Low Dimensional Contact Geometry

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1. Summary

My work centers on contact geometry — an odd dimensional analog of symplectic geometry. Specifically, I have produced results concerning the existence and classification of contact structures on 3-manifolds. In addition, I have discovered some, and studied other, surprising relations between contact geometry and topology and dynamics.

2. Goals

Classify contact structures on "simple" 3-manifolds. Building on past work I shall better understand the relationship between tight and fillable contact structures and obtain a (at least, crude) classification on contact structures on Seifert fibered spaces, Haken manifolds and possibly non-Haken manifolds with strongly irreducible Heegaard splittings.

Understand transversal and Legendrian knots in tight contact structures. Extending the work of myself and others I shall develop a general framework for understanding Legendrian and transversal knots: for example understand how they behave under connected sums and satellite constructions. I shall also classify Legendrian knots in various knot types and explore the algebraic structure and geometric meaning of the contact homology/symplectic field theory of a link.

Investigate the relations between topological hydrodynamics and contact geometry. I will be continuing work (with Ghrist) relating contact geometry to fluid dynamics by addressing problems in energy minimization and hydrodynamic instability.

Explore relations between classical Riemannian geometry and contact geometry. In this recent endeavor I shall study relations between curvature in classical Riemannian geometry and properties such as tightness in contact geometry.

3. Research Summary and Background

3.1. **Contact Geometry.** Understanding contact structures on 3-manifolds is important in its own right but can also illuminate topology and dynamics in 3-dimensions. For example Eliashberg [7] has used them to understand diffeomorphisms of S^3 and Rudolph [34] has used them to study slice knots. Moreover, Kronheimer and Mrowka [29] have recently related contact structures to Seiberg–Witten theory on 3-manifolds and Eliashberg and Thurston [11] have found relations between contact structures and foliations. A *contact structure* on a 3-manifold is a 2-plane field ξ in the tangent bundle that is completely nonintegrable. This means that the 2-planes are not tangent, even locally, to a foliation. (Throughout this proposal all contact structures are assumed to be orientable.) Martinet [30] showed that any 3-manifold admits a contact structure, though it soon became apparent that a different existence problem was more interesting. Eliashberg [7], following work of Bennequin [2], distinguished two types of contact structures: tight and overtwisted. Eliashberg [5] has shown that the classification of overtwisted contact structures essentially reduces to algebraic topology, while tight structures are more interesting and subtle. In particular, since the discovery of tight structures over a decade ago, determining which manifolds admit such structures has been a driving and difficult problem in the field. Honda and I have recently proved there is a 3-manifold that does not admit a tight contact structure, providing the first such example.

Theorem (Etnyre and Honda [18]) *The connected sum of the Poincaré homology sphere P with $-P$ does not admit a tight contact structure.*

It is still unknown if there is an irreducible (*i.e.* not a connected sum) 3-manifold with no tight contact structure.

In the opposite direction, Eliashberg [6], Gompf [24] and Eliashberg-Thurston [11] have constructed tight contact structures on many 3-manifolds. Many of these examples involve symplectic fillings. A contact manifold (M,ξ) is *symplectically fillable* if there exists a symplectic manifold (W,ω) for which $M=\partial W$ and $\omega|_\xi$ is a symplectic form on ξ (and the orientations induced on M by ω and ξ agree). A symplectically fillable contact structure is automatically tight and until recently it seemed possible (and even likely!) that any tight contact structure was symplectically fillable. This however is not the case:

Theorem (Etnyre and Honda [20]) *There exist tight but not fillable contact structures.*

For manifolds admitting tight contact structures one would like to classify the tight contact structures. To this end I have investigated the simplest type of 3-manifolds — lens spaces. Recall, a lens space $L(p,q)$ is a three manifold obtained by gluing two solid tori together along their boundaries and is determined by a pair of relatively prime integers (p,q). I have shown:

Theorem (Etnyre [13]) *Each lens space admits only finitely many tight contact structures.*

Moreover, I have classified the tight contact structures on lens spaces when $p<6$. For example:

Theorem (Etnyre [13]) *The lens space $L(3,1)$ admits precisely two tight contact structures up to isotopy, while $L(3,2)$ admits only one. (This covers all lens spaces with $p=3$.)*

Note: a contact structure ξ is an orientable two dimensional bundle and thus has an Euler class $e(\xi)$ in the second integral cohomology of the manifold.

Theorem (Etnyre [13]) *Each lens space has cohomology classes c and c' such that c is not the Euler class of any tight contact structure and c' is the Euler class of precisely one tight contact structure.*

This result was the first (non-classical) nonexistence result for the possible Euler classes of tight contact structures. Recently Honda [26] and Giroux [23] have announced a complete classification of tight contact structures on lens spaces.

3.2. **Legendrian and Transversal Knots.** The study of knots that respect a contact structure in a certain way has illuminated the geometry and topology of three manifolds. For example *Legendrian knots* (those tangent to the contact planes) have been used by Kanda [28] to distinguish homotopic contact structures on T^3, and invariants associated to Legendrian knots have been used by Rudolph [34] to find obstructions to slicing a knot (this is the difficult problem of determining when a knot in $S^3 = \partial B^4$ bounds a 2-disk in B^4). The genesis of the tight vs. overtwisted dichotomy described above was in the work of Bennequin [2] on *transversal knots* (those transverse to the contact planes). Despite their importance, little is known concerning the classification of Legendrian and transversal knots. There is one simple invariant, the *self-linking number*, of transversal knots and there are two, the *Thurston-Bennequin invariant* and *rotation number*, of Legendrian knots. Eliashberg [8] showed that transversal unknots are determined by their self-linking number, while Eliashberg and Fraser [10] showed that Legendrian unknots are determined by their simple invariants. I have extended this classification to certain transversal torus knots [12] (knots that sit on a standardly embedded torus in S^3), and later Honda and I proved

Theorem (Etnyre [12], Etnyre and Honda [19]) *Legendrian and transversal torus and figure eight knots are determined by their knot type and their simple invariants.*

There are examples of Legendrian knots with the same invariants that are not Legendrian isotopic. These were distinguished by Chekanov [3] and Eliashberg and Hofer using *contact homology.* Contact homology, created by Eliashberg and Hofer [9], is a systematic way to bring Gromov's very successful theory of pseudoholomorphic curves in symplectic manifolds [25] into the arena of contact topology. Though properly defined in terms of holomorphic curves, Chekanov [3] has defined a $\mathbb{Z}_2$ (or nonoriented) combinatorial version of this. Sabloff and I [21] have brought the orientations from the analytic theory into the combinatorial theory, thus extending Chekanov's work and obtaining an invariant defined over $\mathbb{Z}$. Ng [33] has also arrived at this invariant from a purely combinatorial approach.

3.3. **Fluid Dynamics.** Ghrist and I have discovered an interesting connection between Reeb fields (*i.e.* vector fields transverse to a contact structure whose flow preserves the contact structure) and fluid mechanics which allows us to transplant many of the powerful techniques and results of Hofer and others into the world of invicid flows. Recall the Euler equations for a perfect incompressible fluid are

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p \quad ; \quad \mathrm{div}(u) = 0. \tag{1}$$

Here, u denotes the velocity field (or Euler field) of the fluid and p the pressure field. I should note that it is quite difficult to obtain general topological information concerning Euler flows, nevertheless we can show:

Theorem (Etnyre and Ghrist [14, 15, 16, 17]) *In the real analytic setting:*

1. *Any time independent Euler flow on a Riemannian S^3 or $S^1 \times D^2$ has a periodic orbit. Moreover, on S^3 there must be an unknotted periodic orbit.*
2. *Any 3-manifold admits a nonsingular Euler flow.*
3. *There is a nonsingular Euler flow on S^3 whose periodic orbits realize all knots and all links.*

Item 2. should be thought of as showing there are no topological restrictions on the existence of an Euler flow. Item 3. should be compared to Moffatt's work [31, 32] demonstrating a similar result in the standard metric on S^3 but yielding a noncontinuous flow and relying on the existence to solutions to the Navier-Stokes equations (which is still conjectural).

4. Research Plans

4.1. **Contact 3-manifolds.** My future research will concentrate on understanding tight contact structures on 3-manifolds. From what is currently known it seems quite possible that irreducible manifolds always admit tight contact structures. By finding new ways of constructing tight contact structures I hope to answer this question. Specifically, based on my work and work of Honda and myself it seems that one might be able to construct tight contact structures using certain cut-and-paste constructions by controlling where an overtwisted disk might appear. Moreover, these techniques should also be useful in further understanding the relation between tight and symplectically fillable contact structures. This, in turn, should illuminate the subtle nature of tight contact structures.

Using techniques we have developed, Honda and I shall investigate the set of all tight contact structures on various manifolds. For example, we can show that all manifolds obtained from Dehn surgery on the figure eight knot (except three) have a finite number of tight contact structures. It also seems we may be able classify all the tight structures on these manifolds. We are also considering Haken 3-manifolds (*i.e.* ones with a nice decomposition into simple pieces). It seems likely that the number of possible tight contact structures on these manifolds can be bounded by topological information. In particular we hope to show that atoroidal Haken manifolds have a finite number of tight structures. We have similar ideas for other large classes of 3-manifolds.

4.2. **Legendrian and Transversal Knots.** It is interesting to note that all of the Legendrian knots which are classified by their simple invariants have fibered complements while Chekanov's examples do not. I shall study whether or not all Legendrian fibered knots are indeed classified by their simple invariants. Honda and I also have been investigating Legendrian knots which are not determined by their simple invariants. We believe we can understand these examples by looking at Legendrian knots in their complements. This program will shed light on Legendrian isotopy.

Sabloff and I are continuing our work on Legendrian knots and are trying to define a combinatorial version of Eliashberg, Hofer and Givental's *symplectic field theory* (*cf.* [9]). Symplectic field theory is a vast generalization of contact homology and its precise definition in the relative case is still somewhat murky. We expect our combinatorial investigations to help clarify the definition and provide powerful invariants of Legendrian (and possibly transversal) knots. I also hope to combine this work with the work of Honda and myself on Legendrian knots to get a better geometric understanding of what contact homology is really telling us about the Legendrian knots. Lastly, I shall study combinatorial versions of contact homology in higher dimensions. Finally, Ekholm and I have been sorting out the details of this combinatorial theory in higher dimensions.

4.3. **Hydrodynamics.** Ghrist and I are continuing our work on fluid dynamics and contact geometry. We will investigate geometric obstructions to the existence of Euler fields. (For example, hyperbolic metrics seem to restrict the nature of Euler fields.) We shall also investigate the topology of nonsingular Euler flows. For example, we should be able to define a computable topological invariant of an Euler field on a solid torus that will provide information about the number and type of periodic orbits. We also aspire to lift the notion of tight/overtwisted from the realm of contact topology to fluid dynamics. Specifically, we will develop and test these notions with the following questions. First, consider *energy minimization.* Motivated by questions in hydrodynamics, Arnold [1] studied divergence-free vector fields v on a closed manifold M that minimize the energy functional

$$E(v) = \frac{1}{2}\int_M (v, v)\, d\mathrm{vol} \tag{2}$$

among vector fields obtained from v by volume preserving diffeomorphisms of M. It is known that the extremizers of the energy functional are all steady solutions to the Euler equations; however, little is known about the *minimizers.* Based on our previous work we have asked: Is every energy minimizing field on a 3-manifold tight? An affirmative answer to this question would provide a concrete physical implication of the notion of tightness for a fluid flow, as well as giving an interesting analytic interpretation of tightness (which could very well provide a connection to Riemannian geometry).

Secondly, we shall consider *hydrodynamic instability.* Intuitively, the stability of a steady fluid flow is determined by perturbing the velocity field and evolving the field via the Euler equations. Recently Friedlander and Vishik [22] have provided a dynamical criterion, in terms of a periodic orbit, for demonstrating the instability of a fluid flow. This work yields the heuristic conclusion that "almost every Euler flow is hydrodynamically unstable." Ghrist and I will try to provide concrete statements concerning the set of unstable Euler fields in the space of all Euler fields. Using work of Hofer and others, Ghrist and I will investigate the question: Is every Beltrami field (a particularly interesting class of Euler fields) associated to an overtwisted contact structure unstable?

4.4. **Riemannian and Contact Geometry.** Along more speculative lines, work of Ghrist and myself [14] involves constructing metrics adapted to contact structures in various ways. Moreover, there has been quite a lot of work showing connections between contact geometry and Riemannian geometry (see, for example, the pioneering work of Chern and Hamilton [4] and the solution to the CR Yamabe problem in [27]). I would like to investigate this connection between Riemannian and contact geometry more deeply. For example, is it possible to "uniformize" contact structures? By this I mean, given a contact structure, when can one find a "nice" Riemannian metric for which the sectional curvature along the contact planes is $-1, 0$ or 1? If you have such a metric, how do notions like tight and overtwisted relate to properties of the metric? Can you use such metrics to investigate Legendrian and transversal knots?

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