Two Lemmas in Łocal Analytic Geometry

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This paper is dedicated to Leon Ehrenpreis.

ABSTRACT. We prove two results about the local properties of generically one to one analytic mappings.

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§1: INTRODUCTION

In this paper we consider two local properties of analytic maps which are generically one to one. First we consider holomorphic maps from open sets in $\mathbb{C}^2$ which behave locally like monoidal transformations: Let (z, w) denote coordinates for $\mathbb{C}^2$ and $D, D' \subset \mathbb{C}^2$ be neighborhoods of $(0,0)$. We call a holomorphic map $f : D \to D'$ a germ of a blowdown if

(1) $f(0, w) = (0, 0)$,
(2) f is injective on $D \setminus \{z = 0\}$.

We prove the following normal form result for such maps:

Lemma 1. *Suppose that $f : D \to D'$ is a germ of a blowdown then there are local coordinates, (ζ, ξ) on a neighborhood of $(0, 0)$ such that in these coordinates the map is either*

$$f(\zeta, \xi) = (\zeta, \zeta^k \xi), \quad k \in \mathbb{N} \text{ or} \tag{1.1}$$

$$\begin{aligned} f(\zeta, \xi) = (\zeta^j, \zeta^{k_1}(\alpha_1 + \zeta^{k_2}(\alpha_2 + \dots \zeta^{k_p}(\alpha_p + \xi) \dots)), \\ \alpha_i \in \mathbb{C}, k_i \in \mathbb{N}, i = 1, \dots, p. \end{aligned} \tag{1.2}$$

As a consequence of the lemma we obtain the following:

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Corollary 1. *If $f : D \to D'$ is a germ of a blowdown then there is a finite sequence of point blow–ups of*

$$D' = D'_0 \underset{\pi_1}{\longleftarrow} D'_1 \underset{\pi_2}{\longleftarrow} \dots \underset{\pi_m}{\longleftarrow} D'_m$$

and lifted maps $f_i : D \to D'_i$ so that

$$f = \pi_1 \circ \cdots \circ \pi_i \circ f_i$$

and $f_m : D \to D'_m$ is a germ of a biholomorphism.

This result shows that Castelnuovo's classical result characterizing a generically one to one map between smooth, compact complex surfaces as a composition of monoidal transformations has a completely satisfactory local analogue, see [GrHa].

The other result we prove is a consequence of the Łojasiewicz inequality. We let $|x|$ denote the Euclidean norm in $\mathbb{R}^m$ for any m. If $F \subset \mathbb{R}^m$ we let

$$d(x, F) = \inf_{y \in F} |x - y|.$$

Let f be a real analytic function in an open neighborhood, $\Omega \subset \mathbb{R}^n$ of 0 and suppose that $f(0) = 0$. Let Z_f denote the zero locus of f. Łojasiewicz proved that for each compact subset $K \subset \Omega$ there exist positive constants, C, N so that

$$|f(x)| \geq C[d(x, Z_f)]^N,$$

see [Ło]. Hörmander proved a similar result but assuming that f is a polynomial. These results were originally used to prove division theorems for distributions. Hömander used his result to prove a tempered version of the Malgrange–Ehrenpreis theorem, see [Hö] and [Eh,Ma].

Now suppose that $D_r \subset \mathbb{R}^n$ is the ball of radius r and that $\psi : D_{1+\epsilon} \to \mathbb{R}^N$, $\epsilon > 0$ is a real analytic mapping. Let $E_\psi = \{x \in D_{1+\epsilon} | \operatorname{rank} \psi_* < n\}$.

Lemma 2. *Suppose that ψ is as above and that E_ψ is a non–empty proper subvariety of $D_{1+\epsilon}$. Suppose further that ψ is one to one on $D_{1+\epsilon} \setminus E_\psi$ and $\psi(D_{1+\epsilon} \setminus E_\psi) \cap \psi(E_\psi) = \emptyset$. Then there exist positive constants, C, N so that*

$$|\psi(x) - \psi(y)| \geq C|x - y|[d(x, E_\psi) + d(y, E_\psi)]^N \text{ for } x, y \in D_1 \tag{1.3}$$

We have found this a useful consequence of the Łojasiewicz inequality. For example we have the following corollary:

Corollary 2. *With ψ as in Theorem 2 there exists an $M > 0$ such that if $f \in \operatorname{Lip}_1(D_1)$ and*

$$|\nabla f(x)| \leq C[d(x, E_\psi)]^M$$

then $\psi_*(f)(y) = f(\psi^{-1}(y))$ is a Lipschitz function on the closed set $\psi(D_1)$. There is a constant C', independent of f such that

$$\|\psi_*(f)\|_{\operatorname{Lip}_1(\psi(D_1))} \leq C' \left(\|f\|_{C_0(D_1)} + \sup_{x \in D_1} \frac{|\nabla f(x)|}{[d(x, E_\psi)]^M} \right).$$

The proof of this corollary can be found in [EpHe].

§2: Proof of Lemma 1

Proof of Lemma 1. Since f maps $\{z = 0\}$ to $(0,0)$ there are positive integers, j, k such that

$$f(z, w) = (z^j \varphi_1(z, w), z^k \varphi_2(z, w)),$$

where φ_1, φ_2 are holomorphic. We suppose that j, k are the maximal such integers. Define the varieties:

$$V_i = \{\varphi_i^{-1}(0)\},\ i = 1, 2.$$

By assumption $V_i \cap \{z = 0\}$ is a finite set for $i = 1, 2$. We claim that $(0,0) \notin V_i$ for at least one value of i.

Suppose that this is not the case. Each irreducible component of V_2 has a local parametrization of the form $t \to (t^a g(t), t^b h(t))$. This map is injective in some neighborhood of $t = 0$ and a and b are positive integers. In particular there is an $\epsilon > 0$ such that $g(t) \neq 0$ for $|t| < \epsilon$. We consider the composition

$$f(t^a g(t), t^b h(t)) = (t^{ja} g(t)^j \varphi_1(t^a g(t), t^b (h(t)), 0).$$

Since $\varphi_1(0,0) = 0$ there is a $c > 0$ such that

$$\varphi_1(t^a g(t), t^b (h(t)) = t^c k(t),\ \text{where } k(0) \neq 0.$$

By assumption this map is injective in a neighborhood of 0, on the other hand it has the form:

$$f(t^a g(t), t^b h(t)) = (t^{ja+c} g(t)^j k(t), 0).$$

From this we conclude that $ja + c = 1$. However this contradicts the assumptions that j, a and c are all at least 1. From this the claim follows. We therefore assume that $\varphi_1(0,0) \neq 0$. If we introduce the new coordinate

$$\zeta = z(\varphi_1(z, w))^{\frac{1}{j}}$$

then in a neighborhood of $(0,0)$ the map has the form:

$$f(\zeta, w) = (\zeta^j, \zeta^k \varphi_2(\zeta, w)).$$

We first dispose of a simple special case. If $\varphi_2(0,0) = 0$ then using the local parametrization of V_2 from above we conclude that $ja = 1$. In particular $j = 1$. Suppose that $\varphi_2(\zeta, 0) = 0$ so that $\varphi_2 = w^l \varphi_2'(\zeta, w)$ for an $l > 0$; where this is the maximal such l. For sufficiently small $\zeta \neq 0$ the map

$$w \to w^l \varphi_2'(\zeta, w)$$

must be injective and therefore $l = 1$. Finally we observe that if $\varphi_2'(0,0) = 0$ then the set $f^{-1}(\{(t,0)\}) \setminus \{(0,0)\}$ would have two distinct components passing through $(0,0)$. As this would contradict the injectivity of f on the complement of $\{z = 0\}$ it follows that $\varphi_2'(0,0) \neq 0$. If we introduce, as second coordinate $\xi = w\varphi_2'(\zeta, w)$, then the map takes the form

$$f(\zeta, \xi) = (\zeta, \zeta^k \xi).$$

Now we treat the general case. There is a fixed $\epsilon > 0$ such that the maps $\{w \to \varphi_2(\zeta, w) : 0 < |\zeta| < \epsilon\}$ are injective in the set $B_\epsilon = \{|w| < \epsilon\}$. Hurwitz's theorem implies that $w \to \varphi_2(0, w)$ is either injective in B_ϵ or constant. Let

$$\alpha_1 = \varphi_2(0, 0).$$

In the former case we introduce the new coordinate $\xi = \varphi_2(\zeta, w) - \alpha_1$ which puts the map into the normal form (1.2):

$$f(\zeta, \xi) = (\zeta^j, \zeta^k(\alpha_1 + \xi)).$$

Note that if $\alpha_1 = 0$ then $j = 1$ as follows from the argument above.

In the latter case we set $k_1 = k$ and let $0 < k_2$ be the largest integer such that

$$\varphi_2(\zeta, w) = \alpha_1 + \zeta^{k_2}\varphi_2^{(1)}(\zeta, w),$$

where $\varphi_2^{(1)}$ is a holomorphic function. The same observation applies in this case: the maps $\{w \to \varphi_2^{(1)}(\zeta, w) : 0 < |\zeta| < \epsilon\}$ are injective in the set B_ϵ. This leads to the same dichotomy: either $\varphi_2^{(1)}(0, w)$ is injective on B_ϵ or constant. If we repeat this argument p–times we obtain sequences of complex numbers, $\{\alpha_1, \dots, \alpha_p\}$, positive integers, $\{k_1, \dots, k_{p+1}\}$ and a holomorphic function, $\varphi_2^{(p)}$ so that

$$\varphi_2(\zeta, w) = \zeta^{k_1}(\alpha_1 + \zeta^{k_2}(\alpha_2 + \zeta^{k_3}(\dots(\alpha_p + \zeta^{k_{p+1}}\varphi_2^{(p)})\dots).$$

As before the maps $w \to \varphi_2^{(p)}(\zeta, w)$ are injective in B_ϵ for $\zeta \neq 0$. Observe that $\partial_w\varphi_2(\zeta, w)$ is divisible by ζ^{p+1}. In order for φ_2 to depend on w in a non–trivial way there must a finite value, p such that $w \to \varphi_2^{(p)}(0, w)$ is injective in B_ϵ. If, for this p we let $\xi = \varphi_2^{(p)}(\zeta, w) - \alpha_{p+1}$ then we obtain the normal form, (1.2) for f :

$$f(\zeta, \xi) = (\zeta^j, \zeta^{k_1}(\alpha_1 + \zeta^{k_2}(\dots + \zeta^{k_{p+1}}(\alpha_{p+1} + \xi)\dots)).$$

Remark. The normal form, (1.2) can be re–expressed as

$$f(\zeta, \xi) = (\zeta^j, q(\zeta) + \zeta^N\xi)$$

where $q(\zeta)$ is a polynomial of degree at most $N = k_1 + \dots + k_p$. The condition that f be injective in some deleted neighborhood of $(0, 0)$ is: for each j^{th} root of unity, $e^{i\omega}$ the polynomial $q(\zeta) - q(e^{i\omega}\zeta)$ is not divisible by ζ^N. For example, if $\alpha_p = 0$ so that the degree of q is less than N this is equivalent to the condition:

$$\gcd(j, k_1, k_1 + k_2, \dots, k_1 + \dots k_p) = 1.$$

We now deduce the corollary:

Proof of Corollary 1. The proof is a simple recursive argument using the normal form and the fact that a blow–up is locally described by either $(z, w) \to (z, \frac{w}{z})$ or $(z, w) \to (\frac{z}{w}, w)$. If the map takes the normal form (1.1) then blowing up the origin in the target k–times and lifting the map f each time leads to a space D'_k and a map $f_k : D \to D'_k$, given by: $f_k(\zeta, \xi) = (\zeta, \xi)$.

If the map takes the normal form, (1.2) then the Jacobian determinant, J_f is easily computed, it is

$$J_f = j\zeta^{j+k_1+\cdots+k_p-1}.$$

Let $\operatorname{ord} J_f$ denote the order of vanishing of J_f along $\{z = 0\} \cap D$, this is of course invariant under biholomorphisms. We obtain a sequence of spaces, $\{D'_j\}$ by the prescription: Define $\pi_1 : D'_1 \to D'$ as the blow–up of $f(\{z = 0\} \cap D) = (0, 0)$. The map, f lifts to define a map,

$$f_1 : D \longrightarrow D'_1$$

which satisfies $f = \pi_1 \circ f_1$. It is evident that $\operatorname{ord} J_{f_1} < \operatorname{ord} J_f$. If J_{f_1} is non–vanishing then we are done as f_1 is then the germ of a biholomorphism. Otherwise $f_1 : D \to D'_1$ is the germ of a blow–down but $f_1(\{\zeta = 0\} \cap D)$ may not be $(0, 0)$. The normal form theorem applies *mutatis mutandis* to this case as well and so we can define D'_2 by blowing up $f_1(\{\zeta = 0\} \cap D)$ to obtain $\pi_2 : D'_2 \to D'_1$ and $f_2 : D \to D'_2$ with $f_1 = \pi_2 \circ f_2$. Apply this process recursively: assume that we have obtained spaces,

$$D'_k \xrightarrow{\pi_k} D'_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} D'$$

and germs of blow–downs

$$f_i : D \to D'_i \text{ with } f_i = \pi_i \circ f_{i-1},\ i = 1 \ldots, k.$$

The space D'_i is obtained by blowing up $f_{i-1}(\{\zeta = 0\} \cap D)$. At each step we see that $\operatorname{ord} J_{f_i} < \operatorname{ord} J_{f_{i-1}}$. If $\operatorname{ord} J_{f_k} > 0$ then f_k is the germ of a blow–down and we define (D'_{k+1}, f_{k+1}) as above otherwise f_k is the germ of a biholomorphism. With each blow–up the $\operatorname{ord} J_{f_k}$ decreases by at least 1 thus this process must terminate after finitely steps.

§3: Proof of Lemma 2

Proof of Lemma 2. In this argument $\psi : D_1 \to \mathbb{R}^N$ is a real analytic map defined and satisfying the hypotheses of the theorem on a neighborhood of $\bar{D}_1$. We let z and w denote points in $\bar{D}_1$. First consider the real analytic function:

$$\rho(z, w) = |\psi(z) - \psi(w)|^2.$$

This function vanishes on $Z_\rho \subset \Delta \cup E_\psi \times E_\psi$ where

$$\Delta = \{(p, p) | p \in D_1\} \text{ and } E_\psi = \{z \in D_1 | \operatorname{rank} \psi_* < n\}.$$

We get containment and not equality whenever E_ψ has several connected components. Apply Łojasiewicz' inequality to obtain positive constants C_1, N_1 so that

$$\rho(z, w) \geq C_1[d((z, w), Z_\rho)]^{N_1}. \tag{3.2}$$

Evidently

$$d((z,w),Z_\rho)\geq \min\{\frac{1}{2}|z-w|,[d(z,E_\psi)^2+d(w,E_\psi)^2]^{\frac{1}{2}}\}.$$

If $d((z,w),\Delta)\geq d((z,w),E_\psi\times E_\psi)$ then (3.2) implies that

$$\rho(z,w)\geq C_1'|z-w|[d(z,E_\psi)+d(w,E_\psi)]^{N_1}, \tag{3.3}$$

for some possibly smaller constant. For $\delta>0$, $L\geq 1$ we define the sets:

$$A_{\delta,L}=\{(z,w)|\quad d((z,w),\Delta)\geq \delta[d(z,E_\psi)+d(w,E_\psi)]^L\}. \tag{3.4}$$

It follows from (3.2) that we have the estimates

$$\rho(z,w)\geq C_1'|z-w|[d(z,E_\psi)+d(w,E_\psi)]^{L(N_1-1)} \text{ for } (z,w)\in A_{\delta,L}. \tag{3.5}$$

We are left to consider the set

$$B_{\delta,L}=\{(z,w)|\quad |z-w|<\delta[d(z,E_\psi)+d(w,E_\psi)]^L\}$$

for a $\delta>0$ and $L\geq 1$ which are yet to be determined. To that end we express

$$\psi(z)-\psi(w)=M(z,w)(z-w)$$

where $M(z,w)$ is the $N\times n$ matrix valued real analytic function given by:

$$M(z,w)=\int_0^1 \nabla\psi(tz+(1-t)w)dt.$$

Observe that

$$M(z,z)=\nabla\psi(z). \tag{3.6}$$

Let $\mathcal{I}_n$ denote set of multi-indices

$$\mathcal{I}_n=\{(i_1,\dots,i_n)|\quad 1\leq i_1<\dots,i_n\leq N\}.$$

If we let $\underline{i}=(i_1,\dots,i_n)$ denote an element of $\mathcal{I}_n$ then $M_{\underline{i}}$ is the $n\times n$ sub-matrix

$$M_{\underline{i}}=\begin{pmatrix} M_{i_1 1} & \dots & M_{i_1 n}\\ \vdots & & \vdots\\ M_{i_n 1} & \dots & M_{i_n n}\end{pmatrix}$$

and $\psi_{\underline{i}}$ the n–vector valued function

$$\psi_{\underline{i}}(z) = \begin{pmatrix} \psi_{i_1}(z) \\ \vdots \\ \psi_{i_n}(z) \end{pmatrix}.$$

For each such multi-index we have the identity:

$$\psi_{\underline{i}}(z) - \psi_{\underline{i}}(w) = M_{\underline{i}}(z, w)(z - w). \tag{3.7}$$

If $\det M_{\underline{i}}(z, w) \neq 0$ then it follows easily from the fact that ψ is smooth, Cramer's rule and the Cauchy–Schwarz inequality that there is a positive constant, C_4 such that

$$|\psi_{\underline{i}}(z) - \psi_{\underline{i}}(w)| \geq C_4 |\det M_{\underline{i}}(z, w)||z - w|. \tag{3.8}$$

Note that $C_4 > 0$ is a fixed constant which is independent of z, w and $\underline{i}$.

Define the real analytic function

$$m(z, w) = \sum_{\underline{i} \in \mathcal{I}_n} |\det M_{\underline{i}}(z, w)|^2.$$

From (3.6) and the hypothesis of the theorem it follows that $m(z, z)$ vanishes exactly on the set E_ψ. Thus we can apply Łojasiewicz' inequality to obtain that there exist positive constants, C_5, N_2 such that

$$m(z, z) \geq C_5[d(z, E_\psi)]^{N_2}. \tag{3.9}$$

For an $L > N_2$ and $\delta > 0$, sufficiently small we now show that there exists a constant $C_5' > 0$ such that

$$m(z, w) > C_5'[d(z, E_\psi) + d(w, E_\psi)]^{N_2} \text{ for } (z, w) \in B_{\delta, L}. \tag{3.10}$$

It follows from the smoothness of $m(z, w)$ and the mean value inequality that there exists a constant, C_6 so that

$$m(z, w) \geq m(z, z) - C_6|z - w|$$

hence (3.9) implies that

$$m(z, w) \geq C_5[d(z, E_\psi)]^{N_2} - C_6\delta[d(z, E_\psi) + d(w, E_\psi)]^L \text{ for } (z, w) \in B_{\delta, L}. \tag{3.11}$$

If $(z, w) \in B_{\delta, L}$ then the triangle inequality implies that

$$d(w, E_\psi) - d(z, E_\psi) \leq d(w, z) \leq \delta[d(z, E_\psi) + d(w, E_\psi)]^L.$$

For sufficiently small $\delta > 0$, this estimate and the binomial theorem imply that there exists a positive constant C_7 so that

$$\frac{1}{C_7} d(z, E_\psi) \leq d(w, E_\psi) \leq C_7 d(z, E_\psi). \tag{3.12}$$

Putting together (3.11) and (3.12) we obtain (3.10). Let K be the cardinality of $\mathcal{I}_n$. Then (3.10) implies that for each $(z, w) \in B_{\delta, L}$ there exists a $\underline{i} \in \mathcal{I}_n$ for which we have the estimate:

$$|\det M_{\underline{i}}(z, w)|^2 \geq \frac{C_5'}{K}[d(z, E_\psi) + d(w, E_\psi)]^{N_2}. \tag{3.13}$$

Combining this with (3.8) completes the proof of Theorem 2.

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