STEADY NONINTEGRABLE HIGH-DIMENSIONAL FLUIDS

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ABSTRACT. We consider the existence of steady incompressible fluids (solutions to the Euler equations) on Riemannian manifolds of dimensions three and higher. We demonstrate that, as in the case of the *ABC fields* in dimension three, there exist chaotic Beltrami fields – eigenfields of the curl operator – in higher dimensions. We give an explicit set of analytic examples on a non-Euclidean five-torus T^5. We also detail a "plug" construction for inserting chaotic vortices into a Beltrami field. These constructions employ contact-topological techniques.

1. INTRODUCTION

The Euler equations for a perfect incompressible fluid velocity field $u(x,t)$,

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p \; ; \; \nabla \cdot u = 0, \tag{1}$$

can be easily interpreted on any manifold M with Riemannian metric g and volume form μ by letting ∇ denote the Riemannian connection induced by g and by restating the incompressibility condition as $\mathcal{L}_u \mu = 0$, where $\mathcal{L}$ denotes the Lie derivative. One says that $u(x,t)$ is a solution to the Euler equations if there exists a function $p : M \times \mathbb{R} \to \mathbb{R}$ (the pressure) for which (1) is satisfied. The problem of understanding solutions to these geometric Euler equations provides a meeting ground for dynamics, geometry, and topology. Even in the case of the steady solutions, the global aspects of the Euler equations are rich. In this note, we illustrate some of the interplay between topological and dynamical complexity in steady Euler flows.

The classical theorem of Arnold [Arn66] implies that, in the real-analytic category, "most" nonvanishing steady Euler fields on an odd-dimensional Riemannian manifold are strongly integrable in the following sense: the velocity and vorticity fields commute, generating an $\mathbb{R}^2$-action on those regions of the manifold where velocity is not colinear with vorticity. The assumption on nonsingularity implies that the manifold is filled almost everywhere by invariant 2-tori on which the flow is conjugate to linear flow. The real-analyticity implies that the "singular" region where velocity and vorticity are parallel is either a variety of codimension one or greater (hence of zero-measure), or else it is the entire manifold.

The condition of an everywhere colinear vorticity and velocity distribution is the *Beltrami* condition: *i.e.*, the velocity field u is an eigenfield of the curl operator. Arnold suggested to Hénon the following set of vector fields as a fruitful starting point for understanding the

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exceptions to the aforementioned theorem in dimension three:

$$\begin{aligned} \dot{x} &= A\sin z + C\cos y \\ \dot{y} &= B\sin x + A\cos z \\ \dot{z} &= C\sin y + B\cos x \end{aligned} \quad . \tag{2}$$

These are the so-called *ABC fields*. One may assume by rescaling [DFG+86] that $0 \leq C \leq B \leq A \leq 1$. By quotienting out the periodicity, these equations are a Beltrami field on a Euclidean three-torus $S^1 \times S^1 \times S^1$. The dynamics of the ABC fields range from integrable (when any coefficient is zero) to highly nonintegrable and "chaotic" (apparently, when all three coefficients are nonzero). Several authors have used the Melnikov perturbation technique to prove the existence of positive topological entropy in the near-integrable case where C is close to zero [Gau85, HZD98], but rigorous results in the fully nonintegrable regime are surprisingly sparse, given the unique role of Beltrami fields in the analysis of Euler flows and MHD. Figure 1 illustrates the dynamics.

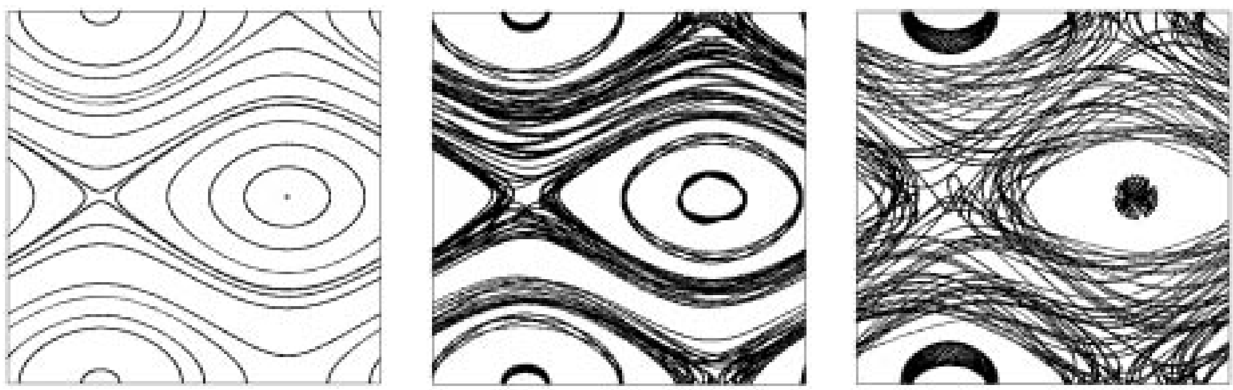

FIGURE 1. The projection of the ABC fields on T^3 onto the (x, z) plane, where $A = 1$, $B = 0.5$ and [left] $C = 0$; [middle] $C = 0.1$; [right] $C = 0.3$. Notice the integrable tori which break into nonintegrable dynamics for $C > 0$.

In this note we consider the analogous case of steady Euler fields on Riemannian manifolds of dimension greater than three. We note the existence of fundamental differences in the behavior of a steady fluid in odd versus even dimensions. In higher even-dimensional flows, instead of having a foliation by tori of dimension two, as in the case of a non-Beltrami steady odd-dimensional fluid, the typical even-dimensional steady fluid has an invariant foliation almost everywhere by manifolds of *codimension* two. For manifolds of dimension six or higher, this indicates that even-dimensional fluids may be dynamically "looser" than their odd-dimensional counterparts. It remains to be seen if four-dimensional fluid dynamics more closely resembles dynamics in dimensions three [very difficult to analyze] or dimension two: see [GK94] for a treatment of steady integrable four-dimensional Euler flows.

The problem stated by Arnold and Khesin [AK98, p. 110-111] is as follows: give an explicit example of a nonintegrable Beltrami field in dimension five or higher and compare their dynamics with those of the ABC fields. We do so in this note.

We first must clarify what is meant by curl on an arbitrary Riemannian manifold. Following [AK98], given a Riemannian manifold (M, g) with volume form μ, the curl of a vector field

X is a vector field for M odd-dimensional and is a function for M even-dimensional. This is most easily expressed in terms of the language of differential forms. Denote by λ to differential 1-form dual to a vector field X via the metric g: *i.e.*, $\lambda := g(X, \cdot)$. Then, if M is of dimension $2n$, the curl of X is the real-valued function on M defined by $(d\lambda)^n/\mu$. This quantity is well-defined since $(d\lambda)^n = d\lambda \wedge \cdots \wedge d\lambda$ is a $2n$-form and the volume form μ is nowhere zero. If M is of dimension $2n+1$, then $\nabla \times X$ is the vector field uniquely defined by the relation

$$\mu(\nabla \times X) = (d\lambda)^n. \tag{3}$$

It can of course be argued that the proper definition of curl should be the 2-form $d\iota_X g$: however, in the context of a fluid, this accurately represents the *vorticity* of the flow, rather than the curl itself. In [GK94] it is shown that, as in the case of dimension three, an odd-dimensional steady Euler flow is nonintegrable (specifically, orbits do not lie on analytic hypersurfaces) only if it is Beltrami: $\nabla \times u$ parallel to u.

Equation (3) is conveniently linear in X in the case $n = 1$. In higher dimensions, however, finding a nonsingular eigenfield is decidedly challenging. To solve such a nonlinear problem, we turn to the study of contact structures — implicitly global objects.

2. Contact forms and Beltrami fields

A *contact form* on a $(2n+1)$-dimensional manifold M (the odd dimension is necessary) is a differential 1-form α on M such that $\alpha \wedge (d\alpha)^n \neq 0$. The *contact structure* associated to α is $\xi := \ker \alpha$, the set of tangent vectors $v \in T_pM$ such that $\alpha_p(v) = 0$. In other words, ξ is a smoothly varying hyperplane field on M.

The difference between the contact form α and the contact structure ξ is encapsulated in the line field $\ker((d\alpha)^n) := \{v \in T_pM : (d\alpha)^n(v) = 0\}$. By the Frobenius Integrability Theorem, the following are equivalent: (1) α is a contact form; (2) ξ is a maximally non-integrable plane field distribution; and (3) $\xi = \ker \alpha$ is everywhere transverse to the line field $\ker((d\alpha)^n)$. By taking a section of the line field $\ker((d\alpha)^n)$ which is of "unit length" with respect to α, we obtain a natural vector field associated to α whose flow preserves both α and ξ. The *Reeb field* of a contact form α is defined to be the unique vector field X on M such that $(d\alpha)^n(X) = 0$ and $\alpha(X) = 1$.

The key step in our constructions is the following correspondence theorem between nonzero Beltrami fields and Reeb fields.

Theorem 2.1. [EG00] *The class of nonsingular vector fields on M^{2n+1} which are reparametrizations of nowhere-vanishing Beltrami fields for some Riemannian metric g and volume form μ is precisely the class of nonsingular fields on M which are reparametrizations of Reeb fields of some contact form α on M.*

In other words, all Beltrami fields (nonsingular with nonsingular curls) are orthogonal to a contact structure; conversely, all nonzero rescalings of arbitrary Reeb fields are, for certain Riemannian structures, Beltrami fields. The proof of this theorem given in [EG00] is valid in all higher odd dimensions with almost no modification.

The problem of constructing simple (steady, nonsingular, smooth) solutions to the Euler equations is inherently geometric: the integrable solutions are very rigid and "rare" [EG99b], and the Beltrami solutions are not amenable to perturbation and the like — *for a fixed Riemannian structure*. However, if one loosens the metric structure, this rigidity disappears. The class of contact forms is open in the space of 1-forms on M (by the definition $\alpha \wedge (d\alpha)^n \neq 0$); hence, there is a great degree of flexibility in building "customized" contact forms and corresponding Reeb fields. Cut-and-paste constructions common in low-dimensional topology may thus be applied to contact forms, and, by the above theorem, to Beltrami fields. For example, one can build a Beltrami field (which is thus a steady Euler field) on a Riemannian $\mathbb{R}^3$ which possesses periodic flowlines of all possible knot and link types simultaneously [EG99].

Our first method, then, for constructing a Beltrami field in higher dimensions is simply to find a contact form. The Reeb field associated to the contact form is necessarily a Beltrami (and hence a steady Euler) field for a Riemannian structure which makes the contact elements orthogonal to the Reeb field. Perhaps not surprisingly, this problem is also highly nontrivial: while contact forms exist on any three-manifold [Mar71], the existence problem for higher-dimensional manifolds is relatively unexplored. There are some known examples and some known obstructions, but not a general theory for existence.

We begin with a simple example in dimension five as an initial exploration of what to expect in the general case.

3. An example on T^5

Consider the following 1-form on the five-torus [Lut77]:

$$\begin{aligned} \alpha_0 \;:=\; & (\sin\theta_2\cos\theta_2)d\theta_1 - (\sin\theta_1\cos\theta_1)d\theta_2 \\ & +(\sin\theta_1\sin\theta_5 + \sin\theta_2\cos\theta_5)d\theta_3 \\ & +(\sin\theta_1\cos\theta_5 - \sin\theta_2\sin\theta_5)d\theta_4 \\ & +(\cos\theta_1\cos\theta_2)d\theta_5. \end{aligned} \tag{4}$$

This form is a contact form since

$$\alpha_0 \wedge (d\alpha_0)^2 = 8 - 8\cos^2\theta_2 - 8\cos^2\theta_1 + 2\cos^4\theta_1 + 6\cos^2\theta_1\cos^2\theta_2 + 2\cos^4\theta_2 > 0. \tag{5}$$

The Reeb field associated to α_0 is thus, by Theorem 2.1 a nonvanishing Beltrami solution to the Euler equations for the appropriate Riemannian structure. Since the contact form is invariant under the T^2-action generated by $(\partial/\partial\theta_3, \partial/\partial\theta_4)$, the Reeb field also possesses an invariance with respect to these directions. One solves the equations defining a Reeb field to obtain:

$$\begin{aligned} \theta_1' &= \sin\theta_2\cos\theta_2/\Phi \\ \theta_2' &= -\sin\theta_1\cos\theta_1/\Phi \\ \theta_3' &= \big(-\sin\theta_1\sin\theta_5 + 2\sin^3\theta_1\sin\theta_5 + 2\sin^3\theta_2\cos\theta_5 \\ &\quad + \sin^2\theta_2\sin\theta_1\sin\theta_5 + \sin^2\theta_1\sin\theta_2\cos\theta_5 - \sin\theta_2\cos\theta_5\big)/\Phi\ , \\ \theta_4' &= \big(-\sin\theta_1\cos\theta_5 + 2\sin^3\theta_1\cos\theta_5 - \sin^2\theta_1\sin\theta_2\sin\theta_5 \\ &\quad + \sin\theta_2\sin\theta_5 - 2\sin^3\theta_2\sin\theta_5 + \sin^2\theta_2\sin\theta_1\cos\theta_5\big)/\Phi \\ \theta_5' &= \cos\theta_2/\Phi \end{aligned} \tag{6}$$

where Φ is the scaling function

$$\Phi = -\sin^2\theta_1 + \sin^4\theta_1 + 3\sin^2\theta_2\sin^2\theta_1 + \sin^4\theta_2 - \sin^2\theta_2 + 1 \tag{7}$$

Figure 2 illustrates the dynamics of this Beltrami field. Note the presence of invariant "rolls" (in this case circles in the (θ_1, θ_2) plane which under the T^2 action yield invariant 3-tori) indicative of integrable dynamics. This compares to the integrable solutions of the ABC equations where one or more coefficients equals zero displayed in Figure 1[left].

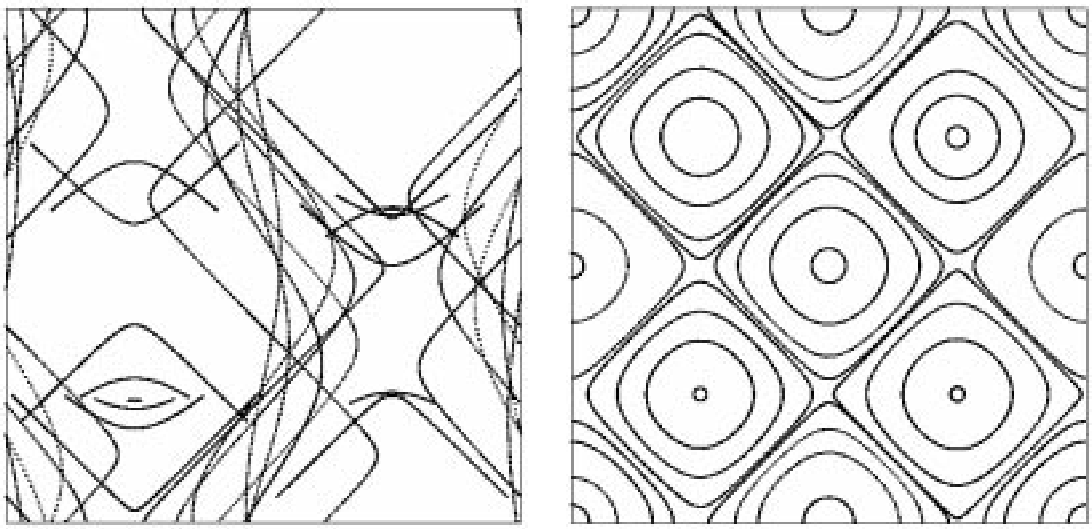

FIGURE 2. The projection of an integrable fluid flow on T^5 onto the (θ_1, θ_5) plane [left] and the (θ_1, θ_2) plane [right]. Several orbits are plotted.

To obtain a more complicated solution, one uses the fact that contact forms are open in the space of 1-forms. Thus, any small perturbation to α will still induce a contact form whose Reeb field yields a Beltrami solution to the Euler equations. By introducing a perturbation which breaks the symmetry induced by the T^2-action, one may force the fluid to mix. Consider the following 1-form:

$$\alpha := \alpha_0 + A\sin(\theta_4)d\theta_1. \tag{8}$$

To verify the contact condition, we compute

$$\alpha \wedge (d\alpha)^2 = \alpha_0 \wedge (d\alpha_0)^2 + A\left(2\cos\theta_1\cos\theta_4\cos\theta_1(\cos^2\theta_1 - 2) + 2\sin\theta_2\cos(\theta_2\sin\theta_4)\right) \tag{9}$$

A computer-algebra routine can be used to show that this function is nonzero for $A \leq \frac{5}{8}$ and that the Reeb field is given as:

$$\begin{array}{rcl}
\theta_1' & = & -\sin\theta_2(-1+\sin^2\theta_2)/\Phi \\
\theta_2' & = & -\left(\sin\theta_1\cos\theta_1 - A\cos\theta_4\sin\theta_1\cos\theta_5 + A\cos\theta_4\sin\theta_2\sin\theta_5\right)\cos\theta_2/\Phi \\
\theta_3' & = & \big(-A\cos\theta_4\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_5 + 2\sin^3\theta_1\sin\theta_5 + \sin^2\theta_1\sin\theta_2\cos\theta_5 \\
& & -\sin\theta_2\cos\theta_5 + 2\sin^3\theta_2\cos\theta_5 + \sin^2\theta_2\sin\theta_1\sin\theta_5\big)\cos\theta_2/\Phi \\
\theta_4' & = & \big(-(\sin\theta_1\cos\theta_5 - 2\sin^3\theta_1\cos\theta_5 + \sin^2\theta_1\sin\theta_2\sin\theta_5 - \sin\theta_2\sin\theta_5 \\
& & +2\sin^3\theta_2\sin\theta_5 - \sin^2\theta_2\sin\theta_1\cos\theta_5)\cos\theta_2\big)/\Phi \\
\theta_5' & = & \left(\cos\theta_1 - A\cos\theta_5\cos\theta_4 - \sin^2\theta_2\cos\theta_1 + A\sin^2\theta_2\cos\theta_5\cos\theta_4\right)/\Phi
\end{array}, \tag{10}$$

where the scaling function is given by

$$\begin{aligned}\Phi \;=\; & -A\sin\theta_4\sin^3\theta_2 + A\sin\theta_4\sin\theta_2 + \cos\theta_2\sin^4\theta_2 - \sin^2\theta_2\cos\theta_2 \\ & +3\cos\theta_2\sin^2\theta_1\sin^2\theta_2 - A\cos\theta_2\cos\theta_1\sin^2\theta_1\cos\theta_4\cos\theta_5 \\ & +\cos\theta_2\sin^4\theta_1 - \cos\theta_2\sin^2\theta_1 + \cos\theta_2 - A\cos\theta_1\cos\theta_2\cos\theta_5\cos\theta_4\end{aligned} \quad . \tag{11}$$

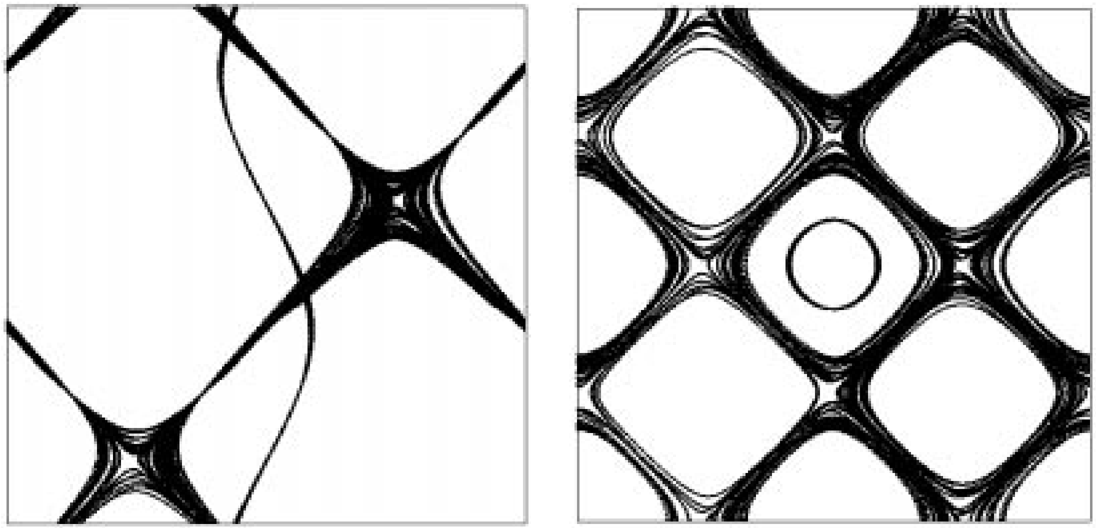

FIGURE 3. The projection of a nonintegrable fluid flow on T^5 onto the (θ_1, θ_5) plane [left] and the (θ_1, θ_2) plane [right], exhibiting diffusion between the near-integrable tubes: $A = 0.05$. Two orbits are plotted.

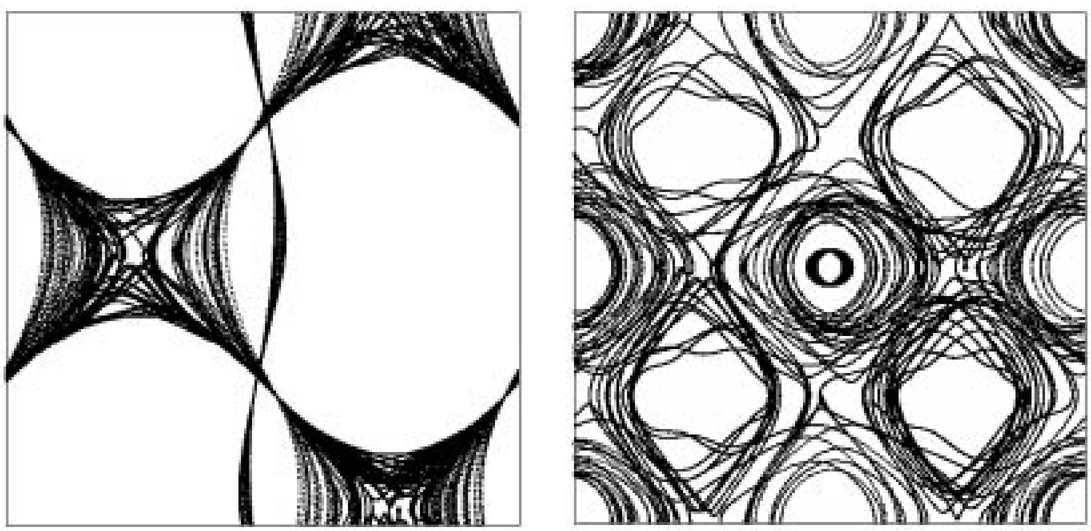

FIGURE 4. The projection of a nonintegrable fluid flow on T^5 onto the (θ_1, θ_5) plane [left] and the (θ_1, θ_2) plane [right]: $A = 0.3$. Two orbits are plotted.

To generate even more complex dynamics, one can perturb the integrable contact form in a manner which entwines all of the coordinates. For the perturbation

$$\alpha := \alpha_0 + A\sin(\theta_4)d\theta_1 + C\sin(\theta_3)d\theta_5, \tag{12}$$

a simple computer algebra routine[1] computes the Reeb field. Figures 5-6 illustrate the qualitative dynamics for parameter values within the contact range.

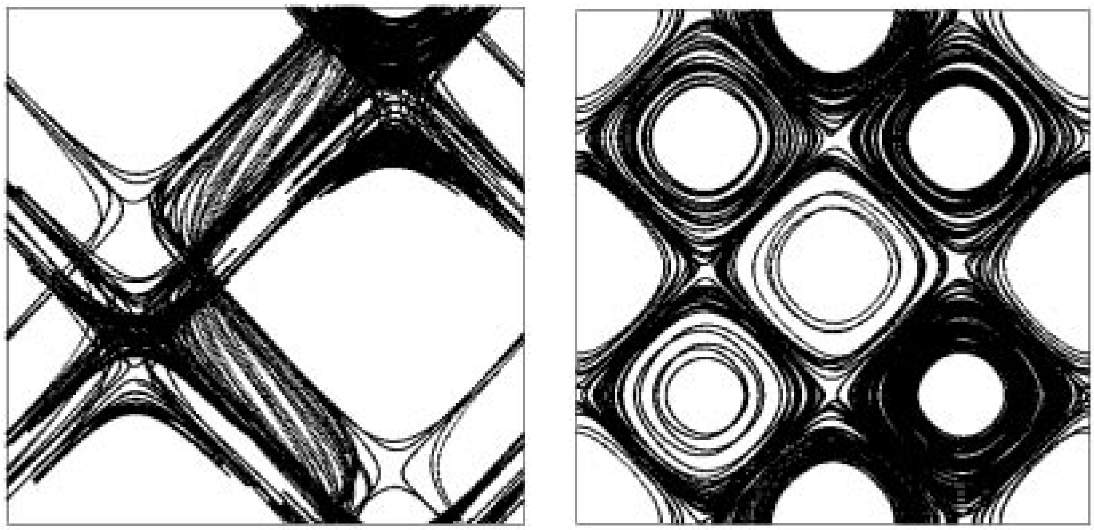

FIGURE 5. One orbit of the Reeb field of the contact form in Equation 12 at the values $A = C = 0.05$. Projections of T^5 onto the (θ_1, θ_5) plane [left] and the (θ_1, θ_2) plane [right].

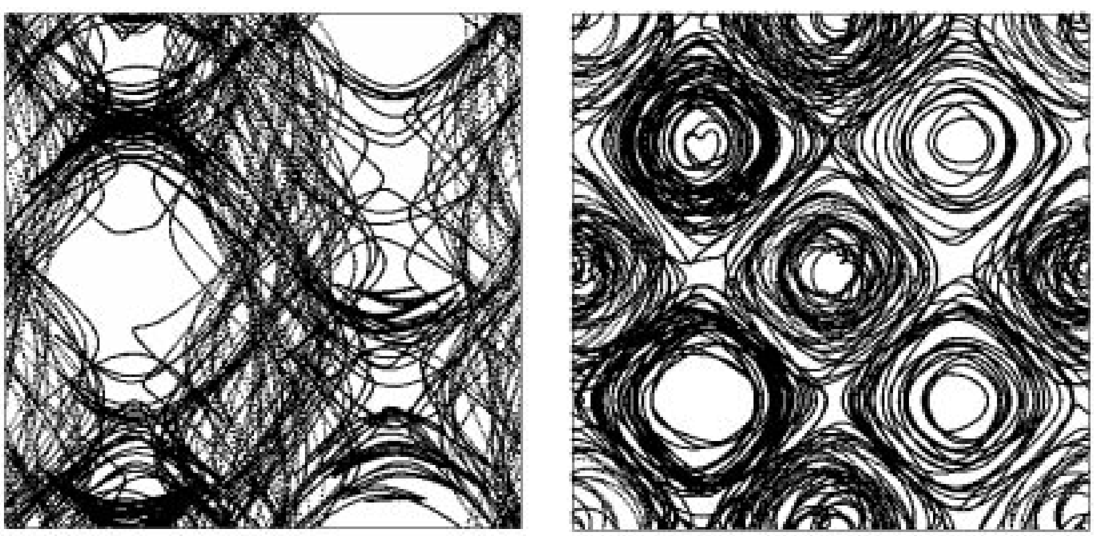

FIGURE 6. One orbit of the Reeb field of the contact form in Equation 12 at the values $A = C = 0.2$. Projections of T^5 onto the (θ_1, θ_5) plane [left] and the (θ_1, θ_2) plane [right].

Although these examples are rigorously Beltrami fields for the appropriate analytic Riemannian structure on T^5, their dynamics are merely qualitatively chaotic: we provide only a numerical indication that the dynamics are complex, not a proof. In the next section, we

[1]We used the `liesymm` package in `Maple` to perform form computations.

detail a method for modifying a Beltrami field which inserts rigorously chaotic "vortices" in the flow.

4. Chaotic vortex plugs

We now outline a general procedure for producing a chaotic steady Euler flow on any manifold M^{2n+1} which admits a Beltrami field. From a well-known result of Gromov (see, *e.g.*, [MS95]), all noncompact manifolds of odd dimension admit a contact form and thus admit a nonsingular Beltrami field by Theorem 2.1. Our construction works as well for Beltrami fields which have singularities (as long as the curl is not identically zero everywhere). It should not be difficult to generalize our construction to the case of all manifolds in all dimensions, but we restrict our attention to the odd-dimensional case for simplicity.

Our plug construction relies on the following theorem:

Theorem 4.1. [EG99] *There exists a tight contact form on $\mathbb{R}^3$ whose Reeb field possesses a compact hyperbolic invariant set conjugate to the suspension of a full shift on two symbols.*

By a "tight" structure is meant that the contact form is equal to some positive rescaling of $dz + x\,dy$. The dynamical consequences are equivalent to saying that there is a compact invariant set possessing positive topological entropy: *i.e.*, the flow is "chaotic." It is fairly nontrivial to construct this contact form: one performs a sequence of careful surgeries on the contact form arising from the ABC fields in the parameter regime which possesses a transverse homoclinic orbit.

It is a simple lemma to show that any positive function times a contact form remains a contact form. Thus, one may modify the contact form on $\mathbb{R}^3$ given by Theorem 4.1 so that it is equal to $dz+x\,dy$ outside of a sufficiently large compact set, the interior of which possesses the chaotic invariant set. This will be of the form $\lambda = f(x,y,z)(dz + x\,dy)$ for some positive $f : \mathbb{R}^3 \to \mathbb{R}$ which tends to 1 outside of a compact set. Since the Reeb field of $dz + x\,dy$ is clearly the field $\partial/\partial z$, we can design a three-dimensional "plug" by restricting λ to the set $P := \{(x,y,z) : x^2+y^2 \leq C, |z| \leq C'\}$ for constants C and C' sufficiently large. The Reeb field of λ enters the cylinder from the bottom, exits from the top, and is tangent to the sides; however, a certain portion of the flow is "trapped" in a chaotic invariant subset as illustrated in Figure 7.

Given any contact form α on M^3, one may insert the chaotic plug at will as follows. The Darboux theorem implies that any contact form is locally expressible[2] as $dz+x\,dy$ with Reeb field locally thus given as $\partial/\partial z$. One may thus directly replace the contact form α with λ on the plug P in local coordinates. This is still a smooth contact form since α and λ agree on a neighborhood of the plug boundary. The Reeb field of the modified form now possesses a small invariant "chaotic vortex" within the plug. After performing this procedure (as many times as desired), the modified contact form has a dynamically chaotic Reeb field which, via Theorem 2.1 defines a Beltrami field on M^3 and hence a steady Euler flow.

[2] A weaker result, often stated in textbooks, is that the contact *structures* are locally contactomorphic; however, the result holds for the contact *form* as well. See, *e.g.*, [BCG$^+$91] for a clear proof of this classical result.

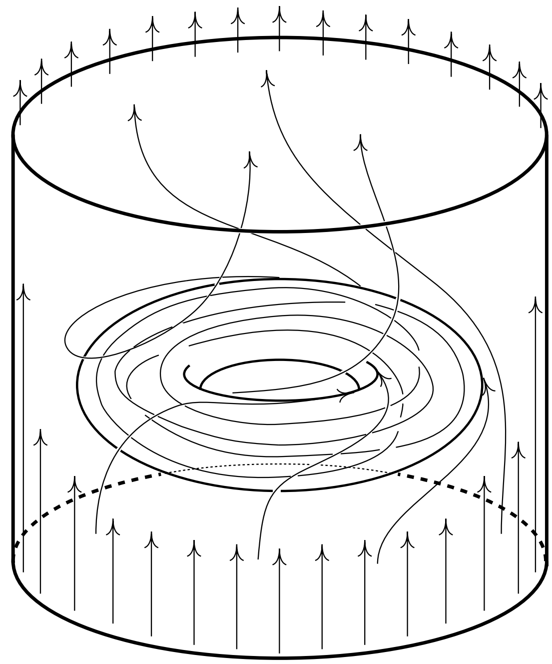

FIGURE 7. A chaotic plug for Reeb/Beltrami fields in 3-d.

Assume now that M is a $2n+1$-dimensional manifold possessing a Beltrami field u which is not identically zero. Choose any neighborhood U in M for which $\nabla \times u$ is nonvanishing. The proof of Theorem 2.1 implies that the 1-form $\alpha := g(u,\cdot)$ is a contact form on U whose Reeb field is a time-reparametrization of u. By the Darboux theorem, it follows that in a smaller neighborhood $U' \subset U$, the form α is equivalent to

$$dz + \sum_{i=1}^{n} x_i \, dy_i,$$

for a suitable coordinate system $\{x_1, y_1, \ldots, x_n, y_n, z\}$. To generate a chaotic plug in the higher dimensional setting, consider the 1-form

$$\tilde{\lambda} := \tilde{f}(x_1, y_1, z)\left(dz + \sum_{i=1}^{n} x_i \, dy_i\right),$$

where $\tilde{f}$ is a smooth positive function equal to 1 outside of a sufficiently large compact set and is equal to f, the scaling of the form λ in (x,y,z)-coordinates, on the interior of the plug $\tilde{P} := \{\sum_i (x_i^2 + y_i^2) \leq C, |z| \leq C'\}$. Since any positive rescaling of a contact form is still a contact form, $\tilde{\lambda}$ is contact. As $\tilde{f}$ depends only on the three variables, the Reeb field is exactly that of the chaotic 3-d plug with product flow in the remaining directions. This yields again chaotic dynamics [shifts, positive topological entropy]. By inserting the high-dimensional plug $\tilde{P}$ at will, one may generate chaotic vortices in any high-dimensional Reeb field. Within the neighborhood $U' \subset U \subset M$, the Reeb field has been modified. By following the reverse implication in Theorem 2.1, one has that for a modified Riemannian metric g the new Reeb field is again Beltrami. Since the modification was performed strictly within U, the local and global metrics match, yielding a globally-defined Beltrami field with chaotic dynamics.

For the high-dimensional plug so constructed, the complicated orbits are constrained to reside within the three-dimensional "sheets" spanned by the local coordinates (x_1, y_1, z). To further increase the mixing, one may modify the plug as follows: define

$$\hat{\lambda} := \left(\sum_{i=1}^{n} a_i \tilde{f}(x_i, y_i, z)\right)\left(dz + \sum_{i=1}^{n} x_i \, dy_i\right). \tag{13}$$

This is clearly a contact form whose Reeb field is the sum of the Reeb fields of the individual terms (since the d operator is linear). Here the a_i terms are constants which control the "amplitudes" of the individual components. Each term in the Reeb field possesses a chaotic vortex in the dimensions (x_i, y_i, z), and their sum generates a vector field which is not a product flow. To rigorously ensure the persistence of positive topological entropy, one may impose the condition that all but one a_i is small and appeal to the structural stability of the hyperbolic invariant set.

5. Concluding remarks

We have not discussed the case of high even-dimensional Beltrami fields. The paper [GK94] deals with the four-dimensional case (and has been erroneously interpreted to mean that all steady 4-d Euler flows are integrable [Güm97]). Although there are no contact forms on even-dimensional manifolds, the techniques outlined here can easily be adapted to produce chaotic four-dimensional steady Euler flows. The curl of such a vector field would of necessity vanish identically; however, the vorticity 2-form can be made to be nondegenerate in $2n-1$ directions. A trivial example would be taking the flat cross product of a 3-manifold with S^1 and placing an irrational flow on this factor.

The examination of high-dimensional fluid flows is in its initial stages: the simplest questions of existence and appearance of steady solutions to the Euler equations are relatively untapped. We have indicated several methods for constructing "customized" chaotic Beltrami solutions based on contact-topological techniques, which have the advantage of relieving the user from dealing with analytic difficulties associated with the [highly nonlinear] curl operator in higher odd dimensions. Conversely, understanding existence of eigenfields of the curl operator in higher odd dimensions may yield new examples of contact forms on manifolds.

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