Tight contact structures and Anosov flows

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Abstract

In this review, we demonstrate how classic and contemporary results on the classification of tight contact structures apply to the problem of existence and uniqueness of Anosov flows on three-manifolds. The ingredients we use are the results of Mitsumatsu on Anosov flows, the homotopy invariant of plane fields as described by Gompf and others, and certain recent classification results of Giroux and Honda. A simple example is a novel proof of the nonexistence of Anosov flows on S^3 using only contact topology (and in particular without use of Novikov's Theorem on foliations).

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1 An introduction to contact structures

The theory of contact structures, though magnificently old, has of late become central to several key questions in, e.g., the study of three-manifolds [4], Seiberg-Witten invariants [25,23], symplectic geometry [8], knot theory [30], and foliation theory [6]. Dynamical systems theory [17,18] and applications [9,10] have also benefitted greatly from a recent influx of contact-topological techniques. In this note, we review a result of Mitsumatsu [26] and present a modest application to a fundamental problem in dynamical systems theory: determining which three-manifolds support an Anosov flow.

For the sake of concreteness and applicability, we restrict all definitions and discussions to the case of contact structures on three-manifolds, noting that several features hold on arbitrary odd-dimensional manifolds. For introductory treatments, see [27,1,4].

A *contact form* on an oriented three-manifold M is a one-form α on M such that $\alpha \wedge d\alpha$ defines an oriented volume form on M. A *contact structure* is a plane field which is the kernel of a (locally defined) contact form:

$$\xi := \ker(\alpha) = \{v \in T_pM : \alpha(v) = 0, p \in M\}. \tag{1}$$

The orientation induced by $\alpha \wedge d\alpha$ is independent of the defining one-form; hence, ξ has a natural orientation which can agree (a *positive* structure) or disagree (a *negative* structure) with that of M. According to the Frobenius integrability condition, a contact structure is thus a maximally nonintegrable plane field. In particular, a contact structure is locally twisted at every point and may be thought of as an "anti-foliation." It is usually sufficient to consider contact structures which are the kernel of a globally defined contact one-form: these are *cooriented* contact structures.

Unlike foliations, contact structures are structurally stable, in the sense that not only is a perturbation of a contact form α still a contact form, but also such a perturbation has kernel isotopic to that of α. In fact, a standard application of the Moser method in this context implies that every contact structure is locally contactomorphic to (or, diffeomorphic via a map which carries the contact structure to the kernel of $dz + x\,dy$ on $\mathbb{R}^3$ (see, e.g., [27]). Note the similarity with codimension-one foliations, which are locally equivalent to the kernel of dz on $\mathbb{R}^3$.

Example 1 The standard positive contact structure on the unit $S^3 \subset \mathbb{R}^4$ is given by the kernel of the one-form

$$\alpha_0 := \frac{1}{2}\left(x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3\right). \tag{2}$$

The contact structure $\xi_0^+ := \ker(\alpha_0)$ is the plane field orthogonal to the fibres of the Hopf fibration (orthogonal with respect to the metric on the unit three-sphere induced by the standard metric on $\mathbb{R}^4$). This contact structure induces the positive orientation on S^3 (i.e., $\alpha_0 \wedge d\alpha_0 > 0$). A negative contact structure on S^3 may be obtained by applying an orientation-reversing diffeomorphism.

As in foliation theory, the global features of a contact structure are closely related to those of the manifold in which it sits. The classification of contact structures follows along lines similar to the Reeb-component versus taut perspective in (codimension-one) foliation theory [6].

Definition 2 Given a three-manifold M with contact structure ξ, let $F \subset M$ be an embedded surface. Then the *characteristic foliation* on F, F_ξ, is the (singular) foliation on F generated by the (singular) line field

$$\mathcal{F} = \{T_pF \cap \xi_p : p \in F\}.$$

A contact structure ξ is *overtwisted* if there exists an embedded disc $D \subset M$ such that the characteristic foliation D_ξ has a limit cycle. A contact structure which is not overtwisted is called *tight*.

A priori, Definition 2 appears arbitrary. However, if one builds an analogy with foliation theory, this definition becomes more natural [6]. Consider a Reeb component in a codimension-one foliation of a three-manifold, as illustrated in Figure 1 (see, e.g., [14] for definitions). The characteristic foliation induced by the Reeb component on a meridional disc is a foliation by circles with one singularity. The intersection of any single $\mathbb{R}^2$-leaf with the meridional disc is a sequence of concentric circles which "limit" onto the boundary torus, which forms a sort of limit cycle. An overtwisted contact structure is the nonintegrable analogue of this object.

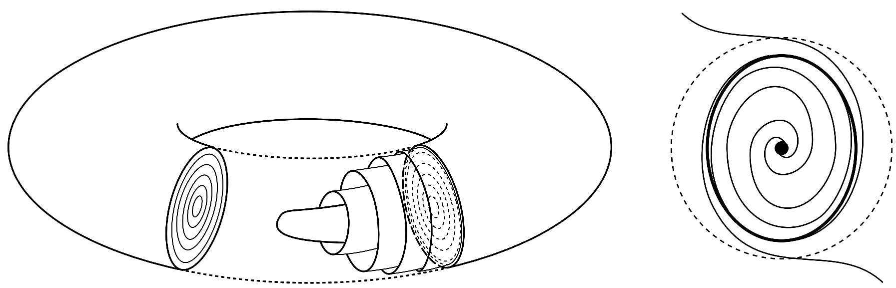

Fig. 1. A Reeb component in a foliation on a three-manifold (left) can be perturbed into an overtwisted contact structure (right).

The classification of overtwisted structures up to contact isotopy coincides with the classification of plane fields up to homotopy [3] and hence reduces to a problem in algebraic topology. The classification of tight structures, on the other hand, is far from complete: for example, it is unknown which three-manifolds admit a tight contact structure (not all do [11]). Like taut foliations, tight contact structures exhibit several "rigid" features which make them relatively rare. The following theorems of Bennequin and Eliashberg are foundational:

Theorem 3 *The contact structure ξ_0^+ of Example 1 is tight [2] and is the unique tight contact structure on S^3 up to orientation and contact isotopy [4].*

2 Anosov flows

Recall that an invariant set Λ of a flow ϕ^t on a Riemannian manifold M is *hyperbolic* if the tangent bundle $TM|_\Lambda$ has a continuous ϕ^t-invariant splitting into $E^c \oplus E^s \oplus E^u$, where E^c is tangent to the flow direction, and $D\phi^t$ uniformly contracts and expands along E^s and E^u respectively: i.e.,

$$\begin{array}{ll} \|D\phi^t(\mathbf{v}^s)\| \leq Ce^{-\lambda t}\|\mathbf{v}^s\| & \text{for } \mathbf{v}^s \in E^s \\ \|D\phi^{-t}(\mathbf{v}^u)\| \leq Ce^{-\lambda t}\|\mathbf{v}^u\| & \text{for } \mathbf{v}^u \in E^u \end{array}, t > 0, \tag{3}$$

for some $C \geq 1$ and $\lambda > 0$. A flow ϕ^t which is hyperbolic on all of M is called an *Anosov flow.* Anosov flows are some of the most important types of flows, from dynamical, topological, and geometric perspectives. Fundamental examples of Anosov flows include geodesic flows on unit tangent bundles of surfaces of constant negative curvature, as well as suspensions of hyperbolic toral automorphisms. It is an open question which three-manifolds support an Anosov flow: obstructions have in the past come primarily from foliation theory, since the plane fields $E^c \oplus E^s$ and $E^c \oplus E^u$ are integrable and tangent to taut (and often minimal) foliations [28]. Group-theoretic obstructions exist [29], but even these rely to some extent on the geometry of the stable and unstable foliations. We will restrict attention to volume-preserving Anosov flows: in dimension three, certain "anomalous" Anosov flows exist which do not preserve volume (and have other unusual properties) [12].

The following beautiful construction was discovered by Mitsumatsu [26] (see also [6]).

Theorem 4 (Mitsumatsu [26]) *Let X be a vector field generating an Anosov flow on a compact M^3. Then X lies in the transverse intersection of a pair of oppositely oriented tight contact structures.*

There is a pair of transverse integrable plane fields containing X given by $E^c \oplus E^s$ and $E^c \oplus E^u$ respectively: these form the (weak-) stable and unstable foliations. For each $p \in M$, define ξ^s_p to be the subspace of TM_p obtained by rotating $E^c_p \oplus E^s_p$ about the E^c_p subspace by a fixed angle, say $\pi/4$. One may define ξ^u likewise by rotating $E^c \oplus E^u$ about the E^c direction by $\pi/4$. Under the action of the flow of X, the transverse directions are always rotated away from the stable section E^s and towards the unstable section E^u; hence, $(\phi^t)^*\xi^s \neq \xi^s$ and $(\phi^t)^*\xi^u \neq \xi^u$ for all t sufficiently small. Any plane field η is integrable if and only if, for every vector field X tangent to η, the pullback of η under the flow of X, $(\phi^t)^*\eta$, equals η for all t sufficiently small. It thus follows that ξ^s and ξ^u define contact structures which are furthermore of opposite orientation (follow the directions of twisting). Mitsumatsu then shows that these structures are tight by appealing to a theorem of Eliashberg and Gromov [5,16] that symplectically semi-fillable structures are tight.

This result is of interest in that it allows one to construct very explicit examples of tight contact structures on those three-manifolds which admit Anosov flows. We consider the converse problem of using existence and uniqueness theorems for tight contact structures as an obstruction to the existence of an Anosov flow. To proceed, we require a bit of knowledge about the homotopy classification of plane fields on three-manifolds.

3 The three-dimensional invariant

We describe an invariant of plane fields on integral homology three-spheres. This invariant was originally defined by Gompf [15] for any closed three-manifold (cf. [24]), but is simplest to define in the restricted case we consider. Given a coorientable plane field ξ on an oriented homology three-sphere M one can always find an oriented almost-complex 4-manifold Y which M bounds (respecting orientations) so that ξ is the field of complex tangencies [15]. Since $H^2(\partial Y;\mathbb{Z}) = 0$ we have $c_1(Y) \in H^2(Y;\mathbb{Z}) \cong H^2(Y,\partial Y;\mathbb{Z})$. Thus we can think of $c_1^2(Y)$ as the integer obtained by pairing $c_1(Y) \smile c_1(Y)$ with the fundamental class $[Y,\partial Y] \in H_4(Y,\partial Y;\mathbb{Z})$. Now define

$$\theta(\xi) := c_1^2(Y) - 2\chi(Y) - 3\sigma(Y), \tag{4}$$

where $\sigma(Y)$ is the signature of Y and $\chi(Y)$ is the Euler characteristic of Y. The invariant $\theta(\xi)$ depends only on the homotopy type of ξ and the orientation on M (not on the coorientation of ξ). To see this, fix Y an almost-complex manifold which bounds $(-M,\xi)$ (i.e., M with reversed orientation). Then, consider Y_0 and Y_1 two almost-complex four-manifolds which bound (M,ξ) with the proper orientation. We can glue Y_0 or Y_1 to Y along their boundaries to obtain a closed almost-complex manifold W. For such a manifold the Hirzebruch signature theorem (see, e.g., [22]) says that

$$c_1^2(W) = 2\chi(W) + 3\sigma(W). \tag{5}$$

This proves, after noting the additivity of all three terms in Equation 4, that (1) the invariant θ is well-defined; and (2) θ reverses sign upon changing the orientation on M. On homology three-spheres, θ is a complete invariant of plane fields.

Theorem 5 *Let ξ_1 and ξ_2 be coorientable plane fields on an oriented homology three-sphere M. Then ξ_1 is homotopic to ξ_2 if and only if $\theta(\xi_1) = \theta(\xi_2)$.*

For a proof of this theorem the reader is referred to [25]. This is a special case of a much more general theorem in [15]. This invariant (and the more general version) yields an invariant of homotopy classes of nonsingular vector fields on three-manifolds by associating to any such vector field a transverse plane field.

The relationships between the dynamics of a nonsingular vector field X and the information encoded in $\theta(X)$ have been almost completely unexplored [1].

Example 6 Let ξ_0^+ denote the standard tight contact structure on S^3 of Example 1. One can realize ξ_0^+ as the set of complex tangencies of the unit $S^3 \subset \mathbb{C}^2$ with the standard complex structure, bounding the trivial 4-ball. Hence,

$$\theta(\xi_0^+) = c_1^2 - 2\chi - 3\sigma = 0 - 2(1) - 3(0) = -2.$$

If, however, we consider ξ_0^-, the unique tight contact structure on S^3 which induces the negative orientation, we can realize this as the image of ξ_0^+ under an orientation-reversing diffeomorphism of S^3. One likewise may compute directly that $\theta(\xi_0^-) = +2$ (or, apply and orientation reversing diffeomorphism and appeal to the results of [15]).

4 A tight obstruction to Anosov flows

Lemma 7 *Let X be a vector field contained in the transversally orientable plane field η on an oriented three-manifold M. Then the three-dimensional invariants of X and η agree.*

PROOF. Choose Z a vector field transverse to η, and let ζ denote the plane field spanned by Z and X. Since Z and X are nowhere collinear, we may homotope Z to X within ζ. □

The classification of tight contact structures on S^3 (Theorem 3) thus yields a simple proof of the nonexistence of Anosov flows on S^3:

Theorem 8 *There are no Anosov flows on S^3.*

PROOF. Assume X is an Anosov flow on S^3. Then X lies in the transverse intersection of a pair of oppositely oriented tight contact structures ξ^+ and ξ^- which are homotopic as they contain a common vector field. Theorem 3 implies that ξ^+ and ξ^- are contact isotopic to ξ_0^+ and ξ_0^- respectively. Therefore, with respect to the positive orientation on S^3, the calculation of Example 6 yields the contradiction $-2 = \theta(\xi^+) = \theta(\xi^-) = 2$. □

[1] The invariant θ is a dynamical invariant in the sense that topologically conjugate vector fields have equal θ-values.

This result, though well-known and easily proved via Novikov's theorem on foliations, provides an alternate motivation for classifying tight contact structures on three-manifolds, as well as extends the range of applications of contact topology to include the field of dynamical systems. Note also that smoothness issues concerning the foliations associated to Anosov flows (which are quite delicate — the foliations are only Hölder continuous in general) are not an issue when working with the associated (smooth) contact structures.

5 Homotopy uniqueness

If a manifold admits one Anosov flow, it admits many: all small perturbations to the vector field yield Anosov flows. These are all dynamically and topologically the same flows. The question of how many "different" Anosov flows can exist is naturally amenable to contact topological methods as follows. Every three-manifold possesses a countable infinity of homotopy classes of cooriented contact structures, or, equivalently, nonsingular vector fields. Thus, the dynamical question of *"How many homotopy classes of Anosov vector fields exist on a given manifold?"* is related to the homotopy classification of tight contact structures.

It has been announced quite recently by Colin, Giroux, and Honda that there are a finite number of homotopy classes of tight contact structures on any compact 3-manifold. This immediately gives a finiteness result for homotopy classes of Anosov fields. More precise bounds can be obtained in certain cases:

Theorem 9 *Given M a torus bundle over S^1, there is at most one homotopy class of Anosov fields on M.*

PROOF. The recent classification for tight contact structures on T^2-bundles over S^1 [13,20] implies that, though there are many different tight contact structures on a torus bundle over S^1, there is a unique homotopy class of universally tight structures — those structures for which no cover yields an overtwisted structure. Given an Anosov vector field, any cover also yields an Anosov vector field, which has the corresponding pair of transverse contact structures which, for a finite cover, are tight by Mitsumatsu. It is known that for torus bundles, contact structures are either universally tight or some finite cover is overtwisted. Thus, the contact structures associated to any Anosov field on this bundle are universally tight, and the result follows by the classification theorems of Giroux and Honda. □

For a hyperbolic torus bundle, there is a natural Anosov flow obtained by the suspension of the hyperbolic monodromy: up to homotopy, this is the unique example.

6 Miscellany

There are several ways in which problems concerning the dynamics of flows on three-manifolds can be assisted by understanding the classification of tight contact structures. As this latter subject is in its infancy and growing rapidly we are optimistic that the following problems may have contact-topological solutions:

Conformally Anosov flows:

These flows, defined independently by Mitsumatsu [26] and Eliashberg and Thurston [6], are flows which have the same dynamics on the projectivized normal bundle to the flow as does an Anosov flow. Such flows are more general than Anosov flows (e.g., they can arise on T^3, whereas Anosov flows cannot); however, they still appear as (and are indeed equivalent to) the intersection of a pair of transverse oppositely oriented contact structures. It is an open problem to classify which manifolds admit conformally Anosov flows. [2] A straightforward adaptation of the proof of Theorem 9 yields that there is at most one homotopy class of conformally Anosov flows on any torus bundle over S^1.

Hyperbolic manifolds

As with most problems in three-manifold topology, finding and classifying Anosov flows and/or tight contact structures is especially challenging on the class of hyperbolic three-manifolds. (with homology three-spheres being of particular interest). Very recently, R. Roberts, J. Shareshian, and M. Stein have announced the existence of closed hyperbolic three-manifolds possessing no taut foliations. This implies there are closed hyperbolic three-manifolds with no Anosov flows. It would be interesting to see if these three-manifolds admit appropriate tight contact structures.

Legendrian flows

Besides the Anosov fields considered thus far, several other important flows in dynamical systems are *Legendrian*, or tangent to a contact structure. It is an interesting question which nonsingular vector fields may be Legendrian, and in particular how the tight/overtwisted dichotomy manifests itself. A

[2] Added in proof: T. Noda and T. Tsuboi have recently made some progress on this problem.

theorem of Honda [19] yields the first set of examples of nonsingular vector fields on S^3 which are not Legendrian. As these examples are all Morse-Smale, it follows that they cannot preserve any volume form. It remains an open problem to find an obstruction for nonsingular volume-preserving Legendrian fields on the three-sphere. [3]

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