Research Area: Birational Geometry and Tropical Compactifications

Tropical Geometry is a rapidly developing field at the interface of algebraic geometry and polyhedral combinatorics. In [14] and [42] tropical geometry was related to constructing compactifications of certain non-compact algebraic varieties. To a subvariety X of an algebraic torus there exist compactifications $\overline{X}$ with boundaries $D = \overline{X} \backslash X$ with desirable properties, and such that the structure of D is controlled by a polyhedral object called the tropical variety of X; see Section 1 below for details; such compactifications are called tropical compactifications.

In my thesis the theme I develop is that it is worthwhile to consider both X and $\overline{X}$ as a pair, or equivalently the pair $(\overline{X}, D)$ where D is the boundary of a tropical compactification. The main results of my thesis relate the *birational* algebraic geometry of such a pair $(\overline{X}, D)$ to the tropical variety associated to X.

1 Motivations and Context:

Recent applications of tropical geometry have been as diverse as intersection theory [30], mirror symmetry [1], computational biology [36, Chp. 3], statistics [35], moduli spaces [14], Diophantine problems [12], and with more certainly on the horizon. The unifying theme in all these cases is to relate algebraic varieties to certain polyhedral complexes called **tropical varieties**. One then tries to describe certain properties of the original variety in terms of the polyhedral geometry of the corresponding tropical variety. These tropical polyhedral complexes have proved to be of independent combinatorial interest as well, see e.g. [39]. My research concerns relating the structure of these tropical polyhedral complexes to classical constructions in birational algebraic geometry. Given the breadth of applications of tropical geometry above, it is not surprising that there are several approaches to the foundations (e.g. [4, 6, 12, 20]); my work builds on an approach due to Tevelev [42] and developed further by Hacking, Keel, and Tevelev [14]. Their techniques were developed mainly to construct compactifications of certain classical moduli spaces, but the theory is more general. The approach is motivated entirely by algebraic geometry, but we emphasize that once formulated it is well-suited to combinatorial study.

Let X be a non-compact variety over the complex numbers. The methods of algebraic geometry are most powerful when a variety is compact, or ideally projective. For algebraic curves constructing compactifications is straightforward as any algebraic curve has a unique smooth compactification, but a general theme in higher dimensional algebraic geometry is to construct geometrically meaningful compactifications. A deep theorem of Nagata says that under very general conditions there always exists a compact[1] variety $\overline{X}$ containing X as a Zariski open subset. Via a resolution of singularities one can assume that the boundary $\overline{X}\backslash X$ is a reduced divisor with simple normal crossings and that $\overline{X}$ is smooth if X is smooth. One could then attempt to study X by studying the compact variety $\overline{X}$, e.g. via techniques discussed in Section 2 below. One practical downside of this approach is that Nagata compactification and resolution of singularities do not give an explicit form for $\overline{X}$, even if X is a solution set to known equations. There is one well-studied situation where some compactifications are easy to understand: if $T \simeq (\mathbb{C}^*)^n$ one can consider equivariant compactifications, i.e. those $\overline{T}$ such that the group action $T \times T \to T$ extends to $T \times \overline{T} \to \overline{T}$. This gives the theory of toric varieties [9, 33], and the normal compactifications have beautiful combinatorial descriptions in terms of fans of cones with almost all aspects of their equivariant geometry admitting polyhedral interpretations. More generally, if $X \subset (\mathbb{C}^*)^n$ is a smooth subvariety of an algebraic torus, one could consider the closure $\overline{X}$ in various toric varieties. Tevelev's observation in [42] was that certain choices of fan will produce "better" compactifications, and that the "best" fans will all have common support; that is, there exists a fixed polyhedral complex depending only on X such that any "best" fan will have the union of all its cones lying exactly on this fixed polyhedral complex. The technical condition is that the induced multiplication

$$(\mathbb{C}^*)^n \times \overline{X} \to (\text{ambient toric variety})$$

be a flat, surjective map and that $\overline{X}$ is proper. One calls a closure $\overline{X}$ obtained in such a fashion a **tropical compactification**. Note in particular that flatness forces the boundary $\overline{X}\backslash X$ to be of pure codimension one. The striking fact proved in [42] is that such a toric variety always exists: given $X \subseteq (\mathbb{C}^*)^n$ there exists a tropical compactification $\overline{X}$. This is never unique; for example, [42] proves that if a fan gives a tropical compactification of X, then so does any refinement of the fan, but as above the union of the cones of any fan giving a tropical compactification of X must lie exactly on one fixed polyhedral complex.

We may define the **tropical variety** associated to X to be the unique polyhedral complex $\mathrm{Trop}(X) \subset \mathbb{R}^n$ afforded by Tevelev's results. It is known that this notion of $\mathrm{Trop}(X)$ agrees with other definitions, e.g. the more traditional approaches via non-Archimedean amoebas as in [6]. More precisely this approach gives **tropical fans**. An important observation is the result of [27] that *any* smooth projective variety $\overline{X}$

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[1] indeed proper

can be given the structure of a tropical compactification by taking X to be the complement of sufficiently many general hyperplane sections of a very ample line bundle. This suggests that it is uninteresting to look only at the compactification $\overline{X}$ alone.

The theme I develop in my thesis work is that one should instead consider the pair $(\overline{X}, D)$ where D is the boundary of the tropical compactification. The main results of my thesis relate the birational geometry of such pairs $(\overline{X}, D)$ to the polyhedral structure of $\mathrm{Trop}(X)$. It natural in birational geometry to not only consider a space along, but to consider pairs of a space with a divisor; this idea has played a central role the Log Minimal Model Program which began with Mori and is still an area of intense research. My thesis work shows that this idea is natural in tropical geometry as well.

Let's end this section with some easy examples. If $S \subset (\mathbb{C}^*)^n$ is a sub-torus, it is well-known [5] that $\mathrm{Trop}(S) \subset \mathbb{R}^n$ is a linear subspace. In particular, if $S \simeq \mathbb{C}^*$, then $\mathrm{Trop}(S)$ is a line passing through $0 \in \mathbb{R}^n$. The compactification $\overline{S} = \mathbb{P}^1$ is tropical and the two rays emanating from the origin correspond to the two points (divisors) added to compactify. If one forms a variety S' by removing points from $\mathbb{C}^*$, then $\mathrm{Trop}(S')$ will have additional rays for each of these missing points. In particular the tropical variety is a linear space if and only if 2 points are removed. We will see in Section 2 that the algebro-geometric construction this corresponds to is the behavior of the *log canonical divisor*. I note that there are relatively few classes of examples where the combinatorial structure of the tropical fan is well-studied. One well-understood class is hypersurfaces and complete intersections [40, Chp. 9], where the tropical theory is related to Newton Polyhedra [10], another is complements to hyperplane arrangements (as in the above example), where the tropical theory is related to matroid polytopes [7]. In my thesis I track some such examples through my main results, making some observations which are hopefully of independent interest.

2 Log Invariants and Tropical Compactifications

There are some general techniques for studying a non-compact variety if a sufficiently nice compactification is known. Let Y be a smooth projective variety and and let D be a *simple normal crossing* divisor on Y. We will call such a pair (Y, D) a *log pair*. To any log pair Iitaka introduced a locally free sheaf $\Omega^1_Y(\log D)$ which is called the sheaf of differentials with logarithmic poles along D. See [17] for details of the construction and foundational results. The crucial fact is that the rank of $\Omega^1_Y(\log D)$ is actually independent of the choice of simple normal crossing compactification, making it an invariant of the incomplete variety $Y\backslash D$. From $\Omega^1_X(\log D)$ one can obtain the log canonical divisor $K_Y + D$ and consider logarithmic versions of classical birational invariants such as plurigenera, irregularity, and Kodaira dimension and these are each also known to be invariants of $Y\backslash D$.

I consider the situation where Y and D arise from a tropical compactification. The boundary of a tropical compactification is not necessarily simple normal crossing, but I note briefly that there is a genericity hypothesis on a tropical compactification known as a **schön** compactification which can be used to circumvent this, see [42] for details; the schön hypothesis is a generalization of the notion of Newton non-degeneracy from the study of hypersurfaces [10]. I have proved that under the hypotheses some log birational invariants can be read off easily from from $\mathrm{Trop}(X)$.

Theorem 2.1. *Let $X \subset \overline{X}$ be an irreducible schön compactification. The log Kodaira dimension of X is equal to the dimension of the largest real linear subspace contained in* $\mathrm{Trop}(X)$*. In particular, the logarithmic Kodaira dimension of X is zero if and only X is a sub-torus and the tropical compactification is a toric compactificaction.*

The methods of the proof can be extended to study compactifications where the log Kodaira dimension is between 0 and $\dim(X)$.

Corollary 2.2. *Let $X \subset \overline{X}$ be an irreducible schön compactification. Let S_X be the connected component of the identity of the stabilizer of X for the torus multiplication $X \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$. Then* $\mathrm{Trop}(X/S_X)$ *is equal to the quotient of* $\mathrm{Trop}(X)$ *by the largest real linear subspace contained in* $\mathrm{Trop}(X)$ *and any schön fan X projects to a schön fan for X/S_X and gives a fibration of schön compactifications $\overline{X} \to \overline{X/S_X}$. This fibration is exactly the Iitaka Fibration associated to $K_{\overline{X}} + D$ where $D = \overline{X}\backslash X$.*

There is another log invariant which can be read off of the tropical fan easily:

Theorem 2.3. *The difference of the logarithmic irregularity of X and the usual irregularity of $\overline{X}$ is equal to the smallest k such that* $\mathrm{Trop}(X)$ *embeds into $\mathbb{R}^k$.*

The idea of the proof of these results and Theorem 2.1 were inspired by an important result of [14] combined with some deep results of Kawamata and Viehweg on the structure of subvarieties of semi-Abelian varieties [22]. The results should also be compared with classical results about the birational geometry of subvarieties of abelian varities [32, Chp. 3]

One can ask if there is a purely polyhedral interpretation of log morphisms and birational log morphisms for tropical compactifications. There seems to be relatively little in the literature regarding morphisms between tropical varieties, we mention the so-called tropical elimination theory of [41] for some results in this direction. Following [24, Chp. 2], one defines a log morphism between log pairs (X, D) and (Y, D') to be a morphism of varieties $f : X \to Y$ such that $f(D) \subseteq D'$ and says that this is a birational log morphism if f is in addition birational. I proved a general result relating tropical varieties related by a birational log morphism; this complements some results of [41]. The precise statement involves some notions developed in [14, 41] which we do not review in this document, and the version below is a more elementary formulation of the general result.

Theorem 2.4. *Let $\overline{X}$ and $\overline{X'}$ be schön compactifications and let Σ and Σ' be the corresponding toric fans supported on $\mathrm{Trop}(X)$ and $\mathrm{Trop}(X')$ respectively. Suppose that $f : \overline{X} \to \overline{X'}$ is a log morphism. Then f induces a surjective map of polyhedral complexes $f_* : \mathrm{Trop}(X) \to \mathrm{Trop}(Y)$ and moreover $\mathrm{Trop}(X)$ can be recovered from $\mathrm{Trop}(Y)$ and the data of the logarithmic ramification divisor of f.*

I call such maps induced on tropical varieties by log morphism **tropical modifications**. This gives a somewhat different perspective on the tropical modifications introduced by Mikhalkin in [31].

It is desirable to have a converse to the above: if one has two schön compactifications and knows a priori that their tropical fans are related by a simple polyhedral operation, one can ask whether there exists a log morphism between the compactifications implementing the map on fans. A simple case is when one tropical fan is refined, say by a sequence of barycentric subdivisions. There are precise connections between schön compactifications and the so-called conical polyhedral complexes introduced by Mumford in [23]. It is easy to show that the tropical fan of a schön compactification coincides with the polyhedral complex of [23]. I have proven a converse that certain conical polyhedral complexes give schön compactifications; this extends some results of [27]. This allows one to realize a refinement of tropical fans as corresponding to a blow-up of an explicit subvariety. This generalizes the well-known fact that equivariant blow-ups of normal toric varieties correspond to refinements of the toric fan. More complicated relations between tropical fans are treated in the next section.

3 Mori Theory and Tropical Flops

In the last thirty or so years the Iitaka philosophy utilized above has been subsumed into the larger framework of the Log Minimal Model Program. That is, it is known that Mori's program of classifying mildly singular varieties birationally via the numerical properties of the canonical divisor extends well to log canonical divisors

$K_X + D$. I refer to [28] for background. Motivated by the results in Section 2, it is natural to ask if there is an accompanying "tropical minimal model program". The loosely analogous case of Mori theory for toric varieties is well-known; a beautiful paper of Miles Reid [37] interprets the contractions, flips, and flops of Mori theory on toric varieties in terms of polyhedral operations of the toric fan. Although classifying toric varieties birationally is trivial (they're all rational), Reid's observation is that the birational morphisms between toric varieties afforded by Mori are quite interesting. See [8, 28, 43] for more recent developments.

The situation for tropical compactifications is slightly different. As observed in Section 1, in order to keep the structure of a tropical compactification constrained one needs to consider the pair $(\overline{X}, D)$. Instead of looking at traditional Mori theory and the canonical divisor of $\overline{X}$, we consider log Mori theory and the log canonical divisor $K_{\overline{X}} + D$ where D is the full boundary of the compactification. In order to keep singularities relatively mild we will typically assume the compactification is schön. [42] notes that in this case $K_{\overline{X}} + D$ is globally generated, hence nef. So any schön compactification is already its own log minimal model. So producing log minimal models in this context is trivial, and instead we can consider birational relations between these log minimal models. We recall the recent result of Kawamata [21] and Birkar, Cascini, Hacon, McKernan [3] which states that any two birational log minimal models are connected by a sequence of log flops. I have obtained the following tropical incarnation of this result:

Theorem 3.1. *If two projective schön compactifications* $(\overline{X}, D)$ *and* $(\overline{Y}, D')$ *are log birational, then* $\mathrm{Trop}(X) = \mathrm{Trop}(Y)$. *If the corresponding fan structures in addition have identical rays (i.e. 1-cones), then* $(\overline{X}, D)$ *and* $(\overline{Y}, D')$ *are related by a sequence of* **tropical flops**.

A **tropical flop** is my terminology for a simple modification of fan which is often called either an elementary modification or a bistellar flip in the combinatorial literature, and is related to the structure of the secondary polytopes of [10, Chp. 7]. The name is justified as, when defined in full detail, it can be shown that a tropical flop between schön compactifications induces a log flop in the traditional Mori theoretic sense. I note that the above theorem should be viewed as a sort of tropical analogue of a well-known theorem of Oda and Park [34] in the context of toric varieties which relates projective toric varieties related by small birational modifications to the vertices of an associated secondary polytope.

4 Work in Progress

I intend to continue research along the general theme of relating tropical geometry to established methods in algebraic geometry. Here are some projects which I intend to develop further during my postdoctoral studies.

A Tropical Gauss Map: In the course of proving some of the results in Section 2, it was natural to consider the Gauss map associated to $X \subset (\mathbb{C}^*)^n$, i.e. the map $\Gamma_X : X \to \mathrm{Gr}$ from the smooth points of X to an appropriate Grassmannian given by mapping a point to its tangent space translated via the group action to the identity. I have proved that if X is smooth and admits a simple normal crossing schön compactification then Γ_X extends to a regular map $\overline{\Gamma_X} : \overline{X} \to \mathrm{Gr}$ and that the log canonical divisor on $\overline{X}$ is the pull-back of the top wedge power of the tautological bundle on the Grassmannian. This extends results of [13, 18] where the case of X the complement of a hyperplane arrangement was considered and [29] where the Gauss map for a hypersurface was considered. The map $\overline{\Gamma_X}$ induces a "tropical Gauss map"

$$\Gamma_{\mathrm{Trop}} : \mathrm{Trop}(X) \to \mathrm{TGr}$$

where TGr denotes the tropical Grassmannian of appropriate dimension. The tropical Grassmannian was introduced in [38]. In particular, it would be interesting to see if there are combinatorial manifestations of some of the deep results about Gauss maps, e.g. Griffiths and Harris' result about generic finiteness [32]. I have computed some examples and expect that the study of this map will be of independent combinatorial interest.

Semistable Minimal Problem: As mentioned in Section 1, most of the results stated above concern tropical fans, but constructions of tropical varieties using Puiseux valuations (as in [6, 40]) or more general valued fields produce not just fans but tropical polyhedral complexes. Given a polyhedral complex of dimension n, one can produce a fan of dimension $n+1$ by taking the cones over the faces of the complex. In the language of toric geometry, this procedure produces a degeneration of toric varieties with general fiber the toric variety corresponding to the fan, and degenerate fiber consisting of a reducible "broken" toric variety consisting of several toric varieties glued along strata in a manner indexed by the original polyhedral complex. This construction occurs quite often in the toric literature, and shows up in particular in work of Alexeev [2] and in the Gross-Siebert program for Mirror Symmetry [11]. It was recognized in [16] that this construction of degenerating toric varieties can be extended to the theory of degenerating tropical compactifications which gives a dictionary between tropical fans and more general tropical polyhedral complexes.

In Mori theory, there is also a well developed theory for treating degenerations called the Semistable (Log) Minimal Model Program which applies to degenerations. See [26, Chp. 7] for details. I intent to extend the results developed in my thesis to the semistable case, thereby relating general tropical polyhedral complexes to constructions in birational geometry, and not just tropical fans.

Chow Polytopes and Geometric Tropicalization: One of the motivating examples in tropical geometry is that the tropical variety of a hypersurface is related to the Newton Polytope of the hypersurface (they are dual). It is reasonable to think of the tropical variety of a higher codimension subvariety as being a generalization of the Newton Polytope. In a slightly different direction, given a subvariety $Z \subset \mathbb{P}^n$ of projective space, there is a polytope $\mathrm{Ch}(Z)$ called the Chow Polytope which also generalizes the notion of a Newton Polytope, see [10, 19] for the construction and fundamental results. It is natural to compare these two notions. A step in this direction was proved in [5] where it was shown that if Z is a projective variety, for simplicity with no components on the toric boundary of $\mathbb{P}^n$, then $\mathrm{Ch}(Z)$ can be recovered via a direct combinatorial algorithm from $\mathrm{Trop}(X)$. In other words, the tropical variety contains at least as much information as the Chow polytope. I propose to study this algorithm from the perspective of algebraic geometry. More precisely, I have some preliminary results confirming that this algorithm is a combinatorial version of the well-studied morphism from the Hilbert Scheme to the Chow Variety [25]. The Chow polytope has beautiful combinatorial structure, but is difficult to approach geometrically. The methods of [13, 14, 42] and my thesis have done much to elucidate the algebraic geometry underlying tropical varieties. I plan to investigate how much of this geometric machinery can be brought over to the structure of the Chow polytope.

Constructing New Modular Compactifications: As mentioned in Section 1, the machinery of [14], which forms the foundation for much of my work, was developed specifically to understand compactifications of certain moduli spaces. The preceeding paper [42] noted that $M_{0,n}$, the famous moduli space of $n > 3$ marked points on $\mathbb{P}^1$ is naturally a subvariety of an algebraic torus and that the well-studied Grothendieck-Knudsen compactification $\overline{M}_{0,n}$ of stable curves has the structure of a schön compactification. [14] considered the situation for the moduli space of marked del Pezzo surfaces of degree $n > 1$. Recall that a del Pezzo surface Y is a smooth Fano surface and the degree refers to the degree of the canonical divisor. Such a Y can be realized as a blow up of $9 - n$ points on $\mathbb{P}^2$ obeying a genericity condition. A marking of a del Pezzo is choice of configuration of points which realize Y as a blow up. Accordingly, a moduli space of marked del Pezzo surfaces of a fixed degree can be constructed as an explicit subvariety of a product of $\mathbb{P}^2$'s, and it is easy to check that this moduli space actually lies inside of a torus. See [14, 15] for details. In particular, one can consider tropical compactifications of these moduli spaces. One of the main results of [14] is that the log canonical divisor of a schön compactification of the moduli space is very ample when the degree is between 1 and 3; in other words, in these cases the log canonical compactification is naturally a tropical compactification. This allows the boundary of the log canonical compactification to be studied combinatorially and compared with other previously studied classical compactifications.

The methods of [14] partially break down in the degree 1 case as the tropical compactifications will not have very ample log canonical divisor. An alternate approach to studying the degree 1 case was suggested to me by Jenia Tevelev and exploits some phenomena specific to the degree 1 case. It is well known [15] that the moduli space of degree 1 marked del Pezzo surfaces can be identified with the moduli space of rational elliptic surfaces with a marked section, which in turn can be interpreted as a moduli space of ramified covers of the projective line (again, see [15] for the details). By associating the relative Jacobian to each cover, the moduli space of degree 1 marked del Pezzo surfaces can be realized as a subvariety of a moduli space of principally polarized abelian varieties. There are several well-studied compactifications for moduli spaces of abelian varieties [2], and the log canonical compactification is known. The goal of this project is to use tropical tools to study these embeddings of the moduli space and compare with previously studied classical compactifications.

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