, 21]. Having fixed a suitable class of tame subsets and mappings on a space $X \subset \mathbb{R}^n$, define the integral over characteristic functions in the obvious manner:

$$\int_X \mathbf{1}_A \, d\chi := \chi(A).$$

Extend by linearity to the class of CONSTRUCTIBLE FUNCTIONS, $\mathsf{CF}(X)$, all finite $\mathbb{Z}$-linear combinations of functions $\mathbf{1}_A$ for $A \subset X$ compact and definable. It follows from Eqn. (2) that the integral $\int_X \cdot \, d\chi : \mathsf{CF}(X) \to \mathbb{Z}$ is well-defined and independent of how the integrand is partitioned into characteristic functions.

2.2. *Euler calculus*

This explicit definition has an implicit formulation in terms of canonical operations: the integral *is* a direct image functor on the sheaf of constructible functions [22]. More concretely, the Euler integral possesses the following properties. Given $P : X \to Y$ a tame mapping between tame spaces, the PULLBACK is the induced mapping $P^* : \mathsf{CF}(Y) \to \mathsf{CF}(X)$ given by

$$(P^* g)(x) = g(P(x)). \tag{3}$$

Correspondingly, the PUSHFORWARD is the induced mapping $F_* : \mathsf{CF}(X) \to \mathsf{CF}(Y)$ given by

$$(P_* h)(y) = \int_{P^{-1}(y)} h \, d\chi. \tag{4}$$

The functoriality of these operations is expressed in the PROJECTION FORMULA,

$$P_*(g(P^* h)) = P_*(g)h, \tag{5}$$

and, respectively, the FUBINI THEOREM:

$$\int_X h \, d\chi = \int_Y P_* h \, d\chi. \tag{6}$$

The Euler integral is thus properly named, as it forms a consistent calculus.

3. Integral transforms

3.1. Convolution

On a vector space V, a convolution operator with respect to Euler characteristic is straightforward. Given $f, g \in CF(V)$, one defines

$$(f * g)(x) = \int_V f(t)g(x-t)\,d\chi. \tag{7}$$

Convolution behaves as expected. In particular,

$$\int_V f * g\,d\chi = \int_V f\,d\chi \int_V g\,d\chi. \tag{8}$$

There is a close relationship between convolution and the MINKOWSKI SUM, as observed in [12]: for A and B convex, $\mathbf{1}_A * \mathbf{1}_B = \mathbf{1}_{A+B}$. See [26, 22, 5] for more on the relationship to $\int d\chi$. Convolution is a commutative, associative operator and provides $CF(V)$ with the structure of an algebra [7].

3.2. Duality

There is an integral transform on $CF(X)$ that is the analogue of Poincaré(-Verdier) duality [24]. Define the DUAL of $h \in CF(X)$ to be:

$$\mathcal{D}h(x) = \lim_{\epsilon \to 0^+} \int_X h\mathbf{1}_{B(x,\epsilon)} d\chi, \tag{9}$$

where $B(x, \epsilon)$ denotes an open ball of radius ϵ about x. This limit is well-defined thanks to the Conic Theorem in o-minimal geometry [25]. Duality provides a de-convolution.

Lemma 3.1 ([23]). *For any convex closed $A \subset V$ with non-empty interior $\mathbf{1}_A * \mathcal{D}\mathbf{1}_{-A} = \delta_0$, where δ is the indicator function of the origin and $-A$ denotes the reflection of A through the origin.*

3.3. Radon transforms

The following general construction comes from [23], see also [7, 9]. Given a locally closed definable set $S \subset W \times X$, let P_W and P_X denote the projection maps of $W \times X$ to their factors.

Definition 3.2. The RADON TRANSFORM is the map $\mathcal{R}_S : CF(W) \to CF(X)$ given by

$$\mathcal{R}_S h = (P_X)_*((P_W^* h)\mathbf{1}_S). \tag{10}$$

Example 3.3. Duality on $CF(X)$ is the Radon transform associated to the relation $S \subset X \times X$ where S is a sufficiently small open tubular neighborhood of the diagonal $\Delta = \{(x,x) : x \in X\}$.

In the context of sensor networks, the Radon transform is entirely natural. Let W denote the WORKSPACE, a topological space where the targets reside; and let X denote the SENSOR SPACE, a topological space modeling the field of sensors (for a finite sensor network, X is, say, a simplicial complex determined by nodes and communication edges between nodes). The SENSOR RELATION is $\mathcal{S} = \{(w,x)$: the sensor at x senses a target at $w\}$. This lies in the product space $W \times X$ as a relation whose 'vertical' fibers $\mathcal{S}_w = P_X(P_W^{-1}(w) \cap \mathcal{S})$ are TARGET SUPPORTS and whose 'horizontal' fibers $\mathcal{S}_x = P_W(P_X^{-1}(x) \cap \mathcal{S})$ are SENSOR SUPPORTS.

Consider the sensor relation $\mathcal{S} \subset W \times X$, and a finite set of targets $T \subset W$ as defining an atomic function $\mathbf{1}_T \in CF(W)$. Observe that the 'counting function' which the sensor field on X returns is precisely the Radon transform $\mathcal{R}_S\mathbf{1}_T$. In this language, the target enumeration theorem of [3] is implied by the following:

Lemma 3.4. *Assume that $\mathcal{S} \subset W \times X$ has vertical fibers $P_W^{-1}(w) \cap \mathcal{S}$ with constant Euler characteristic N. Then, $\mathcal{R}_\mathcal{S} : CF(W) \to CF(X)$ scales integration by a factor of N:*

$$\int_X \circ \mathcal{R}_\mathcal{S} = N \int_W .$$

4. Schapira's inversion formula

The construction of the inverse Radon transform was proved by Schapira [23]. This inversion formula uses a pair of relations $\mathcal{S}$ and $\mathcal{S}'$ and assumes a strict regularity of Euler characteristics of fibers.

Theorem 4.1 (Schapira)**.** *Assume that $\mathcal{S} \subset W \times X$ and $\mathcal{S}' \subset X \times W$ have fibers $\mathcal{S}_w$ and $\mathcal{S}'_w$ in X satisfying (1) $\chi(\mathcal{S}_w \cap \mathcal{S}'_w) = \mu$ for all $w \in W$; and (2) $\chi(\mathcal{S}_w \cap \mathcal{S}'_{w'}) = \lambda$ for all $w' \neq w \in W$. Then for all $h \in CF(W)$,*

$$(\mathcal{R}_{\mathcal{S}'} \circ \mathcal{R}_\mathcal{S})h = (\mu - \lambda)h + \lambda \left(\int_W h \right) \mathbf{1}_W . \tag{11}$$

Recall that the sensor counting function $h : X \to \mathbb{Z}$ is, in the context of this paper, equal to $\mathcal{R}_S\mathbf{1}_T$, where $T \subset W$ is the set of targets. If the conditions of Theorem 4.1 are met and $\lambda \neq \mu$, then the inverse Radon transform $\mathcal{R}_{S'}h = \mathcal{R}_{S'}\mathcal{R}_S\mathbf{1}_T$ is equal to a multiple of $\mathbf{1}_T$ plus a multiple of $\mathbf{1}_W$. Thus, one can localize and identify the targets — determine the exact shape of T — by performing the inverse transform.

5. Examples of Radon inversion

It can be difficult to find concrete examples which satisfy the conditions of Theorem 4.1: the relevant sensor and target supports need to be, in a sense, global, to ensure the independence with respect to $w \neq w'$. Schapira's original paper contained a class of examples based on Grassmannians, with an eye towards reconstruction in imaging.

Example 5.1 (hyperplanes). Let $W = \mathbb{R}^n$ and $X = \mathsf{Gr}_n$ the Grassmannian of hyperplanes in W with $\mathcal{S}$ correlating points in X to points in W contained in the associated hyperplanes. The inverse kernel $\mathcal{S}'$ is the reflection of $\mathcal{S}$ that associates points and hyperplanes via membership. In this case, the fiber $\mathcal{S}_w$ is a projective $(n-1)$-space with $\mu = \chi(\mathbb{RP}^{n-1}) = \frac{1}{2}(1 + (-1)^{n-1})$, whereas $\mathcal{S}_w \cap \mathcal{S}'_{w'}$ is a projective $(n-2)$-space with $\lambda = \chi(\mathbb{RP}^{n-2}) = \frac{1}{2}(1 + (-1)^{n-2})$.

Subsequent papers using Schapira's formula [7, 9] seem to mention only this one example. We introduce some additional families of examples with an eye towards applications to sensor networks.

Example 5.2 (rays). Let the targets T be a finite disjoint collection of points or convex balls in $W = D^n$, the open unit disc in $\mathbb{R}^n$. Assume that the boundary ∂W is lined with sensors, each of which sweeps a ray over W and counts, as a function of bearing, the number of targets intersected by the beam. The sensor space X is homeomorphic to $T_* S^{n-1}$, the tangent bundle of ∂W. (Note that the bearing of a ray at a point $p \in \partial W$ lies in the open hemisphere of the unit tangent bundle $T^1_p W$. This open hemisphere projects to the open unit disc in $T_p \partial W$.)

Any point in W is seen by any sensor in ∂W along a unique bearing angle. Thus, the sensor relation $\mathcal{S}$ has vertical fibers (target supports) which are sections of $T_* S^{n-1}$ and hence spheres of Euler measure $1 + (-1)^n$. The horizontal fibers (sensor supports) are intervals of Euler characteristic 1. Any two disjoint vertical fibers intersect along the subset of rays from ∂W that pass through both points in W: this is a discrete set of cardinality (hence of Euler measure) 2. Theorem 4.1 indicates that the inverse Radon transform of this system is well-defined only for n even. (This obstruction can be visualized easily in the case $n = 1$, where it is clear that you cannot localize the target along a bounded line segment given sensor readings from the two boundary points!) For n even, one has full invertibility with $\lambda = 2$ and $\mu = 0$.

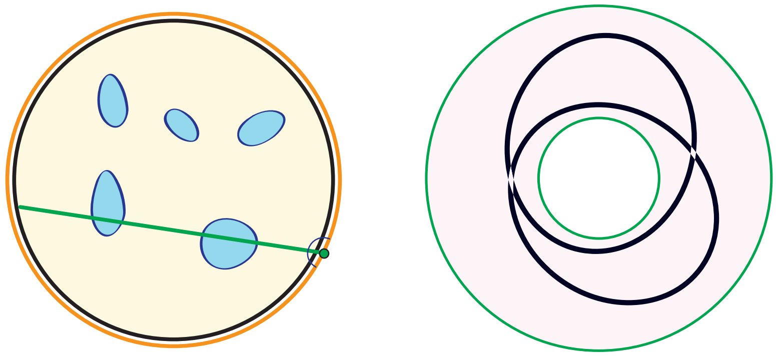


Figure 1. A collection of convex targets in $W = D^2$ is surrounded by beam counting sensors which sweep through the domain [left]. The sensor space X is homeomorphic to an annulus. Target supports are circles with pairwise intersection number 2 [right].

Example 5.3 (distance-based sensing). Non-self-dual examples of Radon inversion can be generated easily with complementary supports. For example, let $X = W = \mathbb{R}^n$ with $\mathcal{S}_w$ a closed ball about w and $\mathcal{S}'_w$ the closure of the complement in $\mathbb{R}^n$. Physically, this means that the sensor relation detects proximity-within-range, and the inverse sensor relation counts targets out-of-range. The inversion formula applies, because of the singular nature of the "eclipse" that occurs when targets coalesce. Specifically: (1) $\chi(\mathcal{S}_w \cap \mathcal{S}'_{w' \neq w}) = 1$; and (2) $\chi(\mathcal{S}_w \cap \mathcal{S}'_w) = \chi(S^{n-1}) \neq 0$.

There are technicalities in applying the inversion formula in settings where the supports are non-compact, since we have defined CF in terms of compactly supported functions. This does not interfere with inversion in this example: all non-compact integrands are still tame. In addition, this example may be easily modified so that the supports (and the corresponding complements) change smoothly from point-to-point within the domain. So long as $\mathcal{S}_w$ is, say, always compact and contractible and varies slowly, so that $\chi(\mathcal{S}_w \cap \mathcal{S}'_{w' \neq w}) = 1$, the inversion formula will apply (see §7 for more about varying supports).

Example 5.4 (discrete distance). Using discrete distances can lead to inversions. Let $W = \mathbb{R}^{2n} = X$ and define $\mathcal{S}_w$ as the unit sphere about w: sensors count targets at a fixed distance. The inverse kernel $\mathcal{S}'$ has fibers equal to concentric spheres about the basepoint of radius $r = 1, 2, 3, \ldots$. Distinct fibers of $\mathcal{S}$ and $\mathcal{S}'$ always intersect in a set homeomorphic either to S^{2n-2} or, if the points line up at integer distances, S^0: either way, $\chi(\mathcal{S}_w \cap \mathcal{S}'_{w' \neq w}) = 2$. In the instance when the basepoints coincide, the intersection is precisely the unit sphere S^{n-1} with $\chi = 0$: full invertibility follows.

6. Weighted Radon inversion

It is possible to perform Radon inversion with weights, as indicated by the following example.

Example 6.1 (weighted kernels). Consider the case of Example 5.4, modified so that $W = X = \mathbb{R}^{2n+1}$. On odd-dimensional spaces, the spheres of intersection have dimension $2n-1$ or 0, depending on position. Thus λ is not a constant, preventing invertibility. However, if we weight the inverse kernel $\mathcal{S}'$ so that concentric spheres of radius $r = 1, 2, 3, \ldots$ have weight $(-1)^{r+1}$, then $\chi(\mathcal{S}_w \cap \mathcal{S}'_{w' \neq w}) = 0$, independent of position. Full inversion is therefore possible, once a weighted version of Schapira's theorem is proved.

Using a weighted kernel is not explicitly covered by Schapira's formula; however, it is a simple matter to generalize from Radon transforms to Fredholm transforms with arbitrary constructible kernels having an Euler regularity in the fibers. Given a kernel $K \in \mathsf{CF}(W \times X)$, one defines the weighted Radon transform as $\mathcal{R}_K h = (P_X)_*((P_W^* h)K)$. Inversion requires an inverse kernel K' as follows:

Theorem 6.2. *Fix $K \in \mathsf{CF}(W \times X)$ and $K' \in \mathsf{CF}(X \times W)$. If there exist constants μ and λ such that $\int_X K(w,x)K'(x,w')d\chi = (\mu - \lambda)\delta_{w-w'} + \lambda$ for all $w, w' \in W$, then for all $h \in \mathsf{CF}(W)$,*

$$(\mathcal{R}_{K'} \circ \mathcal{R}_K)h = (\mu - \lambda)h + \lambda \left(\int_W h \, d\chi \right) \mathbf{1}_W. \tag{12}$$

Proof. For any $w' \in W$,

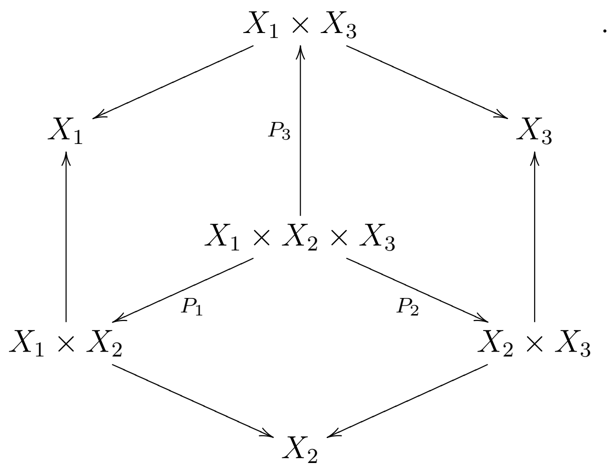
$$\begin{aligned}
(\mathcal{R}_{K'} \circ \mathcal{R}_K h)(w') &= \int_X \left[\int_W h(w) K(w,x) \, d\chi \right] K'(x,w') \, d\chi \\
&= \int_W h(w) \left[\int_X K(w,x)K'(x,w') d\chi \right] d\chi \\
&= \int_W \left[(\mu - \lambda) h(w) \delta_{w-w'} + \lambda h(w) \right] d\chi \\
&= (\mu - \lambda) h(w') + \lambda \int_W h \, d\chi,
\end{aligned}$$

where the Fubini theorem is used in line two. □

A more general proof uses the following COCYCLE CONDITION. Let X_i, $i = 1, \ldots, 3$ be spaces. Supposing the constructible kernels $K_i \in \mathsf{CF}\left(\prod_{j \neq i} X_j\right)$ and the projection maps $P_i : \prod_i X_i \to \prod_{j \neq i} X_j$ satisfy the cocycle condition

$$K_3 = (P_3)_*(P_1^* K_1 \cdot P_2^* K_2). \tag{13}$$

Then, consider the diagram



. (14)

It follows from commutativity and Eqn. (5) that $\mathcal{R}_{K_2} \circ \mathcal{R}_{K_1} = \mathcal{R}_{K_3}$ (see [23] for a version of this argument).

7. Inversion with slowly varying supports

Schapira's inversion formula is by no means the end of the story with regards to Euler integral transforms. The geometric rigidity implicit in all the examples of Section 3.3 can in some cases be relaxed. We consider first the simple situation in which the vertical fibers $\mathcal{S}_w$ are all contractible. In this case, to obtain a generalization of deconvolution (Lemma 3.1), we need that the slices of the support $\mathcal{S}$ satisfy the following assumptions :

Definition 7.1. The support $\mathcal{S}$ is CONTRACTIBLE if it is the closure of an open set in $V \times V$; and all vertical fibers $\mathcal{S}_w$ are contractible. It is called SLOWLY VARYING if all intersections of target supports in V are either contractible or empty, and the intersection of the boundary of any target support with any other target support is either empty or contractible.

The last condition seems to be most restrictive — even if the fibers are all convex, it is not necessarily satisfied. This is an assumption on the rate of change of the fibers as the basepoint varies.

Example 7.2. If $\mathcal{S} = \{|x - w| \leq R(w)\}$ (*i.e.*, a target $w \in W$ is seen at a distance $R(w)$ around it), then the corresponding kernel is slowly varying if R is 1-Lipshitz.

Lemma 3.1 can be generalized as follows.

Lemma 7.3. *For a contractible, slowly varying support $\mathcal{S}$, let $K = \mathbf{1}_{\mathcal{S}}$ denote the* *kernel. Then:*

$$\int_V K(x,w)\mathcal{D}K(x,w')d\chi(x) = \delta_{w-w'}. \tag{15}$$

The proof is a simple exercise for the reader. This gives a reconstruction algorithm for the (constructible) signal h from readings with contractible slowly varying sensing kernels.

8. Pseudo-inversion for annuli

In some cases, inversion is not possible, yet a pseudo-inverse can be constructed. We illustrate a particular pseudo-inverse construction which deals with annular supports in $\mathbb{R}^n$. On even-dimensional spaces, annuli are devilishly difficult to track, since they are sets of χ-measure zero. Figure 2 shows a constructible function in $\mathbb{R}^2$ composed of embedded annuli for which Euler integration fails to enumerate the annuli — rightly so, as the number is not well-defined.

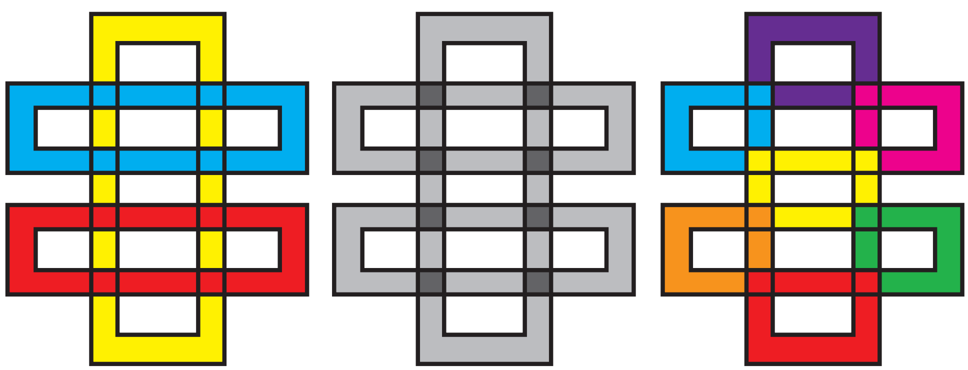


Figure 2. The height function of a collection of an unknown number of annuli. Three? [left] Seven? [right] Even with annuli defined by convex discs, it is ambiguous, since an annulus has $\chi = 0$..

We give a pseudo-inverse for Euclidean combinations of a geometrically-fixed annular set.

Lemma 8.1. *Suppose $J \subset O \subset V$ are compact, convex, and contain the ball $B_C(0)$ for some $C > 0$. Then for any integer $k > 0$, the function $\mathbf{1}_O^{*k} * \mathbf{1}_J^{*k}$ is equal to 1 on the ball $B_{2kC}(0)$.*

Proof. On convex sets, convolution corresponds to Minkowski sum. □

Using techniques as in §7, convexity of O can be relaxed to star-convexity with respect to J.

Theorem 8.2. *Let $I \subset O \subset V$ be convex sets containing 0, such that I is open, O is closed and compact, and the closure of I is contained in the interior of O. Let A denote the annular region $O \setminus I$. Then the formal sum*

$$\Psi = -\mathcal{D}\mathbf{1}_{-I} * \sum_{k=0}^{\infty} (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*k} \tag{16}$$

is a pseudo-inverse to $\mathbf{1}_A$.

Proof. For any $N > 0$, let Ψ_N denote the N^{th} partial sum of Ψ:

$$\Psi_N = -\mathcal{D}\mathbf{1}_{-I} * \sum_{k=0}^{N-1} (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*k}.$$

Note that $\mathbf{1}_A = \mathbf{1}_O - \mathbf{1}_I$. By lemma 3.1, $\mathbf{1}_I * \mathcal{D}\mathbf{1}_{-I} = \delta_0$. Then

$$\begin{aligned}
\mathbf{1}_A * \Psi_N = & -\mathbf{1}_A * \mathcal{D}\mathbf{1}_{-I} * \sum_{k=0}^{N-1} (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*k} \\
&= (\delta_0 - \mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I}) * \sum_{k=0}^{N-1} (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*k} \\
&= \sum_{k=0}^{N-1} (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*k} - \sum_{k=0}^{N-1} (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*(k+1)} \\
&= \delta_0 - (\mathbf{1}_O * \mathcal{D}\mathbf{1}_{-I})^{*N}.
\end{aligned}$$

Let J denote the closure of $-I$. Then $\mathcal{D}\mathbf{1}_{-I} = (-1)^{\dim(V)}\mathbf{1}_J$, so

$$\mathbf{1}_A * \Psi_N = \delta_0 - (-1)^{N \dim V} (\mathbf{1}_O * \mathbf{1}_J)^{*N}.$$

By lemma 8.1, the function $(\mathbf{1}_O * \mathbf{1}_J)^{*N}$ is eventually equal to 1 on any fixed compact set. This implies that the sum in Ψ is divergent; however, on any fixed compact set, for N sufficiently large, we have $\mathbf{1}_A * \Psi_N = \delta_0 \pm \mathbf{1}_V$. In particular, if $h \in \mathsf{CF}(V)$ is supported on the ball of radius R, then for $N > R/2C$ and $|x| < R$,

$$(h * \mathbf{1}_A * \Psi_N)(x) = h(x) - (-1)^{N \dim(V)} \left(\int_V h \, d\chi \right).$$

□

Example 8.3 (beacon counting). In the context of sensor networks, one may have target supports consisting of annuli in the plane. For example, targets may be beacons that can be seen only when the sensor is close-but-not-too-close to the beacon; simple optical sensors perform anonymous beacon counts and return $h \in \mathsf{CF}(\mathbb{R}^2)$ of the form $\mathbf{1}_A * \mathbf{1}_T$, where A is the annular support about each beacon, and T is the discrete set of beacon locations. The convolution $h * \Psi$ reveals the exact locations of the beacons, even though $\int_{\mathbb{R}^2} h = 0$. This pseudo-inversion does not eliminate the ambiguity implied in Figure 2, since the shape of the annulus A must be selected to construct the pseudo-inverse.

9. Conclusions

The limited goals of this short paper include exposition of the Schapira inversion formula and generation of new examples and applications. Several open problems and avenues for exploration remain, including the following:

(i) To what extent can inversion be performed at the hardware level in a sensor network? At the very least, inversion appears to require giving to sensors the ability to emulate targets in order to construct an inverse kernel.

(ii) Can inverse transforms be computed numerically over a discrete network? There appears to be several interesting algorithmic challenges to efficient computation.

(iii) To what extent can real-valued weighted kernels be employed? The measures $\lfloor d\chi \rfloor$ and $\lceil d\chi \rceil$ of [4] seem appropriate. However, the weighted inversion formula proved here relies on the Fubini Theorem, and this does not hold over real-valued continuous integrands.

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Bibliography

[1] R. Adler, *The Geometry of Random Fields*, Wiley, London, 1981. Reprinted by SIAM, 2010.

[2] R. Adler and J. Taylor, *Random Fields and Geometry*, Springer, 2007.

[3] Y. Baryshnikov and R. Ghrist, “Target enumeration via Euler characteristic integrals,” *SIAM J. Appl. Math.*, 70(3), 825-844, 2009.

[4] Y. Baryshnikov and R. Ghrist, “Euler integration for definable functions,” *Proc. National Acad. Sci* ., 107(21), May 25, 9525-9530, 2010.

[5] J. Basch, L. Guibas, G. Ramkumar, and L. Ramshaw, “Polyhedral tracings and their convolutions,” in *Algorithms for Robotic Motion and Manipulation, J.-P. Laumond and M. Overmars, eds.*, A. K. Peters, 1996.

[6] W. Blaschke, *Vorlesungen über Integralgeometrie*, Vol. 1, 2^{nd} edition. Leipzig and Berlin, Teubner, 1936.

[7] L. Bröcker, “Euler integration and Euler multiplication,” *Adv. Geom.* 5, 2005, 145–169.

[8] B. Chen, “On the Euler measure of finite unions of convex sets,” *Discrete and Computational Geometry* 10, 1993, 79–93.

[9] R. Cluckers and M. Edmundo, “Integration of positive constructible functions against Euler characteristic and dimension,” *J. Pure Appl. Algebra*, 208, no. 2, 2006, 691–698.

[10] R. Cluckers and F. Loeser, “Constructible motivic functions and motivic integration,” *Invent. Math.*, 173(1), 2008, 23–121.

[11] K. Dohmen, *Improved Bonferroni Inequalities via Abstract Tubes*, Springer Lecture Notes in Mathematics vol. 1826, Springer-Verlag, 2003.

[12] H. Groemer, “Minkowski addition and mixed volumes,” *Geom. Dedicata* 6, 1977, 141–163.

[13] L. Guibas, L. Ramshaw, and J. Stolfi, “A kinetic framework for computational geometry,” in *Proc. IEEE Sympos. Found. Comput. Sci.*, 1983, 100-111.

[14] H. Hadwiger, “Integralsätze im Konvexring,” *Abh. Math. Sem. Hamburg*, 20, 1956, 136–154.

[15] M. Kashiwara, Index theorem for maximally overdetermined systems of linear differential equations, *Proc. Japan Acad. Ser. A Math. Sci.*, 49 (10), 1973, 803-804.

[16] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Springer-Verlag, 1994.

[17] D. Klain and G.-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, 1997.

[18] R. MacPherson, “Chern classes for singular algebraic varieties,” *Ann. of Math.* 100, 1974, 423-432.

[19] R. Morelli, “A Theory of Polyhedra,” *Adv. Math.* 97, 1993, 1–73.

[20] G. Ramkumar, “Tracings and Their Convolutions: Theory and Tpplications,” Ph.D. thesis, Stanford University, 1998.

[21] G.-C. Rota, “On the combinatorics of the Euler characteristic,” Studies in Pure Mathematics, Academic Press, London, 1971, 221–233.

[22] P. Schapira, “Operations on constructible functions,” *J. Pure Appl. Algebra* 72, 1991, 83–93.

[23] P. Schapira, “Tomography of constructible functions,” in proceedings of *11th Intl. Symp. on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, 1995, 427–435.

[24] J. Schürmann, *Topology of Singular Spaces and Constructible Sheaves*, Birkhäuser, 2003.

[25] L. Van den Dries, *Tame Topology and O-Minimal Structures*, Cambridge University Press, 1998.

[26] O. Viro, “Some integral calculus based on Euler characteristic,” Lecture Notes in Math., vol. 1346, Springer-Verlag, 1988, 127–138.