Shrinking tubes and the $\overline{\partial}$-Neumann problem

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1991 *Mathematics Subject Classification.* All sorts of things.

Contents

Introduction

0.1. The conjecture of Boutet de Monvel and Guillemin

In this monograph we consider analytic questions related to complex manifolds with smooth strictly pseudoconvex boundaries. Our initial aim was to resolve the conjecture of Boutet de Monvel and Guillemin, see [**4**], on the existence of a 'Töplitz isomorphism' for a general compact $\mathcal{C}^\infty$ manifold. Such an isomorphism identifies the $\mathcal{C}^\infty$ functions on the manifold with holomorphic functions (in our formulation holomorphic $(n, 0)$–forms) on a tubular neighbourhood in a complex manifold and its inverse, the Töplitz correspondence, is given by fibre integration. The existence of a Töplitz isomorphism is proved in §25; we show that the Töplitz correspondence is an isomorphism by constructing a rather precise and close approximation to its inverse. In order to do this we need to analyze the 'adiabatic limit' of the $\overline{\partial}$-Neumann problem. This is accomplished by pseudodifferential methods. In Part I we show how to solve the $\overline{\partial}$-Neumann problem itself using these methods. The adiabatic limit is examined in Part II and used in Part III to construct the Töplitz isomorphism.

It was shown by Bruhat and Whitney [**2**] that any compact real analytic manifold, Y, can be embedded as a totally real submanifold of a complex manifold Ω, where $\dim_{\mathbb{C}} \Omega = \dim_{\mathbb{R}} Y$. Since any $\mathcal{C}^\infty$ manifold has a consistent real-analytic structure this means any compact $\mathcal{C}^\infty$ manifold can be so embedded. In fact, Grauert [**7**] showed that there is a neighbourhood of Y, in Ω, which is diffeomorphic to a neighbourhood of Y realized as the zero section of its cotangent bundle T^*Y. Thus a neighbourhood of the zero section in T^*Y has a complex structure with respect to which Y is totally real. Any non-negative $\mathcal{C}^\infty$ function τ on T^*Y which vanishes to exactly second order on Y fixes 'Grauert tubes' around Y, namely

$$X(\epsilon) = \left\{(y, \eta) \in T^*Y; \tau(y, \eta) < \epsilon^2\right\},\ 0 < \epsilon < \epsilon_0.[1.1]$$

Since Y is totally real these are, for small ϵ, strictly pseudoconvex[1] neighbourhoods of Y. A theorem of Kostant and Sternberg (see page 228 of [**8**]) shows that τ and the complex structure can be chosen so that the fibres of T^*Y are totally real and in addition $\mathrm{Im}[\overline{\partial}\tau]$ is a multiple of the contact 1-form of T^*Y. Under this condition Boutet de Monvel and Guillemin [**?**] showed that the map, which we call the Töplitz correspondence,

$$T_\epsilon : \left\{u \in \mathcal{C}^\infty(X(\epsilon); \Lambda^{n,0}),\ \overline{\partial}u = 0\right\} \longrightarrow \mathcal{C}^\infty(Y)[1.2]$$

given by integration over the fibres of T^*Y, is Fredholm. In fact we demand only a somewhat less restrictive condition on the complex structure and exhaustion function. At each $p \in T^*Y$ there is a subspace of $T^*_p(T^*Y)$ defined by those forms

[1] A brief introduction to complex manifolds with boundary is given in §2.

which vanish when restricted to the fiber of $\pi : T^*Y \longrightarrow Y$. Since this subspace is simply π^*T^*Y we call such forms *basic forms.* One of our main results is:

THEOREM 0.1. *Suppose* $X(\epsilon)$ *is fixed by* (0.1) *where the complex structure on* T^*Y *and the* $\mathcal{C}^\infty$ *function* τ, *with non-degenerate minimum of* 0 *at* Y, *are such that* $\mathrm{Im}[\overline{\partial}\tau]$ *is a basic form, then for some* $\epsilon_0 > 0$ *and all* $0 < \epsilon < \epsilon_0$ *the map* T *in* (0.1) *is an isomorphism.*

In fact we prove rather more than this since we construct (separate) calculi of pseudodifferential operators which contain the inverse of T and the Bergman projection on $X(\epsilon)$, uniformly as $\epsilon \downarrow 0$. The methods are described in more detail below but in general terms the idea is that as $\epsilon \downarrow 0$ the complex structure on the tube degenerates to vector fields tangent to the fiber. In this degenerate limit the holomorphic functions are those which are constant on the fibres, i.e. are simply pullbacks of functions from the base. In this limiting case the Töplitz isomorphism is obvious. However the limit is so singular that one should not expect to use standard perturbation methods to extend the isomorphism to small values of ϵ. While the limit is quite singular it is also quite uniform and this allows us to construct a calculus of pseudodifferential operators which contains the Bergmann projectors as $\epsilon \downarrow 0$. This calculus of operators embodies a 'singluar perturbation theory' which is powerful enough to allow us to obtain the Töplitz isomorphism for sufficiently small ϵ.

0.2. The $\overline{\partial}$-Neumann problem

Since our proof of Theorem 0.1 depends on the uniform solution, in a shrinking tube, to the $\overline{\partial}$-Neumann problem we shall start with a brief appraisal of methods used to treat such global analytic questions on strictly pseudoconvex manifolds. There are three basic methods (or four if we include 'kernel-type' methods).

One of the most successful of these methods is the use of L^2 estimates, often in weighted spaces. In particular this allowed Kohn ([**10**], see [**6**]) to solve the $\overline{\partial}$-Neumann problem and show that the system of equations

$$\overline{\partial}u = f \text{ where } \overline{\partial}f = 0[1.4]$$

has a solution $u \in \mathcal{C}^\infty(\Omega)$ if f is a $\mathcal{C}^\infty$ $(0,1)$–form. It also follows from this that the Bergman projection,

$$K : L^2(\Omega) \longrightarrow L^2_{\mathrm{hol}}(\Omega)[1.5]$$

onto the holomorphic square-integrable functions, on a smoothly bounded strictly pseudoconvex manifold maps $\mathcal{C}^\infty(\Omega)$ into $\mathcal{C}^\infty(\Omega)$. More generally Hörmander, [**?**], had used weighted L^2 estimates to show the solvability of (0.2) with (optimal) interior regularity but without smoothness up to the boundary.

Such L^2 methods do not give much explicit information about the Bergman kernel, i.e. the Schwartz kernel of K, or on the solution operator, N, to the $\overline{\partial}$-Neumann problem itself. Fefferman [**?**] showed that the singularity of the Bergman kernel can be described by an asymptotic expansion. Boutet de Monvel and Sjöstrand ([**?**]) used the calculus of Fourier integral operators with complex phase functions to give even more precise information on the kernel of K; see also [**?**]. This same analytic approach allowed Boutet de Monvel and Guillemin to analyze the Töplitz correspondence and show that it is Fredholm. They also showed the essential equivalence of Töplitz operators and pseudodifferential operators under this correspondence. That the Töplitz correspondence is an isomorphism in the case of a sphere was shown by Lebeau [**?**], making explicit use of spherical harmonic expansions. It should also be noted that the treatment of analytic wavefront set given by Hörmander ([**?**]) is closely related to the Töplitz isomorphism.

A third basic approach, quite closely related to Fourier integral operators, is that of the Hermite calculus ([**?**]) and what is essentially the same, the calculus of singular integral operators based on the Heisenberg group. This latter calculus has been brought to a substantial degree of refinement, see [**1**] and [**13**] and the references cited therein. Reduction to the boundary gives operators, such as $\Box_b$ and the Szegö projector, which can be conveniently analyzed using these methods. Just such ideas have been used by [**?**], [**?**] to represent the solution operator to Kohn's $\overline{\partial}$-Neumann problem; from this representation it can be seen that the operator is quite complicated in structure - much more so than the Bergman kernel.

Despite all the successes of these methods, and the far-reaching extension of some of them to weakly pseudoconvex domains, there remain some inadequacies in the standard treatment. For instance the fact that the Schwartz kernel of the operator N is quite complicated means that its mapping properties have not been deduced satisfactorily from a general operator calculus. There is good reason for this, namely there is a fundamental complexity to Kohn's $\overline{\partial}$-Neumann problem which leads to the complexity of N but is essentially unrelated to its utility. Let us examine this briefly.

Kohn's approach to the solution of (0.2) is to look for the solution which is orthogonal to the holomorphic space $L^2_{\text{hol}}(\Omega)$. Here, if $\Omega \subset \mathbb{C}^n$, orthogonality is with respect to the Euclidean metric. It follows that if $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ then

$$\overline{\partial}^*\overline{\partial}u = g,\ g = \overline{\partial}^* f.[1.6]$$

Conversely if (0.2) holds and $u \in \mathcal{C}^\infty(\Omega)$ satisfies the $\overline{\partial}$-Neumann boundary condition, i.e.

$$i_\nu \cdot \overline{\partial}u = 0 \text{ on } \partial\Omega, [1.7]$$

where ν is the outward unit normal, then (0.2) holds.

The complexity of the kernel of N, giving the solution $u = Ng$ to (0.2), subject to (0.2), arises from the fact that $\overline{\partial}^*\overline{\partial}$ is essentially the standard Laplacian, in fact it is $\Delta/2$. Near the boundary Δ has a natural homogeneity which can be seen, for example, in the Poisson kernel for the Dirichlet problem, see [**?**]. The homogeneity of the boundary conditions (0.2), on the other hand, is of the 'parabolic' type characteristic of the boundary behaviour of strictly pseudoconvex domains and reflected, on the boundary itself, in the Heisenberg calculus mentioned above. In the hyperbolic homogeniety all directions have the same weight whereas in the parabolic homogeneity there is a distinguished complex line of higher weight. This 'mixed' homogeneity is responsidbel for the rather complicated local structure for the solution operator.

Now, as noted above, the main use of N is to establish the regularity of solutions to the $\overline{\partial}$ system. Certainly the $\overline{\partial}$-Neumann problem (0.2) and (0.2) is not the only way such an operator can be found, since only the restriction of N to $\overline{\partial}$-closed $(0, 1)$–forms is relevant. This leads us to modify Kohn's problem and to consider another operator which gives the same solution properties to (0.2), since it gives the same solution! This modified operator has only the one dominant 'parabolic homogeneity.' This approach is explained in full in §6 below. In brief it consists in replacing the problem (0.2), for functions and $(0,1)$–forms, by the equivalent problem for $(n,0)$–forms and $(n,1)$–forms and then replacing the (incomplete) Euclidean metric by a Bergman type metric. This is a Kähler metric, g_r, with Kähler form

$$\omega_r = -\partial\overline{\partial}\log r[1.8]$$

where $r \in \mathcal{C}^\infty(\Omega)$ is a plurisubharmonic defining function for Ω. Thus the metric is obtained from (0.2) through the isomorphism between Hermitian symmetric bilinear forms and real antisymmetric $(1,1)$–forms. This leads to the replacement of the non-elliptic boundary condition (0.2), for the elliptic operator (0.2), by the equation

$$\varDelta_r u' = g'[1.9]$$

where $\varDelta_r$ is the Laplacian of g_r in (0.2) acting on $(n,0)$–forms. The metric g_r is complete and the 'boundary conditions' for (0.2) become the square-integrability of the $(n,0)$–form u' and orthogonality to the null space of $\varDelta_r$. It might seem surprising that this gives the same solution to (0.2) by taking $g' = \overline{\partial}^*_r f$ but the null space just consists of the square-integrable holomorphic $(n,0)$–forms and the L^2-norm on $(n,0)$–forms is actually independent of the metric. Thus the null space of (0.2) can be identified with $L^2_{\text{hol}}(\Omega)$ and orthogonal projection onto it, or off it, is independent of the metric. To find the desired solution of (0.2) we use the corresponding problem for $(n,1)$–forms.

Thus the method used here to treat these analytic questions for strictly pseudoconvex domains is to invert the Bergman Laplacian. The invertibility properties were discussed by Donnelly and Fefferman [**?**] but we are interested in more precise information, in particular more precise mapping properties. The paper [**5**] is closely related to the present work. In it we constructed a calculus of pseudodifferential operators[2] on the pseudoconvex domain itself which includes the inverse of $\varDelta_r$ acting on functions. In the first sections below we recall this calculus and generalize it to (p,q)–forms, with special emphasis on the two cases, $(n,0)$–forms and $(n,1)$–forms, of most immediate interest.

The method used here, which we think of as 'pseudodifferential,' is of course related to the three approaches discussed briefly above, L^2-estimates, Hermite-Heisenberg calculus and complex Fourier integral operators, particularly to the last of these. Nevertheless it is different in important ways which we exploit in the second part of the monograph to examine the adiabatic limit of the $\overline{\partial}$-Neumann problem. To define the class of pseudodifferential operators used, and describe their properties, we rely heavily on the properties of polyhomogeneous conormal distributions on manifolds with corners. Many of the relevant results were discussed in [**5**]. A more detailed exposition of this theory can be found in [**12**] but the most crucial properties are recalled in Appendix B. In [**12**] a rather general procedure is described by which a calculus of pseudodifferential operators can be associated to a Lie algebra of vector fields with appropriate properties. Below we work out several examples of this general approach in detail; two main, and several ancillary, calculi are discussed. In fact these examples have strongly guided the general theory.

[2]We use the term 'calculus' of pseudodifferential operators to indicate that these spaces do not (always) form algebras but that the composition properties are nevertheless fully established, i.e. conditions are given under which the operators can be composed and then the composite is again a pseudodifferential operator in the same sense.

The first calculus, introduced in [**5**], is that of Θ–pseudodifferential operators. This is intrinsic to any smooth complex manifold with strictly pseudoconvex boundary[3]. In fact, it is defined on any manifold with a contact structure on its boundary. Its 'boundary values' are in the Hermite-Heisenberg calculus as described in [**?**], [**13**] and [**1**]. The main results we give on the $\overline{\partial}$-Neumann problem follow from the fact that the resolvent of Δ_r on $(n,1)$–forms for the metric g_r, as given by (0.2), is a holomorphic family of Θ–pseudodifferential operators. As a consequence we conclude that the Bergman projector, and the solution operator to (0.2), are also Θ–pseudodifferential operators. The basic mapping properties can then be deduced by standard techniques for the operator calculus. In this way we also recover the representation of the Bergman kernel given by Boutet de Monvel and Sjöstrand [**?**]. We shall describe this calculus in more detail, after first briefly reviewing the standard calculus of pseudodifferential operators.

0.3. A geometric approach to pseudodifferential operators

By virtue of the Schwartz kernel theorem any continuous linear operator, A, from $\mathcal{C}^\infty$ functions of compact support on a manifold, X, to distributions on the same manifold can be identified with a distributional density on the product, $X^2 = X \times X$, the (Schwartz) kernel k_A of A. If X is a compact manifold without boundary then $A \in \Psi^m_{\mathrm{KN}}(X)$ is a "classical" pseudodifferential operator of order m (as introduced by Kohn and Nirenberg in [**11**] and Hörmander [**?**]) if k_A is a 1-set polyhomogeneous conormal distribution (really a section of the right density bundle) of order m with respect to the diagonal $\Delta \subset X^2$. The symbol of A is the leading singularity of its kernel identified, via the Fourier transform, with a function on the cotangent bundle to X (which is canonically isomorphic to the conormal bundle to Δ).

A fundamental property of pseudodifferential operators is that they form a (symbol filtered) ring. Composition of operators,

$$C = A \cdot B, [1.10]$$

corresponds to the integral formula for their kernels

$$k_C(x,x') = \int_X k_A(x,x'')k_B(x'',x')dx''.[1.11]$$

This can be rewritten in terms of pull-back, push-forward and product operations as

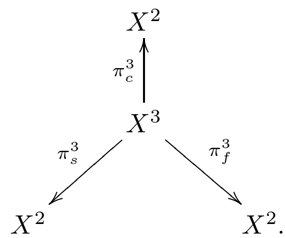
$$k_C = (\pi_c)_* \left[\pi_s^* k_A \cdot \pi_f^* k_B\right].[1.12]$$

Here π^3_o are, for $o = f, c, s$, the three projections from the triple product X^3 back to X^2 by dropping, respectively, the first, second and third factor of X. The notation is supposed to suggest that π^3_o are the projections corresponding to the first, composite and second operators (i.e. A, C, and B in (0.3)). Thus the fact that

$$\Psi^{m'}(X) \cdot \Psi^m(X) \subset \Psi^{m+m'}(X)[1.13]$$

[3] Actually we define such a calculus for *any* compact complex manifold with smooth boundary. In other than strictly pseudoconvex cases we do not have applications of the calculus.

can be deduced from the functorial properties of conormal distributions under the maps in the diagram



[1.14]

Namely in X^3 there are three partial diagonals, Δ_o^2 for $o = f, c, s$ each being the preimage of $\Delta \subset X^2$ under π_o^3. These intersect in the triple diagonal

$$\Delta_t = \Delta_o \cap \Delta_{o'},\ o \neq o' \text{ for } o, o' = f, c, s.\text{[1.15]}$$

The composition formula (0.3) can then be proved in three steps. First show that under pull-back by π_o^3 a conormal distribution at Δ lifts to a conormal distribution at Δ_o^2. Next show that the product of conormal distributions at Δ_f and Δ_s is a conormal distribution at the intersection system $\Delta_f \cup \Delta_s$, which includes the intersection Δ_t. Finally show that push-forward under π_c^3 annihilates the singularities away from Δ_t and gives a conormal distribution at $\Delta = \pi_c^3(\Delta_t)$.

We construct two main calculi, a Θ-calculus and an α-calculus which are both generalizations of the standard calculus on manifolds with boundary and corner respectively. The first is designed to analyze operators on complex manifolds, with strictly pseudoconvex boundary, with degeneracy at the boundary characteristic of Bergman type metrics. The second is used to analyze degenerating families of operators. To orient the reader, unfamiliar with these concepts, we present elementary model cases of these constructions in Appendix A.

0.4. The Θ-calculus

The Θ-calculus is defined on a compact manifold with boundary X. The Schwartz kernel of an operator, A, is still a distributional density k_A on X^2 which is now a manifold with corners. It has boundary components of codimensions one and two; the component of codimension two is called 'the corner.' In the interior the Θ–pseudodifferential operators just reduce to pseudodifferential operators in the usual sense, so k_A should be a conormal distribution at $\Delta \subset X^2$. Since the resolvent of the Bergman Laplacian is essenitally a function of the Riemannian distance it follows that the kernel should also have a parabolic homogeneity at the boundary, B, of the diagonal. This means it will not be a conormal distribution in the ordinary sense, i.e. A should not be the restriction to X of a pseudodifferential operator on a manifold containing X as a relatively compact subset. Heuristically one can think of the non-conormality as a result of the 'interaction' between the singularities of the kernel along the boundary of X^2 and the diagonal.

The homogeneity structure is fixed by a line bundle in $T^*_{\partial X}X$ which is nowhere normal to ∂X, it is spanned by a form Θ, hence the term Θ-structure (see §2). To handle the parabolic homogeneity we replace X^2 by a new 'Θ-stretched product' denoted X^2_Θ, which is obtained by the S-parabolic blow-up of X^2 along B. Here S is defined by lifting Θ from the two factors of X. In the notation of Appendix C,

$$X^2_\Theta = \left[X^2; B, S\right],\ \beta^2_\Theta : \left[X^2; B, S\right] \longrightarrow X^2.\text{[1.16]}$$

This blow-up gives X^2_Θ a new boundary hypersurface, making it into a compact manifold with corners up to codimension three. The diagonal in X^2 lifts (in the sense of blow-up explained in Appendix C) to a submanifold Δ_Θ which meets the boundary transversally. By definition, A is a Θ–pseudodifferential, written

$$A \in \Psi^{m,\mathcal{E}}_\Theta(X)[1.17]$$

if and only if its kernel lifts from X^2 to a distributional section, κ_A, of the appropriate density bundle (the kernel bundle) over X^2_Θ which is polyhomogeneous with respect to the boundary and Δ_Θ. The notation of orders, m, and index families, $\mathcal{E}$, is explained in Appendix B. Basically these orders represent the powers which appear in the expansion of the kernel at the corresponding submanifolds, m corresponds to the 'diagonal' order, $\mathcal{E}$ to the various boundary orders.

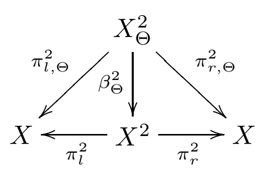
We think of X^2_Θ as a replacement for X^2 in quite a strong sense. Consider for a moment how an operator is defined by its Schwartz kernel, $k_A \in \mathcal{C}^{-\infty}(X^2; \Omega_R)$. Namely, formally,

$$A\phi(x) = \int_X k_A(x,y)\phi(y)dy,\ \ \phi \in \mathcal{C}^\infty(X).[1.18]$$

In terms of the two horizontal projections, in (0.4), this is expressed by

$$A\phi = (\pi^2_l)_*[k_A \cdot (\pi^2_r)^*\phi].[1.19]$$

Here all the operations, pullback, product and pushforward are in the sense of distributions. Replacing X^2 by X^2_Θ we can take a distributional density, κ_A on X^2_Θ and again define an operator by considering the two diagonal maps in the diagram



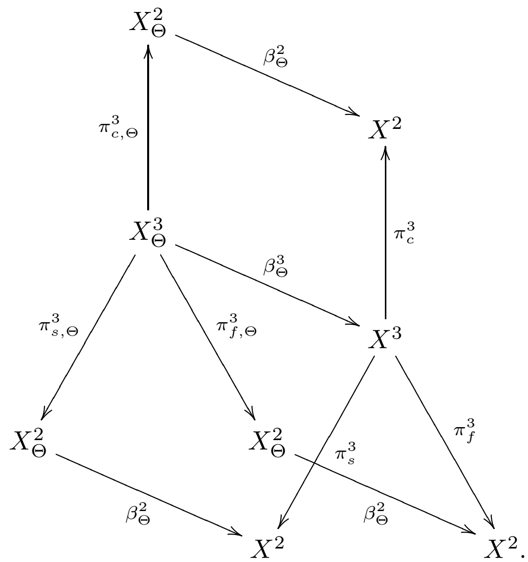
[1.20]

and write

$$A\phi = (\pi^2_{l,\Theta})_* \left[\kappa_A \cdot (\pi^2_{r,\Theta})^*\phi\right].[1.21]$$

It then becomes important to consider the properties of $\pi^2_{h,\Theta} = \pi^2_h \cdot \beta^2_\Theta$, $h = l, r$ where β^2_Θ is the blow-down map. These are no longer fibrations as are the π^2_h. The fibration property is used crucially in the study of the push-forward and it is therefore important that it can be replaced by a slightly weaker property, that of being a b-fibration, which still allows similar results to be proved, see §b5, §B6.

Thus to examine the composition properties of Θ–pseudodifferential operators, and show they form a 'calculus' we follow the general scheme of [**12**] and construct a Θ-triple product. This is obtained by (iterated) parabolic blow-up of X^3. The diagram of fibrations (0.3) extends to a commutative diagram, with the 'stretched projections' $\pi^3_{o,\Theta}$, for $o = f, s, c$ all b-fibrations:



[1.22]

This allows the composition of Θ–pseudodifferential to be discussed by the same approach as described above for the ordinary case. Formally the kernel of the composition $C = A \circ B$ is given by

$$\kappa_C = \pi^3_{c,Theta\,*}(\pi^{3\,*}_{f,\Theta}\kappa_A \cdot \pi^{3\,*}_{s,\Theta}\kappa_B).$$

Since the kernels are conormal with respect to four different submanifolds of X^2_Θ, namely the three boundary hypersurfaces and the lifted diagonal, there are four different symbol maps, measuring the singularity at each. A Θ–pseudodifferential operators is compact on a weighted L^2-space if and only if the order of its kernel at each of these submanifolds has the correct (strict) bound. To construct a parametrix, $E(\lambda)$, as an approximation to the resolvent family, $(\varDelta_r - \lambda)^{-1}$, of the Laplacian we need to choose the symbols of $E(\lambda)$ so that the error term in

$$(\varDelta_r - \lambda)E(\lambda) = \mathrm{Id} - R(\lambda)[1.23]$$

is compact. The symbol at the diagonal behaves much as the standard symbol. A Θ–pseudodifferential operator is elliptic provided this diagonal symbol is elliptic in the usual sense, i.e. is an isomorphism. The symbol at the front face is particularly important. It is called the normal homomorphism and its value is the normal operator. It is a homomorphism into a related calculus of "model" operators. Namely the front face of X^2_Θ has the structure of a bundle (over ∂X) of (compactified) solvable Lie groups on which the normal operator acts as a convolution operator. The normal operator of the composite operator on the left in (0.4) is the composite of the normal operators. In the case under discussion the normal operator of $\varDelta_r$ turns out to be just the Laplacian of the true (bi-invariant) Bergman metric on the complex ball. Thus an important part of the construction of E is the solution of this model problem and the analysis of the kernel of its inverse, to show that it is a Θ–pseudodifferential operator. On functions this was carried out in [**5**] and is extended here to higher degree forms for reasons already noted.

0.5. The adiabatic limit

In fact, we do not carry out the construction of the Θ-calculus here. One reason being that it was done in [**5**]. However there is another reason, namely the second main calculus we investigate, the α-calculus, includes the Θ-calculus as a special case. Indeed the α-calculus is the Θ-calculus with an additional parameter which has a singular (adiabatic) limit. It is designed to allow the direct investigation of the $\overline{\partial}$-Neumann problem, Bergman kernel etc, on tubular strictly pseudoconvex domains as the radius of the tube shrinks to zero. This analysis eventually leads to the resolution of the conjecture of Boutet de Monvel and Guillemin, i.e. the proof of Theorem 0.1.

To construct the α-calculus we start out on a compact manifold with corners, $X \simeq \Omega \times [0, \epsilon_0]$, $\epsilon_0 > 0$. This has three boundary hypersurfaces, one a Θ boundary, one the boundary where the adiabatic parameter, ϵ, is singular (i.e. vanishes) and the third a less interesting regular surface for the parameter. Indeed this third surface can be identified with the manifold with boundary on which the Θ-calculus takes place. The brief discussion above of the Θ-calculus can also serve as an introduction to the α-calculus. The α-calculus contains families of operators in the Θ-calculus which degenerate, in a controlled way, as the parameter tends to zero. Thus we need analogues of the space X^2_Θ and X^3_Θ, and of the maps discussed above. This is carried out in detail below starting in §7. An α-pseudodifferential operator has a kernel which is conormal on the space X^2_α with respect to the lifted diagonal. This manifold with corners has six boundary hypersurfaces. Of these one is the regular value of the parameter, the 'free' boundary face, two are Θ boundary faces and there is one 'adiabatic' boundary face. Finally, there are two 'front faces,' one adiabatic and the other the Θ front face. The intersection of the lifted diagonal with the boundary of the blown-up space is compactly contained within the two 'front faces' and the free boundary. The operators we consider here all have kernels which are rapidly vanishing at the adiabatic boundary, this somewhat simplifies the discussion.

At each of the front faces a symbol map gives a normal operator for the α-calculus. At the Θ front face it is essentially the same as in the Θ-calculus. At the adiabatic front face the normal operator becomes a parameterized convolution operator on Euclidean space. Thus, following the general discussion above, to solve the problem analogous to (0.4) we need to solve, rather explicitly, a new model problem. This turns out to be the same 'Bergman' Laplacian for the region

$$\mathbb{R}^n \times \mathbb{B}^n \subset \mathbb{C}^n[1.24]$$

with the defining function for the boundary being a quadratic form. The inversion of this Laplacian allows us to describe the behaviour of the solution operator to the $\overline{\partial}$-Neumann problem on a tube as the diameter shrinks to zero.

To construct an approximate inverse to the Töplitz correspondence in (0.1), as $\epsilon \downarrow 0$, we examine further spaces of operators from the tube, G, to Y, from Y to Y and from Y to G. The calculus on Y consists of a trivial case (in the sense that there is no fibration) of the adiabatic calculus examined first in [**?**].

The operators from Y to G, in which the inverse of T_ϵ lies, exhibit some rather subtle behaviour. In particular the kernels of these operators are defined on $(G\widetilde{Y})_\alpha$, a blown-up version of the fibre product of the tube and Y. To examine the composition of such an operator with say the Bergman projector on the tube we follow the ideas outlined above. This leads us to consider the blown-up triple product $(G^2\widetilde{Y})_\alpha$. The subtlety arises because one of the stretched projections from this space is *not* a b-fibration (see §19). This means that the calculus 'does not work' i.e. certain compositions do not give operators of the same type; which is to say, the operators with kernels defined on $(G\widetilde{Y})_\alpha$, do not form a left module over the α-calculus.

In the problem which is considered here we use the special features of the Bergman kernel, namely its holomorphy properties, to overcome this obstruction. In §23 and §24, we show that appropriate composition formulæ do hold in this restricted class. This allows us to construct the inverse of T_ϵ by singular perturbations from the model case at $\epsilon = 0$. In fact, this is reduced to the invertibility of an elliptic element in the very simple, adiabatic, calculus on Y.

A more specific guide to the contents of individual sections is contained in the introductions to the three parts below, describing the Θ-calculus, the α-calculus and the Töplitz isomorphism respectively. In the appendices we cover background material, not readily accessible in the literature and certain related developments which are tangential to the main topic of this monograph. Any reader unfamiliar with the language of conormal distributions or the geometric approach to pseudodifferential operator calculi is strongly urged to peruse Appendix A (in which some simple examples are considered) and Appendices B and C before plunging into the body of the text.

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Part 1

Bergman metrics and the $\overline{\partial}$-Neumann problem

In this first part we show how the Θ-calculus defined and described in [**5**] can be used to solve the $\overline{\partial}$-Neumann problem in any compact complex manifold with $\mathcal{C}^\infty$ strictly pseudoconvex boundary. The basic approach to this calculus, as outlined in the introduction, is described in §2 and §3; although the composition properties are not proved again in detail here. Indeed they follow from the more general results of Part II and so are relegated to corollaries of the discussion there. In §4 the examination of the model problem for the the Bergman Laplacian, namely the invariant Bergman Laplacian on the complex ball acting on (p, q)–forms is discussed, particularly the cases of $(n, 0)$–forms and $(n, 1)$–forms. The general case is treated more fully in Appendix F. This analysis is used, together with the calculus, to give a detailed description of the resolvent (including analytic extension in the eigenparameter) in §5. Then in §6 the application to the $\overline{\partial}$-Neumann problem is made, the $\mathcal{C}^\infty$ regularity of solutions is deduced and the structure of the kernels of both the solution operator, N, and the Bergman projection are examined closely. This recaptures earlier results of Boutet de Monvel and Sjöstrand ([**?**]) and refines results of Stein ([**?**]) on the operator N. We also show how the standard regularity properties for the solution follow from the mapping properties of Θ–pseudodifferential operators.

CHAPTER 1

Θ-structures

After briefly discussing the basic structures on a complex manifold with boundary we show how a Θ-structure can be associated to such a manifold and then review the construction of the Θ-calculus. The details of this material can be found in [**5**]. A Θ-structure is an example of a boundary-fibration structure; the general definition of this notion is given in [**12**], it is a Lie algebra of $\mathcal{C}^\infty$ vector fields with certain additional properties. We shall not use the general theory here although many of the constructions from it are used and conversely the results below serve to motivate the general case.

By a compact complex manifold with boundary we mean that Ω is a compact $\mathcal{C}^\infty$ manifold with boundary which has a complex structure. Thus the complexified tangent bundle of Ω has a specified smooth splitting

$$\mathbb{C}T_pM = \mathbb{C}\otimes_{\mathbb{R}} T_pM,\ \mathbb{C}T\Omega = T^{1,0}\Omega \oplus T^{0,1}\Omega,\ T^{0,1}\Omega = \overline{T^{1,0}\Omega}[2.1]$$

which is integrable in the sense that the space of vector fields of type $0,1$:

$$\mathcal{V}_{0,1}(\Omega) = \mathcal{C}^\infty(\Omega; T^{0,1}) \text{ is a Lie algebra}[2.2]$$

under the commutator bracket. The dual, cotangent, bundle has an induced splitting which is written

$$\mathbb{C}T^*\Omega = \Lambda^{1,0}\Omega \oplus \Lambda^{0,1}\Omega.[2.3]$$

and this splitting extends to all form bundles in the sense that

$$\Lambda^k\Omega = \bigoplus_{p+q=k} \Lambda^{p,q}\Omega,\ \Lambda^{p,q}\Omega = \Lambda^p(T^{1,0}\Omega)\otimes\Lambda^q(T^{0,1}\Omega).[2.4]$$

One can also define complex manifolds in terms of open covers and local coordinates. That is we suppose that there is a cover of Ω by open sets $\{U_\alpha\}$ and homeomorphisms into $\mathbb{C}^n$ $\{\phi_\alpha\ \phi_\alpha : U_\alpha \longrightarrow \mathbb{C}^n\}$ such that the maps

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta),$$

are biholomorphisms. Informally, one can introduce local holomorphic coordinates. The equivalence of these two definitions, at least away from the boundary, is the content of the Newlander-Nirenberg theorem. As was pointed out by Spencer, a solution of the $\overline{\partial}$-Neumann problem which uses only the formal intergrability of the complex structure can be used to prove this fundamental result. Recently Catlin has shown that the Newlander-Nirenberg theorem extends to manifolds with boundary provided the boundary satisfies an appropriate convexity hypothesis, see [**?**]. The reader unfamiliar with the fundamentals of complex geometry should consult Chapter ??? of [**?**] or Chapter ??? of [**?**].

The exterior differential decomposes into $d = \partial + \overline{\partial}$, the integrability condition on (1) is equivalent to the fact that for each p

$$\mathcal{C}^\infty(\Omega; \Lambda^{p,0}) \xrightarrow{\overline{\partial}} \mathcal{C}^\infty(\Omega; \Lambda^{p,1}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{C}^\infty(\Omega; \Lambda^{p,n}) [2.5]$$

is a complex, i.e. $\overline{\partial}^2 = 0$.

The complex structure on Ω induces a CR-structure (Cauchy-Riemann structure) on $\partial\Omega$. This is defined by the subbundles

$$T^{1,0}_p\partial\Omega = T^{1,0}_p\Omega \cap \mathbb{C}T_p\partial\Omega,\ T^{0,1}_p\partial\Omega = T^{0,1}_p\Omega \cap \mathbb{C}T_p\partial\Omega \quad \forall\ p \in \partial\Omega. [2.6]$$

The real part of the direct sum of these two subbundles is denoted $H \subset T\partial\Omega$, it is a bundle of hyperplanes. Let r be a defining function for $\partial\Omega$, i.e. $r \in \mathcal{C}^\infty(\Omega)$, $r \geq 0$ with $r = 0$ exactly on $\partial\Omega$ and $dr \neq 0$ at $\partial\Omega$. The $(0,1)$–form $i\overline{\partial}r$ is real on $T\partial\Omega$, set

$$L_p = \mathrm{sp}_{\mathbb{R}}\{i\partial r\} \subset T\partial\Omega. [2.7]$$

These lines form a smooth line bundle in $T\partial\Omega$, independent of the choice of r. Moreover and $H = \mathrm{null}\, i\overline{\partial}r = L^\circ$ is the annihilator of L. The $(1,1)$–form

$$\partial\overline{\partial}r \in \mathcal{C}^\infty(\Omega; \Lambda^{1,1}) [2.8]$$

induces an Hermitian form, the Levi form, on $T^{1,0}\partial\Omega$ by

$$\lambda(v, w) = -\partial\overline{\partial}r(v, \overline{w}),\ v, w \in T^{1,0}\partial\Omega [2.9]$$

where the Hermitian symmetry follows from the fact that r is real and that $\overline{\partial}\partial + \partial\overline{\partial} = 0$ on functions. The condition that the boundary of Ω be strictly pseudoconvex is that the Levi form be positive definite:

$$\lambda >> 0 \text{ on } T^{1,0}\partial\Omega. [2.10]$$

Near any point $p \in \Omega$ the bundle $\Lambda^{1,0}\Omega$ is spanned by n forms, $\zeta^1, \dots, \zeta^n$. If $p \in \partial\Omega$ we shall always take bases of $\Lambda^{1,0}_p\Omega$ with

$$\zeta^1 = \partial r \Longrightarrow \iota^*_{\partial\Omega}\zeta^1 \text{ is pure imaginary on } T\partial\Omega \text{ and spans } L_p, [2.11]$$

whereas the remaining $n-1$ forms have independent real and imaginary parts, even after restriction to the boundary.

Now consider the 2-form defined in (0.2). Carrying out the differentiation

$$\omega_r = -\frac{\partial\overline{\partial}r}{r} + \frac{\partial r}{r} \wedge \frac{\overline{\partial}r}{r}. [2.12]$$

LEMMA 1.1. *If Ω is a complex manifold with strictly pseudoconvex boundary and $r \in \mathcal{C}^\infty(\Omega)$ is a boundary defining function, then the Hermitian form associated to* (1)*:*

$$g_r(v, w) = \omega_r(v, \overline{w}) \quad \forall\ v, w \in T^{1,0}_p\Omega,\ p \notin \partial\Omega, [2.14]$$

is positive definite for p in a neighbourhood of the boundary.

PROOF. From (1) and (1.1)

$$g_r(v, w) = \frac{\lambda(v, w)}{r} + \frac{\partial r(v) \cdot \overline{\partial r(w)}}{r^2}. [2.15]$$

The second term has rank one as an Hermitian form and is non-negative, so it is enough to check the positivity of the first quadratic form on the annihilator of ∂r. At the boundary this is (1), so the form is positive near the boundary. □

It is now rather well understood that pseudoconvexity is an essential hypothesis if one expects the $\overline{\partial}$-equation to be "well posed." In our approach strict pseudoconvexity is quite essential. It makes the local geometry at the boundary uniform from point to point. For example it is well known that the metrics g_r have bounded geometry and in fact their curvatures tend, as one approaches the boundary, to that of the standard Bergman metric on the unit ball. An analytic consequence of this uniformity is that the associated Laplace operators degenerate in a very specific way and can be osculated to high order at boundary points by a single model operator, the Laplacian for the standard Bergman metric on the unit ball. This is described in great detail in §3.

Anoher way to describe this uniformity is through the Moser normal form: by a local holomorphic change of variables every strictly pseudoconvex hypersurface can be locally defined by an equation of the form

$$\operatorname{Im}(z_1) = z_2\bar{z}_2 + \cdots + z_n\bar{z}_n + O(|z_1|^2 + |z|^3).[2.100]$$

In other words the leading terms are fixed, non-degenerate and specify the general character of the local analytic geometry of the hypersurface. In fact in the strictly pseudoconvex case the Moser normal form or Chern connection provide complete local invariants, see [**?**],[**?**].

Any compact pseudoconvex hyersurface contains strictly pseudoconvex points. At points where the hypersurface is not strictly pseudoconvex the invariant theory is far more complicated and presently not well understood. One expects that these gross changes in geometric structure will be reflected in much more complicated singularities for the Bergman kernel and other operators closely connected to the complex geometry, see [**?**], [**?**]. Thus if one desires a reasonably simple uniform treatment in an analytically interesting case, strict pseudoconvexity becomes a necessary assumption. It is likely that our methods can be extended to include certain non-strictly pseudoconvex examples though it will be at the cost of considerable additional complexity.

In the next sections we shall study the invertibility of the Laplacian for a metric of the form of (1.1). So far the metric is only defined in a neighbourhood of the boundary. In the special case that Ω is a smoothly bounded strictly pseudoconvex domain in a Stein manifold we can extend the definition globally:

LEMMA 1.2. *If $\Omega \subset X$ is a compact and smoothly bounded strictly pseudoconvex domain in a Stein manifold X there are boundary defining functions $r \in \mathcal{C}^\infty(\Omega)$ such that the Hermitian form g_r in* (1.1) *is positive definite in the interior of Ω.*

PROOF. See [**?**] ****maybe a more precise reference****. □

In the general case of a compact complex manifold with strictly pseudoconvex boundary it may not be possible to find such a global potential. This can be seen from the fact that the metric g_r is Kähler and there are non-trivial obstructions to the existence of Kähler metrics. In that case we simply extend g_r into the interior as an Hermitian metric.

DEFINITION 1.1. *A Bergman metric on a compact complex manifold with strictly pseudoconvex boundary is an Hermitian metric which is of the form* (1.1) *near the boundary.*

1.1. The Θ-tangent bundle

The presence of the single power of r in the first term in (1) means that, for a Bergman metric, there cannot be a local orthonormal frame which is smooth up to the boundary. Since our methods for studying the Laplacian depend on writing it as a polynomial with $\mathcal{C}^\infty$ coefficients in a Lie algebra of vector fields we need such a smooth frame. To obtain it we simply "introduce $r^{\frac{1}{2}}$ as a $\mathcal{C}^\infty$ function". This is a very special case of the general process of parabolic blow-up as described in Appendix C and used extensively in the inversion of the Laplacian. In this case we are just replacing Ω by another manifold with boundary, actually diffeomorphic to it,

$$\mho = \Omega_{\frac{1}{2}} = [\Omega; \partial\Omega, {}_+N^*\partial\Omega]\,, [2a.18]$$

where we use the notation of §C2. One can view this as simply changing the $\mathcal{C}^\infty$ structure (i.e. the algebra of $\mathcal{C}^\infty$ functions) on the manifold. The new $\mathcal{C}^\infty$ structure is obtained by adjoining to $\mathcal{C}^\infty(\Omega)$ the square root of a defining function for the boundary. Thus a $\mathcal{C}^\infty$ function on $\mho$ is simply a (continuous) function on Ω which can be written locally as a $\mathcal{C}^\infty$ function of local coordinates and $r^{\frac{1}{2}}$. In the interior $r^{\frac{1}{2}}$ is $\mathcal{C}^\infty$ so this changes nothing but it admits more functions near the boundary. The natural (blow-down) map

$$\beta_{\frac{1}{2}}\; \mho \longrightarrow \Omega, [2a.19]$$

which is the identity map of sets, has a fold singularity at $\partial\mho \cong \partial\Omega$.

On $\mho$ consider the n singular forms

$$\widetilde{\zeta}^1 = \rho^{-2}\beta^*_{\frac{1}{2}}\zeta^1,\ \widetilde{\zeta}^k = \rho^{-1}\beta^*_{\frac{1}{2}}\zeta^k,\ k = 2, \dots, n[2a.20]$$

where $\rho \in \mathcal{C}^\infty(\mho)$ is a defining function for the boundary. We can, and generally do, take $\rho = \sqrt{\beta^*_{\frac{1}{2}} r}$. From (1) it follows directly that

$$\beta^*_{\frac{1}{2}} g_r = \sum_{j,k=1}^n h_{jk}\widetilde{\zeta}^j\overline{\widetilde{\zeta}^k}[2a.21]$$

where the coefficient matrix h^{jk} has entries smooth up to the boundary. Set

$$\mathcal{V}_\Theta(\mho) = \{V \in \mathcal{C}^\infty(\mho; T\mho); \widetilde{\zeta}^j(V), \overline{\widetilde{\zeta}^j}(V) \in \mathcal{C}^\infty(\mho), j = 1, \dots, n\}.[2a.22]$$

From (1.1) this is just the space of smooth vector fields on $\mho$ which have bounded length, uniformly up to the boundary. To show that there is a smooth orthonormal frame on $\mho$ for $\beta^*_{\frac{1}{2}} g_r$ we need to show the existence of

$$\begin{gathered} V_j \in \mathcal{V}_\Theta(\mho),\ j = 1, \dots, n,\ \text{satisfying} \\ \widetilde{\zeta}^k(V_j) = \delta_{jk},\ \overline{\widetilde{\zeta}^j}(V_j) = 0 \quad \forall\ j, k = 1, \dots, n. \end{gathered} [2a.23]$$

To check this we study the structure of the space $\mathcal{V}_\Theta(\mho)$ more closely. In fact $\mathcal{V}_\Theta(\mho)$ is the basic geometric object. It is more fundamental than the metric g_r since it describes the degeneracy of all Bergman metrics on Ω and is therefore intrinsically associated to $\mho$, and hence Ω.

We use the notation $\mathcal{V}_b(\mho)$ for the Lie algebra of $\mathcal{C}^\infty$ vector fields on $\mho$ (any manifold with boundary, or even corners) which are tangent to the boundary (see Appendix B). First note that

$$\mathcal{V}_\Theta(\mho) \subset \mathcal{V}_b(\mho) \text{ is a Lie subalgebra and } \mathcal{C}^\infty(\mho)\text{-module.}[2a.24]$$

That it is a $\mathcal{C}^\infty(\mho)$-module follows directly from the definition. That it is a Lie algebra under the commutator bracket follows from the properties of the exterior derivative. Thus, on Ω, we can choose the the ζ^j to be exact for $j > 1$ and $\zeta^1 = i\partial r$. Then

$$\begin{aligned} d\widetilde{\zeta}^k &= -\frac{1}{2}\frac{d\rho^2}{\rho^2}\wedge\widetilde{\zeta}^k = -\frac{1}{2}(\widetilde{\zeta}^1 + \overline{\widetilde{\zeta}^1})\wedge\widetilde{\zeta}^k,\ k > 1 \\ d\widetilde{\zeta}^1 &= -\frac{d\rho^2}{\rho^2}\wedge\rho^{-2}\beta^*_{\frac{1}{2}}\zeta^1 + \rho^{-2}\beta^*_{\frac{1}{2}}(d\zeta^1) = -\overline{\widetilde{\zeta}^1}\wedge\widetilde{\zeta}^1 + \sum_{q,p=1}^n a^{qp}\widetilde{\zeta}^q\wedge\overline{\widetilde{\zeta}^p} \end{aligned} \quad [2a.25]$$

where a^{qp} is a smooth matrix of functions locally on $\mho$. This shows that the $\widetilde{\zeta}^j$ and their complex conjugates span a differential ideal. This is the dual form of the condition that $\mathcal{V}_\Theta(\mho)$ be a Lie algebra in view of Cartan's identity

$$\widetilde{\zeta}^j([V, W]) = V(\widetilde{\zeta}^j(W)) - W\widetilde{\zeta}^j(V) - d\widetilde{\zeta}^j(V, W).[2a.26]$$

We call the Lie algebra $\mathcal{V}_\Theta$ a Θ-structure because, as a real algebra, it is determined by the conformal class of the 1-form

$$\Theta = \beta^*_{\frac{1}{2}}\zeta^1 \text{ at } \partial\mho.[2a.27]$$

One of the important properties of a Θ-structure is that it determines, and is determined by, a $\mathcal{C}^\infty$ vector bundle over $\mho$ and bundle map $\iota_\Theta\ {}^\Theta T\mho \longrightarrow T\mho$ such that

$$\iota^*_\Theta\mathcal{V}_\Theta = \mathcal{C}^\infty(\mho; {}^\Theta T\mho).[2a.28]$$

We usually write this in the abbreviated form $\mathcal{V}_\Theta(\mho) = \mathcal{C}^\infty(\mho; {}^\Theta T\mho)$. To see that such a bundle exists note that

$$\widetilde{\zeta}^1 = \frac{d\rho}{\rho} + i\frac{\Theta}{\rho^2},\ \widetilde{\zeta}^j = \frac{\alpha^j + i\beta^j}{\rho},\ j > 1[2a.29]$$

where $d\rho, \Theta, \alpha^2, \dots, \alpha^n, \beta^2, \dots, \beta^n$ give a smooth real coframe for $\mho$ up to the boundary. Thus if $N, T, u_2, \dots, u_n, v_2, \dots v_n$ is the dual frame of $\mho$ then

$$\rho N, \rho^2 T, \rho u_2, \dots, \rho u_n, \rho v_2, \dots \rho v_n[2a.30]$$

is a real basis of $\mathcal{V}_\Theta(\mho)$ and

$$\widetilde{Z}_1 = \rho N + i\rho^2 T, \widetilde{Z}_2 = \rho(u_2 + iv_2), \dots, \widetilde{Z}_n = \rho(u_n + iv_n)[2a.31]$$

along with their conjugates is a complex basis. That is the fibres, defined abstractly by

$$\begin{gathered} {}^\Theta T_p = \mathcal{V}_\Theta(\mho)/\sim_p,\ V \sim_p W \Longleftrightarrow \\ \lim_{q\to p \text{ from } \mho\setminus\partial\mho}[\widetilde{\zeta}^j(V - W)] = 0,\quad \lim_{q\to p \text{ from } \mho\setminus\partial\mho}[\overline{\widetilde{\zeta}^j}(V - W)] = 0,\ j = 1, \dots, n, \end{gathered} \quad [2a.32]$$

are real vector spaces of dimension $2n$ for each $p \in \mho$, including boundary points and the basis (1.1) gives, to disjoint union of fibers

$${}^\Theta T\mho = \bigsqcup_{p\in\mho} {}^\Theta T_p, [2a.33]$$

the structure of a $\mathcal{C}^\infty$ vector bundle. The map $\iota_\Theta\ {}^\Theta T\mho \longrightarrow T\mho$ is then obtained by weakening the equivalence relation in (1.1) to that defining the tangent bundle.

The 'structure bundle' ${}^\Theta T\mho$ is an effective replacement for the usual tangent bundle in the treatment of differential operators associated to the Θ-structure. In particular the dual, Θ-cotangent bundle, ${}^{\Theta}T^*\mho$, and associated exterior powers, the Θ-form bundles ${}^{\Theta}\Lambda^{p,q}\mho$, play an essential rôle.

Although $\mathcal{V}_{\Theta}(\mho)$ is determined by Θ this 1-form does not fix the complex structure on ${}^{\Theta}T\mho$ which is determined from (1.1):

$$ {}^{\Theta}\Lambda^{1,0}_p\mho = \operatorname{sp}\{\widetilde{\zeta}^j(p), j = 1, \dots, n\},\ \mathbb{C}{}^{\Theta}T\mho = {}^{\Theta}\Lambda^{1,0}\mho \oplus {}^{\Theta}\Lambda^{0,1}\mho. [2a.34] $$

With this complex structure there is associated a Dolbeault complex, given by identifying the Θ-form bundles over the interior with the usual form bundles. On functions set

$$ {}^{\Theta}\overline{\partial} f = \sum_{j=1}^{n} (\overline{\widetilde{Z}_j} f)\overline{\widetilde{\zeta}^j} \in \mathcal{C}^{\infty}(\mho; {}^{\Theta}\Lambda^{0,1}) \text{ for } f \in \mathcal{C}^{\infty}(\mho) [2a.35] $$

where $\widetilde{Z}_j$ is the basis of ${}^{\Theta}T\mho$ dual to the basis $\widetilde{\zeta}^j$ of ${}^{\Theta}T^*\mho$ given by (1.1). Now ${}^{\Theta}\Lambda^{0,1}\mho$ is locally spanned by the singular forms $\widetilde{\zeta}^j$ in (1.1), which is to say the forms $\rho^{-1}{}^{\Theta}\overline{\partial} f$, for $f \in \mathcal{C}^{\infty}(\mho)$. Thus to analyze the action of ${}^{\Theta}\overline{\partial}$ on 1-forms it suffices to note that

$$ {}^{\Theta}\overline{\partial}[\rho^{-1}{}^{\Theta}\overline{\partial} f] = -\frac{{}^{\Theta}\overline{\partial}\rho}{\rho} \wedge \frac{{}^{\Theta}\overline{\partial} f}{\rho} \in \mathcal{C}^{\infty}(\mho; {}^{\Theta}\Lambda^{0,2}). [2a.36] $$

Thus ${}^{\Theta}\overline{\partial}$ extends to forms of higher degree by Leibniz' rule and we conclude that

$$ \mathcal{C}^{\infty}(\mho; {}^{\Theta}\Lambda^{p,0}) \xrightarrow{{}^{\Theta}\overline{\partial}} \mathcal{C}^{\infty}(\mho; {}^{\Theta}\Lambda^{p,1}) \xrightarrow{{}^{\Theta}\overline{\partial}} \cdots \xrightarrow{{}^{\Theta}\overline{\partial}} \mathcal{C}^{\infty}(\mho; {}^{\Theta}\Lambda^{p,n}) [2a.37] $$

is a differential complex.

The main reason that the Θ-structure is important is that it captures the boundary singularity of the Bergman metrics. Indeed we can restate (1.1) in the form:

LEMMA 1.3. *A Bergman metric, in the sense of Definition 1.1 , on a compact complex manifold with strictly pseudoconvex boundary,* Ω, *lifts via* $\beta_{\frac{1}{2}}$ *to* ℧, *to define an Hermitian fibre metric on the* Θ*-structure bundle* ${}^{\Theta}T\mho$, $\mathcal{C}^{\infty}$ *and positive definite up to the boundary.*

1.2. Θ-differential operators

Let $\operatorname{Diff}^m_{\Theta}(\mho)$ be the space of differential operators of order m generated by $\mathcal{V}_{\Theta}(\mho)$ and $\mathcal{C}^{\infty}(\mho)$, i.e. the order filtration for the enveloping algebra of $\mathcal{V}_{\Theta}(\mho)$. That is $P \in \operatorname{Diff}^m_{\Theta}(\mho)$ is an operator

$$ P\, \mathcal{C}^{\infty}(\mho) \longrightarrow \mathcal{C}^{\infty}(\mho) \text{ of the form} $$

$$ Pu = \sum_{\substack{j=1,\dots,N \\ k\le m}} a_j V_{j,1} \dots V_{j,k} u,\ V_{j,r} \in \mathcal{V}_{\Theta}(\mho),\ a_j \in \mathcal{C}^{\infty}(\mho). [2b.39] $$

As a space of Schwartz kernels this is a local $\mathcal{C}^{\infty}(\mho^2)$-module so we can directly extend the definition to operators acting between sections of arbitrary smooth vector bundles, E and F, over ℧ with the resulting space denoted $\operatorname{Diff}^m_{\Theta}(\mho; E, F)$ or $\operatorname{Diff}^m_{\Theta}(\mho; E)$ if $E = F$. Expressed more prosaically this just requires that the expression of $P \in \operatorname{Diff}^m_{\Theta}(\mho; E, F)$ in common local trivializations of E and F be given by matrices with entries in $\operatorname{Diff}^m_{\Theta}(\mho)$.

The fact that $\mathcal{V}_{\Theta}(\mho)$ is a Lie algebra means that the part of P of order m, in (1.2), is well-defined. For $m = 1$ we think of the part of order 1 as a linear function on the fibres of ${}^{\Theta}T^*\mho$. Extending multiplicatively to the general case this gives the symbol map:

$$\sigma^m_\Theta \ \mathrm{Diff}^m_\Theta(\mho) \ni P \longmapsto \sum_{\substack{j=1,\dots,N\\ k=m}} a_j V_{j,1}\cdots V_{j,k} \in \mathcal{C}^\infty({}^{\Theta}T^*\mho)[2b.40]$$

which associates to each $P \in \mathrm{Diff}^m_\Theta(\mho)$ a homogeneous polynomial on the fibres of ${}^{\Theta}T^*\mho$. In the case of an operator between sections of vector bundles the symbol is a homogeneous polynomial on ${}^{\Theta}T^*\mho$ with values in the bundle of homomorphisms of the lifts to ${}^{\Theta}T^*\mho$ of the bundles E and F :

$$\sigma^m_\Theta \ \mathrm{Diff}^m_\Theta(\mho; E, F) \longrightarrow P^m({}^{\Theta}T^*\mho; \mathrm{Hom}(\pi^*_\Theta E, \pi^*_\Theta F).[2b.41]$$

An element of $\mathrm{Diff}^m_\Theta(\mho; E, F)$ is said to be elliptic if the symbol is invertible off the zero section of ${}^{\Theta}T^*\mho$. By the Laplacian of a Bergman metric we shall mean the Kohn-Spencer Laplacian of the complex (1.1), i.e.

$$\varDelta_g = {}^{\Theta}\overline{\partial}^*\,{}^{\Theta}\overline{\partial} + {}^{\Theta}\overline{\partial}\,{}^{\Theta}\overline{\partial}^*[2b.42]$$

where the adjoint is taken with respect to the metric g.

Proposition 1.1. *The Laplacian,* (1.2)*, of a Bergman metric on a compact complex manifold with strictly pseudoconvex boundary acting on the* Θ-(p,q)*–forms bundles is an elliptic element*

$$\varDelta_g \in \mathrm{Diff}^2(\mho; {}^{\Theta}\varLambda^{p,q}).[2b.44]$$

Proof. Directly from the definition of the spaces of Θ-differential operators it is clear that

$$\mathrm{Diff}^m_\Theta(\mho; F, G)\cdot \mathrm{Diff}^{m'}_\Theta(\mho; E, F) \subset \mathrm{Diff}^{m+m'}_\Theta(\mho; E, G).[2b.45]$$

From (1.1) it is clear that ${}^{\Theta}\overline{\partial} \in \mathrm{Diff}^1_\Theta(\mho; {}^{\Theta}\varLambda^{p,q}; {}^{\Theta}\varLambda^{p,q+1})$ for any p and q. From the definition, (1.2), and (1.2) it is therefore enough to show that

$${}^{\Theta}\overline{\partial}^* \in \mathrm{Diff}^1_\Theta(\mho; {}^{\Theta}\varLambda^{p,q}, {}^{\Theta}\varLambda^{p,q-1}).[2b.46]$$

Again from the definition it is clear that the adjoint with respect to a non-degenerate fibre metric and a non-vanishing smooth density of an element of $\mathrm{Diff}^m_\Theta(\mho; E, F)$ is in $\mathrm{Diff}^m_\Theta(\mho; F, E)$. In this case we certainly have a smooth and non-degenerate fibre metric, by Lemma 1.3 , but the metric density is of the form

$$\nu = \rho^{-2n-1}\nu',\ 0 \neq \nu' \in \mathcal{C}^\infty(\mho; \Omega)[2b.47]$$

as follows from (1.1). Thus it is enough to observe that conjugation by any power of ρ gives an isomorphism

$$\rho^{-a}\,\mathrm{Diff}^m_\Theta(\mho; E, F)\rho^a = \mathrm{Diff}^m_\Theta(\mho; E, F)[2b.48]$$

to conclude that (1.1) holds.

The standard computation of the symbol of the Laplacian gives

$$\sigma^2(\varDelta_g)(x,\xi) = \frac{1}{2}|\xi|^2[2b.49]$$

at interior points, $x \in \mho\backslash\partial\mho$, and $\xi \in T^*_x\mho$. The symbol in the sense of (1.2) reduces to the standard symbol over the interior, where ${}^{\Theta}T\mho$ is canonically identified with $T\mho$. Thus, by continuity it follows that

$$\sigma^2_\Theta(\varDelta_g)(x,\zeta) = \frac{1}{2}|\zeta|^2,\ x \in \mho,\ \zeta \in {}^{\Theta}T^*\mho.[2b.50]$$

This completes the proof of the lemma. □

In the next several sections we describe a calculus of pseudodifferential operators which quantizes the Lie algebra $\mathcal{V}_\Theta(\mho)$; we call it the Θ-caluclus. This description of the Laplacian as an elliptic element of the algebra $\mathrm{Diff}^*_\Theta(\mho)$ allows us to use the Θ-calculus to construct a close approximation to its resolvent family. From the general properties of this calculus we can deduce, for example:

THEOREM 1.1. *The Laplacian for a Bergman metric acting on its natural domain has closed range and null space*

$$H^{p,q}_{\mathrm{Do}}(\Omega) = \{u \in L^2_\Theta(\mho; {}^\Theta\Lambda^{p,q}); {}^\Theta\overline{\partial}^* u = {}^\Theta\overline{\partial} u = 0\}.[2b.52]$$

The null space (1.1) *is finite dimensional unless* $p+q=n$. *If* $\Omega \subset \mathbb{C}^n$ *is a pseudoconvex domain then* Δ_r *is invertible for* $p+q \neq n$.

A proof of this result (which is mostly due to Donnelly and Fefferman [**?**]) is described in §5 and Appendix F. In fact we give, following [**5**], a rather detailed description of the Schwartz kernel of the resolvent of Δ_g as a holomorphic family of Θ–pseudodifferential operators.

1.3. The model algebra

We recall the constructions, made in [**5**], leading to the calculus of Θ–pseudodifferential operators which contains the inverse of the Laplacian, Δ_r, of Theorem 1.1 acting on functions. In §5 these results are extended to $(n,1)$–forms and in Appendix F to general (p,q)–forms. This, in turn, is used in §6 to analyze the Bergman projector and the $\overline{\partial}$-Neumann problem, by way of the case of $(n,0)$–forms.

At interior points of $\mho$ the vector fields in $\mathcal{V}_\Theta$ span the tangent space. At boundary points, as is clear from (1.1), they all vanish. It follows (cf. [**5**]§3) that the fibre ${}^\Theta T_p\mho$ is a Lie algebra for each $p \in \partial\mho$. Indeed if $p \in \partial\mho$ and $\mathcal{I}_{\{p\}} \subset \mathcal{C}^\infty(\mho)$ is the ideal of functions vanishing at p then

$$\mathcal{I}_{\{p\}} \cdot \mathcal{V}_\Theta(\mho) \subset \mathcal{V}_\Theta(\mho)[3.1]$$

is an ideal, precisely because all the elements of $\mathcal{V}_\Theta(\mho)$ vanish at p. The quotient, by definition ${}^\Theta T_p\mho$, is therefore a Lie algebra. It is a solvable algebra, being just the homogeneous extension of the Heisenberg algebra. An important step in our construction is the realization of ${}^\Theta T_p\mho$ as a Lie algebra of vector fields on a compact space. It is rather easy to construct a open manifold with boundary which has a ${}^\Theta T_p\mho$-action. To orient the reader we first give a coordinate based discussion and then a more functorial description using the theory of parabolic blow-ups developed in Appendix C.

Since Ω is a strictly pseudoconvex domain we can introduce real coordinates in a neighborhood of a boundary point $p \in \partial\mho$, $(\rho, t, x_2, \dots, x_n, y_2, \dots, y_n)$ such that p corresponds to the origin, the $\partial\mho$ is $\rho = 0$ and

$$\Theta = dt + \sum_{i=2}^n x_i dy_i - y_i dx_i.[3.100]$$

These coordinates can also be though of a linear coordinates on the tangent space at p, so that ${}_+T_p\mho$ is identified with $\rho \geq 0$. From the form of Θ it is clear that we can introduce a local frame field for ${}^\Theta T\mho$ of the form

$$T = \rho^2\partial_t,\ N = \rho\partial_\rho,\ X_i = \rho(\partial_{x_i} + y_i\partial_t),\ Y_i = \rho(\partial_{y_i} - x_i\partial_t),\ i = 2, \dots, n.[3.101]$$

We define a dilation structure in terms of these coordinates by

$$M_\delta(t,\rho,x,y) = \delta^2 t, \delta(\rho,x,y).[3.102]$$

Observe that the vector fields in (1.3) are homogeneous of degree zero relative to this action. We can think of this action as identifying larger and larger parts of the half space ${}_+T_p\mho$ with smaller and smaller neighborhoods of p in $\mho$.

If $V \in \mathcal{V}_\Theta$ then near p is can be expressed as

$$V = aT + bN + \sum_{i=2}^{n}(c_iX_i + d_iY_i).[3.103]$$

The coefficients in (1.3) are smooth. Since the vector fields are homogeneous of degree zero it follows that

$$N_p(V) = \lim_{\delta\downarrow 0} M_{\delta\,*}^{-1}V$$

exists and equals

$$N_p(V) = a(0)T + b(0)N + \sum_{i=2}^{n}(c_i(0)X_i + d_i(0)Y_i).[3.104]$$

The vector fields on the right hand side of (1.3) can be thought of as acting on ${}_+T_p\mho$. The vector field T, N, X_i, Y_i define a finite dimensional Lie algebra and the map N_p is a Lie algebra homomorphism from $\mathcal{V}_\Theta$ onto this Lie algebra. From (1.3) it follows that the kernel of N_p is exactly $\mathcal{I}_{\{p\}}\mathcal{V}_\Theta$ and thus we have realized ${}^\Theta T_p\mho$ as a Lie algebra of vector fields acting on an open manifold with boundary. It is easy to show that though this construction involved a choice of coordinates the resultant isomorphism is actually canonical. We give instead a more functorial approach to the same construction and finally a construction which produces a compact space on which ${}^\Theta T_p\mho$ acts. The reader unfamiliar with these concepts is urged to read §C1-§C2 before proceeding.

At each $p \in \partial\mho$ the form Θ in (1.1) fixes a line in $T_p^*\mho$:

$$S_p = \operatorname{sp}\Theta \subset {}_+T_p^*\mho; [3.2]$$

Since Θ is non-zero even when pulled back to $\partial\mho$ the annihilator $S_p^\circ \subset {}_+T_p\mho$ is a p-subspace and so defines a p-subbundle. The inward-pointing S-parabolic normal space, ${}_+N\{\mho;p,S\}$, to $\{p\}$ in $\mho$ is defined in §C2. In the notation of (C.2)

$$\mathcal{V}_\Theta \subset \mathcal{V}_b\{\mho;p,S\}, [3.3]$$

so, by Lemma C.1 , $\mathcal{V}_\Theta$ lifts to define a Lie algebra of $\mathcal{C}^\infty$ vector fields on ${}_+N\{\mho;p,S\}$ all of which are invariant under the $\mathbb{R}^+$-action (C.3). The null space of (C.1) is $\mathcal{I}_{\{p\}}\cdot\mathcal{V}_\Theta$ so ${}^\Theta T_p\mho$ acts as a Lie algebra on ${}_+N\{\mho;p,S\}$. Some of the conclusions of [**5**] can be summarized as follows:

Lemma 1.4. *The half-space ${}_+N\{\mho;p,S\}$, with the complex structure induced by ${}^\Theta T_p\mho$, is diffeomorphic, by a map which is holomorphic in the interior, to the region*

$$Q^+ = \left\{(z', z_n) \in \mathbb{C}^n; \operatorname{Im} z_n \geq \frac{1}{2}|z'|^2\right\}[3.5]$$

with the square root differential structure.

One can give a coordinate based proof of this lemma by using the Moser normal form presented in (1) and an argument similar to that presented above.

Now we obtain a compact space on which ${}^{\Theta}T_p\mho$ acts. To that end we extend $\mho$ by appending a parameter:

$$\widetilde{\mho} = \mho \times [0,1]_\epsilon [3.6]$$

and then blow up the point $\widetilde{p} = (p, 0)$, $\widetilde{S}$-parabolically with $\widetilde{S} = S \times \{0\}$. By Proposition C.2 , $\mathcal{V}_\Theta$ lifts to the blown-up space

$$\mathcal{V}_\Theta \subset \mathcal{V}_b\bigg(\Big[\widetilde{\mho}; \widetilde{p}, \widetilde{S}\Big]\bigg).[3.7]$$

Now $\mathcal{V}_\Theta$ annihilates ϵ so when lifted to ${}_+N\{\widetilde{\mho}; \widetilde{p}, \widetilde{S}\}$ it acts on the subspaces on which ϵ is constant. By identifying ${}_+N\{\mho; p, S\}$ with the subspace $d\epsilon = 1$ in ${}_+N\{\widetilde{\mho}; \widetilde{p}, \widetilde{S}\}$ these two actions of ${}^{\Theta}T_p\mho$ coincide. The homogeneity of the action means that it descends to $S_+N\{\widetilde{\mho}; \widetilde{p}, \widetilde{S}\}$ and this is just the restriction of the action in (1.3) to the front face. Thus with ${}_+N\{\mho; p, S\}$ identified with the complement of the boundary $\epsilon = 0$ in this front face the action of ${}^{\Theta}T_p\mho$ from (1.3) agrees with that defined above. Thus the front face of $\Big[\widetilde{\mho}; \widetilde{p}, \widetilde{S}\Big]$ is a canonical compactification of ${}_+N\{\mho; p, S\}$ to which the action of ${}^{\Theta}T_p\mho$ smoothly extends.

By Lemma 1.4 these constructions, for different p or even different domains Ω of the same dimension, are all equivalent. In particular consider a point, $(0, 1)$, in the complex ball $\mathbb{CB}^n$. As is well-known the ball minus a boundary point is biholomorphic to Q^+ in (1.4):

$$\mathbb{CB}^n \setminus \{(0, -1)\} \ni w = (w', w_n) \longmapsto \left(2^{\frac{1}{2}} \frac{w'}{1 + w_n}, \frac{i(1 + w_n)}{1 + w_n}\right) [3.8]$$

(*cf.* [**5**],§A2). The $\mathbb{R}^+$-action on $\mathbb{CB}^n \setminus \{(0, 1), (0, -1)\}$ given in the hyperquadric model by

$$Q^+ \ni (z', z_n) \longmapsto (tz', t^2 z_n), [3.9]$$

gives a natural isomorphism

$$\Big[\mathbb{CB}^n_{\frac{1}{2}}; (0, 1), \widetilde{S}\Big] \simeq S_+N\{\widetilde{\mho}; \widetilde{p}, \widetilde{S}\}.[3.10]$$

Thus we can refine Lemma 1.4 to

LEMMA 1.5. *At any point $p \in \partial\mho = \partial\Omega$ the Lie algebra ${}^{\Theta}T_p\mho$ acts on the model space $\Big[\mathbb{CB}^n_{\frac{1}{2}}; (0, 1), \widetilde{S}\Big]$ with the complex structure consistent with that from the ball.*

The Lie algebra homomorphism

$$\mathcal{V}_\Theta(\mho) \longrightarrow {}^{\Theta}T_p\mho,\ \ p \in \partial\mho[3.12]$$

extends to a homomorphism of the enveloping algebras, defining the normal operator at p :

$$N_p\ \mathrm{Diff}^m_\Theta(\mho) \longrightarrow \mathcal{D}^m({}^{\Theta}T_p\mho), [3.13]$$

in particular one has a composition formula

$$N_p(P \circ Q) = N_p(P) \circ N_p(Q).$$

Not only does the complex structure in Lemma 1.5 arise from that on the ball but

LEMMA 1.6. *Under the action of* ${}^\Theta T_p \mho$ *on* $S_+N\{\widetilde{\mho};\widetilde{p},\widetilde{S}\} \simeq \left[\mathbb{CB}^n_{\frac{1}{2}};(0,1),\widetilde{S}\right]$ *the normal operator* $N_p(\Delta_r)$ *is identified with the lift from* $\mathbb{CB}^n$ *of the Bergman Laplacian associated to the metric with Kähler form* $-\partial\overline{\partial}\log(1-|z|^2)$.

The use of the calculus of Θ–pseudodifferential operators developed in [**5**] rests on Lemma 1.6 which shows that the basic step in the inversion of Δ_r (on functions) is the inversion of the 'model problem' which is the Bergman Laplacian on the ball. Extending these ideas to the action on (p,q)–forms involves no new ideas. Since the blow-ups involved here are of points, any vector bundle lifts to be canonically trivial over the front face. We discuss these routine matters in (painful) detail in Appendix D. The conclusion is that Lemma 1.6 extends unchanged to (p,q)–forms.

To orient the reader for the later more complicated (!) discussions of a similar nature we briefly recall the features of the Θ-calculus and how the solution to this model problem is actually used in the construction of a parametrix. In constructing the Θ-calculus the fundamental object is a parabolically blown-up version of the product $\mho^2$ along the boundary of the diagonal:

$$B = \left\{(p,p) \in \mho^2; p \in \partial\mho\right\}.[3.15]$$

The parabolic direction is defined by the form

$$\Theta_2 = \pi_L^*\Theta - \pi_R^*\Theta,\ S_2 = \operatorname{sp}\Theta_2 \subset N^*B[3.16]$$

(see [**5**],§7). In the notation of Appendix B the blown-up space is denoted by

$$\mho^2_\Theta = \left[\mho^2; B, S_2\right].[3.17]$$

We let $\beta^{(2)}_\Theta$ denote the natural projection map from $\mho^2_\Theta$ to $\mho^2$. The blown-up space has three boundary components $\Theta\mathrm{l}, \Theta\mathrm{r}, \Theta\mathrm{f}$, the left and right boundaries and the front face respectively. In addition to the boundaries we require the lift of the diagonal

$$\Delta_\Theta = \overline{{\beta^{(2)}_\Theta}^{-1}(\Delta\setminus\partial\Delta)}.[3.110]$$

We can lift vector fields in $\mathcal{V}_\Theta$ from the left factor to $\mho^2_\Theta$. The lifted vector fields are tangent to the fibres of $\Theta\mathrm{f}\left[\mho^2;B,S_2\right]$ over $B = \partial\Delta \equiv \partial\mho$ and represent an action of ${}^\Theta T_p\mho$ on each fibre. Moreover

LEMMA 1.7. *The fibre of* $\Theta\mathrm{f}\left[\mho^2;B,S_2\right]$ *over* $(p,p) \in B$ *is isomorphic to the model space* $S_+N\{\widetilde{\mho};\widetilde{p},\widetilde{S}\}$, *with the isomorphism intertwining the actions of* ${}^\Theta T_p\mho$.

In fact we can extend the lifted action of vector fields on the front face to an action of the enveloping algebra of $\mathcal{V}_\Theta$. The restriction to the front face defines a new normal operator $N(P)$ for $P \in \operatorname{Diff}^m_\Theta(\mho)$. In light of Lemma 1.7 it is immediate that this definition of the normal operator agrees with the definition given earlier.

CHAPTER 2

The Θ-calculus

The Θ-calculus is defined in terms of the singularities of the Schwartz kernels of operators when pulled back to $\mho^2_\Theta$. If A is a linear operator from $\dot{\mathcal{C}}^\infty(\mho)$ to $\mathcal{C}^{-\infty}(\mho)$ then it has a Schwartz kernel $k_A \in \mathcal{C}^{-\infty}(\mho^2)$. We let $\kappa_A = {\beta^{(2)}_\Theta}^* k_A$ denote its pullback to $\mho^2_\Theta$. This distribution has its singular support contained in $\Theta\mathrm{l} \cup \Theta\mathrm{r} \cup \Theta\mathrm{f} \cup \Delta_\Theta$. The operator A belongs to the Θ-calculus provided that κ_A has a standard Kohn-Nirenberg type singularity along Δ_Θ, and is polyhomogeneous conormal along the $\partial\mho^2_\Theta$. The 'order filtration' of such operators is given by a pair $m, \mathcal{E}$ where m is real number corresponding to the symbol class along the lifted diagonal and $\mathcal{E}$ is an index family describing the powers appearing in the asymptotic expansions along the various boundary hypersurfaces. Our convention is to represent

$$\mathcal{E} = \{E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}\}$$

where the components on the right hand side are index sets. For definitions of these concepts see §B2. The notation for these classes of operators is

$$A \in \Psi^{m;\mathcal{E}}_\Theta(\mho).$$

There are several special cases that warrant a separate discussion. It is often the case that the kernel of a Θ-operator is actually smooth up to the front face, aside from the singularity along the diagonal. In this situation we leave off the last index set from $\mathcal{E}$. A subalgebra of this calculus is defined by operators whose Schwartz kernels actually vanish to infinite order at $\Theta\mathrm{l}$ and $\Theta\mathrm{r}$. This is call the 'small Θ-calculus' and its order filtrations are denote by $\Psi^m_\Theta(\mho)$, $m \in \mathbb{R}$. There are four symbol maps defined for Θ-operators. The first is an immediate generalization of the classical symbol and describes the singularity along the diagonal. It is denoted by ${}^\Theta\sigma_m(A)$ and is a polynomial function defined on ${}^\Theta T^*\mho$. There are left and right indicial operators denoted by ${}^\Theta\sigma_L(A)$ and ${}^\Theta\sigma_R(A)$ respectively; these are homogeneous functions defined on ${}_+N(\Theta\mathrm{l})$ and ${}_+N(\Theta\mathrm{r})$ respectively. For differential Θ-operators it is traditional to denote the indicial operator by $I(P)$. A differential operator can be represented as a polynomial in a normalized basis like that introduced in (1.3),

$$\mathcal{P} = P(p; \rho\partial_\rho, T, X_2, , \dots, X_n, Y_2, , \dots, Y_n).$$

The indicial operator is then the polynomial in $\rho\partial_\rho$ given by

$$I_p(\mathcal{P}) = P(p; \rho\partial_\rho, 0, 0, \dots, 0).$$

In doing so one implicitly identifies ρ with a linear fiber variable for the b-normal bundle of the boundary.

Finally for operators whose kernels are smooth up to the front face we can extend the definition of the normal operator. It is denoted $N(A)$. Each fiber of

$\Theta\mathrm{f}\,\big[\mho^2; B, S_2\big]$ has a natural 'identity', the unique point where it meets the lifted diagonal Δ_Θ. This along with the Lie algebra of vector fields defined by the normal operators of elements in $\mathcal{V}_\Theta(\mho)$ defines a group structure on each fiber. For each $p \in \partial\mho$, $N_p(A)$ defines a convolution operator relative to this group structure on the fiber of $\Theta\mathrm{f}$ lying over (p,p). Thus we can think of $N_p(A)$ as a distribution defined on the fiber with polyhomogeneous conormal singularities along the identity in the fiber and the intersection of the fiber with $\Theta\mathrm{l} \cup \Theta\mathrm{r}$. The later subset is simply the boundary of the fiber. The distributions $N_p(A)$ then vary smoothly as p varies over $\partial\mho$.

A very important fact about normal operators is that they have a composition formula. If $P \in \mathrm{Diff}^m_\Theta$ and $A \in \Psi^{m';\{E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}\}}_\Theta(\mho)$ then one easily shows that

$$P \circ A \in \Psi^{m+m';\{E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}\}}_\Theta(\mho)$$

and it is also true that

$$N(P \circ A) = N(P)N(A).[3.111]$$

Since $N(P)$ is tangent to the fibers of $\Theta\mathrm{f}$ one can interpret the right hand side of (2) as the differential operator $N_p(P)$ acting in the usual way on the distribution $N_p(A)$.

More generally one can ask when two Θ-operators have a composition which is again a Θ-operator. This question is treated in considerable detail in [**5**]. The method used to establish such results is outlined in the Introduction and is carried through in great detail for several other calculi in Parts II and III of this monograph. For the moment we content ourselves with stating a few representative 'composition formulæ ' for the Θ-calculus

The first is for the composition between a differential Θ-operator and a arbitrary Θ-operator with kernel smooth up to the front face. Following our the normalization introduced in [**5**] we work with operators acting on half densities. The half density bundle is denoted by $\Omega^{\frac{1}{2}}$.

THEOREM 2.1. *Suppose that* $A \in \Psi^{m;E_{\Theta\mathrm{l}},E_{\Theta\mathrm{r}}}_\Theta(\mho;\Omega^{\frac{1}{2}})$ *and* $P \in \mathrm{Diff}^{m'}_\Theta(\mho;\Omega^{\frac{1}{2}})$ *then*

$$P \cdot A \in \Psi^{m+m';E_{\Theta\mathrm{l}},E_{\Theta\mathrm{r}}}_\Theta(\mho;\Omega^{\frac{1}{2}})[3.113] \tag{2.1}$$

$${}^\Theta\sigma_{m+m'}(P \cdot A) = {}^\Theta\sigma_{m'}(P) \cdot {}^\Theta\sigma_m(A), [3.114] \tag{2.2}$$

moreover

$$N_p(P \circ A) = N_p(P) \cdot N_p(A), [3.115]$$

here $N_p(P)$ *acts as a differential operator tangent to the fibres of the front face and*

$${}^\Theta\sigma_{\Theta\mathrm{l}}(P \cdot A) = I(P) \cdot {}^\Theta\sigma_{\Theta\mathrm{l}}(A), [3.116]$$

here $I(P)$ *acts as a differential operator on the fibers of the b-normal bundle to* $\Theta\mathrm{l}$.

To consideration more general compositions it is useful to have a basic mapping result for Θ-operators. The simplest such results describes the action of Θ-operators on spaces of polyhomogeneous conormal distributions. For a manifold with boundary $\mho$ and E an index set for $\partial\mho$ we let $\mathcal{A}^E_{\mathrm{phg}}(\mho,\Omega^{\frac{1}{2}})$ denote the space of distribution half densities polyhomogeneously conormal relative to $\partial\mho$ with index set E.

PROPOSITION 2.1. *For any index family $\mathcal{E}$ for $\mho^2_\Theta$ and index set E_I for $\mho$ an operator $A \in \Psi^{m;\mathcal{E}}_\Theta(\mho; \Omega^{\frac{1}{2}})$ defines a bounded map*

$$A : \mathcal{A}^{E_I}(\mho, \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}^{E_O}(\mho, \Omega^{\frac{1}{2}}),$$

provided that

$$E_I + E_{\Theta\mathrm{r}} > -1 \text{ and } E_O = E_{\Theta\mathrm{l}} \overline{\cup} (E_{\Theta\mathrm{f}} + E_I).$$

The general composition result is as follows. Note that not every pair of Θ-operators has a well defined composition; the sufficient condition follows from Proposition 2.1 and is given in (2.2).

THEOREM 2.2. *If $E = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$ and $E' = E'_{\Theta\mathrm{l}}, E'_{\Theta\mathrm{r}}, E'_{\Theta\mathrm{f}}$ are index families for $\mho^2_\Theta$ such that*

$$E_{\Theta\mathrm{r}} + E'_{\Theta\mathrm{l}} > -1 [3.118]$$

then for any $A \in \Psi^{m;E}_\Theta(\mho; \Omega^{\frac{1}{2}})$ and $B \in \Psi^{m';E'}_\Theta(\mho; \Omega^{\frac{1}{2}})$ the composite operator $A \cdot B$ is well defined and

$$A \circ B \in \Psi^{m+m';E''}_\Theta(\mho; \Omega^{\frac{1}{2}}) \text{ where } E'' = E''_{\Theta\mathrm{l}}, E''_{\Theta\mathrm{r}}, E''_{\Theta\mathrm{f}},$$
$$E''_{\Theta\mathrm{l}} = E_{\Theta\mathrm{l}} \overline{\cup} [E_{\Theta\mathrm{f}} + E'_{\Theta\mathrm{l}}], E''_{\Theta\mathrm{r}} = [E'_{\Theta\mathrm{f}} + E_{\Theta\mathrm{r}}] \overline{\cup} E'_{\Theta\mathrm{r}}, E''_{\Theta\mathrm{f}} = [E'_{\Theta\mathrm{f}} + E_{\Theta\mathrm{f}}] \overline{\cup} [E'_{\Theta\mathrm{r}} + E_{\Theta\mathrm{l}} + N]. [3.119]$$

The diagonal symbol of the composition is given by

$$^\Theta\sigma_{m+m'}(A \cdot B) = {}^\Theta\sigma_m(A) \cdot {}^\Theta\sigma_{m'}(B); [3.120]$$

the composition on the right hand side is simply pointwise product of polynomials.

The binary operations on index sets $+$ and $\overline{\cup}$ are defined in ???????.

We can now give a short explanation of the central role played by the 'model problem.' Suppose that the resolvent operator, $R(\lambda)$ for the $P \in \mathrm{Diff}^m_\Theta$ belongs to $\Psi^{-m;\mathcal{E}}_\Theta$, and has a kernel which is smooth up to the front face. Then the normal operator $N(R)$ is defined. From the operator equation

$$(P - \lambda) \circ R(\lambda) = \mathrm{Id}$$

we deduce, applying (2.1), that

$$(N_p(P) - \lambda) \cdot N_p(R(\lambda)) = N_p(\mathrm{Id}). [3.121]$$

The normal operator of the identity is simply a δ-function concentrated at p. Thus (2) tells us that the normal operator of the resolvent is simply a fundamental solution for the left invariant operator $N(P) - \lambda$. If we can solve this equation then we can determine, a priori the boundary value of the resolvent kernel at the front face. If P is the Laplace operator defined by a bergman type metric on a strictly pseudoconvex domain then Lemma 1.6 implies that there is only one model operator and it is the Laplace operator on the unit ball relative the standard Bergman metric. In the next section we analyze this problem in detail for the case of $n, 1$-forms.

We conclude this section with a discussion of the 'residual part' of the Θ-calculus. These are operators whose kernels are smooth in the interior of $\mho$ and already polyhomogeneous conormal on the product space $\mho^2$. In other words they do need to be lifted to $\mho^2_\Theta$ in order to be polyhomogeneous conormal. We denote these spaces of operators by $\Psi^{\mathcal{E}}(\mho)$ where $\mathcal{E} = \{E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}\}$ is an index family for $\mho^2$. These space of operators are left and right ideals in the full Θ-calculus.

THEOREM 2.3. *If* $A \in \Psi^{\mathcal{E}}(\mho)$ *and* $B \in \Psi_{\Theta}^{m;\mathcal{E}'}(\mho)$ *provided*

$$E_{\Theta\mathrm{r}} + E'_{\Theta\mathrm{l}} > -1$$

the composition $A \circ B$ *is defined and satisfies*

$$A \circ B \in \Psi^{\mathcal{E}''}(\mho) \text{ where}$$
$$E''_{\Theta\mathrm{l}} = ?????, E''_{\Theta\mathrm{r}} = ??????.$$

If

$$E'_{\Theta\mathrm{r}} + E_{\Theta\mathrm{l}} > -1$$

then $B \circ A$ *is defined and satisfies*

$$B \circ A \in \Psi^{\mathcal{E}''} \text{ where}$$
$$E''_{\Theta\mathrm{l}} = ?????, E''_{\Theta\mathrm{r}} = ??????.$$

*****more should go here but I'm not sure what*********

CHAPTER 3

Bergman Laplacian on the ball

The unit ball is an Hermitian symmetric space with the group $SU(n,1)$ acting as the group of isometries of the Bergman metric. Let

$$E^{p,q}(x,y;s) \in \mathcal{C}^{-\infty}(\mathbb{CB}^n \times \mathbb{CB}^n; \Lambda^{q,p} \times (\Lambda^{p,q})^*)[4.1]$$

be the kernel of the resolvent of $\Delta^{\mathrm{B}}_{p,q}$, the Bergman Laplacian acting on (p,q)–forms. Since the inner product is Hermitian the conjugate bundle is taken in the left factor and the operator acts according to the identity

$$E^{p,q} \cdot f = \overline{< E^{p,q}, f >}.[4.2]$$

The diagonal action of $SU(n,1)$ on $\mathbb{CB}^n \times \mathbb{CB}^n$ lifts to an action on these bundle coefficients. If $A \in \mathrm{SU}(n,1)$ we denote this action by $A^\flat$. Since the Laplace operator commutes with the group action it follows that

$$A^\flat \cdot E^{p,q}(s) = E^{p,q}(s) \quad \forall\ A \in SU(n,1).[4.3]$$

Furthermore as the group action is transitive is suffices to construct a fundamental solution with pole at a given point and the correct asymptotic behaviour at the boundary.

To find the solution we shall use the ball model. We are reduced to solving the equation:

$$(\Delta^B_{q,p} - \mu)E^{p,q}(x) = \delta_{p,q}(x), [4.4]$$

where $\delta_{p,q}(x) \in \mathcal{C}^{-\infty}(X, \Lambda^{q,p})$ is defined by

$$\delta_{p,q}(x) \cdot \omega = \omega(x)$$

and Δ^B is the invariant Bergman Laplacian. The stabilizer of the point 0 in $SU(n,1)$ is $SU(n)$. According to (3) the fundamental solution $E^{p,q} \in \mathcal{C}^{-\infty}(\mathbb{CB}^n; \Lambda^{q,p} \otimes (\Lambda^{p,q}_0)^*)$ is invariant under this action.

Using the coordinate basis $\{dz^1, \dots, dz^n, d\bar{z}^1, \dots d\bar{z}^n\}$ we can identify the action of $SU(n)$ with the linear representation of this group on $\Lambda^{q,p}\mathbb{C}^n \otimes \Lambda^{p,q}_0\mathbb{C}^n$. The metric is used to identify the dual of the form bundle with the bundle itself. Since the right bundle factor is simply a vector space we may think of $E^{p,q}(x)$ as a vector of (q,p)–forms. If $\rho_{p,q}(A)$ denotes the representation of $SU(n)$ on $\Lambda^{p,q}\mathbb{C}^n$ then

$$A^* \cdot E^{p,q} = \rho_{p,q}(A^{-1}) \cdot E^{p,q}.[4.5]$$

On the left hand side of (3) the (q,p)–form components of $E^{p,q}$ are pulled back whereas on the right hand side the group element acts linearly on the vector of forms. We shall henceforth call vector valued (q,p)–forms which satisfy (3) radial (q,p)–forms.

From (3) it follows that the problem of finding the fundamental solution reduces to solving a system of ordinary differential equations along a geodesic ray starting at 0. The number of independent equations depends upon first splitting $\Lambda^{q,p}(\mathbb{CB}^n)$ into parts normal and tangential to the spheres centered at 0. The subgroup $SU(n-1) \subset SU(n)$ which stabilizes the ray through zero acts on the tangential components. The number of independent equations is given by the number of irreducible components in these actions.

As we are most interested in the special case of $(n,1)$–forms we will not pursue the general case here. This analysis is carried out in Appendix F. The Hodge star operator and complex conjugation give a unitary equivalence between $\Delta_{n,1}$ and $\Delta_{n-1,0}$. We shall construct the resolvent kernel for the latter operator as the algebra is a little simpler.

Consider the basis

$$\widetilde{dz^i} = (-1)^{i-1}(dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n)[4.6]$$

of $\Lambda^{n-1,0}$, in terms of which we have

$$E^{n-1,0} = \sum_{i,j=1}^{n} e_{ij}\widetilde{d\bar{z}^i} \otimes \widetilde{dw^j}.$$

It follows from the equivariance, (3), that $E(A \cdot x) = \overline{A}E(x)A^t$ and hence

$$e_{ij}(z) = \frac{\bar{z}^i z^j}{|z|^2}(e_n(|z|^2) - e_t(|z|^2)) + \delta_{ij}e_t(|z|^2).[4.7]$$

An elementary calculation shows that the L^2-norm and Dirichlet form are given by

$$\|e\|^2 = \int_0^1 \left[\frac{\|e_n(r)\|^2}{(1-r)^2} + \frac{(n-1)\|e_t(r)\|^2}{(1-r)}\right] r^{n-1}\,dr,$$

$$D(e,e) = \int_0^1 \big[r(\|e_n'(r)\|^2 + (n-1)(1-r)\|e_t'(r)\|^2)^{[4.8]}$$

$$+ \frac{(n-1)\|e_n(r) - e_t(r)\|^2}{r(1-r)}\big]r^{n-1}\,dr,$$

where $r = \|z\|^2$. The system of equations satisfied by (e_n, e_t) follows easily from (3):

$$\begin{aligned} r(1-r)^2e_n'' + n(1-r)^2e_n' - \frac{(n-1)(1-r)}{r}(e_n - e_t) + \mu e_n &= \delta_0 \\ r(1-r)^2e_t'' + n(1-r)^2e_t' - r(1-r)e_t' - \frac{(e_t - e_n)}{r} + \mu e_t &= \delta_0. \end{aligned} \quad [4.9]$$

This system is real and self-adjoint, as follows from (3). Moreover if we set $e_n = (1-r)^{\frac{1}{2}}f$ in $D(e,e)$ and use the identity

$$-\int_0^1 r^n\frac{d}{dr}f^2dr = \int_0^1 nf^2dr[4.10]$$

we find that

$$D(e,e) \geq \int_0^1 \{\frac{1}{4}rf^2 + \frac{1}{2}n(1-r)f^2 + (n-1)[(1-r)^{\frac{1}{2}}f - e_t]^2\}r^{n-1}\frac{dr}{1-r} \geq \frac{1}{4}\|e\|^2, \quad [4.11]$$

where the last inequality follows by point-wise estimation of the quadratic form.

The resolvent kernel is determined as that solution to (3) which is square-integrable near $r = 1$ and has the correct singularity at $r = 0$. The behaviour of solutions at these two regular singular points is determined by the respective indicial systems. The indicial roots at $r = 0$ are independent of μ, they are $-n, 1-n, 0, 1$. To study the analytic dependence of the resolvent on the eigenvalue it is convenient to introduce a parameter which simultaneously uniformizes all the indicial roots at $r = 1$. For each eigenvalue, μ there are two 'tangential' roots γ_t^-, γ_t^+ and two 'normal' roots, γ_n^-, γ_n^+.

The indicial roots are uniformized by a parameter $\tau \in \mathbb{C} \setminus \{0\}$ where

$$\mu = 1 - \frac{3}{16}\left(\frac{1}{\tau} + \tau\right)^2 \Longrightarrow \gamma_t^{\pm} = \pm\frac{\sqrt{3}}{4}\left(\frac{1}{\tau} + \tau\right),\ \gamma_n^{\pm} = \frac{1}{2} \pm \frac{\sqrt{3}}{4}\left(\frac{1}{\tau} - \tau\right). \quad [4.12]$$

The 'physical region' of the parameter space is

$$\mathrm{Ph} = \{\tau \in \mathbb{C}; \mathrm{Re}\,\tau > 0, 0 < |\tau| < 1\} \quad [4.13]$$

in which μ is single valued and in fact maps precisely onto the resolvent set

$$\mu\ \mathrm{Ph} \longleftrightarrow \mathrm{res}\{\varDelta_{n-1,0}\} = \mathbb{C} \setminus [\frac{1}{4}, \infty), \quad [4.14]$$

see fig$\dot{1}$. To see this we construct appropriate solutions to (3). Notice from (3) that

$$\mathrm{Re}\,\gamma_t^+ \geq \mathrm{Re}\,\gamma_t^- \Longleftrightarrow \mathrm{Re}\,\tau \geq 0 \quad [4.15] \tag{3.1}$$

$$\mathrm{Re}\,\gamma_n^+ \geq \mathrm{Re}\,\gamma_n^- \Longleftrightarrow |\tau| \leq 1 \quad [4.16] \tag{3.2}$$

and that the physical region, which is the intersection of the interiors of the regions in (3.1) and (3.2) is exactly the region in which both γ_t^+ and γ_n^+ correspond to (formal) solutions near $r = 1$ which are square-integrable.

The equations in (3) are analytic and have regular singular points at zero and one. We can construct two formal solutions near $r = 1$, fixed uniquely by the properties

$$\begin{aligned} u^1(r;\tau) &\sim (1-r)^{\gamma_n^+(\tau)} \sum_{j=0}^{\infty} u_j^1(\tau)(1-r)^j,\ u_0^1(\tau) = \begin{pmatrix} (1-\sqrt{3}\tau)(\tau+\sqrt{3}) \\ 4\tau \end{pmatrix} \\ u^2(r;\tau) &\sim (1-r)^{\gamma_t^+(\tau)} \sum_{j=0}^{\infty} u_j^2(\tau)(1-r)^j,\ u_0^2(\tau) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad [4.17]$$

and which have coefficients which are meromorphic functions of τ in $\mathbb{C} \setminus \{0\} \cup A$ where A is the set where 'accidental multiplicities' occur. Thus

$$\begin{aligned} A &= A_t \cup A_n \cup A_{t,n} \cup A_{n,t} \cup \{\frac{1}{\sqrt{3}}\} \\ A_t &= \{\tau \in \mathbb{C} \setminus \{0\}; \gamma_t^- \in \gamma_t^+ + \mathbb{N}\} \subset \{\operatorname{Re} \tau \le 0\} \\ A_n &= \{\tau \in \mathbb{C} \setminus \{0\}; \gamma_n^- \in \gamma_n^+ + \mathbb{N}\} \subset \{|\tau| \ge 1\} \\ A_{t,n} &= \{\tau \in \mathbb{C} \setminus \{0\}; \gamma_n^+ \in \gamma_t^+ + \mathbb{N}\} = \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}\mathbb{N} \\ A_{n,t} &= \{\tau \in \mathbb{C} \setminus \{0\}; \gamma_t^+ \in \gamma_n^+ + \mathbb{N}\} = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\mathbb{N} \end{aligned} \quad [4.18]$$

where $\mathbb{N} = \{1, 2 \dots\}$. The point $\frac{1}{\sqrt{3}} \in A$ because at this value the tangential and normal indicial roots are equal and the characteristic directions coincide. Clearly A is a discrete subset of $\mathbb{C} \setminus \{0\}$ which only meets Ph, at $\frac{1}{\sqrt{3}}$. As we shall see this is important for the spectral theory.

The standard theory of analytic regular singular points shows that formal solutions, such as those defined in (3), actually converge to true solutions in the maximal regular disk, here of radius one, about the singular point, see [**?**]. Such a solution has an expansion about $r = 0$ with leading term r^{-n} :

$$\begin{aligned} u^1(r, \tau) \sim r^{1-n} \big[(a(\tau) + g(\tau) \log(r)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ + e(\tau) \begin{pmatrix} 1-n \\ 1 \end{pmatrix} \big] + r^{-n} b(\tau) \begin{pmatrix} 1-n \\ 1 \end{pmatrix} + O(\log(r) r^{2-n}), \\ u^2(r, \tau) \sim r^{1-n} \big[(c(\tau) + h(\tau) \log(r)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ + f(\tau) \begin{pmatrix} 1-n \\ 1 \end{pmatrix} \big] + r^{-n} d(\tau) \begin{pmatrix} 1-n \\ 1 \end{pmatrix} + O(\log(r) r^{2-n}). \end{aligned} \quad [4.19]$$

The solution $\widetilde{e} = d(\tau)u^1 - b(\tau)u^2$ has an asymptotic expansion of the form:

$$\widetilde{e} \sim r^{1-n} m(\tau) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(r^{2-n}).$$

From (3) it follows that for

$$\tau \in \text{Ph};\ m(\tau) = 0 \Leftrightarrow \widetilde{e}(\tau) = 0$$

since there can be no square-integrable solutions in this region. That $\widetilde{e}(\tau) \neq 0$ for $\tau \neq \frac{1}{\sqrt{3}}$ is an immediate consequence of the linear independence of $u_0^i(\tau)$, $i = 1, 2$. Thus $1/m(\tau)$ is a meromorphic function in $\mathbb{C} \setminus A \cup \{0\}$.

Set $e = c_n m(\tau)^{-1} \widetilde{e}$; the constant, c_n is chosen to make

$$E^{n-1,0}(x; \tau) = \frac{e_n(\|x\|^2; \tau)}{\|x\|^2} \bar{x}^i \widetilde{d\bar{x}^i} \otimes x^j \widetilde{dw^j} + e_t(\|x\|^2; \tau) \left[\delta_{ij} - \frac{\bar{x}^i x^j}{\|x\|^2} \right] \widetilde{d\bar{x}^i} \otimes \widetilde{dw^j} \quad [4.20]$$

a fundamental solution for $\Delta^B_{0,n-1}$ with pole at $z = 0$.

The absence of square-integrable solutions in the interior of the physical region shows that, as an extendible distribution, $e(r; \tau)$ is meromorphic in $\mathbb{C} \setminus A \cup \{0\}$ and holomorphic in Ph. On the other hand, the coincidence of the characteristic directions leaves open the possibility that $m(\frac{1}{\sqrt{3}})$ vanishes and therefore $\widetilde{e}(\frac{1}{\sqrt{3}}) = 0$

as well. This would prevent $e(r;\tau)$ from being holomorphic at $\frac{1}{\sqrt{3}}$ as a supported distribution. A simple family displaying the same behaviour is

$$g(r;\tau) = \frac{(1-r)^{\tau} - (1-r)^{-\tau}}{\tau}.$$

As a supported distribution on $[0,1]$, $\lim_{\tau\to 0} g(r;\tau) = 2\log(1-r)$. The index set of $g(\tau)$ is given by $\{\tau, 0\}\overline{\cup}\{-\tau, 0\}$. Thus, at $\tau = 0$ the multiplicity jumps to 1.

Though $e(r;\tau)$ may not be holomorphic at $\frac{1}{\sqrt{3}}$ as a supported distribution, the Mellin transform provides a simple way to describe its continuity properties at this point and at other points of accidental multiplicity. The general properties of the Mellin transform on spaces of conormal distributions are discussed in [**12**]. Let $\psi(r)$ be a function in $\mathcal{C}^\infty(0,1]$, compactly supported near $r = 1$. Suppose further that $\psi(r) = 1$ in a neighborhood of one. We define the local Mellin transform of the fundamental solution by

$$\widehat{e}(z;\tau) = \int_0^1 \psi(r)e(r;\tau)(1-r)^{-z}\frac{dr}{1-r}.[4.21]$$

As is easily shown, a different choice of ψ leads to a Mellin transform which differs from $\widehat{e}$ by an entire function.

Since $e(r;\tau)$ is analytic in Ph as an extendible distribution, the Mellin transform is jointly holomorphic for $\{(z,\tau); \tau \in \text{Ph}, \operatorname{Re} z > M\}$ for some sufficiently large value of M. On the other hand, given $K > 0$,

$$\widehat{e}_K(z;\tau) = \widehat{e}(z;\tau) \prod_{0\le j,k\le} (\gamma_t^+(\tau) + j - z)(\gamma_n^+(\tau) + k - z)[4.22]$$

is holomorphic for

$$0 < |\tau - \frac{1}{\sqrt{3}}| < 1, \quad \operatorname{Re} z > 2 - K.[4.23]$$

To show that $\widehat{e}_K(z;\tau)$ extends holomorphically to $\tau = \frac{1}{\sqrt{3}}$ we use a Hartog extension lemma:

LEMMA 3.1. *Let $f(w,z)$ be holomorphic in the domain*

$$D = \{(w,z); |w| < 1K < \operatorname{Re} z\} \setminus \{(0,z);\ K < \operatorname{Re} z \le L\},$$

for $K < L$, then $f(w,z)$ has a holomorphic extension to

$$\{(w,z);\ |w| < 1, K < \operatorname{Re} z\}.$$

PROOF. One simply expresses $f(w,z)$ using the Cauchy integral formula. By making different choices of contour in the z-plane one obtains the desired holomorphic extension. □

As K is arbitrary it follows that $e(r;\frac{1}{\sqrt{3}})$ is polyhomogeneous conormal with index set

$$\{\gamma_t^+(\frac{1}{\sqrt{3}}), 0\}\overline{\cup}\{\gamma_n^+(\frac{1}{\sqrt{3}}), 0\}.$$

At other points of accidental multiplicity, where $e(r;\tau)$ does not have a pole, a similar discussion applies.

In addition to giving the exact index set at $\tau = \frac{1}{\sqrt{3}}$, this discussion also gives a precise description of the analyticity of the fundamental solution at points of accidental multiplicity. While the asymptotic expansion exhibits rather discontinuous behavior at such points, the Mellin transform has a simple, uniform description.

It follows directly that the resolvent set is given by (3). The spectrum of the operator corresponds to the boundary of the resolvent set. From (3) and (3)

$$\varrho(\Delta_{n,1}) = \varrho(\Delta_{n-1,0}) = \mathbb{C} \setminus [\frac{1}{4}, \infty).[4.25]$$

Setting

$$E^{n,1}(x;\tau) = \overline{{}^{*}E^{n-1,0}(x;\tau)}$$

gives a fundamental solution for $\Delta^{\mathrm{B}}_{n,1}$, the resolvent, $E^{n,1}(x,y;\tau)$ is constructed using (3). Recalling the notation for index sets of conormal distributions from [**5**] we have proved:

THEOREM 3.1. *For each $\tau \in \mathrm{Ph}$ there is a unique solution, $e(r;\tau)$, of (3) with index set $\{\gamma_t^+(\tau), 0\}\overline{\cup}\{\gamma_n^+(\tau), 0\}$ at $r = 1$ and an asymptotic expansion at $r = 0$ of the form:*

$$e(r;\tau) \sim c_n\, r^{1-n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(r^{2-n}).[4.27]$$

As a supported distribution, this solution is an analytic function of τ in $\mathrm{Ph} \setminus \left\{ \frac{1}{\sqrt{3}} \right\}$ and has a meromorphic extension to the complement of A in the punctured plane. As an extendible distribution, $e(r;\tau)$ is holomorphic in the whole resolvent set.

As in [**5**] we need to introduce the 'square root' differential structure at $\partial\mathbb{CB}^n$ giving the new manifold (still diffeomorphic to $\mathbb{CB}^n$) which we denote by

$$Z^n = [\mathbb{CB}^n; \partial\mathbb{CB}^n, N^*\partial\mathbb{CB}^n]\,.[4.28]$$

Next we lift the resolvent kernel to the blown-up product space $[Z^n]^2_\Theta$, which we denote as Z^{2n}_Θ. Finally to conform with our chosen normalization we add in density factors to obtain a kernel which acts on half-density sections of ${}^{\Theta}\Lambda^{n,1}$ i.e. the sections of the tensor product of this bundle with the half-density bundle. It follows from Theorem 3.1 and the construction of the resolvent kernel that if

$$E^{n,1}_\Theta(\tau) = \beta^{(2)^*}_\Theta E^{n,1}(\tau) \text{ and } \mathcal{I}_\tau = \{2\gamma_t^+(\tau) + n, 0\}\overline{\cup}\{2\gamma_n^+(\tau) + n - 1, 0\}[4.29]$$

then

$$E^{n,1}_\Theta(\tau) \in \Psi^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau,0}_\Theta(Z^n; {}^{\Theta}\Lambda^{n,1}, {}^{\Theta}\Lambda^{n,1}).[4.30]$$

The resolvent kernel for the action on singular, half-density sections of ${}^{\Theta}\Lambda^{n,1}$ is given by

$$\widehat{E}^{n,1}(\tau) = E^{n,1}(\tau)(dV^L_{\mathrm{B}} dV^R_{\mathrm{B}})^{\frac{1}{2}}.$$

To obtain a kernel for the action on nonsingular half density sections we need to conjugate by appropriate powers of a defining function for $\partial\mathbb{CB}^n_{\frac{1}{2}}$:

$$\widetilde{E}^{n,1}(\tau) = \rho_L^{\frac{1}{2}N} \widehat{E}^{n,1}(\tau) \rho_R^{-\frac{1}{2}N}, \quad N = 2n + 1.[4.31]$$

Lifting $\widetilde{E}^{n,1}$ to the stretched product we obtain

$$\beta^{(2)^*}_\Theta \widetilde{E}(\tau) = \widetilde{E}^{n,1}_\Theta(\tau)\nu, \quad \nu \in \mathcal{C}^\infty(\left[\mathbb{CB}^n_{\frac{1}{2}}\right]^2_\Theta; \rho_{\mathrm{ff}}^{-\frac{1}{2}N}\Omega^{\frac{1}{2}}).[4.32]$$

Using the arguments presented in §13 in [**5**] we conclude

Proposition 3.1. *The resolvent family for $\Delta^{\mathrm{B}}_{n,1}$ acting on half-density sections extends to a meromorphic family of* Θ*–pseudodifferential operators*

$$\widetilde{E}^{n,1}_{\Theta}(\tau) \in \Psi^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau-N,0}_{\Theta}(\mathbb{CB}^n_{\frac{1}{2}}; {}^{\Theta}\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})$$

in the punctured plane $\mathbb{C} \setminus A \cup \{0\}$.

As is clear from the case of $(n,1)$–forms, the construction of the resolvent kernel for $\Delta^{\mathrm{B}}_{p,q}$, in the resolvent set, does not actually require an exact determination of the 'radial' system of ordinary differential operators. Rather it is only necessary to observe that this system has regular singular points at $r = 0$ and $r = 1$ and to determine the indicial roots at these points. The roots at $r = 0$ are independent of the eigenvalue (and of the metric), the roots at 1 are easily determined by working in the hyperquadric model. This analysis is carried out, in detail, in §F2.

One also needs to find the lower bound on the spectrum as in (3); this has already been done by Donnelly and Fefferman [**?**]. The characterization of the resolvent kernel as a Θ–pseudodifferential operator on Z^n follows from this data and the group invariance of the kernel. Thus, one can generalize (3.1) to arbitrary values of p and q although the case $p+q=n$ is distinctly different from the others since there is an isolated eigenvalue of infinite multiplicity at 0. The discussion of §6, corresponding to $p = n$, $q = 0$ can then be extended to the general case of $p+q=n$. For general p, q the spectral theory has more structure than that arising from the Hodge theory. The Laplace operator commutes with multiplication by the Kähler form. In algebraic geometry, this analysis leads to the Hard Lefschetz Theorem. In general the Dolbeault complex splits into 'primitive' summands which are invariant subspaces for the Laplace operator. This is discussed in §F2.

******We need some discussion of the notions of analyticity for families of polyhomogeneous conormal dist.******

CHAPTER 4

Resolvent of the Laplacian

In this section we show that the analytic continuation of the resolvent family of $\Delta_{n,1}$ defined by a metric of the form (1.1) on a strictly pseudoconvex manifold, Ω, is a meromorphic family of Θ–pseudodifferential operators. As the construction of a parametrix is very similar to that given in the scalar case examined in [**5**] many of the details are left to the reader.

THEOREM 4.1. *Let Ω be a strictly pseudoconvex manifold and let g be an Hermitian metric on Ω which is of the form* (1.1) *in a neighbourhood of $\partial\Omega$. If $\mho = \Omega_{\frac{1}{2}}$ is the domain with the square root differential structure at the boundary then the resolvent family of ${}^g\Delta_{n,1}$ satisfies:*

$$\begin{aligned}({}^g\Delta_{n,1} - \mu(\tau))^{-1} \in \Psi_{\Theta}^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau - N}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) \\ + \Psi^{-\infty;\mathcal{I}_\tau,\mathcal{I}_\tau - N}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})\end{aligned} \qquad [5.2]$$

where the eigenvalue, $\mu(\tau)$, is given by (3) *and the index set, $\mathcal{I}_\tau$, is defined in* (3) *with $N = 2n+1$. As a supported distribution on $\mho^2_\Theta$, the resolvent is an analytic function for $\tau \in \mathrm{Ph} \setminus \left\{\frac{1}{\sqrt{3}}\right\}$ as defined in* (3) *except possibly for a finite number of real poles of finite multiplicity.*

REMARK 4.1. *The 'Laplacian' is normalized by*

$${}^g\Delta = {}^\Theta\overline{\partial}^{*}\,{}^\Theta\overline{\partial} + {}^\Theta\overline{\partial}\,{}^\Theta\overline{\partial}^{*}.$$

PROOF. The proof is constructive. Using an iterative procedure we construct a parametrix which is a distribution of the type described in (4.1) with the desired analyticity properties in τ. The error term is of the form $(\mathrm{Id} - E(\tau))$ where $E(\tau)$ is analytic in τ, in an appropriate sense, and a compact operator on a weighted L^2 space. Using analytic Fredholm theory we then complete the construction of a parametrix which is analytic as a family of conormal, though not necessarily polyhomogeneous, distributions. In the set $\mathrm{Ph} \setminus \{\frac{1}{\sqrt{3}}\}$ we can obtain analyticity as polyhomogeneous conormal distributions. Arguing separately one can show that the the kernel is polyhomogeneous conormal at the exceptional point. Using the Mellin transform and Lemma 3.1 we compute the index set at this point and obtain a precise continuity statement. In the sequel we let $\varrho(\Delta_{n1})$ denote the L^2 resolvent set of $\Delta_{n,1}$ in the τ-parameter.

The first step in the construction is to obtain a kernel with the correct singularity along the diagonal and the correct normal operator along the front face. This is easily accomplished using (D.2), (D.2) and Proposition 3.1 . We then apply (D.3) to remove the Taylor series along the left boundary component. As all these operations are formal the resultant kernel, $P_0(\tau)$, is analytic as a family of conormal distributions in the same set in which $\widetilde{E}^{n,1}_{\Theta}(\tau)$ is. It is moreover analytic as a family of polyhomogeneous conormal distributions for $\tau \in \mathrm{Ph} \setminus \{\frac{1}{\sqrt{3}}\}$. We have

$$\begin{aligned} P_0(\tau) &\in \Psi_{\Theta}^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}, {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}), \\ E_0(\tau) = \mathrm{Id} - (\Delta_{n,1} - \mu(\tau))P_0(\tau) &\in \Psi_{\Theta}^{-\infty;\emptyset,\mathcal{I}_\tau - N,1}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}, {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}). \end{aligned} \tag{5.4}$$

We now apply the general composition formula as described in (D.3) to show that the Neumann series for $(\mathrm{Id} - E_0(\tau))^{-1}$ is asymptotically summable at the front face

$$E_0^k(\tau) \in \Psi_{\Theta}^{-\infty;\emptyset,\mathcal{I}_{\tau,k} - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}, {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}). \tag{5.5}$$

The index set $\mathcal{I}_{\tau,k}$ is defined by

$$\mathcal{I}_{\tau,k} = \begin{cases} \mathcal{I}_\tau,\ k=1 \\ \{\mathcal{I}_{\tau,k-1}+1\}\overline{\cup}\mathcal{I}_\tau. \end{cases} \tag{5.6}$$

This sequence of index sets has a well defined limit as $k \to \infty$, which is an index set; denote it by $\mathcal{I}_{\tau,\infty}$.

For the moment it is more convenient to think of the Neumann series as a formal sum as we will wish to reorganize it in a moment. That is

$$\begin{gathered} (\mathrm{Id} + E_0(\tau))^{-1} \sim \sum_{k=0}^{\infty} E_0^k(\tau) \\ E_0^k(\tau) \in \Psi_{\Theta}^{-\infty;\emptyset,\mathcal{I}_{\tau,k} - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}, {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}). \end{gathered} \tag{5.7}$$

This series may of course be summed leading to a kernel of the form $(\mathrm{Id} + E_1(\tau))$. As a family of conormal distributions it depends analytically on $\tau \in \mathrm{Ph}$. It also satisfies:

$$E_1(\tau) \in \Psi_{\Theta}^{-\infty;\emptyset,\mathcal{I}_{\tau,\infty} - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}, {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}).$$

Composing the summed kernel with $P_0(\tau)$ we obtain $P_2(\tau)$ which satisfies:

$$\begin{aligned} P_2(\tau) &\in \Psi_{\Theta}^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}) + \Psi_{\Theta}^{-\infty;\emptyset,\mathcal{I}_{\tau,\infty} - N,\mathcal{J}_{\tau,\infty}}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}) \\ E_2'(\tau) = (\Delta_{n,1} - \mu(\tau))P_2(\tau) - \mathrm{Id} &\in \Psi_{\Theta}^{-\infty;\emptyset,\mathcal{I}_{\tau,\infty} - N,\emptyset}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}, {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}). \end{aligned} \tag{5.8}$$

Let $P''(\tau)$ denote the second term in (4). Here the index set at the front face $\mathcal{J}_{\tau,\infty}$ is the limit, as $k \to \infty$ of

$$\mathcal{J}_{\tau,k} = \{1,0\}\overline{\cup}\mathcal{I}_{\tau,k}.$$

The kernel, $P_2(\tau)$, in (4) is a parametrix, which depends analytically on $\tau \in \mathrm{Ph}$ as a family of conormal distributions and analytically as a family of polyhomogeneous conormal distributions in $\varrho(\Delta_{n,1}) \cap A^{\complement}$. The error term $E_2(\tau)$ is an analytic family of compact operators on a weighted L^2 space. Using a standard argument, one can show that for τ in a neighborhood of any point where $(\Delta_{n,1} - \mu(\tau))^{-1}$ exists, $P_2(\tau)$ can be replaced by a kernel in the same symbol class for which the corresponding error term, $(\mathrm{Id} + E_2(\tau))$, is invertible. Let $P_3(\tau) = P_2(\tau)(\mathrm{Id} + E_2(\tau))^{-1}$. Then $P_3(\tau)$ is analytic in the resolvent set as a family of conormal distributions and

$$\begin{gathered} P_3(\tau) \in \Psi_{\Theta}^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}) + \Psi_{\Theta}^{-\infty;\mathcal{I}_\tau,\mathcal{I}_{\tau,\infty} - N,\mathcal{J}_{\tau,\infty}}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}) + \\ \Psi^{-\infty;\mathcal{I}_\tau,\mathcal{I}_{\tau,\infty} - N}(\mho; {}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac12}). \end{gathered} \tag{5.9}$$

This, in a sense, completes the construction of the resolvent. We have established that the resolvent is analytic as a family of conormal distributions in $\varrho(\Delta_{n,1})$ and as a family of polyhomogeneous conormal distributions in $\varrho(\Delta_{n,1}) \cap A^{\complement}$ with a meromorphic extension to the complement of A. However (4) is not optimal in several ways. First the index set is a great deal more complicated than it need be. Secondly, the kernel has more refined continuity properties than simply as a family of conormal distributions. The sense in which the kernel is 'analytic' at points of accidental multiplicity can be described in several different ways. For latter applications we show that the Mellin transform is an analytic function of τ with values in meromorphic functions defined on $\mathbb{C}$.

By using the Lemma 3.1 , as in the study of the model problem, and the regularity already established we are reduced to explicitly studying the kernel on $A^{\complement}$. To do so we return to (4). The first task is to remove the complicated index set which arose at the front face. In light of (4) the composition, $P_0(\tau)(\operatorname{Id} + E_0(\tau))^{-1}$, has an expansion near the front face of the form

$$P''(\tau) \sim \sum_{j,k=0}^{\infty} \left[a_{jk}(\tau) \rho_{\Theta\mathrm{f}}^{j+k+\gamma_t^+(\tau)} \rho_{\Theta\mathrm{r}}^{k+\gamma_t^+(\tau)} + b_{jk}(\tau) \rho_{\Theta\mathrm{f}}^{j+k+\gamma_n^+(\tau)} \rho_{\Theta\mathrm{r}}^{k+\gamma_n^+(\tau)} (\log \rho_{\Theta\mathrm{f}})^k \right]. \quad [5.10]$$

The coefficients a_{jk}, b_{jk} are polyhomogeneous conormal distributions on Θf with index sets uniformly bounded below.

By setting $\rho_{\Theta\mathrm{f}} \rho_{\Theta\mathrm{r}} = \rho_r$ we can re-sum the series to to obtain $P''(\tau)$ as a sum of two terms

$$P''(\tau) = P_1''(\tau) + \log(\rho_r) P_2''(\tau), \quad [5.11]$$

The symbol class of the first term is

$$P_1''(\tau) \in \Psi_\Theta^{-\infty;\emptyset,\mathcal{I}'_{\tau,\infty}-N,1}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).$$

Here $\mathcal{I}'_{\tau,\infty}$ is a slightly more complicated set than $\mathcal{I}_{\tau,\infty}$. It has same leading term as $\mathcal{I}_{\tau,\infty}$; beyond this, its exact description is not important. Applying the operator one easily determines that

$$(\Delta_{n1} - \mu(\tau))(\log(\rho_r) P_2''(\tau)) = \log(\rho_r)(\Delta_{n,1} - \mu(\tau)) P_2''(\tau)$$

and therefore this term vanishes to infinite order at the front face. As such it can be dropped from the parametrix without changing the error term at the front face. Let $P_2(\tau)$ denote this new parametrix,

$$\begin{aligned} P_2(\tau) \in {} & \Psi_\Theta^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau-N}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) + \\ & \Psi_\Theta^{-\infty;\emptyset,\mathcal{I}'_{\tau,\infty}-N}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) \end{aligned} \quad [5.12]$$

$$(\Delta_{n1}(\tau) - \mu(\tau)) P_2(\tau) - \operatorname{Id} = E_2''(\tau) \in \Psi_\Theta^{-\infty;\emptyset,\mathcal{I}'_{\tau,\infty}-N,\emptyset}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).$$

At this point we can construct the inverse in the resolvent set using the analytic Fredholm argument described above. The result we denote by $P_3(\tau) = P_2(\tau)(\operatorname{Id} + E_2''(\tau))^{-1}$. It belongs to the symbol class

$$\begin{aligned} P_3''(\tau) \in {} & \Psi_\Theta^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau-N}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) + \\ & \Psi_\Theta^{-\infty;\mathcal{I}_\tau,\mathcal{I}'_{\tau,\infty}-N,1}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) + \quad [5.13] \\ & \Psi^{-\infty;\mathcal{I}_\tau,\mathcal{I}'_{\tau,\infty}-N}(\mho; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}, {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}). \end{aligned}$$

Of course the two kernels agree on their common domain of definition. The self adjointness of $\Delta_{n,1}$ on L^2 implies that the resolvent kernel must also be symmetric. This implies that the index set at the right boundary must actually be $\mathcal{I}_\tau - N$. Thus the log-terms in the second and third summands in (4) must cancel identically. A priori, this might not be possible maintaining a splitting as in (4). However since the index set at the front face arising from the third term is disjoint from that arising from the second term, the log terms, from each summand, must vanish separately, to infinite order at this boundary component. From these considerations it is apparent that the kernel actually satisfies

$$P_3(\tau) \in \Psi_\Theta^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau-N}(\mho;{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})+\Psi^{-\infty;\mathcal{I}_\tau,\mathcal{I}_\tau-N}(\mho;{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}}),\ \tau\in\varrho(\Delta_{n,1})\cap A^{\complement}. \tag{5.14}$$

This completes the refinement of the class on $A^{\complement}$. As with the model case, the kernel does not generally extend to be analytic as a polyhomogeneous conormal distribution onto A even though, wherever it is defined, it is polyhomogeneous conormal on this set as well. For latter applications we need to refine the index set and continuity properties at $\tau = \frac{1}{\sqrt{3}}$. The results of Donnelly and Fefferman imply that this point is in the resolvent set.

We use the Mellin transform as in the study of the model kernel. The analysis is complicated by the fact that $\mho^2_\Theta$ has more than one bounding hypersurface. This difficulty is easily circumvented because we already know that $P_3(\frac{1}{\sqrt{3}})$ is a polyhomogeneous conormal distribution. Thus we can restrict our attention to the interior of the boundary components and the a priori regularity takes care of the behavior of the kernel in the corners.

Let ψ be a smooth function with support meeting $\partial\mho^2_\Theta$ in a relatively compact subset of either Θf or Θr. Let $\widehat{P}_\psi(\tau;z)$ be the Mellin transform of $\psi P_3(\tau)$. It takes values in a space of smooth functions on $\operatorname{supp}\psi\cap\partial\mho^2_\Theta$. By construction, it is jointly holomorphic in (τ,z) for τ in the resolvent set and $\operatorname{Re} z$ sufficiently large. As in the model case the function

$$\widehat{P}_\psi(\tau;z)\prod_{0\le j,k\le K}(2\gamma_t^+(\tau)-n-1+j-z)(2\gamma_n^+(\tau)-n-2+k-z),$$
$$\text{if } \operatorname{supp}\psi\cap\partial\mho^2_\Theta\subset\Theta\mathrm{r}$$
$$\widehat{P}_\psi(\tau;z)\prod_{0\le j,k,l,m\le K}(2(2\gamma_t^+(\tau)+n)+j-z)(2(2\gamma_n^+(\tau)+n-1)+k-z)\times$$
$$(2\gamma_t^+(\tau)+2\gamma_n^+(\tau)+2n-1+l-z)(m-z),$$
$$\text{if } \operatorname{supp}\psi\cap\partial\mho^2_\Theta\subset\Theta\mathrm{f}$$

has an analytic extension to the half plane $\operatorname{Re} z > K-2$ for $0<|\tau-\frac{1}{\sqrt{3}}|<1$. Thus, by Lemma 3.1 , it actually continues holomorphically to $\tau=\frac{1}{\sqrt{3}}$. Using the freedom in the choice of ψ, K and the polyhomogeneous conormality of the kernel at the exceptional point we obtain that

$$\begin{gathered}P_3(\frac{1}{\sqrt{3}})\in\Psi_\Theta^{-2;\{n+1,1\},\{-n,1\},E_{\Theta\mathrm{f}}}(\mho;{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}}),\\ \text{where } E_{\Theta\mathrm{f}}=\{(0,0),(2(n+1),1),(2n+3,2),(2n+4,3)\}.\end{gathered}\tag{5.15}$$

Moreover the Mellin transform at $\tau=\frac{1}{\sqrt{3}}$ is the limit of the Mellin transforms at nearby points. For subsequent applications this observation is actually more important than (4). ☐

For later reference we record:

COROLLARY 4.1. *If $\Omega \subset \mathbb{C}^n$ is a smoothly bounded strictly pseudoconvex domain then the Laplacian, Δ_r, of a metric* (1.1) *acting on* $(n,1)$*–forms is invertible and:*

$$\Delta_{n,1}^{-1} \in \Psi_{\Theta}^{-2;\{n+1,1\},\{-n,1\},E_{\Theta\mathrm{f}}}(\mho;{}^{\Theta}\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) \quad [5.17]$$

The Mellin transforms on the interiors of the boundary faces of Δ_{n1}^{-1} are the limits of the Mellin transforms for nearby values of the spectral parameter.

In the next section we will use the resolvent kernel on $n,1$-forms to study the solution operator for the $\overline{\partial}$-Neumann problem. As also happens in the more traditional approach, we do not require all of the resolvent kernel for $n,1$-forms to understand this problem. Retaining the full kernel leads to the conclusion that the Bergman projector might have higher powers of the $\log(\rho_{\Theta\mathrm{f}})$ in its asymptotic at the front face. This, of course is ruled out by well known results of Fefferman and Boutet de Monvel and Sjöstrand. In light of this fact we observe that by using the Hodge theory for the Dolbeault complex we can split the resolvent into a 'tangential' and a 'normal' piece. Each such piece has simpler analyticity properties than the whole resolvent. By using the Hodge decomposition,

$$L^2(\Omega;{}^{\Theta}\Lambda^{p,q}) = {}^{\Theta}\overline{\partial}\,\mathrm{Dom}_{p,q-1}({}^{\Theta}\overline{\partial}) \oplus {}^{\Theta}\overline{\partial}^*\,\mathrm{Dom}_{p,q+1}({}^{\Theta}\overline{\partial}^*) \oplus \mathrm{null}\,\Delta_{p,q}.$$

For the case of $n,1$-forms it can be seen that the projectors onto the ranges of ${}^{\Theta}\overline{\partial}$ and ${}^{\Theta}\overline{\partial}^*$ are given by

$$\Pi_{{}^{\Theta}\overline{\partial}{}^{\Theta}\Lambda^{n,0}} = {}^{\Theta}\overline{\partial}{}^{\Theta}\overline{\partial}^*\Delta_{n,1}^{-1}, \quad \Pi_{{}^{\Theta}\overline{\partial}^*{}^{\Theta}\Lambda^{n,2}} = {}^{\Theta}\overline{\partial}^*{}^{\Theta}\overline{\partial}\Delta_{n,1}^{-1}$$

and therefore

$$\begin{aligned}(\Delta_{n,1}-\mu)^{-1}\restriction_{{}^{\Theta}\overline{\partial}^*{}^{\Theta}\Lambda^{n,2}} &= {}^{\Theta}\overline{\partial}^*{}^{\Theta}\overline{\partial}\Delta_{n,1}^{-1}(\Delta_{n,1}-\mu)^{-1} = R_-^{n,1}\\ (\Delta_{n,1}-\mu)^{-1}\restriction_{{}^{\Theta}\overline{\partial}{}^{\Theta}\Lambda^{n,0}} &= {}^{\Theta}\overline{\partial}{}^{\Theta}\overline{\partial}^*\Delta_{n,1}^{-1}(\Delta_{n,1}-\mu)^{-1} = R_+^{n,1}\end{aligned} \quad [5.18]$$

A computation with the indicial operator shows that the leading order tangential terms in $(\Delta_{n,1}-\mu(\tau))^{-1}$ are annihilated by ${}^{\Theta}\overline{\partial}^*$. On the other hand we have

$$\begin{aligned}(\Delta_{n,0}-\mu){}^{\Theta}\overline{\partial}^*(\Delta_{n,1}-\mu)^{-1} &= {}^{\Theta}\overline{\partial}^*(\Delta_{n,1}-\mu)(\Delta_{n,1}-\mu)^{-1}\\ &= {}^{\Theta}\overline{\partial}^*\,\mathrm{Id}\,.\end{aligned} \quad [5.19]$$

Since the leading tangential term is annihilated, it follows from (4) that the complete expansion arising from $\gamma_t^+(\tau)$ must actually be absent in ${}^{\Theta}\overline{\partial}^*(\Delta_{n,1}-\mu(\tau))^{-1}$. This in turn implies that, if we set

$$\mu = \gamma_n^+(1-\gamma_n^+),$$

then $R_-^{n,1}$ has a meromorphic extension as a function of

$$\gamma_n^+ \in \mathbb{C}\setminus A_n.$$

Similar remarks apply to $R_+^{n,1}$; it has an analytic extension as a function of γ_t^+.

4.1. Quotients of $\mathbb{CB}^n$

If the Hermitian manifold Ω, g arises as the quotient of the unit ball by a discrete subgroup of $\mathrm{Aut}(\mathbb{CB}^n)$ then the construction of the resolvent kernel on (p,q)–forms is greatly simplified. This is because we can directly construct a parametrix without any iterative step. In the next section we will see that this also leads to a simplification of the asymptotic expansion of the Bergman kernel.

We suppose that Γ is a convex co-compact subgroup of $\mathrm{Aut}(\mathbb{CB}^n)$ so that $M = \Gamma\backslash\mathbb{CB}^n$ is a compact manifold with boundary. We further suppose that M is endowed with the constant biholomorphic sectional curvature metric. Such a manifold can be covered by a finite collection of open balls $\{U_i; i = 1, \dots, N\}$ on each of which a local isometry, ϕ_i, is defined to a ball contained in $\mathbb{CB}^n$. By shrinking the balls if necessary, we can assume that the sets $U_{ij} = U_i \cap U_j$ are either empty or are topologically balls. Since the maps are local isometries, there are elements $\vartheta_{ij} \in \mathrm{Aut}(\mathbb{CB}^n)$ such that

$$\phi_i \restriction_{U_{ij}} = \vartheta_{ij} \circ \phi_j \restriction_{U_{ij}} \quad \text{if } U_{ij} \neq \emptyset.$$

The sets $Z_i = U_i \times U_i$ define an open cover of a neighborhood of $\Delta \subset M^2$. Let

$$V = \cup_{i=1}^N Z_i.$$

Note that

$$Z_{ij} = Z_i \cap Z_j = (U_i \cap U_j) \times (U_i \cap U_j).$$

The maps $\Phi_i(x,y) = (\phi_i(x), \phi_i(y))$ define local isometries from Z_i into $[\mathbb{CB}^n]^2$. If we let ϑ_{ij} act diagonally on $[\mathbb{CB}^n]^2$, $\vartheta_{ij}(x,y) = (\vartheta_{ij}x, \vartheta_{ij}y)$, then on $Z_{ij} = Z_i \cap Z_j \neq \emptyset$ we have

$$\Phi_j \restriction_{Z_{ij}} = \vartheta_{ij} \circ \Phi_i \restriction_{Z_{ij}} .[q.1]$$

The resolvent kernels $E^{p,q}(x,y;\tau)$ are invariant under the diagonal action of $\mathrm{Aut}(\mathbb{CB}^n)$. In the sequel τ denotes a local uniformizing parameter for the indicial roots, see §F1. We shall omit the p,q from the notation where it should cause no confusion. Thus if we define $P_0'(\tau)$ on U_i by

$$P_0'(\tau) \restriction_{Z_i} = \Phi_i^*(E^{p,q}(\tau)), [q.2]$$

then (4.1) implies that $P_0'(\tau)$ is well defined on the overlaps and therefore on V. From (4.1) it follows that, on the interior of V,

$$(\varDelta_{p,q} - \mu(\tau))P_0'(\tau) - \delta_\Delta^{p,q} = 0.[q.3]$$

Let $\psi \in \mathcal{C}_c^\infty(V)$ such that $\psi = 1$ in a neighborhood of Δ. Setting $P_0(\tau) = \psi P_0'(\tau)$ we obtain

$$\begin{gathered} P_0(\tau) \in \Psi_\Theta^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau - N}(M_{\frac{1}{2}}; {}^\Theta\!\varLambda^{p,q} \otimes \Omega^{\frac{1}{2}}) \\ (\varDelta_{p,q} - \mu(\tau))P_0(\tau) - \mathrm{Id} = R_0(\tau) \in \Psi^{-\infty;\mathcal{I}_\tau+1,\mathcal{I}_\tau - N}(M_{\frac{1}{2}}; {}^\Theta\!\varLambda^{p,q} \otimes \Omega^{\frac{1}{2}}). \end{gathered} [q.4]$$

In addition to being a 'residual' term, $R_0(\tau)$ actually vanishes in a neighborhood of the front face. By solving the indicial equation at Θl we can obtain a kernel $P_1(\tau)$ such that

$$\begin{gathered} P_1(\tau) \in \Psi^{-\infty;\mathcal{I}_\tau+1,\emptyset}(M_{\frac{1}{2}}; {}^\Theta\!\varLambda^{p,q} \otimes \Omega^{\frac{1}{2}}) \\ (\varDelta_{p,q} - \mu(\tau))(P_0(\tau) + P_1(\tau)) - \mathrm{Id} = R_1(\tau) \in \Psi^{-\infty;\emptyset,\mathcal{I}_\tau - N}(M_{\frac{1}{2}}; {}^\Theta\!\varLambda^{p,q} \otimes \Omega^{\frac{1}{2}}). \end{gathered} [q.5]$$

The standard analytic Fredholm argument shows that we can invert a slightly modified version of $(\mathrm{Id} + R_1(\tau))$ in a neighborhood of any point in the resolvent set.

Moreover, $(\mathrm{Id}+R_1(\tau))^{-1} = (\mathrm{Id}+R_2(\tau))$ with $R_2(\tau)$ in the same symbol class as $R_1(\tau)$. Thus the resolvent kernel for a quotient is given by

$$R_\Gamma^{p,q}(\tau) = P_0(\tau) + P_0(\tau)R_2(\tau) + P_1(\tau)R_2(\tau) + P_1(\tau)R_2(\tau).[q.6]$$

As is clear from the construction, each of the terms in (4.1) has the same analyticity properties as the resolvent kernel for the model space. That is to say, off the set of accidental multiplicities each term defines a holomorphic map into a space of polyhomogeneous conormal distributions; at points of accidental multiplicity, where the resolvent for the model problem does not have a pole, the Mellin transforms of each term define holomorphic mappings into spaces of meromorphic functions. The first term in (4.1) belongs to

$$P_0(\tau) \in \Psi_\Theta^{-2;\mathcal{I}_\tau,\mathcal{I}_\tau-N}(M_{\frac{1}{2}}; {}^\Theta\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}}),$$

the other terms are residual belonging to

$$\Psi^{-\infty;\mathcal{I}_\tau,\mathcal{I}_\tau-N}(M_{\frac{1}{2}}; {}^\Theta\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}}).$$

Thus we see that for a quotient of the unit ball the construction of the resolvent requires only trivial iterations, a series resulting form the indicial operator and an absolutely convergent Neumann series. This in turn simplifies the form of the Bergman kernel.

4.2. Mapping properties of the Θ-calculus

The basic mapping result is a statement that certain Θ–pseudodifferential operators define a bounded mapping from $L^2(\mho;\Omega^{\frac{1}{2}})$ to itself and that residual operators are compact. These results are easily extended to mapping results for the Θ-calculus acting between weighted L^2-Sobolev spaces defined by operators in $\mathrm{Diff}^*_\Theta(\mho)$. More interesting mapping results can be obtained using vector fields which do not vanish at the boundary. To prove these results we study the commutation properties of smooth vector fields on $\mho$ and Θ–pseudodifferential operators. One can also give mapping results between Hölder spaces.

We begin with the fundamental L^2-mapping theorem, the proof can be found in §13 of [**5**].

THEOREM 4.2. *If $m \le 0$ and the index sets satisfy*

$$E_{\Theta\mathrm{l}} > -\frac{1}{2},\ E_{\Theta\mathrm{r}} > -\frac{1}{2},\ E_{\Theta\mathrm{f}} \ge 0,$$

then each $A \in \Psi_\Theta^{m;\mathcal{E}}$ defines a bounded operator from $L^2(\mho;\Omega^{\frac{1}{2}})$ to itself; each $A \in \Psi^{-\infty;\{E_{\Theta\mathrm{l}},E_{\Theta\mathrm{r}}\}}(\mho;\Omega^{\frac{1}{2}})$ defines a compact map of $L^2(\mho;\Omega^{\frac{1}{2}})$ to itself.

Using the calculus, this result is easily extended to the $\mathcal{V}_\Theta$-based Sobolev spaces. For real m these are defined by

$$\mathcal{H}^m_\Theta(\mho;\Omega^{\frac{1}{2}}) = \{u \in L^2(\mho;\Omega^{\frac{1}{2}});\quad \Psi^m_\Theta u \in L^2(\mho;\Omega^{\frac{1}{2}})\}\quad m \ge 0$$

$$\mathcal{H}^m_\Theta(\mho;\Omega^{\frac{1}{2}}) =$$

$$\{u \in \mathcal{C}^{-\infty}(\mho;\Omega^{\frac{1}{2}});\quad u = \sum_i P_i u_i, P_i \in \Psi_\Theta^{-m}(\mho;\Omega^{\frac{1}{2}}),\ u_i \in L^2(\mho;\Omega^{\frac{1}{2}})\},\quad m < 0.$$

We add weighting by real powers of a defining function for $\partial\mho$ by setting

$$\rho^s\mathcal{H}^m_\Theta(\mho;\Omega^{\frac{1}{2}}) = \{u \in \mathcal{C}^{-\infty}(\mho;\Omega^{\frac{1}{2}}); \rho^{-s}u \in \mathcal{H}^m_\Theta(\mho;\Omega^{\frac{1}{2}})\},\quad \forall s,m \in \mathbb{R}.$$

Analogous spaces are defined relative to the $\mathcal{C}^\infty$-Lie algebra $\mathcal{V}_b(\mho)$ these are denoted by $\rho^s\mathcal{H}_b^m(\mho;\Omega^{\frac{1}{2}})$.

The mapping results for the $\mathcal{V}_\Theta$-based Sobolev spaces are

THEOREM 4.3. *If r, r', m, s, s' are real numbers and $\mathcal{E}$ is an index family such that*

$$E_{\Theta\mathrm{l}} > s' - \frac{1}{2}, E_{\Theta\mathrm{r}} + s > -\frac{1}{2}, E_{\Theta\mathrm{f}} \geq s' - s, r' \leq r - m[6a.3]$$

then each element of $\Psi_\Theta^{m;\mathcal{E}}(\mho;\Omega^{\frac{1}{2}})$ defines a continuous linear opearator

$$\rho^s\mathcal{H}_\Theta^r(\mho;\Omega^{\frac{1}{2}}) \longrightarrow \rho^{s'}\mathcal{H}_\Theta^{r'}(\mho;\Omega^{\frac{1}{2}}).$$

If the strict inequalities hold

$$E_{\Theta\mathrm{f}} > s' - s, r' < r - m$$

then the map is compact.

Now we consider the mapping results for Θ–pseudodifferential operators acting on $\mathcal{V}_b$-Sobolev spaces and standard Sobolev spaces. Our results here are somewhat less complete than those above. This is simply to shorten our exposition a little as the full results require considerably more discussion. For non-negative integral s the standard Sobolev space of order s is defined by

$${}^sH^s(\mho;\Omega^{\frac{1}{2}}) = \{u \in L^2(\mho;\Omega^{\frac{1}{2}}); Pu \in L^2(\mho;\Omega^{\frac{1}{2}}), \forall P \in \mathrm{Diff}^s(\mho)\}.$$

The spaces for positive real numbers can then be defined by interpolation and for negative values of s by duality. That is

$${}^sH^s(\mho;\Omega^{\frac{1}{2}}) =$$

$$\{u \in \mathcal{C}^{-\infty}(\mho;\Omega^{\frac{1}{2}}); u \text{ extends to define a bounded functional on } {}^sH^{-s}(\mho;\Omega^{\frac{1}{2}})\}, s < 0.$$

The following commutation result suffices to prove the mapping results for the small Θ-calculus. Let $\mathcal{V}$ denote all smooth vector fields on $\mho$

PROPOSITION 4.1. *For any real m we have*

$$\mathcal{V}\Psi_\Theta^m \subset \Psi_\Theta^m\mathcal{V} + \Psi_\Theta^m.[6a.32]$$

PROOF. The inclusion is a local statement which is trivial away from the diagonal in $\mho^2$. Near to Δ_Θ but far from the front face in $\mho_\Theta^2$ the proposition follows using the argument for standard pseudodifferential operators. Thus we are left to consider operators whose Schwartz kernels are supported near to the front face. We therefore introduce local coordinates near a point (p,p) on $\partial\Delta$. Starting with coordinates, $r, t, x_2, \dots, x_n, y_2, \dots, y_n$ on $\mho$ in which Θ is given by

$$\Theta = dt + \sum x_i dy_i - y_i dx_i,$$

local coordinates on $\mho^2$ are then given by $(r, t, x_2, \dots, x_n, y_2, \dots, y_n; r', t', x_2', \dots, x_n', y_2', \dots, y_n')$ and local coordinates on $\mho_\Theta^2$ by

$$R = (r - r')/r, T = (t - t' + \sum x_i(y_i - y_i') - y_i(x_i - x_i'))/(r)^2,$$
$$X_i = (x_i - x_i')/r, Y_i = (y_i - y_i')/r;$$
$$r', t', x_i', y_i', \ i = 2, \dots, n.$$

If $A \in \Psi_\Theta^m(\mho;\Omega^{\frac{1}{2}})$ then its Schwartz kernel on $\mho^2$ is of the form $F_0\mu$ with

$$\mu = |drdtdxdydr'dt'dx'dy'|^{\frac{1}{2}}.$$

It lifts to $\mho^2_\Theta$ to define a distribution of the form

$$F\rho_{\Theta\mathrm{f}}^{-N/2}\nu, [6a.4]$$

where

$$\beta_\Theta^{(2)^*}(F_0) = \rho_{\Theta\mathrm{f}}^{-N} F [6a.5]$$

and ν is a smooth half density on the blown-up space. F is a conormal distribution vanishing to infinite order at $\Theta\mathrm{l} \cup \Theta\mathrm{r}$ with a classical singularity along Δ_Θ. In the local coordinates F depends smoothly on r', t', x', y' has a singularity where $R = T = X_i = Y_i = 0$, and vanishes to infinite order as any of $R, T, X_2, \dots, X_n, Y_2, \dots, Y_n$ approach ∞.

We are interested in comparing $\mathcal{V}$ acting on the lifted kernel of A on the left and, by integration by parts on the right. To that end we compute the left and right lifts of the coordinate vector fields:

$$\begin{gathered}\partial_r \rightsquigarrow \frac{1-R}{r}\partial_R - \frac{1}{r}(X\cdot\partial_X + Y\cdot\partial_Y + 2T\partial_T), \quad \partial_{r'} \rightsquigarrow \partial_{r'} - \frac{1}{r}\partial_R,\\ \partial_t \rightsquigarrow \frac{1}{r^2}\partial_T, \quad \partial_{t'} \rightsquigarrow -\frac{1}{r^2}\partial_T + \partial_{t'},\\ \partial_{x_i} \rightsquigarrow \frac{1}{r}\partial_{X_i} - \frac{y_i'}{r^2}\partial_T, \quad \partial_{x_i'} \rightsquigarrow -\frac{1}{r}\partial_{X_i} + \frac{y_i}{r^2}\partial_T + \partial_{x_i'},\\ \partial_{y_i} \rightsquigarrow \frac{1}{r}\partial_{Y_i} + \frac{x_i'}{r^2}\partial_T, \quad \partial_{y_i'} \rightsquigarrow -\frac{1}{r}\partial_{Y_i} - \frac{x_i}{r^2}\partial_T + \partial_{y_i'}.\end{gathered} [6a.6]$$

Using these formulæ the statement of the proposition is immediate for all the basis vector fields besides ∂_r. For example the kernel of $\partial_{x_i}A$, $A \in \Psi^m_\Theta(\mho, \Omega^{\frac{1}{2}})$ is obtained by replacing F in (4.2) by

$$(\partial_{x_i})_l F = \frac{1}{r}\partial_{X_i}F - \frac{y_i'}{r^2}\partial_T F. [6a.7]$$

Here the subscript $(V)_o$, $o = l, r$ indicates the left or right action, respectively of the vector field V. Using the formulæ in (4.2) we obtain that

$$(\partial_{x_i})_l F = (\partial_{x_i})_r F + (\partial_t)_r(rY_iF) + \partial_{x_i'}F + \partial_{t'}(rY_iF). [6a.8]$$

In the last two terms $\partial_{y_i'}, \partial_{t'}$ are acting as coordinate vector fields in $\mho^2_\Theta$; the kernels rY_iF and F depend smoothly on these parameters. The expression on the right hand side is therefore the kernel of an operator of the form

$$A_1V_1 + A_2V_2 + A_3, A_i \in \Psi^m_\Theta(\mho, \Omega^{\frac{1}{2}}), i = 1, 2, 3, V_1, V_2 \in \mathcal{V}. [6a.9]$$

Similar calculations prove the result for $\partial_t, \partial_{y_i}$. We give the commutation formulæ for latter applications

$$\begin{gathered}(\partial_t)_l F = (\partial_t)_r F + \partial_{t'}F,\\ (\partial_{y_i})_l F = (\partial_{y_i})_r F - (\partial_t)_r(rX_iF) + \partial_{y_i'}F - \partial_{t'}(rX_iF).\end{gathered} [6a.10]$$

All that remains is to consider the action of ∂_r.

The kernel of $\partial_r A$ in the representation given in (4.2) is

$$\frac{1-R}{r}\partial_R + F\frac{N-1}{r}F - \frac{1}{r}(2\partial_T T + \partial_X\cdot X + \partial_Y\cdot Y)F,\ N = \dim\mho + 1. [6a.11]$$

The last term in (4.2) can be treated by the argument given above for the tangential vector fields. Now suppose that B is a Θ operator with Schwartz kernel of the form

$$G_0\mu, \quad \beta_\Theta^{(2)^*}(G_0) = \rho_{\Theta\mathrm{f}}^{-N} G. [6a.12]$$

It follows from (4.2) and integration by parts that the kernel of $B\partial_r$ is given by

$$\frac{1}{r}\partial_R G + \frac{N}{r(1-R)}G + \partial_{r'}G.[6a.13]$$

If we take $G = (1-R)F$, then G is the Schwartz kernel of an operator in $\Psi^m_\Theta(\mho;\Omega^{\frac{1}{2}})$ and

$$(\partial_r)_l F = (\partial_r)_r G + H$$

where H is a sum of terms of the type given in (4.2) and (4.2). □

Before stating the mapping results that follow from Proposition 4.1 we state a commutation result for the boundary terms in the Θ-calculus. If $\mathcal{E} = \{E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}\}$ is an index family for $\mho^2_\Theta$ then it is apparent that

$$\partial_r \Psi^{-\infty;\mathcal{E}}_\Theta \not\subseteq \Psi^{-\infty;\mathcal{E}}_\Theta \mathcal{V} + \Psi^{-\infty;\mathcal{E}}_\Theta,$$

as the desired inclusion already fails away from the front face. A careful examination of (4.2) and (4.2) and the fact that rX_i, rY_i are smooth functions on $\mho^2$ imply that we can commute vector fields tangent to the boundary with operators in $\Psi^{-\infty;\mathcal{E}}_\Theta$ in the sense that

$$\mathcal{V}_b \Psi^{-\infty;\mathcal{E}}_\Theta \subset \Psi^{-\infty;\mathcal{E}}_\Theta \mathcal{V}_b + \Psi^{-\infty;\mathcal{E}}_\Theta.[6a.14]$$

There is no need to check the behavior of $r\partial_r$ as this belongs to $\mathcal{V}_\Theta(\mho)$ and thus is covered by the composition formula, Theorem 2.1.

Proposition 4.2. *If $P \in \mathrm{Diff}^m_b(\mho;\Omega^{\frac{1}{2}})$ and $A \in \Psi^{-\infty;\mathcal{E}}_\Theta$ then there exists operators $P_i \in \mathrm{Diff}^m_b(\mho;\Omega^{\frac{1}{2}})$, $A_i \in \Psi^{-\infty;\mathcal{E}}_\Theta$, $i = 1,\ldots,q$ and an operator $A_{q+1} \in \Psi^{-\infty;\mathcal{E}}_\Theta$ such that*

$$PA = \sum_{i=1}^{q} A_i P_i + A_{q+1}.[6a.16]$$

Since we do not need it in the sequel we omit the detailed comparision of the left and right actions of ∂_r on the full calculus. In applications such as the $\overline{\partial}$-Neumann problem, one uses the $\mathcal{V}_b$-mapping results to obtain regularity in the tangent directions at the boundary and then, using essentially the classical method, employs the equation directly to obtain normal regularity.

From Proposition 4.1 and Theorem 4.2 we deduce the mapping results for the small calculus acting on standard Sobolev spaces.

Theorem 4.4. *If $m \leq 0$ and $s \in \mathbb{R}$ then every $A \in \Psi^m_\Theta(\mho;\Omega^{\frac{1}{2}})$ defines a bounded linear map from ${}^sH^s(\mho;\Omega^{\frac{1}{2}})$ to itself.*

Note that even when $m < 0$ we do not get a improvement in regularity uniformly throughout $\mho$. The mapping results for the $\mathcal{V}_\Theta$-based Sobolev spaces do imply an improvment in the interior regularity. In order to get uniform improvements in regularity one generally needs to assume that the Schwartz kernel vanishes at the front face to some order.

Finally we give the mapping results for the boundary terms acting on $\mathcal{V}_b$-based Sobolev spaces.

Theorem 4.5. *If $\mathcal{E}$ is an index family for $\mho^2_\Theta$ r, m, s, s' are real numbers with $m \leq 0$ and*

$$E_{\Theta\mathrm{l}} > s' - \frac{1}{2},\, E_{\Theta\mathrm{r}} + s > -\frac{1}{2}, E_{\Theta\mathrm{f}} \geq s' - s,$$

then each element of $\Psi_{\Theta}^{-\infty;\mathcal{E}}(\mho;\Omega^{\frac{1}{2}})$ *defines a continuous linear map*

$$A : \rho^{s}\mathcal{H}_b^r(\mho;\Omega^{\frac{1}{2}}) \longrightarrow \rho^{s'}\mathcal{H}_b^r(\mho;\Omega^{\frac{1}{2}}).$$

Proof. The theorem follows immediately from Proposition 4.2 and Proposition 4.2 by observing that the operator

$$\rho_l^{-s'}A\rho_r^{s} \in \Psi_{\Theta}^{-\infty;\mathcal{E}'}(\mho;\Omega^{\frac{1}{2}}) \text{ where}$$
$$E'_{\Theta\mathrm{l}} = E_{\Theta\mathrm{l}} - s',\ E'_{\Theta\mathrm{r}} = E_{\Theta\mathrm{r}} + s,\ E'_{\Theta\mathrm{f}} = E_{\Theta\mathrm{f}} + s - s'.$$

□

CHAPTER 5

The $\overline{\partial}$-Neumann problem and the Bergman kernel

If the canonical bundle of Ω is trivial (which it always is for a domain) then the operator

$$K = K_{-}^{n,1} = {}^{\Theta}\overline{\partial}^{*}\Delta_{n,1}^{-1} \in \Psi_{\Theta}^{-1;\{n+1,1\},\{-n,1\},E_{\Theta\mathrm{f}}}(\mho;{}^{\Theta}\Lambda^{n,1},{}^{\Theta}\Lambda^{n,0}),[6.01]$$

$E_{\Theta\mathrm{f}}$ is defined in (4) can be used to solve the $\overline{\partial}$-Neumann problem. Let ϖ be a non-vanishing holomorphic section of $\Lambda^{n,0}\Omega$. If η is a $\overline{\partial}$-closed $\mathcal{C}^{\infty}$-$(0,1)$–form then $\widetilde{\eta} = \eta \wedge \varpi$ is also $\overline{\partial}$-closed, and conversely. Let ρ be a defining function for $\partial\mho$. Since η and ϖ are smooth it follows easily that

$$\|\widetilde{\eta}\| = O(\rho^{n+2})[6.1]$$

as a section of ${}^{\Theta}\Lambda^{n,1}$. From (5) it follows that for every real r

$$\widetilde{\eta} \in \rho^{n+2}\mathcal{H}_{b}^{r}(\mho;{}^{\Theta}\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})$$

and therefore Theorem 4.5 along with (5) imply that for any $r \in \mathbb{R}$, $\epsilon > 0$,

$$K\widetilde{\eta} \in \rho^{n+3/2-\epsilon}\mathcal{H}_{b}^{r}(\mho;{}^{\Theta}\Lambda^{n,0}\otimes\Omega^{\frac{1}{2}})[6.100]$$

If we set set $K\widetilde{\eta} = \varphi\varpi$ it follows that

$$\varphi \in \rho^{\frac{1}{2}-\epsilon}\mathcal{H}_{b}^{r}(\mho;\Omega^{\frac{1}{2}}).[6.101]$$

A simple application of Hodge theory shows:

$$\overline{\partial}K\widetilde{\eta} = \eta\varpi \text{ and } \overline{\partial}(\varphi\varpi) = (\overline{\partial}\varphi)\varpi.[6.2]$$

and thus φ solves the $\overline{\partial}$-equation

$$\overline{\partial}\varphi = \eta.[6.3]$$

Furthermore $\varphi\varpi$ is orthogonal to the holomorphic $(n,0)$–forms with respect to the natural Hermitian pairing

$$\Lambda^{n,0}\otimes\Lambda^{n,0} \longrightarrow \mathbb{C}.[6.4]$$

Using (5) along with (5) iteratively to obtain normal regularity one can easily show that

$$\phi \in \mathcal{C}^{\infty}(\mho;\Omega^{\frac{1}{2}}).$$

On the other hand one can also use the polyhomogeneous conormal mapping theorem, (D.2), to conclude that

$$\varphi \in \mathcal{A}_{\mathrm{phg}}^{0,E'}(\mho;\Omega^{\frac{1}{2}})$$

where E' is an index set consisting of a finite collection of pairs $\{n,k\}$, $n,k \in \mathbb{N}$. Since the map $\mho \longrightarrow \Omega$ is just a fold it follows that φ is polyhomogeneous conormal on Ω with respect to the boundary. The boundary of Ω is non-characteristic for $\overline{\partial}$ acting on functions so (from the 'ABC' theorem, see[**12**]) it follows that φ is actually smooth up to the boundary, $\varphi \in \mathcal{C}^{\infty}(\Omega)$. Thus we have proved

THEOREM 5.1. *If* $\eta \in \mathcal{C}^\infty(\Omega; \Lambda^{0,1})$ *is* $\overline{\partial}$*-closed then* $\varphi = (\varpi)^{-1}K(\eta\varpi) \in \mathcal{C}^\infty(\Omega)$ *is the solution to*

$$\overline{\partial}\varphi = \eta[6.6]$$

such that $\varphi\varpi$ *is orthogonal to the holomorphic* $(n,0)$*–forms with respect to the canonical pairing.*

*****Richard: these mapping results seem a little feeble to me. Should we put in results for data with finite order of differentiability, Holder regularity, etc.? On the other hand this stuff is pretty tangential to our main concerns. ******

We can also we use the discussion of the Laplacian on $(n,1)$–forms above to investigate the Laplacian on $(n,0)$–forms. We define the operators

$$\begin{aligned} R^n_+(\gamma) &= {}^\Theta\overline{\partial}^*(\varDelta_{n,1} - \gamma(1-\gamma))^{-1}\varDelta^{-1}_{n,1}{}^\Theta\overline{\partial} \\ B &= K^n_0 = \mathrm{Id} - K^\Theta\overline{\partial}. \end{aligned} [6.7]$$

PROPOSITION 5.1. *Under the same hypotheses as Theorem 4.1 the Laplacian on* $(n,0)$*–forms has* 0 *as an isolated eigenvalue of infinite multiplicity and continuous spectrum* $[\frac{1}{4},\infty)$ *and possibly a finite number of eigenvalues of finite multiplicity in* $(0,\frac{1}{4})$*. The orthogonal projection on the null space of* $\varDelta_{n,0}$ *is given by* B *in* (5)*. The operator,* $R^n_+(\gamma)$*, satisfies*

$$R^n_+(\gamma) = (\varDelta_{n,0} - \gamma(1-\gamma))^{-1} - \gamma^{-1}(1-\gamma)^{-1}K^n_0[6.9]$$

It extends from $\operatorname{Re}\gamma > \frac{1}{2}$ *to be meromorphic in the* γ*-plane, in the complement of the set of accidental multiplicities, and takes values in the space*

$$\begin{aligned} R^n_+(\gamma) \in \Psi^{-2;2\gamma+n,2\gamma+n}_\Theta(\mho; {}^\Theta\!\varLambda^{n,0}) +& \\ \Psi^{-\infty;2\gamma+n,2\gamma+n}(\mho; {}^\Theta\!\varLambda^{n,0})& \end{aligned} .[6.10]$$

The operator K^n_0 is the ‘Bergman’ projector, while R^n_+ is the resolvent family on the orthogonal complement of the holomorphic $(n,0)$–forms. It follows easily from the identity

$${}^\Theta\overline{\partial}^*\varDelta_{n,1} = \varDelta_{n,0}{}^\Theta\overline{\partial}^*$$

that

$$(\varDelta_{n,0} - \gamma(1-\gamma))R^n_+(\gamma) = (\mathrm{Id} - K^n_0).$$

The computation of the index family for R^n_+ follows as

$$R^n_+(\gamma) = {}^\Theta\overline{\partial}^* R^{n,1}_+(\gamma)\varDelta^{-1}_{n,1}{}^\Theta\overline{\partial}$$

where $R^{n,1}_+$ is defined in (4).

From (5) we conclude that

$$\overline{\partial}\cdot B(f\varpi) = 0.[6.11]$$

and thus B projects an $(n,0)$–form onto its holomorphic part orthogonally relative to the canonical pairing, (5.1). In light of (3), (5) implies that

$$\partial\kappa_B = 0, [6.12]$$

where κ_B is the kernel of B. Since B is an Hermitian operator (5) implies that

$$\overline{\partial}^t\kappa_B = 0[6.13]$$

as well. From (5)-(5) we immediately conclude that

$$\kappa_B \in \mathcal{C}^\infty(\overset{\circ}{\to} \mho^2_\Theta).[6.14]$$

By combining (4.1) with (5) we deduce that

$$B \in \Psi_{\Theta}^{-\infty;\{n+1,1\},\{-n,1\}}(\mho; {}^{\Theta}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}) + \Psi^{-\infty;\{n+1,1\},\{-n,1\}}(\mho; {}^{\Theta}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}). \quad [6.15]$$

We can obtain a more precise conclusion by considering the limit of

$$K_0^n(\tau) = {}^{\Theta}\overline{\partial}^* R^{n,1}(\tau) {}^{\Theta}\overline{\partial}$$

as $\tau \to \frac{1}{\sqrt{3}}$. The discussion after (4) and (4) shows that, for some $0 < \delta$,

$$0 < |\tau - \frac{1}{\sqrt{3}}| < \delta,$$

$K_0^n(\tau)$ is analytic as a family of polyhomogeneous conormal distributions and

$$K_0^n(\tau) \in \Psi_{\Theta}^{0;2\gamma_n^+(\tau)+n-1,2\gamma_n^+(\tau)-n-2,0}(\mho; {}^{\Theta}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}) + \Psi^{-\infty;2\gamma_n^+(\tau)+n-1,2\gamma_n^+(\tau)-n-2}(\mho; {}^{\Theta}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}) \subset \Psi_{\Theta}^{0;2\gamma_n^+(\tau)+n-1,2\gamma_n^+(\tau)-n-2,\{4\gamma_n^+(\tau)+2(n-1)\}\overline{\cup}\{0\}}(\mho; {}^{\Theta}\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}). \quad [6.16]$$

From (5) and Corollary 4.1 it follows that as $\tau \to \frac{1}{\sqrt{3}}$ the index set of $K_0^n(\tau)$ at the front face tends to

$$E'_{\Theta\mathrm{f}} = \{(0,0),(2n+2,1)\}. [6.17]$$

Thus we obtain

THEOREM 5.2. *The Bergman kernel as defined by* (5) *satisfies*

$$B \in \Psi_{\Theta}^{-\infty;\{n+1,0\},\{-n,0\},E'_{\Theta\mathrm{f}}}(\mho; {}^{\Theta}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}, {}^{\Theta}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}) [6.19]$$

with $E'_{\Theta\mathrm{f}}$ *given by* (5).

The usual Bergman kernel is defined as an orthogonal projection of functions onto holomorphic functions, where orthogonality is with respect to an L^2 structure defined by a nonsingular measure on Ω. If as above the canonical bundle is trivial one can choose a non-vanishing holomorphic section, ϖ and define a Bergman projector on functions by

$$\widetilde{B}(f) = (\varpi)^{-1} B(f\varpi). [6.20]$$

This is clearly an orthogonal projection onto holomorphic functions relative to $L^2(\Omega; \varpi \wedge \bar{\varpi})$.

Boutet de Monvel and Sjöstrand [?] obtained a more precise expansion than (5.2). Their result can be deduced from our description by the use of a regularity theorem for polyhomogeneous conormal distributions which are holomorphic-antiholomorphic on the product, as in (5)-(5). To state the result we need to introduce the 'almost-analytic extension' of a function defined on Ω or on $\partial\Omega$. We identify Ω with the diagonal in Ω^2 and the $\partial\Omega$ with the diagonal in $(\partial\Omega)^2$.

DEFINITION 5.1. *Let f be a smooth function defined on Ω, the almost-analytic extension of f is a function F defined on Ω^2 such that*

$$\begin{aligned} &1) \quad F\restriction_{\Delta} = f \\ &2) \quad \partial_R F \text{ and } \overline{\partial}_L F \text{ vanish to infinite order along } \Delta. \end{aligned} \quad [6.22]$$

If f is real then we can obtain a solution, unique in Taylor series at the diagonal, by requiring $F(p,q) = \bar{F}(q,p)$.

For a function, g defined on $\partial\Omega$, the almost-analytic extension is obtained by first extending g to a function on Ω and applying (5.1) to the extended function. The uniqueness statement then holds along the diagonal in the $\partial\Omega^2$. For details on these constructions see [**?**]. Let ρ be a defining function for $\partial\Omega$ and R its almost-analytic extension. In [**?**] it is shown that R can be chosen to have positive real part. Thus $\log R$ is defined as the principal branch and any power R^z is defined by $R^z = e^{z\log R}$.

Proposition 5.2. *Let G be a distribution,*

$$\begin{aligned} G \in I^{\mathcal{E}}_{\text{phg}}(\mho^2_\Theta) \text{ such that} \\ \overline{\partial}_L G = \partial_R G = 0. \end{aligned} [6.24]$$

Suppose further that $E_{\Theta\mathrm{f}} = (n, m), n \in \mathbb{Z}, m \in \mathbb{N}_0$, then

$$G = \sum_{i=n}^{\infty}\sum_{j=0}^{m}(R)^{\frac{i}{2}}(\log R)^j g_{ij} \text{ where } g_{ij} \in \mathcal{C}^\infty(\Omega^2).[6.25]$$

Proof. The different logarithmic terms do not interact; we give the complete proof only in the simplest case, $m = 0$.

Since the Θ-boundary of $\mho$ is strictly pseudoconvex, $\overline{\partial}$ is transversely elliptic along $\Theta\mathrm{l}$ and similarly ∂ is transversely elliptic along $\Theta\mathrm{r}$. From this we conclude that a holomorphic-antiholomorphic, polyhomogeneous conormal kernel on $\mho^2_\Theta$ is actually smooth at $\Theta\mathrm{l} \cup \Theta\mathrm{r}$ in the standard $\mathcal{C}^\infty$-structure. Thus we can consider K as a kernel on Ω^2_Θ.

From the definition of R it is immediate that $|R|^{\frac{1}{2}}$ lifts to Ω^2_Θ as a defining function for $\Theta\mathrm{f}$. Therefore $F_0 = R^{\frac{-m}{2}}G$ satisfies (5.1) with index set $E_{\Theta\mathrm{f}} = (0, 0)$. We have the following asymptotic expansion at $\Theta\mathrm{f}$:

$$F_0 \sim \sum_{j\geq 0} a_i \rho^i_{\Theta\mathrm{f}}.[6.26]$$

The front face is a fibration over $\partial\mho$, by working in local coordinates one can show that

$$\text{span}\{\pi^*_L({}^\Theta T^{0,1}\Omega), \pi^*_R({}^\Theta T^{1,0}\Omega)\},$$

restricted to the front face, is the complete tangent space to the fiber. Thus (5.1) implies that the function a_0, in (5), is actually constant on each fibre. Therefore $a_0 \restriction_{\Theta\mathrm{f}} = \beta^{2\,*}_\Theta(b_0)$, where b_0 is in $\mathcal{C}^\infty(\partial\Delta)$. Let B_0 denote the almost holomorphic-antiholomorphic extension of b_0 to a neighbourhood of $\partial\Delta$ in Ω^2 and its lift to Ω^2_Θ. The difference $F_0 - B_0$ satisfies(5.1) with $E_{\Theta\mathrm{f}} = (1, 0)$, thus we can apply the above argument to $F_1 = (F - B_0)R^{-\frac{1}{2}}$. Continuing recursively in this fashion we obtain the desired expansion. □

By applying Proposition 5.2 we can refine the expansion in (5.2) to

$$\widetilde{B} = \frac{B_1}{R^{n+1}} + (\log R)B_2, \quad B_1, B_2 \in \mathcal{C}^\infty(\Omega^2).$$

This is the asymptotic expansion obtained in [**?**].

The Szegö kernel is obtained by appropriately scaling and allowing the point of evaluation in the Bergman kernel to tend to $\partial\mho \times \partial\mho \setminus \Theta\mathrm{f}$ in $\mho^2_\Theta$ in a manner very much like that used to construct the Poisson kernel in [**5**]. One can obtain an asymptotic expansion for this kernel along the diagonal using this observation and (5.2).

In case Ω arises as the quotient of the unit ball by a convex, co-compact discrete subgroup we can further refine the description of the Bergman kernel

THEOREM 5.3. *If* $\Gamma \subset \mathrm{Aut}(\mathbb{CB}^n)$ *is convex co-compact subgroup of infinite volume then asymptotic expansion of the Bergman kernel for* $M = \Gamma\backslash\mathbb{CB}^n$, *with the induced constant holomorphic sectional curvature metric, has no log term at the front face.*

PROOF. This follows from (4.1) and (5). The resolvent kernel for $(n,1)$–forms can be written as a sum

$$R_\Gamma^{n,1}(\tau) = P_0(\tau) + P_1'(\tau),\ \lambda = \mu(\tau).[6.28]$$

The kernel $P_0(s)$, is a exact fundamental solution in a neighborhood of the front face. This implies that, in this neighborhood, the tangential indicial roots are absent from each term of

$$R_{+\Gamma}^n(s) = {}^\Theta\overline{\partial}^* R_\Gamma^{n,1}(s){}^\Theta\overline{\partial}, \lambda = s(1-s).$$

Moreover each term is separately holomorphic, in a neighborhood of $s = 1$, as described at the end of section 5. The two terms do not interact and therefore

$$\begin{aligned} R_{+\Gamma}^n(1) \in \Psi_\Theta^{-2;n+2,n+2}(M_{\frac{1}{2}}; {}^\Theta\Lambda^{n,0}, {}^\Theta\Lambda^{n,0}) +& \\ \Psi^{-\infty;n+2,n+2}(M_{\frac{1}{2}}; {}^\Theta\Lambda^{n,0}, {}^\Theta\Lambda^{n,0}).& \end{aligned} \quad [6.29]$$

The theorem follows immediately from (5) $\square$

Part 2

Adiabatic limit of the $\overline{\partial}$-Neumann problem

In this second part the discussion of the $\overline{\partial}$-Neumann problem in terms of the Θ-calculus is extended to a treatment of the adiabatic limit of the problem. This terminology arises from [?] and similar problems for families of Dirac operator and Laplacians have been discussed by Cheeger [?], by Bismut and Cheeger [?], in [?] and by Dai [?]. As described in the Introduction, we consider the special case of pseudoconvex domains arising as tubular neighbourhoods of totally real submanifolds of complex manifolds. By the adiabatic limit we mean the singular limit as the radius of the tube shrinks to zero. The rather explicit representations of the Bergman kernel and the solution operator to the $\overline{\partial}$-Neumann problem given above extend uniformly, in the appropriate sense, in this singular limit. These results are used in Part III to resolve the conjecture of Boutet de Monvel and Guillemin.

In §6 we describe the adiabatic limit for the $\overline{\partial}$-complex around a totally real submanifold of a complex manifold in terms of the a-structure. This captures the adiabatic quality of the limit, without the complications caused by the presence of the boundary. Then in §7 the closely related α-structure which captures the uniformity inherent in a shrinking family of pseudoconvex ('Grauert') tubes is described. The integral leaves of the associated Lie algebra of vector fields, and normal homomorphism, are described in §7.1. This singles out the model problems which need to be solved to invert the Bergman type Laplacian, uniformly in the adiabatic limit.

The geometric part of the construction of the α-calculus is contained in §7.2 and §7.3, in which the double and triple α-products are defined. The α-pseudodifferential operators are defined in §8 and their elementary mapping properties are described. The structure of the triple stretched products is used to obtain composition and symbolic properties leading to the full calculus of α-pseudodifferential operators. The new model problem formulated in §7.1, associated to the adiabatic front face of the stretched product, is analyzed in §8.1 and this analysis is used in §8.2 to show that the inverse of the α-type Bergman Laplacian on $(n,1)$–forms is an element of the α-calculus. This construction gives a detailed asymptotic description of the adiabatic limit. Finally, in §9, this is used to analyze the asymptotic behaviour of the Bergman kernel and the solution to the $\overline{\partial}$-Neumann problem.

CHAPTER 6

Adiabatic limit for $\overline{\partial}$

Let Ω be a complex manifold of (complex) dimension n. Consider an embedded, compact, totally real submanifold $Y \hookrightarrow \Omega$ of real dimension n. Thus, the tangent bundle of Y sits as a subbundle of $T_Y\Omega$ in such a way that

$$T_xY \oplus (J_x \cdot T_xY) = T_x\Omega \quad \forall\, x \in Y.[7.1]$$

Here, $J_x : T_x\Omega \longrightarrow T_x\Omega$ is the almost-complex structure (i.e. multiplication by i.)

We shall denote by ϵ a real parameter with $0 \le \epsilon \le \epsilon_0$ for some $\epsilon_0 > 0$ and set

$$\begin{gathered}\widetilde{\Omega} = \Omega \times [0,\epsilon_0] \xrightarrow{\pi_\epsilon} [0,\epsilon_0]\\ \widetilde{Y} = Y \times [0,\epsilon_0] \subset \widetilde{\Omega}.\end{gathered}[7.2]$$

Thus $\widetilde{\Omega}$ is a compact manifold with boundary; the boundary has the two components

$$\mathrm{sf}(\widetilde{\Omega}) = \{\epsilon = 0\},\ \mathrm{eb}(\widetilde{\Omega}) = \{\epsilon = \epsilon_0\}.$$

Consider the submanifold of $\mathrm{sf}(\widetilde{\Omega})$

$$Y_0 = Y \times \{0\}.[7.3]$$

Since Y is an embedded submanifold of the boundary Y_0 is a p-submanifold of $\widetilde{\Omega}$. We take the radial blow-up of $\widetilde{\Omega}$ along Y_0 (see [**12**] and particularly §C.2 below for the notation) and consider

$$\Sigma = \left[\widetilde{\Omega}; Y_0\right] \xrightarrow{\beta_0} \widetilde{\Omega}[7.4]$$

where $\beta_0 = \beta\left[\widetilde{\Omega}; Y_0\right]$ is the blow-down map. Let $\mathrm{ab}(\Sigma) = \beta^*(Y_0) = \beta_0^{-1}(Y_0)$ be the front face of Σ, to be thought of below as the adiabatic boundary of Σ. This is a half n-sphere bundle over $Y_0 \equiv Y$ with the fibration given by β_0. The other boundary hypersurface over $\epsilon = 0$ will be denoted $\mathrm{sf}(\Sigma) = \beta_0^*(\partial\widetilde{\Omega})$; by definition this is the closure of $\beta_0^{-1}(\mathrm{sf}(\widetilde{\Omega}) \setminus Y_0)$ in Σ. In the discussion below of the $\overline{\partial}$-Neumann problem we shall only be interested in a relatively small part of Σ. The part of interest does not meet $\mathrm{sf}(\Sigma)$ at all.

On Σ consider the space of vector fields

$$\begin{gathered}\mathcal{V}_a(\Sigma) = \big\{V \in \mathcal{V}_b(\Sigma); V \text{ is tangent to the fibres of}\\ \widetilde{\pi}_\epsilon = \beta_0 \circ \pi_\epsilon : \Sigma \longrightarrow (0,\epsilon_0] \text{ and to the fibres of } \mathrm{ab}(\Sigma)\big\}.\end{gathered}[7.5]$$

Certainly $\mathcal{V}_a(\Sigma)$ is a $\mathcal{C}^\infty(\Sigma)$-module and Lie algebra; it is also locally free:

PROPOSITION 6.1. *The Lie algebra $\mathcal{V}_a(\Sigma)$ fixes a vector bundle, ${}^aT\Sigma$, over Σ with a vector bundle map $\iota_a\ {}^aT\Sigma \longrightarrow {}^bT\Sigma$ such that*

$$\mathcal{C}^\infty(\Sigma; {}^aT\Sigma) = \iota_a^* \mathcal{V}_a(\Sigma)[7.7]$$

and under the composite map $\iota_a \circ (\widetilde{\pi}_\epsilon)_*$ the complex structure on Ω lifts to a complex structure on the fibres of ${}^aT\Sigma$.

PROOF. Away from $\mathrm{ab}(\Sigma)$ the second condition in (6) is void so when the identification of $\Sigma \setminus \mathrm{ab}(\Sigma)$ with $\widetilde{\Omega} \setminus Y_0$ is made, ${}^aT\Sigma$ is naturally identified with $T_\Sigma\Omega$.

Near any point $p \in Y$ holomorphic coordinates $z_1, \dots, z_n$ can be introduced in Ω so that $p = \{z_1 = \cdots = z_n = 0\}$ and locally Y is just $\{y_i = 0, j = 1, \dots, n\}$ where $z_j = x_j + iy_j$ is the decomposition into real and imaginary parts. Thus $x_1, \dots, x_n, y_1, \dots, y_n, \epsilon$ give local coordinates in $\widetilde{\Omega}$ in which

$$Y_0 = \{y_1 = \cdots = y_n = \epsilon = 0\}.[7.8]$$

Near any point of $\mathrm{ab}(\Sigma)$ we can use projective coordinates based on these local coordinates. Thus if p' is a point in the interior of $\mathrm{ab}(\Sigma)$ with $\beta_0(p') = p$ then

$$x_1, \dots, x_n, Y_1 = \frac{y_1}{\epsilon}, \dots, Y_n = \frac{y_n}{\epsilon}, \epsilon[7.9]$$

give local coordinates, with $\mathrm{ab}(\Sigma) = \{\epsilon = 0\}$ and the fibres of $\mathrm{ab}(\Sigma)$ given locally by $x = \mathrm{const}$. Then $\mathcal{V}_a(\Sigma)$ is locally spanned by

$$\epsilon\partial_{x_j}, \partial_{Y_j} \quad j = 1, \dots, n[7.10]$$

as follows immediately from (6).

Similarly at a boundary point of $\mathrm{ab}(\Sigma)$ one of the y_j, which we relabel as y_1, can be used as 'radial' variable. Thus

$$x_1, \dots, x_n, \rho = y_1, Y_2 = \frac{y_2}{y_1}, \dots, Y_n = \frac{y_n}{y_1}, \eta = \frac{\epsilon}{y_1}[7.11]$$

can be used as local coordinates. The space of vector fields tangent to the boundaries and to the fibres of $\mathrm{ab}(\Sigma)$ is spanned by

$$\rho\partial_{x_j},\ j = 1, \dots, n,\ \rho\partial_\rho, \partial_{Y_k},\ k = 2, \dots, n \text{ and } \eta\partial_\eta.[7.12]$$

The subalgebra $\mathcal{V}_a(\Sigma)$ is characterized by the annihilation of $\epsilon = \rho\eta$ so is spanned by

$$\rho\partial_{x_j},\ j = 1, \dots, n,\ \rho\partial_\rho - \eta\partial_\eta, \partial_{Y_k},\ k = 2, \dots, n.[7.13]$$

Together (6) and (6) show that $\mathcal{V}_a(\Sigma)$ has, everywhere, a local basis. This proves the existence of the vector bundle ${}^aT\Sigma$. At $q \in \Sigma$ the fibre ${}^aT_q\Sigma$ is simply the quotient

$${}^aT_q\Sigma = \mathcal{V}_a(\Sigma)/\mathcal{I}_q \cdot \mathcal{V}_a(\Sigma)$$

where $\mathcal{I}_q \subset \mathcal{C}^\infty(\Sigma)$ is the ideal of functions vanishing at q.

Consider the dual bundle ${}^aT^*\Sigma$. In the same local coordinates (6), (6) we have the local (dual) bases:

$$\frac{dx_j}{\epsilon},\ dY_j \quad j = 1, \dots, n[7.14] \tag{6.1}$$

$$\frac{dx_j}{y_1},\ j = 1, \dots, n,\ \frac{d\rho}{\rho},\ dY_k, \quad k = 2, \dots, n[7.15] \tag{6.2}$$

since $d\epsilon \equiv 0$ as a linear form on ${}^aT\Sigma$. In the respective coordinate systems

$$\frac{1}{\epsilon}dz_j \equiv \frac{dx_j}{\epsilon} + idY_j,\ j = 1, \dots, n$$
$$\frac{1}{y_1}dz_k \equiv \frac{dx_k}{\rho} + idY_k + \frac{1}{2}Y_k(\frac{d\rho}{\rho} + \frac{d\eta}{\eta}),\ k = 2, \dots, n, [7.16]$$
$$\frac{dz_1}{\rho} = \frac{dx_1}{\rho} + \frac{1}{2}i(\frac{d\rho}{\rho} + \frac{d\eta}{\eta})$$

have independent real and imaginary parts in ${}^aT^*\Sigma$. This shows that the subspaces of $(1, 0)$ and $(0, 1)$ vectors over $\Sigma \setminus \mathrm{ab}(\Sigma)$ in ${}^aT\Sigma$ extend to be smooth and independent up to $\mathrm{ab}(\Sigma)$. □

In the two types of coordinate systems (6) and (6) we have the following bases of ${}^aT^{0,1}\Sigma$:

$$\epsilon\partial_{x_j} + i\partial_{Y_j},\ j = 1, \dots, n$$
$$\rho\partial_{x_1} + i(\rho\partial_\rho - \eta\partial_\eta) \text{ and } \rho\partial_{x_k} + i\partial_{Y_k},\ k = 2, \dots, n. [7.17]$$

Even though it is not, over $\mathrm{ab}(\Sigma)$, the tangent space of a $2n$-dimensional manifold the complex structure on ${}^aT\Sigma$ induces an associated, degenerate, Dolbeault complex. Let

$$\bigoplus_{p+q=k} {}^a\Lambda^{p,q} = {}^a\Lambda^k\Sigma = \Lambda^k({}^aT^*\Sigma)[7.18]$$

be the decomposition of the exterior powers of the bundle ${}^aT^*\Sigma$, i.e. the a-form bundles, over Σ. Then the complexes

$$\mathcal{C}^\infty(\Sigma; {}^a\Lambda^{q,0}) \xrightarrow{{}^a\overline{\partial}} \mathcal{C}^\infty(\Sigma; {}^a\Lambda^{q,1}) \xrightarrow{{}^a\overline{\partial}} \cdots \xrightarrow{{}^a\overline{\partial}} \mathcal{C}^\infty(\Sigma; {}^a\Lambda^{n,q}) \ [7.19]$$

for $q = 1, \dots, n$, are fixed by the identification, over $\Sigma \setminus \mathrm{ab}(\Sigma)$, with the usual $\overline{\partial}$-complex on the fibres of $\widetilde{\Omega}$, i.e. the Dolbeault complex of Ω. Locally if $\overline{Z_1}, \dots, \overline{Z_n} \in \mathcal{C}^\infty(\Sigma; {}^aT^{0,1})$ is a basis and $\overline{\zeta_1}, \dots, \overline{\zeta_n}$ is the dual basis of $\mathcal{C}^\infty(\Sigma; {}^a\Lambda^{0,1})$ then

$${}^a\overline{\partial}\phi = \sum_{j=1}^n (\overline{Z_j}\phi)\overline{\zeta_j} \in \mathcal{C}^\infty(\Sigma; {}^a\Lambda^{0,1}) \quad \forall\ \phi \in \mathcal{C}^\infty(\Sigma).[7.20]$$

For any Lie algebra of vector fields, $\mathcal{V}$, the part of the enveloping algebra of order m, $\mathcal{D}^m(\mathcal{V})$, is the space of differential operators which can be written everywhere locally as finite sums of up to m-fold products of elements of $\mathcal{V}$, with $\mathcal{C}^\infty$ coefficients. Since the definition is local it extends immediately to fix $\mathcal{D}^m(\mathcal{V}; E, F)$, the space of such differential operators acting from sections of one vector bundle, E, to another, F. For $\mathcal{V} = \mathcal{V}_a(\Sigma)$ we write $\mathcal{D}^m(\mathcal{V}; E, F)$ as $\mathrm{Diff}^m_a(\Sigma; E, F)$. The formula (6) shows that ${}^a\overline{\partial} \in \mathrm{Diff}^1_a(\Sigma; {}^a\Lambda^{0,0}, {}^a\Lambda^{0,1})$. Since ${}^a\overline{\partial}(dz_j) = 0$, using the local bases (6) and (6) we see (respectively) that

$$
\begin{gathered}
{}^a\overline{\partial}(\frac{1}{\epsilon}dz_j) = 0,\ j = 1, \dots, n \\
{}^a\overline{\partial}(\frac{1}{y_1}dz_k) = -\frac{1}{\rho}{}^a\overline{\partial}\rho \wedge \frac{dz_k}{\rho} = -i\frac{d\overline{z_1}}{\rho} \wedge \frac{dz_k}{\rho},\ k = 2, \dots, n, [7.21] \\
{}^a\overline{\partial}(\frac{1}{y_1}dz_1) = 0.
\end{gathered}
$$

The extension of (6) to higher order forms can now be made using the Leibniz formula, showing that

$$
{}^a\overline{\partial} \in \mathrm{Diff}^1_a(\Sigma; {}^a\Lambda^{p,q}, {}^a\Lambda^{p,q+1}) \quad \forall\ p, q.[7.22]
$$

We think of the operator ${}^a\overline{\partial}$ as the giving the adiabatic limit of $\overline{\partial}$. If $\phi \in \mathcal{C}^\infty(\Sigma)$ then

$$
{}^a\overline{\partial}\phi \in \mathcal{C}^\infty(\Sigma; {}^a\Lambda^{0,1})
$$

is *not* in general a smooth form in the ordinary sense. However $\epsilon \cdot {}^a\overline{\partial}\phi$ is smooth up to the front face. Moreover

$$
\iota^*_{\mathrm{ab}(\Sigma)}(\epsilon {}^a\overline{\partial}\phi) = \sum_k \partial_{Y_k}\phi\, dY_k[7.23]
$$

shows that on $\mathrm{ab}(\Sigma)$ $\epsilon {}^a\overline{\partial}\phi$ is just the fiber derivative of ϕ. Thus as $\epsilon \downarrow 0$ the operator ${}^a\overline{\partial}$ captures the degeneration of the complex derivative to a real exterior derivative. It is this singular limit which we study in detail below.

6.1. Almost-analytic continuation

Next we note the relationship between ${}^a\overline{\partial}$ and the almost analytic continuation of functions from Y. The submanifold $\widetilde{Y}$ in (6) lifts under β_0 to a $\mathcal{C}^\infty$ submanifold $\beta_0^*(\widetilde{Y}) \subset \Sigma$ such that $\beta_0 : \beta_0^*(\widetilde{Y}) \longleftrightarrow \widetilde{Y}$ is a diffeomorphism. Thus we can denote the lift by $\widetilde{Y}$ as well.

Lemma 6.1. *If $f \in \mathcal{C}^\infty(\widetilde{Y})$ there exists $u \in \mathcal{C}^\infty(\Sigma)$ such that*

$$
\begin{gathered}
u_{\restriction \widetilde{Y}} = f \\
{}^a\overline{\partial}u \in \mathcal{C}^\infty(\Sigma; {}^a\Lambda^{0,1}) \text{ vanishes to all orders at } \mathrm{ab}(\Sigma).
\end{gathered}
[7.25]
$$

The solution to (6.1) *is unique up to a term vanishing to all orders at* $\mathrm{ab}(\Sigma)$ *and if $v \in \mathcal{C}^\infty(\widetilde{\Omega})$ satisfies*

$$
\begin{gathered}
v_{\restriction \widetilde{Y}} = f \\
\overline{\partial}v \text{ vanishes to all orders at } Y_0 \subset \widetilde{\Omega}
\end{gathered}
[7.26]
$$

then $u = \beta_0^* v$ satisfies (6.1) *and conversely.*

Proof. For a detailed discussion of the almost analytic continuation of a smooth function off a totally real submanifold see [**?**]. Using a sufficiently fine partition of unity it suffices to note that a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$ with support in $|x| < 1$ has an almost analytic continuation to $\mathbb{C}^n$ with support in $|z| < 1$. The conditions in (6.1) just fix the Taylor series of the function at $\mathbb{R}^n$. Summing the

Taylor series using Borel's lemma gives the almost analytic continuation. The addition of the parameter ϵ makes essentially no difference, so that one can actually find a solution to (6.1) with the second condition strengthened to vanishing to all orders everywhere on $\widetilde{Y}$, not just at Y_0. Clearly the lift of the solution to (6.1) gives a solution to (6.1), since $\beta_0^{-1}(Y_0) = \text{ab}(\Sigma)$.

Thus it remains only to prove the uniqueness of the solution to (6.1). Thus suppose that u' satisfies (6.1) with $f = 0$. From (6) it follows that u' must be constant along the fibres of $\text{ab}(\Sigma)$. Since each fibre meets $\widetilde{Y}$ the vanishing of f in the first condition in (6.1) implies that u' vanishes on $\text{ab}(\Sigma)$. Away from the part, $\text{sf}(\Sigma)$, of the boundary of Σ above $\epsilon = 0$ which is the complement of the interior of $\text{ab}(\Sigma)$ we can divide by ϵ (which is a parameter) and repeat the argument. It therefore follows that u' must vanish to infinite order at the interior, and hence the whole of, $\text{ab}(\Sigma)$. A smooth function with this property is necessarily of the form $u' = \beta^* v'$ with $v' \in \mathcal{C}^\infty(\widetilde{\Omega})$ vanishing to all orders at Y_0. This shows the uniqueness and so completes the proof of the lemma. □

CHAPTER 7

The α-structure

The principal topic of interest in this monograph is the combination of the two structures considered so far, i.e. the Θ-structure of §1 and the a-structure of §6. As in §6 let Y be a closed embedded totally real submanifold of a complex manifold Ω, with $\dim_{\mathbb{R}} Y = \dim_{\mathbb{C}} \Omega$. Let $\psi \in \mathcal{C}^\infty(\Omega)$ be a quadratic defining function for Y in the sense that it has a non-degenerate minimum of 0 at Y, i.e.

$$\psi \geq 0,\ Y = \{\psi = 0\},\ \operatorname{Hess}\psi >> 0 \text{ on } NY.[8.1]$$

For example ψ could be, near Y, the distance from Y measured with respect to some Riemannian metric on Ω.

The shrinking tubes of the title of the paper are the strictly pseudoconvex neighbourhoods of Y :

$$X_\epsilon = \{z \in \Omega; \psi(z) \leq \epsilon^2\},\ 0 < \epsilon \leq \epsilon_0.[8.3]$$

In particular we wish to analyze the behaviour, as $\epsilon \downarrow 0$, of the solution to the $\overline{\partial}$-Neumann problem on X_ϵ. The neighbourhoods of Y (7) fix a conical subset of $\widetilde{\Omega}$:

$$\widetilde{X} = \{(z, \epsilon) \in \widetilde{\Omega} = \Omega \times (0, \epsilon_0]; \psi(z) \leq \epsilon^2\}.[8.4]$$

Applying the blow-up construction of §6 to Y_0 in $\widetilde{\Omega}$, we consider the lift of $\widetilde{X}$ to the resulting space Σ :

$$M = \operatorname{cl} \beta_0^{-1}(\widetilde{X}) \subset \Sigma.[8.5]$$

Lemma 7.1. *The lift, M, of $\widetilde{X}$ to Σ is, for $\epsilon_0 > 0$ sufficiently small, a compact $2n+1$ dimensional submanifold, with corners of codimension two, fibering over $[0, \epsilon_0]$ with fibres isomorphic to the normal ball bundle of Y in Ω*

$$\begin{array}{ccc} \mathbb{B}NY & \longrightarrow & M \quad [8.7] \\ & & \downarrow \\ & & [0, \epsilon_0]. \end{array}$$

The fibre over $\epsilon = 0$ is naturally a ball bundle over Y.

Proof. The collar neighbourhood theorem allows us to identify a neighbourhood of $Y \subset \Omega$ with the unit ball bundle in NY. Then (7) shows that ψ has a non-degenerate critical point at 0 on each fibre. If $z_j = x_j + iy_j$, $j = 1, \dots, n$ are local complex coordinates in which Y is given by $y = 0$ the y_j are local coordinates on the fibres, while $x_1 \dots, x_n$ are local coordinates on Y, so

$$\psi(x, y) = \sum_{j,k=1}^{n} a_{jk}(x, y) y_j y_k,\ a_{jk}(x, 0) >> 0.[8.8]$$

The functions ϵ, x_j and $Y_j = y_j/\epsilon$ give local coordinates in Σ as in (6) and in terms of these

$$\tilde{\psi} = \epsilon^{-2}\beta_0^*\psi = \sum_{j,k=1}^{n} a_{jk}(x, \epsilon Y)Y_jY_k[8.9]$$

so that, provided ϵ_0 is small,

$$M = \{\tilde{\psi} \leq 1\} \subset \Sigma[8.10]$$

is a ball bundle as stated. Notice that M is indeed contained in the region of validity of the coordinates (6), i.e. meets the boundary of Σ above $\epsilon = 0$ only in the adiabatic face:

$$M \cap \partial\Sigma = [M \cap \mathrm{ab}(\Sigma)] \cup [M \cap \{\epsilon = \epsilon_0\}]\,.[8.11]$$

The final statement of the lemma, that over $\epsilon = 0$ M has the structure of a ball bundle (unlike the other fibres which are just diffeomorphic to a ball bundle) follows from the definition of blow-up, in that the interior of the front face of Σ is canonically a vector bundle over Y. □

Since Y is totally real it follows from (7) that

$$\partial\overline{\partial}\psi >> 0 \text{ at } Y[8.2]$$

and therefore $-\psi$ is plurisubharmonic near Y. On M we have the restriction of the boundary-fibration structure $\mathcal{V}_a$. Because of (7) we only need to consider, near $\mathrm{ab}(M) = M \cap \mathrm{ff}(\Sigma)$, the coordinate systems (6), not (6). From (6) we see, just as in §1, that the complex structure on ${}^aT_M\Sigma$ is spanned by any n singular forms $\zeta_1, \ldots, \zeta_n$ which are independent linear combinations of those in (6). Naturally we wish to consider the extension down to $\epsilon = 0$ of the contact structure defined in §1. This is a line bundle over the 'Θ boundary' of M :

$$\Theta\mathrm{b}(M) = \mathrm{cl}\,\beta_0^{-1}\{\tilde{\psi} = 1; 0 < \epsilon \leq \epsilon_0\}.[8.12]$$

LEMMA 7.2. *The complex bundle* ${}^aT^{(1,0)}_{\Theta\mathrm{b}(M)}$ *has a line subbundle,* L*, which is the span (over* $\mathbb{R}$*) of those elements which are real when paired with those real elements of* $\mathcal{V}_a$ *which are tangent to* $\Theta\mathrm{b}(M)$*. If* ϕ *is any defining function for* $\Theta\mathrm{b}(M)$ *then* $i^a\partial\phi$ *spans* L*.*

PROOF. Away from the corner of M over $\epsilon = 0$ this follows from the discussion in §1, in particular (**??**) and (1). At a point of the corner, $\Theta\mathrm{b}(M) \cap \mathrm{ff}(\Sigma)$ we see from (6) that if ϕ is a defining function for $\Theta\mathrm{b}(M)$ then the real elements of ${}^aT_pM$ tangent to $\Theta\mathrm{b}(M)$ are the combinations

$$\sum_{j=1}^{n}[a_j\epsilon\partial_{x_j} + b_j\partial_{Y_j}] \text{ s.t. } \sum_{j=1}^{n} b_j\partial_{Y_j}\phi(p) = 0.[8.14]$$

Thus from (6) any element of ${}^aT_p^*M$ which is real when paired with these is a multiple of

$$\sum_{j=1}^{n}\partial_{Y_j}\phi[\frac{dx_j}{\epsilon} + idY_j] = -i^a\partial\phi \text{ at } p.[8.15]$$

This proves the lemma. □

In particular this means that L is spanned by ζ_1 where

$$i\zeta_1 = {}^a\partial\tilde{\psi} = {}^a\partial\phi,\ \phi = \tilde{\psi} - 1.[8.16]$$

Next we apply to M the procedure of §1, i.e. we introduce the square root differential structure along the Θ-boundary by blowing up this boundary parabolically. The naturality of this construction means that we are really applying it fibre by fibre for M as a bundle over $[0, \epsilon_0]$. Setting $G = M_{\frac{1}{2}}$, (7.1) becomes

$$\begin{array}{ccc} \mathbb{B}_{\frac{1}{2}}NY & \text{———} & G \quad [8.17] \\ & & \downarrow \\ & & [0, \epsilon_0]. \end{array}$$

Thus G is again a manifold with corner of codimension two. There are three boundary hypersurfaces, one being $\mathrm{eb}(G) = \{\epsilon = \epsilon_0\}$; the other two will be denoted $\Theta\mathrm{b}(G)$ and $\mathrm{ab}(G)$. By the corner we mean

$$\mathrm{co}(G) = \Theta\mathrm{b}(G) \cap \mathrm{ab}(G)[8.18]$$

even though there is another (uninteresting) corner, *viz* $\mathrm{eb}(G) \cap \Theta\mathrm{b}(G)$. As is evident from (7) and (7) $i\zeta_1$ is singular at $\epsilon = 0$ in directions conormal to the fibres of G over $\widetilde{Y}$.

LEMMA 7.3. *The form (in the ordinary sense)*

$$^a\theta = \epsilon \operatorname{Im} \zeta_1 \ \text{on}\ \Theta\mathrm{b}(M)[8.20]$$

lifts to a non-vanishing form $^a\Theta$ on G over the boundary $\Theta\mathrm{b}(G)$ so defining a Θ-structure on G. The restriction of $^a\Theta$ to each ϵ-fibre is a Θ-structure and, at each point of $\mathrm{co}(G)$, $^a\Theta$ is the lift of a form from the base, Y.

PROOF. We have already observed that the real form $^a\theta$ defined by (7.3) is non-vanishing when restricted to $\Theta\mathrm{b}(M)$, it therefore defines a Θ-structure as claimed. In fact the form $^a\theta$ is in the span of the dx_j at each point of the corner, so is the lift of a form from the base (depending on p). □

We denote the Lie algebra of vector fields arising from this Θ-structure by $\mathcal{V}_\Theta(G)$. The Θ-structure on a ϵ-fibre, $G(\epsilon)$, gives rise to a Lie algebra which is just the restriction to the fibre of the subspace of $\mathcal{V}_\Theta(G)$ consisting of the elements tangent to that fibre, or all fibres. Note that the rank of the exterior derivative $d^a\Theta$ on $\Theta\mathrm{b}(G)$, or equivalently of $d^a\theta$ on $\Theta\mathrm{b}(M)$, drops to 0 at the corner. This means that this Θ-structure on $G(\epsilon)$ degenerates as $\epsilon = 0$; however it is still a Θ-structure in the sense of [**5**].

The Lie algebra $\mathcal{V}_\Theta(G)$ consists of those $\mathcal{C}^\infty$ vector fields on G which pair smoothly with the singular forms

$$\rho_\Theta^{-2}\Theta, \rho_\Theta^{-1}\nu,\ \nu \in \mathcal{C}^\infty(G; \Lambda^1).[8.21]$$

If $p \in G$ let ζ_j for $j = 1, \dots, n$ be a basis for $^aT^*M$ near $\beta_{\frac{1}{2}}(p) \in M$. If $p \in \Theta\mathrm{b}(G)$ we always take $\zeta_1 = {}^a\partial r$ with r a defining function for $\Theta\mathrm{b}(M)$. Now consider the space

$$\mathcal{V}_\alpha = \{V \in \mathcal{V}_b(G); V\epsilon = 0, \beta_{\frac{1}{2}}^*(\overline{\zeta}_1)(V) \in \rho_\Theta^2\mathcal{C}^\infty(G),\ \beta_{\frac{1}{2}}^*(\overline{\zeta}_j)(V) \in \rho_\Theta\mathcal{C}^\infty(G), \forall\, j\}.[8.22]$$

This just consists of the elements of $\mathcal{V}_\Theta(G)$ which are tangent to the ϵ-fibres and at ab(G) to the fibres of ab(G) as a ball bundle. Let us also introduce the bundle with fibre

$$^{\alpha}T_p^{(1,0)}G = \mathrm{sp}\left\{\rho_\Theta^{-2}\beta_{\frac{1}{2}}^*\zeta_1,\ \rho_\Theta^{-1}\beta_{\frac{1}{2}}^*\zeta_k,\ k=2,\dots,n\right\} \text{ at } p\in G[8.23]$$

then directly from the definition, (7),

$$\mathcal{V}_\alpha(G) = \mathcal{C}^\infty(G;{}^{\alpha}TG)[8.24]$$

where $^{\alpha}T$ is simply the dual bundle to (7).

Proposition 7.1. *There is a naturally defined vector bundle map* $\iota_\alpha : {}^{\alpha}T(G) \longrightarrow TG$, *over* $G(\epsilon)$ *for* $\epsilon > 0$ ι_α *extends to an isomorphism of* $^{\alpha}TG$ *and* $^{\Theta}TG(\epsilon)$ *and over* $G \setminus \Theta\mathrm{b}(G)$ *it extends to an isomorphism of* $^{\alpha}TG$ *and* ^{a}TM; *in both cases in the sense of complex bundles.*

Proof. Directly from (7) the bundle $^{\alpha}T^*G$ is naturally isomorphic with the bundles $^{\Theta}T^*G(\epsilon)$ on each $G(\epsilon)$ with $\epsilon > 0$. Similarly when $\rho_\Theta \neq 0$, i.e. away from $\Theta\mathrm{b}(G)$ the definition (7) just reduces to that of ^{a}TM. □

Consider the form of a general element of $\mathcal{V}_\alpha(F)$ near co(G). From (6) and (7) it follows that $\mathcal{V}_\alpha(G)$ has a basis

$$\rho\partial_\rho, \rho\partial_{Y_j}, \epsilon\rho Z_j, \epsilon\rho^2 Z',\ j=1,\dots,n-1,[8.26]$$

where ρ is a defining function for $\Theta\mathrm{b}(G)$ and the Z_j, Z' span a transversal subspace to the fibres of ab(G) in ab(G). The complex structure on $\mathcal{V}_\alpha(G)$ is locally spanned by

$$\epsilon\rho^2 Z' + i\rho\partial_\rho, \epsilon\rho Z_j + i\rho\partial_{Y_j}.[8.27]$$

By combining the discussion in the two earlier cases we conclude directly that G has its own Dolbeault complex, acting along the fibres over $[0,\epsilon_0]$:

$$\mathcal{C}^\infty(G,{}^{\alpha}\Lambda^{p,0}G) \xrightarrow{{}^{\alpha}\overline{\partial}} \mathcal{C}^\infty(G;{}^{\alpha}\Lambda^{p,1}G) \xrightarrow{{}^{\alpha}\overline{\partial}} \cdots \longrightarrow \mathcal{C}^\infty(G;{}^{\alpha}\Lambda^{p,n}G)\ [8.28]$$

for each p.

We remark, without proof, that $\mathcal{V}_\alpha$ is an example of a boundary fibration structure, as defined in [**12**]. We do not make explicit use of this fact, but it acts as a guiding principle. In particular we are immediately led to expect:

Proposition 7.2. *The lift to* G *of the Kähler metric on the fibres of* $\widetilde{X}$ *over* $(0,\epsilon_0]$

$$g_\epsilon = -\beta_{\frac{1}{2}}^*\left[\partial\overline{\partial}\log\left(\psi-\epsilon^2\right)\right]\ [8.30]$$

extends to an Hermitian bundle metric on $^{\alpha}T^{1,0}G$, *smooth and non-degenerate up to all boundaries and the corners.*

Proof. It is only necessary to reappraise the discussion surrounding (**??**), from which the result follows away from ab(G). On M the $(1,1)$–form in (7.2) can be written

$$\frac{{}^{a}\partial\phi}{\phi}\wedge\frac{{}^{a}\overline{\partial}\phi}{\phi} - \frac{{}^{a}\partial{}^{a}\overline{\partial}\phi}{\phi}.[8.31]$$

From (7), (6) and (6) it follows that the numerator in the second term in (7) is a non-degenerate Hermitian bilinear form on ^{a}TM over ab(M). The first term is just $\zeta_1 \wedge \overline{\zeta_1}$, and from (7) the second term is non-degenerate as an Hermitian bilinear form on the annihilator L° so indeed (7) is a smooth non-degenerate bilinear form on ${}^\alpha T^{0,1}G$ as claimed. □

We denote by $\mathrm{Diff}^m_\alpha(G)$ the space of differential operators on G which can be locally written as up to m-fold products of the vector fields in $\mathcal{V}_\alpha$. The fact that $\mathcal{V}_\alpha$ is local, as expressed by (7) means that the spaces $\mathrm{Diff}^m_\alpha(G; U, W)$ are also well-defined for any vector bundles U and W over G. The single most important consequence of Proposition 7.2 and one that always follows for a metric on the structure bundle of a boundary-fibration structure is:

COROLLARY 7.1. *The Laplace-Beltrami operator of the metric* (7.2) *on the* ϵ-*leaves of* $\widetilde{X}$ *is an elliptic element of* $\mathrm{Diff}^2_\alpha(G; {}^\alpha\Lambda^{p,q})$ *for each* p, q.

PROOF. Since g is a Kähler metric the Laplace-Beltrami operator can be written

$$\frac{1}{2}\Delta = {}^\alpha\overline{\partial}^*\,{}^\alpha\overline{\partial} + {}^\alpha\overline{\partial}\,{}^\alpha\overline{\partial}^* = {}^\alpha\partial^*{}^\alpha\partial + {}^\alpha\partial{}^\alpha\partial^*.[8.33]$$

Thus it is only necessary to check that $\mathrm{Diff}^m_\alpha(G)$ is closed under the passage to adjoints to conclude that Δ is an α-differential operator. The ellipticity follows as usual, since the symbol of an α-differential operator can be interpreted as a polynomial on ${}^\alpha T^*G$, with values in the homomorphism bundle. For the Laplacian this is always just the scalar given by the metric as a function (i.e. the length squared). □

7.1. Normal operators

We next describe the normal homomorphisms for an α-structure. This is done in an abstract form following [**12**]. Essentially the same discussion as in §1.3 applies to points on the Θ-boundary of the manifold G except at the corner where it meets the adiabatic face. A similar approach is also used to find a realization of the normal operator at the adiabatic boundary of G. This in turn leads to a construction which applies even at the corner.

Each point $p \in G$ lies in a maximal integral submanifold of the Lie algebra $\mathcal{V}_\alpha(G)$ let Q_p be its closure. Then we have:

LEMMA 7.4. *For the Lie algebra* $\mathcal{V}_\alpha$ *the closures of the maximal integral submanifolds through points are:*

$$Q_p = \{p\} \text{ if } p \in \Theta\mathrm{b}(G)$$
$$Q_p = \text{leaf of } \mathrm{ab}(G) \text{ as a ball bundle over } Y \text{ if } p \in \mathrm{ab}(G) \setminus \mathrm{co}(G)[9.2]$$
$$Q_p = \text{leaf of } (7.1) \text{ if } p \in G \setminus [\Theta\mathrm{b}(G) \cup \mathrm{ab}(G)].$$

One of the requirements of a boundary-fibration structure is that for each fixed boundary face, F, the closures of the integral submanifolds through points in the interior of F should be leaves of a b-fibration of F. In the present case all the b-fibrations are in fact fibrations and we can check this property directly. Starting with interior points of G we see that the closure of the $\mathcal{V}_\alpha$-integral submanifolds through such points are just the ϵ-leaves, i.e. the corresponding fibration is just (7.1); the same is true of the boundary face $\epsilon = \epsilon_0$, the closure of the leaf through any point is the whole boundary face. One the other hand the fibration of $\Theta\mathrm{b}(G)$ is just the point by point fibration (i.e. given by the identity map). Finally the fibration of $\mathrm{ab}(G)$ is clearly as a ball bundle over Y; this arises from the blow-up of $Y \times \{0\}$ in (6).

With each of these leaves, Q, obtained by closing an integral submanifold, is associated a 'restriction algebra'. This arises in the obvious way:

$$\mathcal{U}(Q) = \mathcal{V}_\alpha(G)_{\restriction Q} \subset \mathcal{C}^\infty(Q; {}^bTQ).[9.3]$$

It is easy to determine the structure of these Lie algebras:

LEMMA 7.5. *For the three cases in* (7.4)

$$\mathcal{U}(Q_p) = \{0\},\ p \in \Theta\mathrm{b}(G)$$
$$\mathcal{U}(Q_p) = \mathcal{V}_0(Q_p) = \{V \in \mathcal{C}^\infty(Q_p; TQ_p); V = 0 \ \textit{at}\ \partial Q_p\},\ p \in \mathrm{ab}(G) \setminus \mathrm{co}(G)[9.5]$$
$$\mathcal{U}(Q_p) = \mathcal{V}_\Theta(Q_p),\ p \in G \setminus [\Theta\mathrm{b}(G) \cup \mathrm{ab}(G)].$$

PROOF. Only the middle of these three cases requires any comment. All the elements of $\mathcal{V}_a(G)$ certainly vanish at $\Theta\mathrm{b}(G)$, so it is only necessary to check that there is no further constraint on the elements of $\mathcal{U}(Q)$. This follows from (7). □

Now $\mathcal{U}(Q_p)$ does not capture the full structure of $\mathcal{V}_\alpha$ near Q_p, except for the last case in (7.5). More of the local structure is captured by considering the quotient

$$\mathcal{W}(Q) = \mathcal{V}_\alpha(G)/\mathcal{I}_Q \cdot \mathcal{V}_\alpha(G) = \mathcal{C}^\infty(Q; {}^\alpha T_Q G).[9.6]$$

Here $\mathcal{I}_Q \subset \mathcal{C}^\infty(G)$ is the ideal of functions vanishing on Q so $\mathcal{I}_Q \cdot \mathcal{V}_\alpha(G) \subset \mathcal{V}_\alpha(G)$ is an ideal. Thus $\mathcal{W}(Q)$ is a Lie algebra and it is also a $\mathcal{C}^\infty(Q)$-module. Moreover the restriction map (7.1) factors through (7.1) so induces a restriction homomorphism (of Lie algebras) the null space of which we call the 'normal algebra' at Q :

$$0 \longrightarrow \mathcal{N}(Q) \longrightarrow \mathcal{W}(Q) \longrightarrow \mathcal{U}(Q) \longrightarrow 0.[9.7]$$

In fact there is always a subbundle

$${}^\alpha NQ \subset {}^\alpha T_Q G \text{ s.t. } \mathcal{N}(Q) = \mathcal{C}^\infty(Q; {}^\alpha NQ).[9.8]$$

This is always a Lie algebra bundle, i.e. each fibre ${}^\alpha N_p Q$ has the structure of a finite dimensional Lie algebra, with the structure constants varying smoothly if computed with respect to a smooth local basis. Again it is easy to compute the form of these algebras:

LEMMA 7.6. *If $Q = Q_p$ is the closure of the maximal integral submanifold of the Lie algebra $\mathcal{V}_\alpha$ through p then, corresponding to the three cases in* (7.4)

$${}^\alpha N_p Q = {}^\alpha T_p G \ \textit{if}\ p \in \Theta\mathrm{b}(G)$$
$${}^\alpha N_p Q \subset {}^\alpha T_p G \ \textit{is abelian of dimension}\ \dim Y \ \textit{if}\ p \in \mathrm{ab}(G) \setminus \mathrm{co}(G)[9.10]$$
$${}^\alpha N_p Q = \{0\} \ \textit{if}\ p \in G \setminus [\Theta\mathrm{b}(G) \cup \mathrm{ab}(G)].$$

PROOF. Again the first and last cases are quite clear and the middle one follows from (7). The last two sets of vector fields in (7) span ${}^\alpha N_p Q$ since these are the ones vanishing at Q. In local coordinates the Z_j and Z' can be taken to be vector fields on Y so clearly they induce an abelian Lie algebra structure on ${}^\alpha N_p Q$. □

The part of order m of the enveloping algebra of $\mathcal{W}(Q)$, denoted $\mathcal{D}^m(\mathcal{W}(Q)) \subset \mathcal{D}(\mathcal{W}(Q))$, is generated by up to m-fold products of elements of $\mathcal{W}(Q)$. There is a natural quotient map

$$N_Q : \mathrm{Diff}^m_\alpha(G) \longrightarrow \mathcal{D}^m(\mathcal{W}(Q)) \text{ for } Q = Q_p[9.11]$$

which is the 'normal homomorphism' for Q, (or p) i.e. the image of an α-differential operator P is called the normal operator of P at Q.

The Lie algebra $\mathcal{W}(Q)$ is defined as a quotient of $\mathcal{V}_\alpha(G)$, in (7.1). Since we wish to invert the normal operator of an element $P \in \mathrm{Diff}^m_\alpha(G)$ we wish to have $\mathcal{W}(Q)$ act as a space of vector fields on a manifold, so that $N_Q(P)$ becomes a differential operator. For $\mathcal{V}_\Theta$ this is done in §1.3. We can summarize the results there, with a view to generalizing them, by noting that there is an action of $\mathcal{W}(Q)$ on the "appropriate" normal bundle to Q and a related 'compactified' action on the spherical normal bundle in a parameter-extended space (because the first action is homogeneous). In §1.3 the appropriate normal bundle is the S-parabolic bundle. The general case examined in [**12**] is more complicated. We consider four cases for the integral leaves of $\mathcal{V}_\alpha(G)$:

$$p \in G \setminus [\Theta\mathrm{b} \cup \mathrm{ab}][9.12] \tag{7.1}$$

$$p \in \Theta\mathrm{b} \setminus \mathrm{co}[9.13] \tag{7.2}$$

$$p \in \mathrm{ab} \setminus \mathrm{co}[9.14] \tag{7.3}$$

$$p \in \mathrm{co}\,.[9.15] \tag{7.4}$$

The diffeomorphism type of Q_p is the same for (7.2) and (7.4) but the hierarchy of the leaves is different in these two cases. Thus

$$p \in \Theta\mathrm{b} \setminus \mathrm{co},\ Q_p = \{p\} \subset Q_{p'} \Longleftrightarrow p' \in G \setminus [\Theta\mathrm{b} \cup \mathrm{ab}][9.16] \tag{7.5}$$

$$p \in \mathrm{co},\ Q_p = \{p\} \subset Q_{p'} \Longleftrightarrow p' \in \mathrm{ab} \setminus \mathrm{co}\,.[9.17] \tag{7.6}$$

This affects the construction of the representation of $\mathcal{W}(Q)$, even though these algebras are the same for the two cases (7.2) and (7.4).

First consider the case (7.1). This is trivial in the sense that $\mathcal{W}(Q) = \mathcal{U}(Q)$ so we already have a representation, namely on $Q_p = G(\epsilon)$, $\epsilon > 0$. Nevertheless let us briefly consider the analogue of the construction in §1.3. The normal bundle to Q_p is just $G(\epsilon) \times \mathbb{R}_{d\epsilon}$ to which $\mathcal{V}_\alpha(G)$ lifts (by Proposition (C.1)); clearly it annihilates $d\epsilon$ and so acts on each of the surfaces $d\epsilon$ =const. On each of these it represents $\mathcal{U}(Q) = \mathcal{V}_\Theta(G(\epsilon))$. Adding the extra parameter and projecting to the spherical normal bundle from $d\epsilon = 1$ gives exactly the same representation.

The second case to consider is (7.2). Of course this *is* the case considered in §1.3 except for the (essentially trivial) parameter $\epsilon > 0$. Again if we make an S-parabolic blow-up and restrict to $d\epsilon = 1$ we recover the representation of $\mathcal{W}(Q) = {}^\alpha T_p G$ on ${}_+N\{G(\epsilon); p, S\}$, or the compactified representation on $S_+N\{\widetilde{G}(\epsilon); \widetilde{p}, \widetilde{S}\}$ where $\widetilde{G}(\epsilon) = G \times [0,1]$. These spaces can be identified with blown-up versions of the ball, exactly as in §1.3. The only point to note is that we could also obtain this representation by actually applying the construction of §1.3 to the Θ-algebra obtained from the leaf $G(\epsilon) \ni p$ above. This commutativity of the passage to the normal representation becomes more significant below.

Next we pass to the first new case, (7.3). The appropriate normal space here is just the usual normal space, i.e. there are no parabolic directions. By (7.4) $Q_p = F_y$ is a ball which is a leaf of the fibration of $\mathrm{ab}(G)$ over Y. The normal bundle is just

$${}_+N\{G; Q_p\} = N\{\mathrm{ab}(G); F_y\} \times [0,\infty)_{d\epsilon},\ p \in \mathrm{ab} \setminus \mathrm{co}\,.[9.18]$$

Again, by Proposition (C.1), $\mathcal{V}_\alpha(G)$ lifts to ${}_+N\{G;Q_p\}$ to a space of homogeneous vector fields on this vector bundle over F_y, all annihilating $d\epsilon$. Recall however from §7 that the fibration of ab over Y is obtained by blowing up. Thus

$$N\{\mathrm{ab}(G);F_y\} = F_y \times T_yY[9.19]$$

is canonically trivial. Then we find

Lemma 7.7. *For $p \in \mathrm{ab}\setminus\mathrm{co}$ the action of $V \in \mathcal{W}(Q_p)$ on $N\{\mathrm{ab}(G);F_y\}$, where $F_y = Q_p$, by restriction from* (7.1) *to $d\epsilon = 1$ represents it as a translation-invariant vector field with respect to the linear structure on the second factor of* (7.1) *and these vector fields span the tangent space over the interior the interior of F_y.*

Proof. This follows directly from (7). The vector fields $\rho\partial_\rho$ and $\rho\partial_{Y_j}$ generate $\mathcal{U}(Q_p)$ and lift to be trivial on the linear, second, factor in (7.1). Under the blow-up the vector fields ϵZ, $\epsilon Z'$ lift to span T_yY at $d\epsilon = 1$ so the result follows (since ρ lifts to be constant on the linear fibres of (7.1)). □

Since G was obtained from M by blowing up the Θ boundary parabolically the fibre F_y can be identified with the square-root of the ball in the normal space to Y at y in Ω :

$$F_y = (B_y)_{\frac12},\ B_y = \{v \in N_y\{\Omega;Y\}; \mathrm{Hess}(\psi)(v,v) \le 1\}[9.21]$$

where $\mathrm{Hess}(\psi)$ is the Hessian of ψ in (7), i.e. just the quadratic form a_{jk}. Then from (7) we see that

Lemma 7.8. *Under the identification* (7.1) *the complex structure on $T_yY \times N_y\{\Omega;Y\}$ (arising from the fact that Y is totally real) lifts to identify*

$$F_y \times T_yY \equiv (\mathbb{B}^n \times \mathbb{R}^n)_{\frac12},\ (\mathbb{B}^n \times \mathbb{R}^n) \subset \mathbb{C}^n[9.23]$$

so that $\mathcal{W}(Q_p)$ for $p \in \mathrm{ab}\setminus\mathrm{co}$, with $F_y = Q_p$, is identified as the translation-invariant part of the algebra $\mathcal{V}_\Theta(F_y \times T_yY)$.

Thus we can say that the model space at a point in the interior of the adiabatic boundary is just the tubular region of $\mathbb{C}^n$:

$$\Omega_a = \mathbb{B}^n \times \mathbb{R}^n \subset \mathbb{C}^n[9.24]$$

where by this we really mean with the boundary blown-up parabolically, i.e.

$$\mho_a = \mathbb{B}^n_{\frac12} \times \mathbb{R}^n.[9.25]$$

In fact even more is true for the metrics we are interested in. We can introduce linear coordinates in $N_y\{\Omega;Y\}$ so that

$$\mathrm{Hess}(\psi) = \mathrm{Id}\,.[9.26]$$

Proposition 7.3. *For $p \in \mathrm{ab}(G)\setminus\mathrm{co}(G)$ the representation of $\mathcal{W}(Q_p)$ on $\mho_a$ in* (7.1) *given by the identification* (7.1)*, for linear coordinates in which* (7.1) *holds, identifies the normal operator of the Laplacian of the metric* (7.2) *with the lift from $\mathbb{B}^n_x \times \mathbb{R}^n_y$ of the Laplacian for the potential $1-|y|^2$ i.e.*

$$g_a = -\beta^*_{\frac12}[\partial\overline{\partial}\log(1-|y|^2)].[9.28]$$

PROOF. We have already identified the algebra $\mathcal{W}(Q)$ as the translation-invariant part of $\mathcal{V}_\Theta$ on the model space. This identifies the lift of the structure bundle, ${}^\alpha TG$, to the normal bundle to Q_p with the Θ-structure bundle. The metric (7.2) induces a non-degenerate fibre metric on ${}^\alpha T(M)$ and induces the metric (7.3) on the Θ-structure, so the proposition follows from the naturality of the definition of the Laplacian. □

This result extends directly to identify the normal operator for the action of the Laplacian on (p,q)–forms as well as functions.

This completes the identification of the normal operator for the third case, (7.3), of points of G. It remains to analyze the last case, (7.4). In fact $\mathcal{W}(Q_p) = {}^\alpha T_p G$ in this case, and from Lemma 7.8 we see that we can apply the discussion of §1.3 but now to any point (p,y) where $p \in F_y$ for $y = \phi(p)$. That is, we note that $p \in Q_{p'}$ is in the boundary of one of the adiabatic leaves of $\mathcal{V}_\alpha(G)$ so $\mathcal{W}(Q_p)$ can be obtained as a quotient of the algebra $\mathcal{W}(Q_{p'})$. Thus from Lemma 7.8 and Proposition 7.3 we see that the normal operator at $p \in \mathrm{co}(G)$ is again just the lift of the Laplacian for the Bergman metric on the ball.

These representations of the normal operators at the various integral leaves of $\mathcal{V}_\alpha(G)$ are a very strong guide to the definition of the α-stretched product discussed next. In particular to obtain the representation of the normal operator at $p \in \mathrm{co}(G)$ as the Laplacian on the ball we have first considered the normal bundle to that fibre of the adiabatic face containing the point in its boundary and then passed to the (parabolic) normal space of a preimage of the point in this normal bundle. In simple terms, we observe that as a point tends to the adiabatic boundary along the Θ-boundary the Θ-structure degenerates. In particular the rank drops from $n-1$ to 0. If we first blow-up a leaf of the fibration of the adiabatic boundary and then allow a point on the Θ-boundary to approach the interior of the new front face then no such degeneration occurs. Considering the close relation between these normal spaces and blow-up of the corresponding submanifolds we can interpret this as saying "First blow up the adiabatic face and then the Θ face". This indeed is how we shall proceed.

7.2. α-stretched product

The α-stretched (fibre) product of G with itself is the manifold with corners on which the kernels of α-pseudodifferential operators, such as the resolvent of the Laplacian of the metric (7.2), take a simple form.

The space G in (7) is fibred over $[0,\epsilon_0]$ by $\widetilde{\pi}_\epsilon = \beta_0 \circ \pi_\epsilon$; set $G(\epsilon) = \widetilde{\pi}_\epsilon^{-1}(\{\epsilon\})$. Since all the operators we are interested in depend on ϵ only as a parameter we shall not consider the full product $G \times G$. Rather we consider the fibre product:

$$G^2 = \pi_2^{-1}\left[\Delta_\epsilon\right],\ \pi_2 = \widetilde{\pi}_\epsilon \times \widetilde{\pi}_\epsilon : G \times G \longrightarrow [0,\epsilon_0]^2,\ \Delta_\epsilon = \{(\epsilon,\epsilon) \in [0,\epsilon_0]^2\}; [10.1]$$

it is clearly fibred over $[0,\epsilon_0]$ by the restriction of either map $\widetilde{\pi}_\epsilon$. Notice here the somewhat unconventional notation, namely that $G^2 = GG$ stands for the fibre product, i.e.

$$G^2 \neq G \times G. [10.2]$$

This convention of writing ϵ-fibre products will be adopted throughout; if the true product is intended it will be indicated with an explicit product notation as in (7.2).

There are two projections

$$\pi_l^2, \pi_r^2 : G^2 \longrightarrow G[10.3]$$

obtained by restriction of the projections onto the factors of $G \times G$. The four hypersurface boundary faces of G^2 are

$$\begin{aligned} \mathrm{ab}(G^2) &= \widetilde{\pi}_\epsilon^{-1}(0) \cong G_0^2 \\ \Theta\mathrm{l}(G^2) &= (\pi_l^2)^{-1}(\Theta\mathrm{b}(G)) \cong \Theta\mathrm{b}(G) \times G_0 \\ \Theta\mathrm{r}(G^2) &= (\pi_r^2)^{-1}(\Theta\mathrm{b}(G)) \cong G_0 \times \Theta\mathrm{b}(G) \\ \mathrm{eb}(G^2) &= \widetilde{\pi}_\epsilon^{-1}\{\epsilon = \epsilon_0\} \cong \mathrm{ab}(G^2). \end{aligned}[10.4]$$

From G^2 there are also the two fibration maps to $\widetilde{Y}$ and the double fibration to $\widetilde{Y}^2$:

$$\begin{aligned} \phi_l^2 &: G^2 \longrightarrow \widetilde{Y} \\ \phi_r^2 &: G^2 \longrightarrow \widetilde{Y} \quad [10.5] \\ \phi^2 &: G^2 \longrightarrow \widetilde{Y}^2. \end{aligned}$$

Consider the submanifold:

$$B_a = \left\{p \in \mathrm{ab}(G^2); \phi_l^2(p) = \phi_r^2(p)\right\} .[10.6]$$

As a submanifold of $\mathrm{ab}(G^2)$ this is just the fibre diagonal. It is a compact boundary p-submanifold in the sense of [**12**], see also §C.2. Thus we can blow up G^2 along this submanifold, we denote the resulting manifold with corners

$$G_a^2 = \left[G^2; B_a\right], \ \beta_a^2 = \beta\left[G^2; B_a\right] : G_a^2 \longrightarrow G^2[10.7]$$

using the notation for blow-up of §C.2. Note that B_a is the flow-out of the diagonal in G_0^2 under the lift to G^2, from either the left or right factors, of $\mathcal{V}_a$. Although we have blown up a submanifold of the boundary of G^2, G_a^2 still has corners only up to codimension three since the face of G^2 of codimension 3 is actually a submanifold of B_a. The boundary hypersurfaces of G_a^2 consist of

$$M_1(G_a^2) = \{\mathrm{af}, \mathrm{ab}, \Theta\mathrm{l}, \Theta\mathrm{r}, \mathrm{eb}\}\,[10.8]$$

where $\mathrm{af}(G_a^2)$ is the new 'adiabatic front' face, and the others are the lifts of the corresponding boundary hypersurfaces of G^2.

In G^2 consider the diagonal of the Θ-boundaries, again a compact p-submanifold:

$$B_\Theta = \left\{p \in \Theta\mathrm{l}(G^2) \cap \Theta\mathrm{r}(G^2); \pi_l^2(p) = \pi_r^2(p)\right\}[10.9]$$

The difference of the pull-backs of the 1-form Θ, under the two projections π_l^2 and π_r^2 fixes a line subbundle

$$S_2 = \mathrm{sp}\left\{(\pi_l^2)^*\Theta - (\pi_r^2)^*\Theta\right\} \subset N^*B_\Theta[10.10]$$

of the conormal bundle to B_Θ.

Next we show that B_a and B_Θ meet parabolically cleanly in the sense of (C.3) and (C.3). Certainly the clean intersection condition of Bott, (C.3), holds for B_a and B_Θ, since

$$B_a \cap B_\Theta = B_\Theta \pitchfork \mathrm{ab}(G^2)[10.11]$$

is just the boundary of the diagonal in G_0^2. Furthermore, over $B_a \cap B_\Theta$, $\widetilde{S} = N^* B_a \cap N^* B_\Theta$ is just the range of the pull-back for the projection from G_0^2 to Y^2, i.e. of the fibre normals. Then (C.3) follows from

$$S_2 \subset (\phi^2)^*[N^*\Delta_Y] \text{ over } B_a \cap B_\Theta, [10.12]$$

meaning it is spanned by differentials lifted from the base. Lemma 7.3 shows that (7.2) does indeed hold.

The cleanness of this intersection allows Lemma (C.2) to be applied, so B_Θ lifts to a p-submanifold in G_a^2, with S_2 lifting to a p-subbundle of the conormal bundle of the lift. Indeed the lift is isomorphic to

$$[B_\Theta; B_a \cap B_\Theta, S_2]\,.[10.13]$$

Now we can define the α-stretched product to be the S_2-parabolic blow-up of G_a^2 along B_Θ :

$$G_\alpha^2 = \left[G_a^2; B_\Theta, S_2\right] = \left[G^2; B_a; B_\Theta, S_2\right]\,,\ G_\alpha^2 \xrightarrow{\beta_\alpha^2} G^2[10.14]$$

using the notation for iterated blow-ups of §C.4.

The boundary of G_α^2 consists of six hypersurfaces. Namely, the new 'Θ front' face, and the lifts of the five boundary hypersurfaces of G_a^2, which we call the adiabatic front face, the adiabatic side face, left Θ face, the right Θ face and the extension boundary:

$$M_1(G_\alpha^2) = \{\mathrm{af}, \mathrm{ab}, \Theta\mathrm{f}, \Theta\mathrm{l}, \Theta\mathrm{r}, \mathrm{eb}\}\,.[10.15]$$

Lemma 7.9. *Under the overall blow-down β_α^2 the diagonal*

$$\Delta = \left\{p \in G^2; \pi_l^2(p) = \pi_r^2(p)\right\} [10.17]$$

lifts to a compact interior p-submanifold $\Delta_\alpha \subset G_\alpha^2$ which meets the boundary only in the two front faces $\mathrm{af}(G_\alpha^2) \cup \Theta\mathrm{f}(G_\alpha^2)$, *and the extension boundary* $\mathrm{eb}(G_\alpha^2)$. *The restriction of β_α^2 to Δ_α is an isomorphism onto Δ.*

Proof. In G^2 the diagonal is only a b-submanifold. However its image in the normal bundle to B_a is a p-subbundle away from the intersection of B_a with B_Θ. Moreover (C.3) clearly holds (and (C.3) is trivial). Thus Lemma (C.2) applies and shows that the diagonal lifts to G_a^2 as a b-submanifold which meets $\mathrm{af}(G_a^2)$ transversally. From the structure of S_2, as a difference of the lift of forms, it follows that, in G_a^2, the image of the lift of Δ in the S_2-parabolic normal bundle to the lift of B_Θ is a p-subbundle and again (C.3) and (C.3) hold for the intersection of the lifts. Thus in G_α^2 the lifted diagonal is actually a p-submanifold. Since Δ meets B_a in codimension one, and the lift of Δ to G_a^2 meets the lift of B_Θ in codimension one too, Lemma (C.2) actually shows that the lifted diagonal in G_α^2 is isomorphic with the diagonal in G^2. $\square$

By composing the two blow-down maps with the projections we get 'α-stretched' projections:

$$G_\alpha^2 \pi_{r,\alpha}^2 = \pi_r^2 \cdot \beta_\alpha^2 \qquad [10.18]$$

$$G \xleftarrow{\pi_{l,\alpha}^2 = \pi_l^2 \cdot \beta_\alpha^2} \qquad \xrightarrow{\text{Ł}} G$$

PROPOSITION 7.4. *The two maps* (7.2) *are b-fibrations, under either of them the Lie algebra* $\mathcal{V}_\alpha$ *lifts to a Lie subalgebra of* $\mathcal{V}_b(G^2_\alpha)$ *which is transversal to* Δ_α.

PROOF. Both blow-down maps, and the projections, are b-maps hence so are the stretched projections. We need to show that the b-differential

$$^b(\pi^2_{l,\alpha})_* : {}^bT_pG^2_\alpha \longrightarrow {}^bT_{p'}G,\ p' = \pi^2_{l,\alpha}(p)[10.20]$$

is surjective for all $p \in G^2_\alpha$. To do so we use Proposition (B.4), so we first check that the stretched projections are b-normal maps. By definition, (B.5), this is the combinatorial problem of tracking the lifts, to G^2_α, of the boundary hypersurfaces of G. The exponents in the lifts of defining functions for the boundaries of G from left and right, as in (7.2), are summarized in Table 1, the McDonald table for these maps (eb is excluded since it behaves trivially). The entries in the part of this table corresponding to one of the maps, f, in (7.2) are the coefficients in

$$f^* \log \rho'_{h'} = \sum_{h \in M_1(G^2_\alpha)} e(h', h) \log \rho_h, \quad \mod \mathcal{C}^\infty, [10.21]$$

where $\rho'_{h'}$ and ρ_h are defining functions for the boundary hypersurfaces of the domain and target of f. Clearly (B.5) follows from Table 1. The 'density row' labelled ν is discussed below.

The second part of the hypothesis of Proposition (B.4) which needs to be checked is the surjectivity of the ordinary differential as a map (B.4). This is trivially true except at points in blown-up boundary faces, i.e. in $\mathrm{af}(G^2_\alpha) \cup \Theta\mathrm{f}(G^2_\alpha)$. Away from the intersection $\mathrm{af}(G^2_\alpha) \cap \Theta\mathrm{f}(G^2_\alpha)$ (which is a boundary face of $\Theta\mathrm{f}(G^2_\alpha)$) only one blow-up is involved. The respective front faces fibre over B_a or $B_\Theta \setminus B_a$. Since these submanifolds of G^2 project onto $\mathrm{ab}(G)$ and $\Theta\mathrm{b}(G) \setminus \mathrm{ab}(G)$ respectively the surjectivity in (B.4) is clear at these points. Similarly $\mathrm{af}(G^2_\alpha) \cap \Theta\mathrm{f}(G^2_\alpha)$ fibres over the front face of the lift of B_Θ to G^2_a under the second blow-up and this in turn fibers over $B_a \cap B_\Theta$ under the first blow-up. This intersection projects from G^2 to $\mathrm{co}(G)$ under π^2_l or π^2_r so (B.4) again holds. Thus both stretched projections are b-fibrations.

That elements of $\mathcal{V}_\alpha$ lift from left or right to be smooth vector fields on G^2_α follows the fact that they are tangent to both B_a and to the B_Θ, and at the latter annihilate the subbundle S_2 quadratically, so Proposition (C.2) applies. It remains only to show that these lifted algebras are transversal to the lifted diagonal.

In the interior of G^2_α this is trivial, since this is also the interior of G^2 where $\mathcal{V}_a$ spans the fibres on the left or right and so is transversal to the (fibre) diagonal. From Lemma 7.3 we deduce that the Θ-structure along the Θ-boundary of the lifted fibre diagonal in G^2_a is uniformly non-degenerate down to $\epsilon = 0$. As the lifted diagonal does not intersect ab, ϵ is a pure parameter in a neighbourhood of the Θ-boundary of the lifted diagonal and thus the transversality in G^2_α follows from Proposition 7.14 in [**5**]. □

As was remarked at the end of the previous section the order in which the blow-ups are performed in the construction of the space G^2_α is dictated, in part, by the fact that blowing up the adiabatic boundary first desingularizes the Θ-structure along B_Θ. Performing the blow-ups in the opposite order would lead to a space of limited utility. The usual 'yoga' recommends first blowing up the intersection of two submanifolds which are to be blown up and do not meet cleanly. This can have the disastrous consequence of producing spaces which are not b-fibrations, as in the case at hand. The usual yoga requires modification in the presence of a degenerating structure on one of the submanifolds.

Next we turn to the consideration of densities. The basic operators we consider will be those acting on half-densities. The identity operator, with ϵ regarded as a parameter, has Schwartz kernel of the form

$$\delta(x-x')\delta(y-y')\delta(z-z')|dxdydzdx'dy'dz'|^{\frac{1}{2}}[10.22]$$

in local coordinates, y in the base and x, z in the fibres with the boundary being $x = 0$. Since we wish to consider the kernels as distributions on G^2_α we first lift to the partial product G^2.

To make the kernel a half-density we write it in the form

$$\delta(x-x')\delta(y-y')\delta(z-z')|dxdx'dydy'dzdz'd\epsilon|^{\frac{1}{2}}\cdot|d\epsilon|^{-\frac{1}{2}}.[10.23]$$

Next consider the lift of this distribution, under the blow-down maps, to G^2_α. From Lemma (C.4) we see that the density bundle can be lifted and

$$\beta^*_\alpha : \mathcal{C}^\infty(G^2;\Omega) \longrightarrow \rho^n_{\mathrm{af}}\rho^{2n+1}_{\Theta\mathrm{f}}\mathcal{C}^\infty(G^2_\alpha;\Omega).[10.24]$$

This gives the 'density row' in Table 1, the exponents for the bundle

$$(\pi^2_{l,\alpha})^*(\Omega^{\frac{1}{2}})\cdot(\pi^2_{r,\alpha})^*(\Omega^{\frac{1}{2}}) = \rho^\nu\Omega^{\frac{1}{2}}|d\epsilon|^{\frac{1}{2}},\ \nu = (\frac{n}{2}, 0, n+\frac{1}{2}, 0, 0).[10.25]$$

Using the homogeneity of the delta function and taking the square root of the bundles in (7.2) allows us to interpret the kernel of the identity as a Dirac delta section, with support Δ_α, of the bundle

$$\mathrm{KD}_\alpha = \rho_{\mathrm{af}}^{-\frac{1}{2}n}\rho_{\Theta\mathrm{f}}^{-n-\frac{1}{2}}\Omega^{\frac{1}{2}}\otimes|d\epsilon|^{-\frac{1}{2}},[10.26]$$

with the tensor product over $\mathcal{C}^\infty(G^2_\alpha)$. We shall use the bundle (7.2) to normalize our kernels. The action of a generalized section of this bundle on a density on G is realized by reversing the discussion above. That is

$$\mathcal{C}^{-\infty}(G^2_\alpha;\mathrm{KD}_\alpha) \ni \kappa_A \longleftrightarrow A_\kappa : \dot{\mathcal{C}}^\infty(G;\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}(G;\Omega^{\frac{1}{2}})[10.27]$$

is given by the identity

$$\langle A\phi,\psi\rangle = \langle\kappa,(\pi^2_{l,\alpha})^*\psi\otimes(\pi^2_{r,\alpha})^*\phi\rangle \quad \forall\ \phi\in\dot{\mathcal{C}}^\infty(G;\Omega^{\frac{1}{2}})[10.28]$$

where we interpret $(\pi^2_{l,\alpha})^*\psi\otimes(\pi^2_{r,\alpha})^*\phi$ as a $\mathcal{C}^\infty$ section of $\mathrm{KD}^*_\alpha\otimes\Omega$, ($\mathrm{KD}^*_\alpha$ being the dual bundle to KD_α) vanishing at all boundary faces.

We need to discuss the identification of the density bundles along the front faces.

The small calculus of α-pseudodifferential operators (acting on half densities) of order m, denoted $\Psi^m_\alpha(G;\Omega^{\frac{1}{2}})$, is then defined as

$$\Psi^m_\alpha(G;\Omega^{\frac{1}{2}}) = \left\{\kappa\in\mathcal{C}^\infty_{\mathrm{Df}}(G^2_\alpha)\cdot I^{m-\frac{1}{4}}(G^2_\alpha,\Delta_\alpha;\mathrm{KD}_\alpha)\right\}.[10.29]$$

Here $\mathcal{C}^\infty_{\mathrm{Df}}(G^2_\alpha)$ is the subspace of $\mathcal{C}^\infty(G^2_\alpha)$ consisting of the functions vanishing in Taylor series at

$$\partial(G^2_\alpha)\setminus\mathrm{Df}(G^2_\alpha),\ \mathrm{Df}(G^2_\alpha) = \mathrm{af}(G^2_\alpha)\cup\Theta\mathrm{f}(G^2_\alpha)\cup\mathrm{eb}(G^2_\alpha).[10.30]$$

Thus $\mathrm{Df}(G^2_\alpha)$ is just the union of those boundary hypersurfaces of G^2_α which meet the lifted diagonal. Since the conormal sections are smooth away from the lifted diagonal (7.2) just means that the kernels vanish to all orders at boundary points outside $\mathrm{Df}(G^2_\alpha)$. The rather odd-looking order normalization in (7.2) is standard for conormal distributions since, if $S \subset X$ is a submanifold of codimension k in a manifold of dimension N, then a Dirac delta distribution at S is of order $\frac{1}{2}k - \frac{1}{4}N$:

$$\delta_S \in I^{\frac{1}{2}k-\frac{1}{4}N}(X,S).[10.31]$$

Following the discussion above we conclude that

$$\mathrm{Id} \in \Psi^0_\alpha(G;\Omega^{\frac{1}{2}}).[10.32]$$

As a consequence of Proposition 7.4 it then follows that for any non-negative integer, k, and $m \in \mathbb{R}$,

$$\begin{gathered}\mathrm{Diff}^k_\alpha(G;\Omega^{\frac{1}{2}}) \subset \Psi^k_\alpha(G;\Omega^{\frac{1}{2}})\\ \mathrm{Diff}^k_\alpha(G;\Omega^{\frac{1}{2}}) \cdot \Psi^m_\alpha(G;\Omega^{\frac{1}{2}}) \subset \Psi^{m+k}_\alpha(G;\Omega^{\frac{1}{2}})\end{gathered}[10.33]$$

since the action of an α-differential operator is obtained by the superposition of the action of up to k elements of $\mathcal{V}_\alpha$, and these lift into $\mathcal{V}_b(G^2_\alpha)$ and hence map $\Psi^m_\alpha(G;\Omega^{\frac{1}{2}})$ into $\Psi^{m+1}_\alpha(G;\Omega^{\frac{1}{2}})$.

7.3. α-triple product

In this section we construct the α-stretched triple product, G^3_α, which is used crucially in the composition formula for general α-pseudodifferential operators in §8. Its construction is straightforward but involved, since four individual blow-up steps are needed. These are all geometrically motivated; the first two are related to the adiabatic boundary (cf [**?**]) and the second two to the Θ-boundary (cf [**5**]). We make systematic use of the notation for iterated blow-ups in §C.4.

As in the construction of G^2_α we start from the appropriate fibre product

$$G^3 = \pi_3^{-1}[\Delta^3_\epsilon],\ \pi_3 = \widetilde{\pi}_\epsilon\times\widetilde{\pi}_\epsilon\times\widetilde{\pi}_\epsilon : G\times G\times G \longrightarrow [0,\epsilon_0]^3,\ \Delta_\epsilon = \left\{(\epsilon,\epsilon,\epsilon) \in [0,\epsilon_0]^3\right\},[11.1]$$

again it is fibred over $[0,\epsilon_0]$. There are three projections onto G, $\pi^3_h : G^3 \longrightarrow G$ for $h = l,m,r$. Similarly there are three projections onto G^2, $\pi^3_o,: G^3 \longrightarrow G^2$ for $o = f,c,s$ obtained, respectively, by ignoring the last, central and first factor of G. The partial product has five boundary hypersurfaces, the adiabatic boundary, the left, middle and right Θ-boundaries and the extension boundary, which we denote

$$\begin{gathered}\mathrm{ab}(G^3) = \pi_3^{-1}\left\{\epsilon = 0\right\} = G^3_0\\ \Theta\mathrm{h}(G^3) = (\pi^3_h)^{-1}(\Theta\mathrm{b}(G)),\ h = l,m,r[11.2]\\ \mathrm{eb}(G^3) = \pi_3^{-1}\left\{\epsilon = \epsilon_0\right\}.\end{gathered}$$

As the triple product of G_0 the adiabatic boundary hypersurface has three fibrations denoted $\phi_h : \mathrm{ab}(G^3) \longrightarrow Y$, for $h = l,m,r$ and we consider in it four submanifolds, the triple fibre diagonal and the three lifts of the fibre diagonals from G^2_0 :

$$\begin{gathered}B^3_a = \left\{p \in G^3_0; \phi_l(p) = \phi_m(p) = \phi_r(p)\right\}\\ B^2_{a,o} = \left\{p \in G^3_0; \phi_h(p) = \phi_{h'}(p)\right\} \text{ where } o = f,c,s \qquad [11.3]\\ \text{and then } \{h,h'\} = \{m,r\},\{l,r\},\{l,m\} \text{ respectively.}\end{gathered}$$

All are compact embedded p-submanifolds of $\mathrm{ab}(G^3)$. In fact each $B^2_{a,o}$ is just the product of B^2_a, in (7.2), with a factor of G_0 and $B^2_{a,o} \cap B^2_{a,o'} = B^3_a$ if $o \neq o'$. In fact this intersection is clearly transversal within the adiabatic boundary:

$$B^2_{a,o} \pitchfork B^2_{a,o'} = B^3_a \text{ within } \mathrm{ab}(G^3) \text{ if } o \neq o'.[11.4]$$

Indeed this is just the obvious independence of the conormals to the two submanifolds at B^3_a.

This intersection property (typical of 'diagonals') is the reason we first blow up B^3_a, to separate the factors of $B^2_{a,o}$. Set

$$G^3_1 = \left[G^3; B^3_a\right] \xrightarrow{\beta^3_1} G^3.[11.5]$$

Since B^3_a is a submanifold of each of the $B^2_{a,o}$ these submanifolds of G^3 lift under the first blow-up to p-submanifolds; we continue to denote the lifts as $B^2_{a,o}$. The lifts of these three double fibre diagonals are disjoint because their images in the spherical normal bundle of B^3_a are disjoint. We denote by B^2_a the union of the three. The new 'front face' of G^3_1, will be called the adiabatic triple face and denoted $\mathrm{at}(G^3_1)$. Using the same notation for the hypersurfaces boundaries of G^3_1 as for the hypersurfaces of which they are lifts we then have

$$M_1(G^3_1) = \{\mathrm{at}, \mathrm{ab}, \Theta\mathrm{l}, \Theta\mathrm{m}, \Theta\mathrm{r}, \mathrm{eb}\}.[11.6]$$

The submanifold $B^2_a \subset \mathrm{ab}(G^3_1)$ is a compact p-submanifold, thus we can blow it up and define

$$G^3_a = \left[G^3_1; B^2_a\right] \simeq \left[G^3; B^3_a; B^2_a\right] \simeq \left[G^3; B^3_a; B^2_{a,o}; B^2_{a,o'}; B^2_{a,o''}\right] \xrightarrow{\beta^3_2} G^3_1.[11.7]$$

Here o, o', o'' is any permutation of f, c, s; these identities follow from the trivial case of Lemma (C.7). The space G^3_a is the 'adiabatic triple product'. We call the three new boundary hypersurfaces so introduced the 'adiabatic side faces' and denote them $\mathrm{aso}(G^3_a)$, for $o = f, c, s$. Thus

$$M_1(G^3_a) = \{\mathrm{at}, \mathrm{asf}, \mathrm{asc}, \mathrm{ass}, \mathrm{ab}, \Theta\mathrm{l}, \Theta\mathrm{m}, \Theta\mathrm{r}, \mathrm{eb}\}\,.[11.8]$$

Before proceeding to the Θ-blow-ups we pause to note the important relationship between G^3_a and G^2_a (cf [?]). Using Lemma (C.8) we give an alternative construction of G^3_a. Thus we first think of G^3_a as obtained by the successive blow-up of B^3_a, then of $B^2_{a,f}$ and finally of $B^2_{a,s} \cup B^2_{a,c}$.

$$G^3_a \simeq \left[G^3; B^3_a; B^2_{a,f}; B^2_{a,s} \cup B^2_{a,c}\right].[11.9]$$

Since $B^3_a \subset B^2_{a,f}$ we can exchange the order of the first two blow-ups, as in (C.8), and obtain

$$G^3_a \simeq \left[G^3; B^2_{a,f}; B^3_a; B^2_{a,s} \cup B^2_{a,c}\right] [11.10]$$

where B^3_a now actually represents the lift of B^3_a under the first blow-up. This gives the iterated blow-downs

$$G^3_a \longrightarrow \left[G^3; B^2_{a,f}\right] \longrightarrow G^3.[11.11]$$

Now observe that the blow-up of G^3 along $B^2_{a,f}$ is a fibre by fibre blow-up with respect to $\pi_f : G^3 \longrightarrow G^2$, the image of $B^2_{a,f}$ being B_a. Thus we find that there are commutative diagrams:

$$\begin{array}{ccc} G^3_a & \xrightarrow{\pi^3_{o,a}} & G^2_a \\ \downarrow & & \downarrow \\ G^3 & \xrightarrow[\pi^3_o]{} & G^2 \end{array} \qquad o = f, c, s.[11.12]$$

Here the map on the right is the blow-down map for (7.2) and that on the left is given by (7.3). From (7.3) $\pi^3_{o,a}$ is the product of a blow-down map and a projection. It is straightforward to show that

$$\pi^3_{o,a} : G^3_a \longrightarrow G^2_a \text{ is a } b\text{-fibration for } o = f, c, s.[11.13]$$

Now we return to the construction of G^3_α. To resolve the Θ-boundary we consider the four submanifolds of G^3 analogous to (7.3):

$$B^3_\Theta = \{p \in G^3; \pi^3_l(p) = \pi^3_m(p) = \pi^3_r(p) \in \Theta\mathrm{b}(G)\}$$
$$B^2_{\Theta,o} = \{p \in G^3; \pi^3_h(p) = \pi^3_{h'}(p) \in \Theta\mathrm{b}(G)\} \text{ where } o = f, c, s[11.14]$$
$$\text{and } \{h, h'\} = \{m, r\}, \{l, r\}, \{l, m\} \text{ respectively.}$$

As usual

$$B^2_{\Theta,o} = (\pi^3_o)^{-1}(B_\Theta)[11.15]$$

and

$$B^3_\Theta = B^2_{\Theta,f} \cap B^2_{\Theta,c} = B^2_{\Theta,c} \cap B^2_{\Theta,s} = B^2_{\Theta,s} \cap B^2_{\Theta,f}.[11.16]$$

In view of (7.3) we can define the parabolic (line) bundles over each of these partial diagonals as

$$S_{2,o} = (\pi^3_o)^* S_2,\ o = f, c, s[11.17]$$

and then the parabolic bundle for the triple diagonal is the span

$$S_3 = S_{2,o} \oplus S_{2,o'},\ o \neq o' \text{ over } B^3_\Theta.[11.18]$$

This structure follows from the fact that each of the $S_{2,o}$ can also be viewed as the differences of the pull-backs of the Θ forms under two of the projections π^3_h. It is immediately clear that any pair of these four submanifolds intersect parabolically cleanly in the sense of (C.3) and (C.3); the second condition being a consequence of (7.3).

Since we need to lift them to G^3_a, we need also to check the cleanness of the intersections with the adiabatic diagonals. There are five essentially different intersections, consisting of $B^3_a \cap B^3_\Theta$, $B^3_a \cap B^2_{\Theta,o}$, $B^2_{a,o} \cap B^3_\Theta$, and the two cases of $B^2_{a,o} \cap B^2_{\Theta,o'}$ as $o = o'$ or $o \neq o'$. Although all are clean we note for later reference that

$$B^2_{a,o} \pitchfork B^2_{\Theta,o'} \text{ for } o \neq o'.[11.19]$$

In all cases the Θ-spaces are transversal to $\mathrm{ab}(G^3)$ with the parabolic bundles not containing $d\epsilon$, so we can restrict attention to the intersections within $\mathrm{ab}(G^3)$. The cleanness is then quite clear from the local coordinate expressions for these submanifolds. As before, (C.3) follows from Lemma 7.3 and in particular the fact that the parabolic bundles are spanned by differentials from the base.

From this cleanness it follows that all the Θ-diagonals lift to p-submanifolds of G^3_1, with their parabolic bundles. To examine the intersection properties there it suffices to consider the images in the spherical conormal bundle to B^3_a, since this is $\mathrm{ff}(G_1^3)$. A crude picture of this is given in Figure **??**. In particular all pairs from the lifts of $B^2_{a,o}$, B^3_Θ and the $B^2_{\Theta,o'}$ meet parabolically cleanly. The transversality in (7.3) lifts to

$$B^2_{a,o} \cap B^2_{\Theta,o'} = \emptyset \text{ in } G_1^3 \text{ if } o \neq o'.[11.20]$$

Thus we can make two further parabolic blow-ups:

$$G_3^3 = \left[G_a^3; B^3_\Theta, S_3\right],\ G_\alpha^3 = \left[G_3^3; B^2_\Theta, S_2\right].[11.21]$$

In the second blow-up, B^2_Θ is the disjoint union of the lifts of the three submanifolds $B^2_{\Theta,o}$, this is a p-submanifold by the discussion above and one more application of Lemma (C.3). This defines the triple α-stretched product G^3_α.

The new front face introduced in the blow-up of B^3_Θ will be called the Θ-triple face and denoted $\Theta\mathrm{t}(G^3_3)$. Following the pattern of the adiabatic blow-up the new boundary hypersurfaces introduced in the final step of the construction of G^3_α will be called the Θ side faces, consisting of the three pieces Θsf, Θsc and Θss corresponding to the three partial diagonals. Thus we can list the boundary hypersurfaces as:

$$\begin{aligned} M_1(G_3^3) &= \{\mathrm{at}, \mathrm{asf}, \mathrm{asc}, \mathrm{ass}, \mathrm{ab}, \Theta\mathrm{t}, \Theta\mathrm{l}, \Theta\mathrm{m}, \Theta\mathrm{r}, \mathrm{eb}\} \\ M_1(G_\alpha^3) &= \{\mathrm{at}, \mathrm{asf}, \mathrm{asc}, \mathrm{ass}, \mathrm{ab}, \Theta\mathrm{t}, \Theta\mathrm{sf}, \Theta\mathrm{sc}, \Theta\mathrm{ss}, \Theta\mathrm{l}, \Theta\mathrm{m}, \Theta\mathrm{r}, \mathrm{eb}\}. \end{aligned} [11.22]$$

Consider the three partial diagonals, and the triple diagonal, of G^3 :

(7.7)
$$\begin{aligned} &\Delta_o = \{p \in G^3; \pi_h(p) = \pi_{h'}(p)\}, \\ &\qquad o = f, c, s \text{ and } \{h, h'\} = \{m, r\}, \{l, r\}, \{l, m\} \text{ respectively.} \end{aligned} [11.23]$$

(7.8)
$$\Delta_t = \{p \in G^3; \pi_l(p) = \pi_m(p) = \pi_r(p)\}.[11.24]$$

Each of these submanifolds lifts to an interior p-submanifold of G^3_α. Indeed one can see this step by step since at each stage of the construction (and for each of the four submanifolds, D, in (7.7), (7.8)) the submanifold blown up is either contained in D or disjoint from it. The most important properties of the α-triple product are related to the projections to the α-stretched product, G^2_α and to these lifted diagonals, which we denote $\Delta_{o,\alpha}$ for $o = f, c, s$ and Δ_t.

Proposition 7.5. *There are three stretched projections from G^3_α to G^2_α fixed by the commutative diagrams*

$$\begin{array}{ccc} G^3_\alpha & \xrightarrow{\pi^3_{o,\alpha}} & G^2_\alpha \\ \downarrow{\scriptstyle \beta^3_\alpha} & & {\scriptstyle \beta^2_\alpha}\downarrow \\ G^3 & \xrightarrow[\pi_o]{} & G^2 \end{array} \qquad o = f, c, s; [11.26]$$

all are b-fibrations with $\pi^3_{o,\alpha}$ transversal to $\Delta_{o',\alpha}$ if $o' \neq o$ and such that each $\pi^3_{o,\alpha}$ is a fibration near $\Delta_{t,\alpha}$ and maps it diffeomorphically onto $\Delta_\alpha \subset G^2_\alpha$.

Proof. Since G^3_α is symmetric in the sense that the factor-exchange maps lift to be diffeomorphisms it suffices to consider one case, say $o = f$. To produce the map $\pi^3_{f,\alpha}$ we give an alternative construction of G^3_α starting from the fibre product GG^2_α.

Writing out the definition of the triple α-stretched product from (7.3) and using the notation of §C.4:

$$G^3_\alpha = \left[G^3; B^3_a; B^2_a; B^3_\Theta, S_3; B^2_\Theta, S_2\right].[11.27]$$

Working in G^3 consider the images in NB^3_a of the manifold B^3_Θ and of the three $B^2_{a,o}$. We have already noted that the latter are disjoint, but so indeed are all four. It follows that the lifts of these four submanifolds are disjoint in G^3_1 so the order of blow-up can be exchanged using Lemma (C.7) giving

$$G^3_\alpha \simeq \left[G^3; B^3_a, B^3_\Theta, S_3; B^2_a; B^2_\Theta, S_2\right].[11.28]$$

Again in the normal bundle to B^3_a in G^3 the images of the $B^2_{a,o}$ only meet the $B^2_{\Theta,o'}$ if $o = o'$ and once both B^3_a and B^3_Θ are blown up it follows that the only intersections between the six submanifolds obtained by lifting the $B^2_{a,o}$ and the $B^2_{\Theta,o'}$ for $o, o' = f, c, s$ are between the pairs corresponding to the same projection i.e.

$$\text{in } \left[G^3; B^3_a; B^3_\Theta, S_3\right] \text{ if } o \neq o' \begin{cases} B^2_{a,o} \cap B^2_{a,o'} = \emptyset \\ B^2_{\Theta,o} \cap B^2_{\Theta,o'} = \emptyset \ [11.29] \\ B^2_{a,o} \cap B^2_{\Theta,o'} = \emptyset. \end{cases}$$

Thus we can further manipulate the order of the blow-ups and obtain

$$G^3_\alpha \simeq \left[G^3; B^3_a; B^3_\Theta, S_3; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}; B^2_{a,c}; B^2_{\Theta,c}, S_{2,c}; B^2_{a,s}; B^2_{\Theta,s}, S_{2,s}\right].[11.30]$$

From this it follows that there is a blow-down map

$$G^3_\alpha \longrightarrow \left[G^3; B^3_a; B^3_\Theta, S_3; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}\right].[11.31]$$

Again using the fact that $B^2_{\Theta,f} \subset B^3_\Theta$ and the fact that $B^2_{a,f}$ and B^3_Θ are disjoint in G^3_1 we can reorganize the target in (7.3) and obtain a blow-down map

$$G^3_\alpha \longrightarrow \left[G^3; B^3_a; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}; B^3_\Theta, S_3\right].[11.32]$$

Dropping the last blow-up, of B^3_Θ, and exchanging the order of the first two, which is possible since $B^3_a \subset B^2_{a,f}$ we conclude that there is an iterated blow-down map

$$G^3_\alpha \longrightarrow \left[G^3; B^2_{a,f}; B^3_a; B^2_{\Theta,f}, S_{2,f}\right].[11.33]$$

To complete the argument we need to interchange $B^2_{\Theta,f}$ and B^3_a; this is possible using Lemma (C.8) since

$$B^2_{\Theta,f} \pitchfork B^3_a \text{ in } \left[G^3; B^2_{a,f}\right].[11.34]$$

To see (7.3) note that the second lift is a submanifold of the front face, while the first is certainly transversal to the front face, so it suffices to prove transversality of the intersections within ff $\left[G^3; B^2_{a,f}\right]$ or equivalently of the images in $NB^2_{a,f} \cap \{\epsilon = 0\}$ of the submanifolds B^3_a and $B^2_{\Theta,f} \cap \{\epsilon = 0\}$. The second is a subbundle and the first is the restriction of the bundle to a submanifold so it is enough to show the transversality of B^3_a and $B^2_{\Theta,f} \cap \{\epsilon = 0\}$ as submanifolds of $B^2_{a,f} \cap \{\epsilon = 0\}$. In terms of the ball bundle structure of G_0 the second submanifold is again a subbundle, whilst the first is the restriction of the bundle to a submanifold hence the intersection is certainly transversal.

One can also easily see the transversality in (7.3) by introducing polar coordinates around $B^2_{a,f}$. Let $\{\epsilon; y, x, \rho; y', x', \rho'; y'', x'', \rho''\}$ denote local coordinates with $\rho = 0$ the boundary of the fibration of M_0 and the y's the remaining fibre variables. Polar coordinates are then given by

$$R^2 = \epsilon^2 + |x' - x''|^2; r = \frac{\epsilon}{R}, X'_i = \frac{x'_i - x''_i}{R}.$$

In these coordinates the lifts to G^3_a are

$$B^3_a = \{R = 0, x = x'\} \text{ and } B^2_{\Theta,f} = \{\rho' = \rho'' = 0, y' = y'', X' = 0\}[11.35]$$

and transversality follows

Thus we can interchange the order of the last two blow-ups on the right in (7.3) and so obtain a blow-down map to the fibre product of G^2_α and G :

$$G^3_\alpha \longrightarrow \left[G^3; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}\right] = G^2_\alpha G.[11.36]$$

Thus the desired map,

$$\pi^3_{o,\alpha} : G^3_\alpha \longrightarrow G^2_\alpha[11.37]$$

is defined as a composition of diffeomorphisms, blow-downs and projections; it is therefore a b-map.

To show that each $\pi^3_{o,\alpha}$ is a b-fibration we shall apply Proposition (B.4). Tables 3 and 4 show the boundary exponents for the adiabatic and Θ faces. The exponents linking a- and Θ-faces are all zero. The computation of these exponents is elementary. Thus from (B.5) it follows that the $\pi^3_{o,\alpha}$ are all b-normal maps.

Thus we need to check the surjectivity of the ordinary differential, at the various boundary faces, as in (B.4). Let us concentrate on $\pi^3_{f,\alpha}$. We shall use the description of G^3_α leading to (7.3), namely:

$$G^3_\alpha \simeq \left[G^3; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}; B^3_a; B^2_{a,s}; B^2_{a,c}; B^3_\Theta, S_3; B^2_{\Theta,s}, S_{2,s}; B^2_{\Theta,c}; S_{2,s}\right].[11.38]$$

This allows us to factorize

$$\pi^3_{f,\alpha} = F_1 \circ F_2[11.39]$$

where

$$\begin{gathered} F_1 : G^3_{I,f} \longrightarrow G^2_\alpha,\ F_2 : G^3_\alpha \longrightarrow G^3_{I,f} \\ G^3_{I,f} = \left[G^3; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}; B^3_a; B^2_{a,s}; B^2_{a,c}\right] \end{gathered}[11.40]$$

denote the indicated maps derived from (7.3).

Since F_1 is just the blow-down map for B^3_a, $B^2_{a,s}$ and $B^2_{a,c}$ followed by projection it can be seen to be a b-fibration by applying Proposition (B.4) and deducing (B.4) from the fact that the blow-down part of F_1 is a fibration of the three front faces over their bases, the lifts of B^3_a, $B^2_{a,s}$ and $B^2_{a,c}$ to $G^3_{I,f}$. These fibre over $\mathrm{af}(G^2_\alpha)$, $\mathrm{ab}(G^2_\alpha)$ and $\mathrm{ab}(G^2_\alpha)$ respectively.

If $F_2(p) = q \in \Theta\mathrm{t}$ then $\mathrm{range}(F_2)^* = T_q\,\Theta\mathrm{t}$. In $G^3_{I,f}$, $\Theta\mathrm{t} \cap \{\epsilon = 0\}$ is a half n-sphere bundle over $\Theta\mathrm{f} \cap \{\epsilon = 0\} \subset G^2_\alpha$ relative to F_1 and thus $(\pi^3_{f,\alpha})_*$ is a submersion along $\Theta\mathrm{t}$. Similarly, $\Theta\mathrm{ss} \cap \{\epsilon = 0\} \subset G^3_{I,f}$ is a half n-sphere bundle relative to F_1 over $\Theta\mathrm{l} \cap \{\epsilon = 0\}$ and $\Theta\mathrm{sc} \cap \{\epsilon = 0\} \subset G^3_{I,f}$ is a half n-sphere bundle relative to F_1 over $\Theta\mathrm{r} \cap \{\epsilon = 0\}$. For points p such that $F_2(p) = q \in \Theta\mathrm{ss} \setminus \Theta\mathrm{t}$ and $\Theta\mathrm{sc} \setminus \Theta\mathrm{t}$ the ranges of $(F_2)_*$ are $T_q\,\Theta\mathrm{ss}$ and $T_q\,\Theta\mathrm{sc}$ respectively. This completes the proof that $\pi^3_{f,\alpha}$ is a b-fibration, the other cases follow by symmetry.

To complete the proof of the theorem we need to show establish the transversality and fibration properties near to the partial diagonals and triple diagonal respectively. First we consider the behavior of $\pi^3_{f,\alpha}$ in a neighborhood of $\Delta_{s,\alpha}$. Due to the symmetries of G^3_α the analyses for any other disjoint pair of projection and partial diagonals are identical.

First observe that Δ_s lifts to $G^2_\alpha G$ as graph over G^2_α. Thus it suffices (and is necessary) to prove that the projection,

$$\pi^3_{f,i}; G^3_\alpha \longrightarrow G^2_\alpha G,$$

is injective on $T\Delta_{s,\alpha}$. Here $\Delta_{s,\alpha}$ is considered as a submanifold of G^3_α. This follows from Lemma (C.5), Lemma (C.6) and the description of G^3_α given in (7.3). One easily verifies that, beginning with $G^2_\alpha G = \left[G^3; B^2_{a,f}; B^2_{\Theta,f}, S_{2,f}\right]$, each successive blow-up in (7.3) satisfies the hypothesis of one of the lemmas.

The submanifolds $B^2_{a,o}, B^2_{\Theta,o}, o = s, c, B^3_a, B^3_\Theta$ are interior p-submanifolds of different boundary hypersurfaces of $G^2_\alpha G$. Thus they remain so under successive blow-ups. The conditions (C.5) are essentially immediate for $B^2_{a,o}, B^2_{\Theta,o}, o = s, c$ For $o = c$, the intersections with $\Delta_{s,\alpha}$ are empty whereas

$$B_{o,s} = \Delta_{s,\alpha} \cap M, o = a, \Theta$$

here M is an appropriate boundary hypersurface and $\Delta_{s,\alpha}$ is the appropriate lift of $\Delta_{s,\alpha}$. Since each B-submanifold lies in a different boundary hypersurface and $\Delta_{s,\alpha}$ is an interior p-submanifold these conditions are preserved under the successive blowups.

The intersection with the adiabatic triple diagonal follows in a similar manner as a simple calculations shows that in $G^2_\alpha G$

$$B^3_a = M \cap \Delta_{s,\alpha},$$

as above M is an appropriate boundary hypersurface. However B^3_Θ lies in a codimension 2 boundary component. A calculation in local coordinates shows that the lift of $\Delta_{s,\alpha}$ intersects B^3_Θ like a partial diagonal and thus we can apply Lemma (C.6) to obtain the injectivity of the blow-down restricted to $T\Delta_{s,\alpha}$ at this stage. Recursively applying Lemma (C.5) establishes that $\pi^3_{f,i*}$ restricted to $\Delta_{s,\alpha}$ is injective. The transversality statement follows since

$$\dim G^2_\alpha = \dim \Delta_{s,\alpha}.$$

To establish that $\pi^3_{f,\alpha}$ is a fibration in a neighborhood of $\Delta_{t,\alpha} \subset G^3_\alpha$ we use the description afforded by (7.3). One easily verifies that $\Delta_{t,\alpha} \cap \partial G^3_\alpha$ is compactly contained within $B^3_a \cup B^3_\Theta$. Thus the projection to G^2_α is already defined in a neighborhood of $\Delta_{t,\alpha}$ in the intermediate space $\left[G^3; B^3_a; B^3_\Theta, S^3\right]$. The fibration condition is easily verified by local coordinate calculations analogous to those used in the proof of Lemma (C.6) □

The rows labelled marked 'ν' in Tables 3 and 4 represent important density information used in the composition formula. They arise from the identity of bundles:

$$(\pi^3_{f,\alpha})^*\Omega^{\frac{1}{2}} \otimes (\pi^3_{c,\alpha})^*\Omega^{\frac{1}{2}} \otimes (\pi^3_{s,\alpha})^*\Omega^{\frac{1}{2}} \equiv \rho^\nu \Omega \otimes |d\epsilon|^{\frac{1}{2}}.[11.41]$$

One way to compute the exponents ν is first to compute the exponents in

$$(\pi^3_{f,\alpha})^*\Omega \otimes \pi^3_{l,\alpha}\Omega = \rho^{\nu(f)}\Omega|d\epsilon|[11.42]$$

which is easily done using the alternative construction of G^3_α described in the proof of Proposition 7.5. By relabelling the faces one can deduce the corresponding exponent sets $\nu(s)$ and $\nu(c)$. Then, using the direct construction of G_α^3 one can similarly compute the exponents for

$$(\beta_\alpha^3)^*\Omega = \rho^\mu\Omega.[11.43]$$

The exponents for (7.3) are given by $\nu = \frac{1}{2}(\nu(f) + \nu(c) + \nu(s)) - \mu$.

CHAPTER 8

The α-calculus

We now consider the calculus of general (but always polyhomogeneous) pseudodifferential operators associated to the α-structure on G. In (7.2) we have defined the small calculus of α-pseudodifferential operators. We enlarge this by allowing more general conormal behaviour of the kernels at the boundaries of G^2_α. Let $\mathcal{E}$ be an index family for G^2_α. Thus (see Appendix B and [**12**]) $\mathcal{E}$ associates to each boundary hypersurface of G^2_α an index set. We label and order the boundary hypersurfaces as in (7.2) so

$$\mathcal{E} = (E_{\mathrm{af}}, E_{\mathrm{ab}}, E_{\Theta\mathrm{f}}, E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}})[12.1]$$

where each E_h is an index set, i.e. a discrete subset

$$E_h \subset \mathbb{C} \times \mathbb{N}_0,\ \mathbb{N}_0 = \{0, 1, \dots\}[12.2]$$

with the properties:

$$e_j \in E_h,\ |e_j| \longrightarrow \infty \Longrightarrow e_j = (z_j, \ell_j) \text{ with } \operatorname{Im} z_j \longrightarrow -\infty[12.3] \tag{8.1}$$

$$(z, \ell) \in E_h \Longrightarrow (z - ik, \ell') \in E_h \quad \forall\ k \in \mathbb{N},\ 0 \le \ell' \le \ell,\ \ell' \in \mathbb{N}_0.[12.4] \tag{8.2}$$

The first condition in (8.2) ensures that any formal sum of powers

$$\sum_{(z,k)\in E_h} a(z, k)\rho^{iz}(\log \rho)^k[12.5]$$

can be asymptotically summed using an appropriate form of Borel's lemma. In Appendix B the space, $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X)$, of polyhomogeneous conormal functions on a manifold with corners is defined (as in [**12**]) in terms of such expansions. The second condition in (8.2) ensures that this space is a $\mathcal{C}^\infty$-module. More generally if $Y \subset X$ is an interior p-submanifold then for any real number m the space

$$I^{m,\mathcal{E}}_{\mathrm{phg}}(X, Y; F) \subset \mathcal{C}^{-\infty}(X; F)[12.6]$$

of distributions conormal of 'Kohn-Nirenberg' type and order m at Y and polyhomogeneous with exponents from $\mathcal{E}$ at the boundary of X is defined for any vector bundle F. We generalize (7.2) by setting

$$\Psi^{m,\mathcal{E}}_\alpha(G; F) = I^{m-\frac{1}{4},\mathcal{E}}_{\mathrm{phg}}(G^2_\alpha, \Delta_\alpha; \mathrm{KD}_\alpha \otimes \pi^*_{r,\alpha}[\Omega^{\frac{1}{2}} \otimes F] \otimes \pi^*_{l,\alpha}[\Omega^{-\frac{1}{2}} \otimes F'])[12.7]$$

where F' is the dual bundle to F. The identification (7.2) (generalized to include F) still applies, so these pseudodifferential operators can be regarded as continuous linear maps

$$A : \dot{\mathcal{C}}^\infty(G; F \otimes \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}(G; F \otimes \Omega^{\frac{1}{2}}).[12.8]$$

Of course they have much more refined mapping properties than this. We note one, for conormal functions on G, which is enough to allow the composition properties of these operators to be described.

LEMMA 8.1. *For any $m \in \mathbb{R}$, any index family $\mathcal{E}$ for G^2_α and any index family $\mathcal{M} = (M_a, M_\Theta)$ for G, each $A \in \Psi^{m,\mathcal{E}}_\alpha(G;\Omega^{\frac12})$ defines a linear map*

$$\begin{gathered} A : \mathcal{A}^{\mathcal{M}}_{\text{phg}}(G;\Omega^{\frac12}) \longrightarrow \mathcal{A}^{\mathcal{M}'}_{\text{phg}}(G;\Omega^{\frac12}) \text{ provided} \\ M_\Theta + E_{\Theta\text{r}} > -1 \text{ and } \mathcal{M}' = (M'_a, M'_\Theta) \text{ with} \qquad [12.10] \\ M'_a = (M_a + E_{\text{af}})\overline{\cup}(M_a + E_{\text{ab}}),\ M'_\Theta = (M_\Theta + E_{\Theta\text{f}})\overline{\cup}(M_\Theta + E_{\Theta\text{l}}). \end{gathered}$$

PROOF. The formula (7.2) for the action on $\dot{\mathcal{C}}^\infty(G;\Omega^{\frac12})$ can be rewritten as

$$A\phi \cdot \nu = (\pi^2_{l,\alpha})_*[A \cdot (\pi^2_{l,\alpha})^*\nu \cdot (\pi^2_{r,\alpha})^*\phi] \quad \forall\ \phi \in \dot{\mathcal{C}}^\infty(G;\Omega^{\frac12})[12.11]$$

where the product on G^2_α is to be interpreted as a density using (7.2), (7.2) and (7.2). Using the information from Table 1 for the b-fibrations $\pi^2_{l,\alpha}$ and $\pi^2_{r,\alpha}$ we see that

$$\begin{gathered} A \cdot (\pi^2_{l,\alpha})^*\nu \cdot (\pi^2_{r,\alpha})^*\phi \in I^{m-\frac14,\mathcal{F}}_{\text{phg}}(G^2_\alpha, \Delta_\alpha;\Omega) \text{ with} \\ F_{\text{af}} = E_{\text{af}} + M_a,\ F_{\text{ab}} = E_{\text{ab}} + M_a, \qquad [12.12] \\ F_{\Theta\text{f}} = E_{\Theta\text{f}} + M_\Theta,\ F_{\Theta\text{l}} = E_{\Theta\text{l}},\ F_{\Theta\text{r}} = E_{\Theta\text{r}} + M_\Theta. \end{gathered}$$

Then, using Theorem (B.3) and the assumption $E_{\Theta\text{l}} + M_\Theta > -1$, the push-forward to G is polyhomogeneous conormal since, as shown in Proposition 7.4, the map $\pi^2_{l,\alpha}$ is transversal to Δ_α. The formula for the index family gives (8.1). ☐

Operators in $\Psi^{m,\mathcal{E}}_\alpha(G;\otimes\Omega^{\frac12})$ have a variety of symbol maps corresponding to the diagonal and the boundary components of G^2_α. The diagonal symbol is defined as for the Θ-calculus. In fact, since the diagonal only meets the adiabatic boundary in the interior of the front face, where ϵ is a pure parameter, we can compute the diagonal symbol an ϵ-fiber at a time using the recipe given in (7.3) of [**5**] to obtain a symbol

$$\sigma_m(A) \in S^{\{m\}}(N^*\Delta_\alpha) \otimes \Omega_{\text{fiber}}(N^*\Delta_\alpha) \otimes \rho_{\Theta\text{f}}^{-\frac12 N}\rho_{\text{af}}^{-\frac12 n}.[12.101]$$

All that remains is to identify the density factors to obtain a

$${}^\alpha\sigma_m(A) \in S^{\{m\}}({}^\alpha T^*G).[12.102]$$

I'll leave that to you Richard. We also need to state the exact sequences for the various symbols and the standard consequences of these sequences.

In addition to the diagonal symbol there is also a symbol defined for each boundary component of G^2_α by (B.2), (B.2). These take values in polyhomogeneous distributions defined on the boundary face. These symbols are denoted by

$$\sigma_{\text{af}}(A), \sigma_{\text{ab}}(A), \sigma_{\Theta\text{l}}(A), \sigma_{\Theta\text{r}}(A), \sigma_{\Theta\text{f}}(A),$$

respectively. The symbols at the front faces, are usually called normal operators. They are discussed in some detail below.

From (8.1) we see that the composite $A \circ B$ of two α-pseudodifferential operators, $A \in \Psi^{m,\mathcal{E}}_\alpha(G;\Omega^{\frac12})$ and $B \in \Psi^{m',\mathcal{E}'}_\alpha(G;\Omega^{\frac12})$ is well-defined provided

$$E_{\Theta\text{r}} + E'_{\Theta\text{l}} > -1.[12.13]$$

Proposition 8.1. *If* (8) *holds then*

$$\Psi_\alpha^{m,\mathcal{E}}(G;\Omega^{\frac12})\cdot\Psi_\alpha^{m',\mathcal{E}'}(G;\Omega^{\frac12})\subset\Psi_\alpha^{m+m',\mathcal{F}}(G;\Omega^{\frac12})[12.15]$$

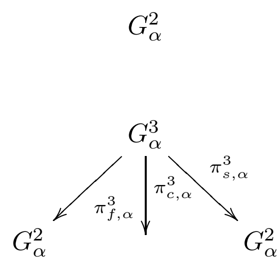
where the index family $\mathcal{F}$ *is composed of the index sets*

$$\begin{aligned}F_{\mathrm{af}}&=(E_{\mathrm{af}}+E'_{\mathrm{af}})\overline{\cup}(E_{\mathrm{ab}}+E'_{\mathrm{ab}}+n)\\F_{\mathrm{ab}}&=(E_{\mathrm{af}}+E'_{\mathrm{ab}})\overline{\cup}(E_{\mathrm{ab}}+E'_{\mathrm{af}})\overline{\cup}(E_{\mathrm{ab}}+E'_{\mathrm{ab}})\\F_{\Theta\mathrm{f}}&=(E_{\Theta\mathrm{f}}+E'_{\Theta\mathrm{f}})\overline{\cup}(E_{\Theta\mathrm{l}}+E'_{\Theta\mathrm{r}}+2n+1)\quad[12.16]\\F_{\Theta\mathrm{l}}&=(E_{\Theta\mathrm{f}}+E'_{\Theta\mathrm{l}})\overline{\cup}E_{\Theta\mathrm{l}}\\F_{\Theta\mathrm{r}}&=(E_{\Theta\mathrm{r}}+E'_{\Theta\mathrm{f}})\overline{\cup}E'_{\Theta\mathrm{r}}.\end{aligned}$$

The diagonal symbol satisfies

$${}^{\alpha}\sigma_{m+m'}(A\cdot B)={}^{\alpha}\sigma_m(A)\cdot{}^{\alpha}\sigma_{m'}(B).[12.161]$$

Proof. For the composition formula we use the diagram of b-fibrations which follows from Proposition 7.5:



[12.17]

If $\nu\in\mathcal{C}^\infty(G_\alpha^2;\mathrm{KD}_\alpha^*\otimes\Omega)$ then the composition kernel can be written

$$C\cdot\nu=(\pi^3_{c,\alpha})_*[(\pi^3_{c,\alpha})^*\nu\cdot(\pi^3_{s,\alpha})^*A\cdot(\pi^3_{f,\alpha})^*B][12.18]$$

where again the bracketed expression is to be interpreted as a density. Using (7.2) and Tables 3 and 4 it can be seen that

$$(\pi^3_{c,\alpha})^*\nu\cdot(\pi^3_{s,\alpha})^*A\cdot(\pi^3_{f,\alpha})^*B\in\mathcal{A}^{\mathcal{G}}_{\mathrm{phg}}[12.19]$$

where the boundary hypersurfaces of G_α^3 are as listed as in (7.3) and

$$\begin{gathered}G_{\mathrm{at}}=E_{\mathrm{af}}+E'_{\mathrm{af}},\ G_{\mathrm{asf}}=E_{\mathrm{ab}}+E'_{\mathrm{af}},\\G_{\mathrm{asc}}=E_{\mathrm{ab}}+E'_{\mathrm{ab}}+n,\ G_{\mathrm{ass}}=E_{\mathrm{af}}+E'_{\mathrm{ab}},\\G_{\mathrm{ab}}=E_{\mathrm{ab}}+E'_{\mathrm{ab}}\ G_{\Theta\mathrm{t}}=E_{\Theta\mathrm{f}}+E'_{\Theta\mathrm{f}},\\G_{\Theta\mathrm{sf}}=E_{\Theta\mathrm{r}}+E'_{\Theta\mathrm{f}},\ G_{\Theta\mathrm{sc}}=E_{\Theta\mathrm{l}}+E'_{\Theta\mathrm{r}}+2n+1,\\G_{\Theta\mathrm{ss}}=E_{\Theta\mathrm{f}}+E'_{\Theta\mathrm{l}},\ G_{\Theta\mathrm{l}}=E_{\Theta\mathrm{l}},\\G_{\Theta\mathrm{m}}=E_{\Theta\mathrm{r}}+E'_{\Theta\mathrm{l}},\ G_{\Theta\mathrm{r}}=E'_{\Theta\mathrm{r}}.\end{gathered}[12.20]$$

Again using the push-forward theorem for polyhomogeneous conormal distributions under b-fibrations, Theorem (B.5), and Tables 3 and 4 we obtain (8.1) provided the integrability conditions are satisfied, proving the proposition.

(say for smoothing operators with compact support in the interior, need to treat diagonal singularity and symbolic composition) □

We also wish to consider the normal operators of these α-pseudodifferential operators, particularly the adiabatic normal operator. We shall suppose for simplicity that the index set at the adiabatic boundary is empty:

$$E_{\mathrm{ab}} = \emptyset.[12.21]$$

It follows from this and (8.1) that under composition orders at the adiabatic front face simply add. Suppose further that

$$E_{\mathrm{af}} = 0, [12.22]$$

so that the kernels are smooth up to the adiabatic front face, as they are for elements of $\mathrm{Diff}^m_\alpha(G; \Omega^{\frac{1}{2}})$. The general discussion of symbols in §B.2 leads to to following result on the normal operator for this small α-calculus:

PROPOSITION 8.2. *If $\mathcal{E}$ is an index set for G^2_α satisfying* (8) *and* (8) *then restriction to the front face defines the normal homomorphism*

$$\sigma_{\mathrm{af}} : \Psi^{m,\mathcal{E}}_\alpha(G; \Omega^{\frac{1}{2}}) \twoheadrightarrow I^{m-\frac{1}{4},\mathcal{E}^{\mathrm{af}}}_{\mathrm{phg}}(\mathrm{af}(G^2_\alpha), \Delta_\alpha; \mathrm{KD}_\alpha)[12.24]$$

The change of order ** in just reflects the change of dimension and codimension of the lifted diagonal in the passage from G^2_α to $\mathrm{af}(G^2_\alpha)$. One useful invertibility property which follows directly from Proposition 8.1 is that given (8) and (8)

$$\mathrm{Id} - A \text{ is invertible if } \in \Psi^{0,\mathcal{E}}_\alpha(G; \Omega^{\frac{1}{2}}) \text{ and } \sigma_{\mathrm{af}}(A) = 0.[12.25]$$

PROPOSITION 8.3. *Normal operators compose.*

yes and how

To conclude this section we establish the basic L^2-mapping properties for operators in the α-calculus.

THEOREM 8.1. *If $A \in \Psi^{m;\mathcal{E}}_\alpha(G; \Omega^{\frac{1}{2}})$ such that*

$$m \leq 0, E_{\mathrm{af}} \geq 0, E_{\Theta\mathrm{f}} \geq 0, E_{\mathrm{ab}} > 0, E_{\Theta\mathrm{l}} > -\frac{1}{2}, E_{\Theta\mathrm{r}} > -\frac{1}{2}, [12.27]$$

then A defines a bounded mapping of $L^2(G; \Omega^{\frac{1}{2}})$ to itself.

PROOF. Using Hörmander's elegant symbolic argument and the composition formula for the diagonal symbol we can construct a 'square root' B such that for some constant, C,

$$C - A^*A = B^*B + R, \text{ where } B \in \Psi^{0;E}_\alpha(G; \Omega^{\frac{1}{2}}), R \in \Psi^{-\infty,E}_\alpha.[12.28]$$

Thus it suffices to prove the theorem for operators of order $-\infty$.

The proof follows, essentially by an application of Schur's theorem. The kernel for B can be thought of as a smooth function on G^2 with certain singularities along the boundaries and along the submanifolds of the boundaries which are blown-up in the construction of G^2_α. Let $f, g \in L^2(G; \Omega^{\frac{1}{2}})$.

alot more goes here □

We define the α-Sobolev spaces

$$\mathcal{H}^s_\alpha(G;\Omega^{\frac12}) = \{f \in L^2(G;\Omega^{\frac12}); A\cdot f \in L^2, A \in \Psi^s_\alpha(G;\Omega^{\frac12})\}, s \geq 0$$
$$\mathcal{H}^s_\alpha(G;\Omega^{\frac12}) = \{u \in \mathcal{C}^{-\infty}(G;\Omega^{\frac12}); u\sum_i P_i u_i, P_i \in \Psi^s_\alpha,\ u_i \in L^2(G,\Omega^{\frac12})\}, s \leq 0. \quad [12.38]$$

We can also add weighting by powers of the defining functions for the boundary hypersurfaces. Let $s = (s_1, s_2)$, we define

$$\rho_a^{s_1}\rho_\Theta^{s_2} = \{u \in \mathcal{C}^{-\infty}(G;\Omega^{\frac12}); \rho_a^{-s_1}\rho_\Theta^{-s_2}u \in \mathcal{H}^s_\alpha(G;\Omega^{\frac12})\}. [12.39]$$

PROPOSITION 8.4. *If $A \in \Psi^{m;\mathcal{E}}_\alpha(G;\Omega^{\frac12})$ such that*

$$s - m \geq t,\ E_{\mathrm{af}} + s_1 - t_1 \geq 0,\ E_{\Theta\mathrm{f}} + s_2 - t_2 \geq 0,$$
$$E_{\mathrm{ab}} + s_1 - t_1 > 0,\ E_{\Theta\mathrm{l}} - t_2 > -\frac12,\ E_{\Theta\mathrm{r}} + s_2 > -\frac12, \quad [12.41]$$

then A defines a bounded mapping of $\rho^{s_1}_{\mathrm{ab}}\rho^{s_2}_{\Theta\mathrm{b}}\mathcal{H}^s_\alpha(G;\Omega^{\frac12})$ to $\rho^{t_1}_{\mathrm{ab}}\rho^{t_2}_{\Theta\mathrm{b}}\mathcal{H}^t_\alpha(G;\Omega^{\frac12})$.

PROOF. This is proved by considering the operator

$$\rho^{-t_1}_{\mathrm{ab}}\rho^{-t_2}_{\Theta\mathrm{b}}A\rho^{s_1}_{\mathrm{ab}}\rho^{s_2}_{\Theta\mathrm{b}}$$

and applying Proposition 8.1 and Theorem 8.1. □

In addition to these mapping results for degenerate Sobolev spaces we can also establish mapping properties with respect to an-isotropic, nondegenerate Sobolev spaces. We call these Heisenberg-Sobolev spaces. They are defined by the Θ-structure and are well known to be the correct context to discuss the mapping properties of operators like the Szegö kernel. We need only extend them to the adiabatic context. These spaces are defined by a weight structure on $\mathcal{V}_a(G)$. Let ${}^a\theta$ be the one form given in (7.3) which defines the adiabatic Θ-structure on G. Elements of weight one are given by

$$\mathcal{V}^1_a(G) = \{V \in \mathcal{V}_a(G); {}^a\theta(V) = O(\rho_{\Theta\mathrm{b}})\}$$

and of weight 2 by

$$\mathcal{V}^2_a(G) = \{V \in \mathcal{V}_a(G); {}^a\theta(V) = O(1)\}$$

Evidently $\mathcal{V}^1_a(G) \subset \mathcal{V}^2_a(G)$, in local coordinates near a Θ-boundary point one easily verifies that that there exists a weight 2 vector field T, tangent to the boundary, such that ${}^a\theta(T) = 1$. Thus every $V \in \mathcal{V}^2_a(G)$ can be locally represented as

$$V = fT + V'; f = {}^a\theta(V),\ \text{forsome}\ V' \in \mathcal{V}^1_a(G).$$

Let $\mathfrak{w}(V)$ denote the weight of the vector field V. This requires a bit of clarification as the weight can vary from point to point, by convention we take the weight to be defined as the maximum weight over all points in the domain of definition of V.

We can extend the weight structure to the universal enveloping algebra, $\mathrm{Diff}^*_a(G)$. First we define the weight of a monomial $V_1 \dots V_l$ to be the sum of the weights of the factors. A little care is required to extend the definition to sums, as a sum of monomials may have lower weight than the maximum of the sums of weights of factors. For example if $V_1, V_2 \in \mathcal{V}^1_a(G)$ then $\mathfrak{w}([V_1, V_2])$ can be either 0,1 or 2. A differential operator in $P \in \mathrm{Diff}^*_a(G)$ is said to be of weight $m = \mathfrak{w}(P)$ if, in a neighborhood of every point, P can be expressed as sum of monomials of weight m and m is the smallest integer with this property. Now we can define the Heisenberg-adiabatic Sobolev spaces for even positive integral $s = 2m$:

$$\mathfrak{H}^s_a(G;\Omega^{\frac{1}{2}}) = \\ \{u \in \mathcal{C}^{-\infty}(G;\Omega^{\frac{1}{2}}); Pu \in L^2(G;\Omega^{\frac{1}{2}}) \ \forall \ P \in \mathrm{Diff}^*_a(G) \text{ with } \mathfrak{w}(P) \leq s\}.$$

A Hilbert space structure can be defined in an obvious way. The following is easily verified:

Proposition 8.5. *A differential operator in $P \in \mathrm{Diff}^*_a(G;\Omega^{\frac{1}{2}})$ of order l and weight m lifts to G^2_α to define an element of*

$$\Psi_\alpha^{-l;0,\emptyset,\emptyset,\emptyset,-m}(G;\Omega^{\frac{1}{2}}).$$

Proof. Introducing local coordinates one shows that there exists a local basis for $\mathcal{V}_a(G)$, $\{X, Y_1, \dots, Y_{2(n-1)}, T\}$ such that

$${}^a\theta(X) = {}^a\theta(Y_i) = 0, i = 1 \dots, 2(n-1);\ {}^a\theta(T) = 1.$$

The vector fields $\{\rho_{\Theta\mathrm{b}}X, \rho_{\Theta\mathrm{b}}Y_i, \rho^2_{\Theta\mathrm{b}}T\}$ define a local basis for $\mathcal{V}_\alpha(G)$. Lifting these vector fields to G^2_α on the left, gives smooth vector fields, tangent to $\Theta\mathrm{f}$ of the form

$$\{\rho_{\Theta\mathrm{l}}\widetilde{X}, \rho_{\Theta\mathrm{l}}\widetilde{Y}_i, \rho^2_{\Theta\mathrm{l}}\widetilde{T}\}.$$

The defining function lifts to $\rho_{\Theta\mathrm{l}}\rho_{\Theta\mathrm{f}}$. From this it follows easily that for a polynomial P of order l and weight m

$$P = \rho^{-m}_{\Theta\mathrm{b}}\widetilde{P}; \widetilde{P} \in \mathrm{Diff}^l_\alpha(G;\Omega^{\frac{1}{2}}).$$

The conclusion is immediate from this. □

To prove the mapping results for the Heisenberg Sobolev spaces we need to prove a commutation result

Lemma 8.2. *If $A \in \Psi^{m;\mathcal{E}}_\alpha(G;\Omega^{\frac{1}{2}})$ and $V \in \mathcal{V}_a(G;\Omega^{\frac{1}{2}})$ then there is an $V' \in \mathcal{V}_a(G;\Omega^{\frac{1}{2}})$and an $A' \in \Psi^{m;\mathcal{E}}_\alpha(G;\Omega^{\frac{1}{2}})$ such that*

$$A \circ V = V' \circ A + A'.[12.46]$$

Or something like this!!!!!!

Proof. □

Using Lemma 8.2 we can easily prove the mapping theorem for positive even integral spaces and using interpolation extend to all positive values. Then using duality we obtain the negative values.

Theorem 8.2. *Let $A \in \Psi^{m;\mathcal{E}}_\alpha(G;\Omega^{\frac{1}{2}})$ then A defines a bounded mapping $A : \mathfrak{H}^s_a(G;\Omega^{\frac{1}{2}}) \longrightarrow \mathfrak{H}^t_a(G;\Omega^{\frac{1}{2}})$ provided*

$$s - m \geq t, E_{\Theta\mathrm{f}} \geq t - s, E_{\Theta\mathrm{l}} > -\frac{1}{2}, E_{\Theta\mathrm{r}} > -\frac{1}{2}, E_{\mathrm{af}} \geq 0, E_{\mathrm{ab}} > 0.$$

Proof. □

8.1. The adiabatic model problem

As shown in Proposition 7.3 the model problem at the adiabatic face is the Laplace operator on $(n,1)$–forms defined by the metric associated to the Kähler form

$$ {}^a\omega = \partial\overline{\partial}(1-|y|^2)[13.1] $$

on ${}^aF = \mathbb{B}^n \times \mathbb{R}^n$. The Riemannian metric is

$$ \begin{gathered} {}^ag = \sum_{i,j=1}^{n} {}^ag_{ij}\, dx^i dx^j + \sum_{i,j=1}^{n} {}^ag_{ij} dy^i dy^j, \\ \text{where } {}^ag_{ij} = \frac{\frac{1}{2}\delta_{ij}}{1-|y|^2} + \frac{y^i y^j}{(1-|y|^2)^2}. \end{gathered} \quad [13.2] $$

The Laplacian will be denoted ${}^a\Delta$.

This operator falls outside the scope of the theory developed in [**5**] because ${}^aF$ is not compact, although its boundary is certainly strictly pseudoconvex. Since the translation group $\mathbb{R}^n$ acting on the x variables is a group of holomorphic isometries we can use the partial Fourier transform to show that ${}^a\Delta_{n,1}$ is invertible and to analyze the Schwartz kernel of its inverse.

By arguing as in §3 we see that ${}^a\Delta_{n,1}$ is unitarily equivalent to ${}^a\Delta_{n-1,0}$ and using the basis defined in (3) we deduce that this operator is determined by the quadratic form

$$ D(f,f) = \int \sum_{i,j=1}^{n} {}^ag_{i\bar{j}}\, {}^ag^{p\bar{q}} \partial_{\bar{z}_p} f_i \overline{\partial_{\bar{z}_q} f_j}\, dx\gamma(y)dy.[13.3] $$

where $dx\gamma(y)dy$ is the volume element. For each $\xi \in \mathbb{R}^n$ set

If $\hat{f}$ denotes the partial Fourier transform of f in the x directions then by the Plancherel relation

$$ D(f,f) = \int\limits_{\mathbb{R}^n} D_\xi(\hat{f},\hat{f})\, d\xi.[13.5] $$

Similarly the L^2 form becomes

$$ N(f,f) = \int\limits_{\mathbb{R}^n} N_0(\hat{f},\hat{f})\, d\xi.[13.6] $$

where

$$ N_0(h,h) = \sum_{i,j}^{n} \int {}^ag_{ij} h_i \overline{h_j}\ \mathrm{dVol}(y).[13.7] $$

Notice that the vector bundle $\Lambda^{n,1}$ is naturally trivial over $\mathbb{R}^n$, so can be considered as a bundle over $\mathbb{B}^n$ and (3) is a translation-invariant basis, so gives a basis over $\mathbb{B}^n$.

Each of the quadratic forms D_ξ defines a second order operator L_ξ acting on sections of $\Lambda^{n,1}$. Then (8.1) shows that

$$ \widehat{{}^a\Delta_{n,1}} f(\xi,y) = L_\xi \hat{f}(\xi,y).[13.8] $$

The operators L_ξ are all $\mathcal{V}_0$-elliptic differential operators on $\mathbb{B}^n$, i.e. could be analyzed using the $\mathcal{V}_0$-calculus developed in [**?**], see also [**12**]. However we do not need to delve into this theory here because we can construct a rather precise parametrix for the L_ξ by using the results of [**5**] and more particularly of §4 above. To do so we use compact quotients of ${}^aF$.

If Γ is a maximal rank lattice (e.g. $\mathbb{Z}^n$) in $\mathbb{R}^n$ the quotient under the translation action, ${}^\Gamma F = \mathbb{T}_\Gamma \times \mathbb{B}^n = {}^aF/\Gamma$, is a compact manifold with strictly pseudoconvex boundary. In particular Theorem (4.1) applies to the Laplacian, ${}^\Gamma\Delta_{n,1}$, of the metric on this quotient induced by the translation-invariant metric ${}^ag$:

Lemma 8.3. *For each lattice $\Gamma \subset \mathbb{R}^n$ of maximal rank the Laplacian ${}^\Gamma\Delta_{n,1}$ is invertible and has inverse*

$$\begin{aligned} E_\Gamma = {}^\Gamma\Delta_{n,1}^{-1} \in \Psi_\Theta^{-2;\{n+1,1\},\{-n,1\}}({}^\Gamma F_{\frac12}; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac12}) \\ +\Psi^{-\infty;\{n+1,1\},\{-n,1\}}({}^\Gamma F_{\frac12}; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac12}). \end{aligned} \quad [13.10]$$

Proof. The invertibility follows from the cohomological interpretation of the null space as discussed in [**?**].

Needs more explanation, or at least a proper reference □

If $f \in \mathcal{C}^\infty(\mathbb{T}_\Gamma; \Lambda^{n,1})$ is lifted to $\Pi_\Gamma^* f \in \mathcal{C}^\infty({}^aF; \Lambda^{n,1})$ as a Γ-invariant section then

$$\pi_\Gamma^*({}^\Gamma\Delta f) = {}^a\Delta\pi_\Gamma^* f.[13.11]$$

This allows us to construct a parametrix for ${}^a\Delta$ from the inverse for any one of the operators ${}^\Gamma\Delta$. Choose some function $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ which is identically equal to 1 near 0 and has support in such a small neighbourhood of 0 that the supports of all Γ-translates of ϕ do not meet the support of ϕ itself. Thus ϕ becomes a smooth function on $\mathbb{T}_\Gamma$ and we can unambiguously set

$$E_1(x, y; 0, y') = \phi(x)E_\Gamma(x, y; 0, y'), [13.12]$$

meaning that $E(x, y; 0, y')$ vanishes outside the support of ϕ. Then

$$E_1(x, y; x', y') = E(x - x', y; 0, y')[13.13]$$

is a parametrix for ${}^a\Delta_{n-1}$. Indeed consider the remainder term in

$${}^a\Delta_{n,1}E_1 = \operatorname{Id} - R_1(x, y; 0, y')[13.14]$$

Here the kernel of R_1 vanishes identically near the diagonal and from the locality of ${}^a\Delta$ we can interpret it as an operator on ${}^\Gamma F$; then

$$R_1 \in \Psi^{-\infty;\{n+1,1\},\{-n,1\}}({}^\Gamma F_{\frac12}; {}^\Theta\Lambda^{n,1} \otimes \Omega^{\frac12}).[13.15]$$

Notice that it is to be lifted to ${}^aF$ using the fact that it has support near 0, not as a Γ-invariant operator.

As a translation-invariant operator R_1 has kernel $R_1(x, y; 0, x')$ which has compact support in x, as does E_1 itself. Taking the partial Fourier transform of (8.1) and using (8.1) gives

$$L_\xi \widehat{E}_1(\xi) = \operatorname{Id} - \widehat{R}_1(\xi).[13.16]$$

Consider the remainder term here. Since x is just a smooth parameter of compact support in R_1 the Fourier transform $\widehat{R}_1(\xi)$ is a polyhomogeneous conormal distribution on $(\mathbb{B}^n)^2$, with values in the kernel bundle, which is smooth in ξ (in fact entire) with all derivatives rapidly decreasing. Thus the kernel of $\widehat{R}_1(\xi)$ is just a finite linear combination of products of powers of the defining functions for the left and right boundaries of $(\mathbb{B}^n)^2$ with $\mathcal{C}^\infty$ sections of the kernel bundle which are rapidly decreasing with all derivatives in ξ. If we compactify $\mathbb{R}^n_\xi$ to $\mathbb{B}^n$ by stereographic projection then the space of these kernels can be written

$$\mathcal{A}^{\emptyset;n+1,-n}_{\mathrm{phg}}(\mathbb{B}^n_\xi \times (\mathbb{B}^n)_{\frac{1}{2}}; {}^{\Theta}\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).[13.17]$$

We know less explicitly about $\widehat{E}_1(\xi)$, but it is certainly smooth in ξ so the kernels $\widehat{E}_1$ do exist. We can use (8.1) and straightforward iteration arguments to prove:

Proposition 8.6. *For each $\xi \in \mathbb{R}^n$ the operator L_ξ is invertible and the inverse has Schwartz kernel*

$$\widehat{E}(\xi) = \widehat{E}_1(\xi) + \widehat{E}'(\xi)[13.19]$$

where $\widehat{E}'$ is in the space (8.1).

Proof. Certainly each $R_1(\xi)$ is compact with respect to the L^2 norm (8.1). Thus from (8.1) (and its adjoint) we conclude that L_ξ has at most finite dimensional null space, consisting of polyhomogeneous conormal sections. However for any $\xi \in \mathbb{R}^n$ a lattice G can be found such that ξ is in the dual lattice. Then, from (8.1) it would follow that the operator ${}^\Gamma\Delta$ had non-trivial null space, contrary to Lemma 8.3. Thus all the L_ξ (for real ξ) are invertible with inverse $\widehat{E}(\xi)$. From (8.1) it follows that

$$\widehat{E}(\xi) - \widehat{E}_1(\xi) = \widehat{E}(\xi)R_1(\xi) = [\widehat{E}(\xi) - \widehat{E}^*_1(\xi)]\widehat{R}_1(\xi) + \widehat{E}^*_1(\xi)R_1(\xi).[13.20]$$

Here, the fact that $\widehat{E}^*_1(\xi)$ and $R_1(\xi)$ can be composed follows from the properness of the supports of both E_1 and R_1, which allows the composition to be carried out on some quotient. It follows that this composite is also in the space (8.1), except possibly for higher multiplicity of the index sets. Using the same argument with $\widehat{E}$ as right inverse and (8.1) gives

$$\widehat{E}(\xi) - \widehat{E}_1(\xi) = \widehat{R}^*_1(\xi)\widehat{E}(\xi)\widehat{R}_1(\xi) + \widehat{E}^*_1(\xi)\widehat{R}_1(\xi).[13.21]$$

This shows that for each fixed ξ the difference $\widehat{E}(\xi) - \widehat{E}_1(\xi)$ is a polyhomogeneous conormal kernel on $(\mathbb{B}^n_{\frac{1}{2}})^2$, the index set must be as indicated in (8.1) since this corresponds to the indicial operator of L_ξ, which is independent of ξ.

The smoothness, and rapid decrease, in ξ does not follow directly from (8.1), since $\widehat{E}(\xi)$ is not known to be smooth in ξ. However already from (8.1) it follows for large ξ, since the norm of $R_1(\xi)$ is rapidly decreasing in ξ. For finite ξ the smoothness in ξ also follows from (8.1) since the addition to $\widehat{E}_1(\xi)$ of a conormal kernel, namely $\widehat{E}(\xi') - \widehat{E}_1(\xi')$, which is independent of ξ allows the operator to be inverted locally smoothly in ξ near each ξ'. The uniqueness of the inverse then implies the result as stated. □

Taking the inverse Fourier transform of (8.6), and recalling the origin of the kernel E_1 shows that we have proved:

THEOREM 8.3. *The operator* ${}^a\Delta_{n,1}^{-1}$ *acting on half-density sections is an element of*

$$\Psi_\Theta^{-2;\{n+1,1\},\{-n,1\}}({}^aF_{\frac{1}{2}};{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}},{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})$$
$$+\Psi^{-\infty;\{n+1,1\},\{-n,1\}}({}^aF_{\frac{1}{2}};{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}},{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}}),$$

is translation-invariant and has kernel rapidly decreasing as $|x-x'|$ *tends to infinity.*

As corollaries of Theorem 8.3 we obtain, by the methods of §5:

COROLLARY 8.1. *The kernel of the solution operator of the* $\overline{\partial}$*-Neumann problem and the Bergman kernel on* ${}^aF$ *are both translation-invariant, rapidly vanishing as* $|x-x'|$ *tends to infinity and satisfy*

$$\begin{aligned}{}^aK=({}^\Theta\overline{\partial}^*{}^a\Delta_{n,1}^{-1})\in\Psi_\Theta^{-1;\{n+1,1\},\{-n,1\}}({}^aF_{\frac{1}{2}};{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})\\+\Psi^{-\infty;\{n+1,1\},\{-n,1\}}({}^aF_{\frac{1}{2}};{}^\Theta\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}}),\end{aligned}[13.24]$$

$$\begin{aligned}{}^aB\in\Psi_\Theta^{-\infty;\{n+1,1\},\{-n,1\}}({}^aF_{\frac{1}{2}};{}^\Theta\Lambda^{n,0}\otimes\Omega^{\frac{1}{2}})\\+\log\rho_{\mathrm{ff}}\Psi^{-\infty;\{n+1,1\},\{-n,1\}}({}^aF_{\frac{1}{2}};{}^\Theta\Lambda^{n,0}\otimes\Omega^{\frac{1}{2}}).\end{aligned}[13.25]$$

Because the L^2-structure on $(n,0)$–forms is independent of the metric it is possible to construct the Bergman projector directly using a Fourier representation. Denote the space of square integrable, holomorphic $(n,0)$–forms by $H^2({}^aF)$. Any $f\in H^2({}^aF)$ has a representation

$$f(z)=\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{ix\cdot\xi}d\xi.[13.26]$$

For the Bergman projector ${}^aB$ for ${}^aF$

$$\langle g,f\rangle=\langle{}^aB\cdot g,f\rangle,\quad\forall\ g\in L^2({}^aF),\ f\in H^2({}^aF)[13.27]$$

we deduce that the operator defined by

$$\widehat{{}^aB\cdot G}(\xi)=\frac{1}{m_n(2\xi)}\int_{{}^aF}G(x,y)e^{-i(x-iy)}\,dxdy[13.1027]$$

is a projection onto $H^2({}^aF)$. A simple application of the Cauchy-Schwarz inequality shows that $\|{}^aB\|\leq 1$. The function $m_n(\xi)$ is defined by

$$m_n(\xi)=\int_{|y|\leq 1}e^{-y\cdot\xi}\,dy=\frac{c_n}{|\xi|^n}\int_0^{|\xi|}I_{\frac{n-2}{2}}(r)r^{\frac{n}{2}}\,dr[13.28]$$

This function is analyzed in Appendix E.

The kernel of ${}^aB$ is given by

$${}^aB(z,\overline{w})=\int_{\mathbb{R}^n}\frac{e^{i(z-\overline{w})\cdot\xi}}{m_n(2\xi)}\,d\xi.[13.29]$$

To establish that ${}^aB(z,\overline{w})$ is the Bergman kernel we observe that as, $m_n(\xi)$ is real valued, it is Hermitian symmetric and by construction it defines a projection onto $H^2({}^aF)$.

From (8.1) and (8.1) it follows that the symbol of ${}^aQ = P \circ {}^aB \circ P^*$ is

$$\sigma[{}^aQ](\xi) = \frac{m_n(\xi)^2}{m_n(2\xi)}.[13.30]$$

PROPOSITION 8.7. *The operator ${}^aQ$ is an invertible elliptic operator of order $\frac{-(n+1)}{2}$.*

PROOF. The asymptotic development, (E.1), of $m_n(\xi)$ shows that

$$\sigma[{}^aQ](\xi) \sim |\xi|^{\frac{-(n+1)}{2}} \sum_{k=0}^{\infty} A_k |\xi|^{-k} \text{ with } A_0 \neq 0.[13.32]$$

From (8.1) and (8.1) it follows that the symbol of ${}^aQ$ does not vanish. This implies that the inverse exists and is an operator of order $\frac{1}{2}(n+1)$. □

8.2. Adiabatic limit of the Laplacian

In this section we construct the resolvent kernel for ${}^{\alpha}\!\Delta_{n,1}$ defined by the metric given in (7.2). This operator is tangent to the fibres ϵ =constant and its restriction to such a fibre, where $\epsilon > 0$, is an operator of the type considered in [**5**]. In light of this, we will concentrate on the part of the construction that pertains to the adiabatic limit and leave some of the other details of the argument to the interested reader. The operator ${}^{\alpha}\!\Delta_{n,1}$ is an elliptic element of $\mathrm{Diff}^2_{\alpha}(G; {}^{\alpha}\!\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})$. We shall construct a parametrix for $({}^{\alpha}\!\Delta_{n,1} - \lambda)^{-1}$ using the symbol maps and calculus defined for α-pseudodifferential operators. Briefly our method is as follows.

The first step in the construction is essentially classical, we use the diagonal symbol map to construct an approximate resolvent in the small calculus Ψ^{-2}_{α}. The error term is smoothing in the interior but is not compact. The next step is to use the normal homomorphism to construct an approximate resolvent which has the correct leading order behaviour at the front faces. This is the step which really distinguishes the adiabatic construction from the previous case as we now have two normal operators and therefore two model problems. To extend the kernel from $\mathrm{af} \cup \Theta\mathrm{f}$ we need to check that the solutions of the two model problems coincide along the intersection, $\mathrm{af} \cap \Theta\mathrm{f}$. This is a consequence of the uniqueness of the resolvent kernel for the Bergman metric on the ball.

Next we use the indicial operator to obtain an error that vanishes to infinite order at $\Theta\mathrm{l}$. The construction of the parametrix is completed by applying the composition formula to iterate the error term obtained at this stage. This allows us to asymptotically sum a Neumann series for the remaining error term leaving only a term which is compact and vanishes to infinity order at the front face. This residual term is plainly invertible for sufficiently small ϵ.

The operator ${}^{\alpha}\!\Delta_{n,1}$ defined on $\dot{\mathcal{C}}^{\infty}(G; {}^{\alpha}\!\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})$ has an extension as a self adjoint operator. If $\langle f, f\rangle_{\epsilon}$ denotes the L^2 inner product defined by the complete

Kähler metric then the quadratic form

$$\mathcal{Q}(f,f)=\int_0^1[<{}^{\alpha}\overline{\partial}f,{}^{\alpha}\overline{\partial}f>_{\epsilon}+<{}^{\alpha}\overline{\partial}^*f,{}^{\alpha}\overline{\partial}^*f>_{\epsilon}]d\epsilon[14.1]$$

is symmetric and non-negative on $\dot{\mathcal{C}}^{\infty}(G;{}^{\alpha}\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})$. Here we identify $\Omega\simeq\Omega_{\text{fiber}}\otimes|d\epsilon|$. The self-adjoint extension of ${}^{\alpha}\!\Delta$ is then defined from $\mathcal{Q}$ by Friedrichs' extension. Formally, iėȧcting on its domain,

$${}^{\alpha}\!\Delta={}^{\alpha}\overline{\partial}{}^{\alpha}\overline{\partial}^*+{}^{\alpha}\overline{\partial}^*{}^{\alpha}\overline{\partial}.$$

As an operator on $L^2(G;{}^{\alpha}\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})$ it is unitarily equivalent to the direct fibre integral

$${}^{\alpha}\!\Delta\simeq\int_{\oplus}\Delta^{\epsilon}_{n,1}\,d\epsilon$$

where

$$\Delta^{c}_{n,1}={}^{\alpha}\!\Delta_{n,1}\restriction_{\epsilon=c}.[14.2]$$

This operator is well defined as $\mathcal{V}_{\alpha}$ is tangent to the fibration of G over $[0,1]$. We will denote the L^2 closure of ${}^{\alpha}\!\Delta$ by the same symbol, as it should not cause confusion.

THEOREM 8.4. *Let ${}^{\alpha}\!\Delta_{n,1}$ be the fibre Laplace operator acting on ${}^{\alpha}\Lambda^{n,1}(G)$ defined by a fibre metric of the form* (7.2) *then*

$${}^{\alpha}\!\Delta^{-1}_{n,1}\in\Psi^{-2;\mathcal{E}}_{\alpha}(G;{}^{\alpha}\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})+\Psi^{-\infty;\mathcal{E}}(G;{}^{\alpha}\Lambda^{n,1}\otimes\Omega^{\frac{1}{2}})[14.4]$$

where the index set is

$$\mathcal{E}=(0,\emptyset,0,\{n+1,1\},\{-n,1\}).$$

PROOF. The first step is to obtain a kernel with the correct singularity along the diagonal. This is easily accomplished as ${}^{\alpha}\!\Delta_{n,1}$ is a formally elliptic operator in that $\sigma^{\alpha}_2({}^{\alpha}\!\Delta_{n,1})_p$ is an isomorphism on ${}^{\alpha}\Lambda^{n,1}_p(G)$ for every $p\in\Delta_{\alpha}$. Repeated use of the composition formula and surjectivity in the symbol exact sequence shows that there exists an operator

$$Q_1\in\Psi^{-2}_{\alpha}(G;{}^{\alpha}\Lambda^{n,1})\text{ such that }{}^{\alpha}\!\Delta_{n,1}\circ Q_1-\mathrm{Id}\in\Psi^{-\infty}_{\alpha}(G;{}^{\alpha}\Lambda^{n,1}).[14.5]$$

The second step is to obtain the correct leading order terms at af and Θf . To do this we use the normal operators $N^{a}({}^{\alpha}\!\Delta_{n,1})$ and $N^{\Theta}({}^{\alpha}\!\Delta_{n,1})$. These operators are defined by simply restricting the left lift of ${}^{\alpha}\!\Delta_{n,1}$ to af and Θf respectively. From this description it is evident that

$$N^{a}({}^{\alpha}\!\Delta_{n,1})\restriction_{\mathrm{af}\,\cap\,\Theta\mathrm{f}}=N^{\Theta}({}^{\alpha}\!\Delta_{n,1})\restriction_{\mathrm{af}\,\cap\,\Theta\mathrm{f}}.[14.6]$$

We will denote the common normal operator by $N({}^{\alpha}\!\Delta_{n,1})$.

The restrictions of $\mathcal{V}_{\alpha}$ to af and Θf define fibrations, the normal operators are tangent to these fibrations. Evidently the intersection af$\cap\Theta$f is itself fibred and thus $N({}^{\alpha}\!\Delta_{n,1})$ is tangent to the intersection. If κ^{-1}_{n1} denotes the kernel of ${}^{\alpha}\!\Delta^{-1}_{n,1}$ then $N(\kappa^{-1}_{n1})$ is determined by the equation

$$N({}^{\alpha}\!\Delta_{n,1})\circ N(\kappa^{-1}_{n1})=\delta_{\partial\Delta_{\alpha}}[14.7]$$

and the condition that $N(\kappa_{n1}^{-1})$ should be in L^2 near the boundary of the fibre.

The Θ-front face is fibred over $\partial_\Theta G$ with fibre isomorphic to the unit ball parabolically blown-up at a point. For

$$p \in \Theta\mathrm{f}, \quad N_p({}^\alpha\!\Delta_{n,1}) = \Delta^{\mathrm{B}}_{n,1},$$

the $\partial\Delta_\alpha$ intersects each such fibre in exactly one point. Thus on each fibre of Θf, $N^\Theta(\kappa_{n1}^{-1})$ is the unique fundamental solution of $\Delta^{\mathrm{B}}_{n,1}$ with index set $\{n+1,1\},\{-n,1\}$ and pole at the point of intersection of the fibre with $\partial\Delta_\alpha$. As the isomorphism between the fibre and the model problem depends smoothly on parameters, $N^\Theta(\kappa_{n1}^{-1})$ does as well. Note that this description extends to af $\cap\,\Theta$f .

As explained in §C.2, af is a fibration over Y with fibres ${}^a\Phi$. The interior of each fibre is in turn a fibration of the interior of the unit ball. The normal operator $N^a({}^\alpha\!\Delta_{n,1})$ is tangent to this fibration and is isomorphic to the operator ${}^a\Delta$ defined in (8.1). The $\mathrm{ff}({}^a\Phi) = {}^a\Phi \cap \Theta\mathrm{f}$ and the normal operator here is imply the $\Delta^{\mathrm{B}}_{n,1}$ as above.

If we let $S^a = \partial\Delta_\alpha \cap {}^a\Phi$, for some fibre, then $N^a(\kappa_{n1}^{-1})$ is determined as the solution of the equation

$$N^a({}^\alpha\!\Delta_{n,1}) \circ N^a(\kappa_{n1}^{-1}) = \delta_{S^a}.[14.8]$$

which has index set $\{n+1,1\},\{-n,1\}$ at ${}^a\Phi \cap \{\Theta\mathrm{l} \cup \Theta\mathrm{r}\}$ and vanishes to infinite order at ${}^a\Phi \cap \mathrm{ab}$. This solution is constructed in §8.1. As above the isomorphism to the model problem depends smoothly on the fibre and thus $N^a(\kappa_{n1}^{-1})$ does as well. It is clear that the restriction of $N^a(\kappa_{n1}^{-1})$ to $\mathrm{ff}({}^a\Phi)$ is well defined.

The restriction of equation (8.2) to ${}^a\Phi\cap\Theta$f agrees with the equation obtained by approaching this intersection along Θf . Moreover the identification of the boundary components is consistent, thus the uniqueness of the solution for the Bergman Laplacian on the ball implies that

$$N^a(\kappa_{n1}^{-1}) \restriction_{\mathrm{af}\,\cap\,\Theta\mathrm{f}} = N^\Theta(\kappa_{n1}^{-1}) \restriction_{\mathrm{af}\,\cap\,\Theta\mathrm{f}} .[14.9]$$

The approximate inverse Q_1 has a normal operator on both front faces. Since the singularity at the diagonal is symbolically determined to infinite order we conclude that $N(\kappa_{n1}^{-1}) - N(Q_1)$ is smooth in the interior of af $\cup\,\Theta$f . Therefore (8.2) and (8.2) imply that this difference can be extended to define an element

$$Q_1' \in \Psi_\alpha^{-\infty;0,\emptyset,0,\{n+1,1\},\{-n,1\}}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).[14.10]$$

Setting $Q_2 = Q_1 + Q_1'$

$${}^\alpha\!\Delta_{n,1} \circ Q_2 - \mathrm{Id} = E_2 \in \Psi_\alpha^{-\infty;1,\emptyset,1,\{n+1,0\},\{-n,1\}}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).[14.11]$$

We could iterate E_2 to obtain a summable Neumann series for $(\mathrm{Id} + E_2)^{-1}$ however it is more convenient to remove the Taylor series of E_2 at Θl . Since the indicial operator $\sigma^{\alpha,l}({}^\alpha\!\Delta_{n,1})$ is elliptic and has constant indicial roots we can obtain a kernel Q_2' such that

$$\begin{aligned} &Q_2' \in \Psi_\alpha^{-2;1,\emptyset,1,\{n+1,1\},\{-n,1\}}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) \text{ such that} \\ &{}^\alpha\!\Delta_{n,1}(Q_2 + Q_2') - \mathrm{Id} = E_2' \in \Psi_\alpha^{-\infty;1,\emptyset,1,\emptyset,\{-n,1\}}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}). \end{aligned} [14.12]$$

Note that $Q_2 + Q_2'$ belongs to the same class of operators as Q_2.

The argument now proceeds along the same lines as that presented in [**5**]: we use the general composition formula to show that the Neumann series for $(\mathrm{Id} - E_2')^{-1}$ is asymptotically summable and that it belongs to a class of α-pseudodifferential operators with higher order logarithmic singularities at the right boundary than one really expects. We proceed to show that these singularities do not in fact arise.

The composition formula Proposition 8.1 implies that

$$E_2'^{\,k} \in \Psi_\alpha^{-\infty;\mathcal{E}_k}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})$$
$$\text{where } \mathcal{E}_k = (k, \emptyset, k, \emptyset, J_k);\ J_1 = (-n, 1) \text{ and } J_k = (J_{k-1} + 1)\overline{\cup}(-n, 1). \quad [14.13]$$

Thus $J_k \subset J_{k+1}$ and furthermore

$$J = \cup_{k \geq 1} J_k$$

is an index set. From (**??**) it is evident that the Neumann series for $(\mathrm{Id} + E_2')^{-1}$ is asymptotically summable at $\mathrm{af} \cup \Theta\mathrm{f}$. Taking

$$F \sim \sum_{k \geq 1} (-1)^k (E_2')^k,\ F \in \Psi_\alpha^{-\infty;1,\emptyset,1,\emptyset,J}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})$$

gives

$$(\mathrm{Id} + E_2')(\mathrm{Id} + F) = (\mathrm{Id} + E_3) \text{ with } E_3 \in \Psi^{-\infty;1,\emptyset,\emptyset,J}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).[14.14]$$

Let $Q_3 = (Q_2 + Q_2')(\mathrm{Id} + F)$. Applying the formula for compositions between residual terms and the α-calculus we obtain

$$Q_3 \in \Psi_\alpha^{-2;0,\emptyset,0,\{n+1,1\},\{-n,1\}}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) + \Psi^{-\infty;\emptyset,\{n+1,1\},J}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}). \quad [14.15]$$

The error term here, E_3, vanishes to infinite order at af and defines a compact operator on $L^2(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}})$. Thus for $\epsilon_0 > 0$ sufficiently small $(\mathrm{Id} + E_3)^{-1}$ exists and is of the form $(\mathrm{Id} + F')$ where

$$F' \in \Psi^{-\infty;\emptyset,\emptyset,J}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}).[14.16]$$

A right inverse for ${}^\alpha\Delta_{n,1}^{-1}$ is therefore given by

$$Q = Q_3(\mathrm{Id} + F') \in \Psi_\alpha^{-2;0,\emptyset,0,\{n+1,1\},\{-n,1\}}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) + \Psi^{-\infty;\emptyset,\{n+1,1\},J}(G; {}^\alpha\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}). \quad [14.17]$$

The operator $\Delta_{n,1}^\epsilon$ is invertible on L^2 for any ϵ and ${}^\alpha\Delta_{n,1}$ is tangent to the sets $\{\epsilon = \text{constant}\}$. Thus the restriction of κ_Q to each leaf, $G(\epsilon)$, for $\epsilon > 0$ defines a kernel which inverts $\Delta_{n,1}^\epsilon$. The L^2 mapping properties of operators in the Θ-calculus and (8.2) imply that $Q \restriction_{G(\epsilon)}$ is bounded on L^2. Thus it follows that the restriction of Q is actually the inverse of $\Delta_{n,1}^\epsilon$ for each $\epsilon > 0$. From this it follows in turn that

$$Q \circ {}^\alpha\Delta_{n,1} = {}^\alpha\Delta_{n,1} \circ Q = \mathrm{Id}\,.[14.18]$$

Note that (8.2) and (8.2) imply that ${}^\alpha\Delta_{n,1}$ is itself invertible which is the same as

$$\inf_{\epsilon \geq 0} \inf(\mathrm{spec}(\Delta_{n,1}^\epsilon)) > 0.$$

The operator ${}^{\alpha}\Delta_{n,1}$ is self adjoint with respect to the singular metric on G and thus the kernel of Q must also be Hermitian symmetric. Since κ_Q has index set $\{n+1,1\}$ at Θl these symmetry considerations imply that the index set at Θr is actually $\{-n,1\}$. This proves (8.4). □

CHAPTER 9

Adiabatic limit of the $\overline{\partial}$-Neumann problem

Using the operator Q and the same approach as in §5 we can solve the $\overline{\partial}$-Neumann problem uniformly as $\epsilon \downarrow 0$.

PROPOSITION 9.1. *The operator*

$$K = ({}^{\Theta}\overline{\partial}^{*} \Delta_{n,1}^{-1}) \in \Psi_{\alpha}^{-1;0,\{n+1,1\},\{-n,1\}}(G; {}^{\alpha}\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) + \Psi^{-\infty;0,\{n+1,1\},\{-n,1\}}(G; {}^{\alpha}\Lambda^{n,1} \otimes \Omega^{\frac{1}{2}}) \quad [15.2]$$

solves the $\overline{\partial}$-Neumann problem:

$$\text{If } f \in L^2(G) \text{ and } {}^{\alpha}\overline{\partial} f = 0 \text{ then } u = K \cdot f, \text{ satisfies } {}^{\alpha}\overline{\partial} u = f \text{ and } u \perp L^2_{\text{hol}}(G). \quad [15.3]$$

PROOF. The first statement follows from (8.4) and the fact that ${}^{\alpha}\overline{\partial}$ is an α-differential operator. The second statement follows from standard Hodge theory and the definition of the L^2-extension of ${}^{\alpha}\Delta_{n,1}$. □

One can also construct the 'Bergman' projector from ${}^{\alpha}\Delta_{n,1}^{-1}$:

$$B = \text{Id} - K {}^{\alpha}\overline{\partial}. [15.4]$$

From (9.1) we conclude that

$${}^{\alpha}\overline{\partial} B(f) = 0 \ \forall \ f \in L^2(G; {}^{\alpha}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}) [15.5]$$

and thus B is just the orthogonal projection from $L^2(G)$ to the holomorphic subspace $L^2_{\text{hol}}(G)$. The L^2 inner product on ${}^{\alpha}\Lambda^{n,0}$ coincides with the natural pairing defined by

$$\langle f, f \rangle = \int_G f \wedge \overline{f}. [15.6]$$

From (9) we conclude that

$${}^{\alpha}\partial \cdot \kappa_B = 0, [15.7]$$

here κ_B is the kernel of B. Since B is an Hermitian operator (9) implies that

$${}^{\alpha}\overline{\partial}^{t} \cdot \kappa_B = 0 [15.8]$$

as well. From (9) and (9) we conclude that $\kappa_B \in \mathcal{C}^{\infty}(\overset{\circ}{\to} G^2_{\alpha})$. Now κ_B is a polyhomogeneous conormal distribution on G^2_{α} with respect to the diagonal, so it follows from its smoothness in the interior that there is no singularity associated to the diagonal at all, i.e.

$$\begin{aligned} B \in \Psi_{\alpha}^{-\infty;0,\emptyset,0,\{n+1,1\},\{-n,1\}}(G; {}^{\alpha}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}) \\ +\Psi^{-\infty;\{n+1,1\},\{-n,1\}}(G; {}^{\alpha}\Lambda^{n,0} \otimes \Omega^{\frac{1}{2}}). \end{aligned} [15.9]$$

The logarithmic singularities must be absent from the left boundary since κ_B satisfies (9) and ∂ is non-characteristic along the left boundary of M^2. As a consequence of the Hermitian symmetry of κ_B the logarithmic terms must also be absent from the right boundary. The log-terms cancel between the first and second terms in (9) however once the cancellation is done the splitting into a primary term and a residual part looses any meaning.

THEOREM 9.1. *The Bergman kernel on half densities as defined by* (9) *is of the form*

$$B = B_0 + \log \rho_{\Theta\mathrm{f}} \rho_{\Theta\mathrm{f}}^{2(n+1)} B_1 [15.11]$$

where B_0 and B_1 have smooth kernels on M^2_{α}.

PROOF. In fact (9) allows for a quadratic log-term at the Θf. From the uniqueness of the Bergman kernel and (5.2) it follows that the expansion in (9.1) is valid on $G^2_{\alpha} \setminus \mathrm{af}$. Since B is polyhomogeneous conormal on G^2_{α} this expansion must hold uniformly down to the front face. □

Part 3

The Töplitz correspondence

In this third part we shall apply the uniform solution of the $\overline{\partial}$-Neumann problem in the adiabatic limit to the solution of the conjecture of Boutet de Monvel and Guillemin described in the Introduction. The conjecture involves a fibration around a totally real submanifold of a complex manifold, Y, as in Part II, with some additional properties which are discussed in §16. Under these additional hypotheses we analyze various classes of operators from the tube to Y, and back, as well as on Y itself. Thus in §17 the stretched product, on which the kernels for operators from the tube to Y are defined, is considered and in §18 the adiabatic stretched product of Y with itself is discussed. The three related stretched triple products are examined in §19, §20 and in §21, in which the adiabatic calculus on Y is presented. The remaining composition formulæ are contained in §22 but it is important to note that not all the compositions which one might hope were well-behaved are in fact so. Thus in §23 and §24 we examine such composition formulæ under the additional hypothesis of holomorphy (or anti-holomorphy) of the kernels. This finally allows us to give a straightforward proof of the invertibility of the Töplitz correspondence, (0.1), in §25.

CHAPTER 10

α-projection structures

Although we have solved the $\overline{\partial}$-Neumann problem uniformly in the context of an α-structure derived from a shrinking tube around a totally real submanifold we are also interested in additional structure. Indeed, the conjecture of Boutet de Monvel and Guillemin concerns the push-forward map along a fibration, over Y, of the tubular neighbourhood:

$$\begin{array}{ccc} \mathbb{B}^n & \longrightarrow & X \quad [16.1] \\ & & \big\downarrow \pi \\ & & Y. \end{array}$$

Just as described in the Introduction, we consider a function $\psi \in \mathcal{C}^\infty(X)$ which is non-negative, vanishes precisely on Y and has non-degenerate transversal Hessian to Y. We demand additional properties of this fibration, the complex structure and their relation to ψ, namely the fibration must be such that

(10.1) the fibres of (10) are totally real and[16.2]

(10.2) $\operatorname{Im} \overline{\partial}\psi$ is a basic form relative to the fibration in (10).[16.3]

That such a function ψ and fibration exist, satisfying (10.1) and (10.2) follows from the results of [**?**] which we shall briefly discuss.

A small tubular neighbourhood of a totally real submanifold Y of maximal dimension in a complex manifold Ω is always (globally) diffeomorphic to a neighbourhood of the zero section in T^*Y (see [**7**]). In particular this gives T^*Y a complex structure near the zero section, Y. In [**?**] it is shown that the complex structure and the function ψ (with non-degenerate minimum at Y) can be chosen so that (10.1) holds and $\operatorname{Im} \partial\psi$ is a multiple of the canonical 1-form on T^*Y. This certainly implies (10.2) since the canonical 1-form vanishes on the fibres (which are homogeneous Lagrangian). In fact it has been shown recently by Guillemin and Stenzil ([**?**]) that ψ can actually be taken to be a quadratic polynomial on the fibres of T^*Y.

The fibration (7.1) extends to give a commutative diagram

$$\begin{array}{ccccccc} \mathbb{B}^n & \text{———} & M & \longrightarrow & M & \text{———} & \mathbb{B}NY \ [16.4] \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ Y & \text{———} & \widetilde{Y} & \longrightarrow & [0, \epsilon_0] & & \end{array}$$

and similarly (7) becomes

$$\begin{array}{ccccccc} \mathbb{B}^n_{\frac{1}{2}} & \text{———} & G & \longrightarrow & G & \text{———} & \mathbb{B}_{\frac{1}{2}}NY \\ & & \Big\downarrow{\scriptstyle \widetilde{\pi}} & & \Big\downarrow & & \\ Y & \text{———} & \widetilde{Y} & \longrightarrow & [0,\epsilon_0] & & \end{array} \quad [16.5]$$

since the parabolic blow-up of the boundary again applies fibre by fibre.

As we shall see in §25 the conditions (10.1) and (10.2) allow us to prove the conjecture of Boutet de Monvel and Guillemin. Set

$$X(\epsilon) = \{z \in X; \psi \leq \epsilon^2\}[16.6]$$

and let $\mathcal{H}^n(X(\epsilon)) \subset \mathcal{C}^\infty(X(\epsilon); \Lambda^{0,n})$ be the space of holomorphic $(0,n)$–forms.

Theorem 10.1. *If π is a fibration* (10)*, with totally real fibres, of a neighbourhood of a compact totally real submanifold of maximal dimension in a complex manifold and ψ is a non-negative function with non-degenerate minimum at Y for which* (10.2) *holds then, for $\epsilon > 0$ sufficiently small, fibre integration*

$$P : \int_{\text{fibre}} : \mathcal{H}^n(X(\epsilon)) \longrightarrow \mathcal{C}^\infty(Y)[16.8]$$

is an isomorphism.

The proof occupies §17-§25.

CHAPTER 11

The space $(G\widetilde{Y})_\alpha$

To invert (10.1) we need to consider operators that carry distributions on $\widetilde{Y}$ to distributions on G depending on ϵ as a parameter. The kernels for such operators are defined on the ϵ-fibre product $G\widetilde{Y}$ and those of interest have singularities on the fibre diagonal. We shall only be interested in such operators under the additional constraints of an α-projection structure. Thus let $\phi : G \longrightarrow \widetilde{Y}$ arise from a fibration as in (10.1)–(10). The fibre diagonal

$$\Delta_\phi(G\widetilde{Y}) = \Big\{(p, y) \in G\widetilde{Y}; y = \pi_\epsilon\phi(p)\Big\} \quad [17.1]$$

is an interior p-submanifold, i.e. each point of Δ_ϕ has a neighbourhood with a product decomposition $[0, 1)^p \times \mathbb{R}^{n-p}$ in which Δ_ϕ is locally a partial product $[0, 1)^p \times \mathbb{R}^{n-p-d}$.

The most important of the operators from $\widetilde{Y}$ to G which we shall consider have singularities only at the boundary of the fibre diagonal, $\partial\Delta_\phi$. This consists of two smooth submanifolds, one in the adiabatic boundary $\mathrm{ab}(G\widetilde{Y}) = \mathrm{ab}(G) \times Y$ and the other in the Θ boundary $\Theta\mathrm{b}(G\widetilde{Y}) = \Theta\mathrm{b}(G) \times Y$. We need to blow-up the adiabatic part and then make a parabolic blow-up of the Θ part. Thus set

$$\widetilde{B}_a = \Delta_\phi \cap \mathrm{ab}(G\widetilde{Y}),\ (G\widetilde{Y})_a = \Big[G\widetilde{Y}; \widetilde{B}_a\Big],\ \beta_a : (G\widetilde{Y})_a \longrightarrow G\widetilde{Y}. \quad [17.2]$$

The Θ-boundary of the partial diagonal

$$\widetilde{B}_\Theta = \Delta_\phi \cap \Theta\mathrm{b}(G\widetilde{Y}) \subset \Theta\mathrm{b}(G\widetilde{Y}) \quad [17.3]$$

is also a p-submanifold. The existence of a natural parabolic subbundle of $\widetilde{B}_\Theta$ depends on the α-projection structure and one effect of the assumption (10.2) is to fix such a direction. Indeed, consider how $\widetilde{B}_\Theta$ is related to the two projections from $G\widetilde{Y}$:

$$\begin{array}{ccccc} \widetilde{B}_\Theta & \longrightarrow & G\widetilde{Y} & \longrightarrow & G \\ & & \downarrow & & \downarrow \\ & & \widetilde{Y} & & \widetilde{Y} \end{array} \quad [17.4]$$

By Proposition 7.3 the form θ, lifted from G to $\widetilde{G}$ on $G\widetilde{Y}$, therefore restricts to a non-vanishing form on $\widetilde{B}_\Theta$. By hypothesis (10.2) $\widetilde{G}$ is, at each point, actually the lift of a form under the map from $(G\widetilde{Y})_a$ through G to $\widetilde{Y}$. Since $\widetilde{B}_\Theta$ is part of the lift of the fibre diagonal this same form can be lifted from $\widetilde{Y}$ under the direct projection; we denote the second form μ. The difference $\widetilde{\Theta} - \mu$ is a 1-form on $G\widetilde{Y}$ at $\widetilde{B}_\Theta$ which vanishes when pulled back to $\widetilde{B}_\Theta$. Thus $\widetilde{\Theta} - \mu$ defines a section of $N^*\widetilde{B}_\Theta$, and hence spans a line subbundle $\widetilde{S}_2 \subset N^*\widetilde{B}_\Theta$.

Naturally we wish to check that $\widetilde{B}_a$ and $\{\widetilde{B}_\Theta, \widetilde{S}_2\}$ meet parabolically cleanly in the sense of (C.3) and (C.3). This follows as in (7.2) and (7.2). Thus we can define the α-stretched product as the $\widetilde{S}_2$-parabolic blow-up of $(G\widetilde{Y})_a$ along the lift of $\widetilde{B}_\Theta$ (see [**5**] and §C4):

$$(G\widetilde{Y})_\alpha = \left[(G\widetilde{Y})_a; \widetilde{B}_\Theta, \widetilde{S}_2\right], \ \beta^2_\alpha : (G\widetilde{Y})_\alpha \longrightarrow (G\widetilde{Y}), [17.5]$$

or in terms of the notation for iterated blow-up of §C4:

$$(G\widetilde{Y})_\alpha = \left[G\widetilde{Y}; \widetilde{B}_a; \widetilde{B}_\Theta, \widetilde{S}_2\right].[17.6]$$

The stretched projections then arise from the commutative diagram

$$\begin{array}{ccccc} & & (G\widetilde{Y})_\alpha & & \\ & \swarrow^{\widetilde{\pi}^2_{l,\alpha}} & \downarrow{\scriptstyle \beta^2_\alpha} & \searrow^{\widetilde{\pi}^2_{r,\alpha}} & \\ G & \xleftarrow[\widetilde{\pi}^2_l]{} & G\widetilde{Y} & \xrightarrow[\widetilde{\pi}^2_r]{} & \widetilde{Y} \end{array} \qquad [17.7]$$

The space $(G\widetilde{Y})_\alpha$ has five boundary hypersurfaces although no more than three meet. They are the adiabatic front face, adiabatic side face, the Θ-front face, the Θ-boundary face and the extension boundary:

$$M_1(G\widetilde{Y})_\alpha = \{\mathrm{af}, \mathrm{ab}, \Theta\mathrm{f}, \Theta\mathrm{b}, \mathrm{eb}\}.[17.8]$$

The entries in the McDonald table for the stretched projections to G and $\widetilde{Y}$ are easily computed, see Table 2.

We then have a partial analogue of Proposition 7.4. Let $\mathcal{V}_a(\widetilde{Y})$ be the space of vector fields on $\widetilde{Y}$ tangent to the ϵ-leaves and vanishing at $\epsilon = 0$.

Lemma 11.1. *The two maps in* (11) *are both b-fibrations and the Lie algebras $\mathcal{V}_\alpha(G)$ and $\mathcal{V}_a(\widetilde{Y})$ lift to $(G\widetilde{Y})_\alpha$ to be transversal to the lift of the fibre diagonal, which is an interior p-submanifold.*

Remark 11.1. *While elements of $\mathcal{V}_\alpha(G)$ lift to be tangent to* Θf *those in $\mathcal{V}_a(\widetilde{Y})$ do not. This has an important effect on the composition properties of calculi modeled on this space.*

Proof. The proof of Proposition 7.4 can be followed closely. Since they are obtained by composition of projections and blow-down maps both stretched projections are b-maps and from Table 2 and (B.5) they are b-normal. Thus, applying Proposition B.4, to prove they are b-fibrations we need to check (B.4). This follows directly from the fibration properties of the blow-down maps, as in the proof of Proposition 7.4. Thus the stretched projections are b-fibrations.

The Lie algebras lift since they consist of vector fields tangent to both submanifolds $\widetilde{B}_a$ and $\widetilde{B}_\Theta$ and they are parabolically tangent to the second along $\widetilde{S}_2$, see Proposition C.2. □

Choosing non-vanishing half-densities on G and $\widetilde{Y}$ we can transfer the transpose of the push-forward operator P, from G to $\widetilde{Y}$, to an operator on half-densities. It has Schwartz kernel of the form

$$a\delta(y-y')|dxdydzdy'd\epsilon|^{\frac{1}{2}}\cdot|d\epsilon|^{-\frac{1}{2}},\ a>0.[17.10]$$

Lifting to $(G\widetilde{Y})_\alpha$ we see that this is a non-vanishing Dirac delta section, over the lifted fibre diagonal $\widetilde{\Delta}_\phi$, of the weighted density bundle

$$\widetilde{\mathrm{KD}}_\alpha=\rho_{\mathrm{af}}^{-\frac{n}{2}}\rho_{\Theta\mathrm{f}}^{\frac{-(n+1)}{2}}\Omega^{\frac{1}{2}}\otimes|d\epsilon|^{-\frac{1}{2}}.[17.11]$$

The small α-calculus, of operators from $\widetilde{Y}$ to G, is then defined to be

$$\Psi^m_\alpha(\widetilde{Y},G;\Omega^{\frac{1}{2}})=\mathcal{C}^\infty_{\mathrm{ff}}(G\widetilde{Y})_\alpha\cdot\left\{\kappa\in I^{m-\frac{1}{4}}((G\widetilde{Y})_\alpha,\widetilde{\Delta}_\phi;\widetilde{\mathrm{KD}}_\alpha)\right\}[17.12]$$

where $\mathcal{C}^\infty_{\mathrm{ff}}((G\widetilde{Y})_\alpha)$ is the subspace of $\mathcal{C}^\infty$ functions vanishing to all orders at boundary faces other than $\mathrm{af}(G\widetilde{Y})_\alpha$ and $\Theta\mathrm{f}(G\widetilde{Y})_\alpha$.

Since Δ_ϕ meets the Θ-boundary transversely the kernel for P lifts to be polyhomogeneous conormal on $(G\widetilde{Y})_a$; it is a Dirac delta section, over the lifted fibre diagonal, of the weighted density bundle

$$\widetilde{\mathrm{KD}}_a=\rho_{\mathrm{af}}^{-\frac{n}{2}}\Omega^{\frac{1}{2}}\otimes|d\epsilon|^{-\frac{1}{2}}.$$

The small a-calculus of operators from $\widetilde{Y}$ to G is defined as consisting of the kernels

$$\Psi^m_a(\widetilde{Y},G;\Omega^{\frac{1}{2}})=\mathcal{C}^\infty_{\mathrm{af}}((G\widetilde{Y})_a)\cdot\left\{\kappa\in I^{m-\frac{1}{4}}((G\widetilde{Y})_a,\widetilde{\Delta}_\phi;\widetilde{\mathrm{KD}}_a)\right\}.[17.13]$$

In addition to the spaces $(G\widetilde{Y})_a$, $(G\widetilde{Y})_\alpha$, G^2_a, and G^2_α it is useful to have analogous spaces without the square root differential structure. These can be constructed from $M\widetilde{Y}$ or M^2 directly using as Θ-structures $\mathrm{sp}\left\{dr_{\Theta\mathrm{b}},\widetilde{\theta}-\mu\right\}$ or $\mathrm{sp}\left\{dr_{\Theta\mathrm{r}},dr_{\Theta\mathrm{l}},\theta_L-\theta_R\right\}$ respectively or by blowing down the Θ-boundary components besides $\Theta\mathrm{f}$ in the spaces already constructed. The commutativity results of Appendix C show that the two constructions lead to canonically isomorphic spaces. We denote these spaces by $(M\widetilde{Y})_a$, $(M\widetilde{Y})_\alpha$, M^2_a, M^2_α.

CHAPTER 12

The space $\widetilde{Y}^2_a$

The next stretched product we consider is for the 'adiabatic' calculus on $\widetilde{Y}$. This corresponds to the boundary fibration structure $\mathcal{V}_a(\widetilde{Y})$ introduced above. Let $\widetilde{Y}^2$ be the fibre product of $\widetilde{Y}$ with itself, i.e. the preimage of the diagonal in the ϵ variable:

$$\widetilde{Y}^2 = \{(p,p');\pi_\epsilon(p) = \pi_\epsilon(p')\} \equiv Y^2 \times [0,\epsilon_0].[18.1]$$

Clearly $\widetilde{Y}^2$ is a compact manifold with boundary, we put $\mathrm{ab}(\widetilde{Y}^2) = \{\epsilon = 0\}$ and $\mathrm{eb}(\widetilde{Y}^2) = \{\epsilon = \epsilon_0\}$. The fibre diagonal, that is the preimage of the diagonal in $\widetilde{Y}^2$, is an interior p-submanifold. Set

$$\widehat{B}_a = \left\{(p,p) \in \widetilde{Y}^2; p \in \mathrm{ab}(\widetilde{Y})\right\},\ \widetilde{Y}^2_a = \left[\widetilde{Y}^2;\widehat{B}_a\right],\ \beta^2_a : \widetilde{Y}^2_a \longrightarrow \widetilde{Y}^2[18.2]$$

being the blow-down map. This is a manifold with corners, the new boundary being denoted $\mathrm{af}(\widetilde{Y}^2_a)$, in which the lifted diagonal is again an interior p-submanifold Δ_a. The algebra $\mathcal{V}_a(\widetilde{Y})$ lifts to be transversal to Δ_a. It is easy to see that the two stretched projections $\widehat{\pi}^2_{h,a} : \widetilde{Y}^2_a \longrightarrow \widetilde{Y}$, $h = l, r$, are b-fibrations.

The identity operator on half-densities on $\widetilde{Y}$ has Schwartz kernel which can be written in local coordinates

$$\delta(y-y')|dydy'd\epsilon|^{\frac{1}{2}} \cdot |d\epsilon|^{-\frac{1}{2}}.[18.3]$$

Lifting to $\widetilde{Y}^2_a$ we see that this is a non-vanishing delta section, over the lifted diagonal, of the bundle

$$\widehat{\mathrm{KD}}_a = \rho_{\mathrm{af}}^{-\frac{n}{2}}\Omega^{\frac{1}{2}} \otimes |d\epsilon|^{-\frac{1}{2}}.[18.4]$$

We therefore take the small a-calculus on $\widetilde{Y}$ to be

$$\Psi^m_a(\widetilde{Y};\Omega^{\frac{1}{2}}) = \mathcal{C}^\infty_{\mathrm{af}}(\widetilde{Y}^2_a) \cdot I^{m-\frac{1}{4}}(\widetilde{Y}^2_a, \Delta_a; \widehat{\mathrm{KD}}_a).[18.5]$$

The identity is therefore an operator of order 0. In general a pseudodifferential operator on $\widetilde{Y}$, independent of the parameter ϵ, is not an element of this small calculus.

We have defined three stretched products, G^2_α, $(G\widetilde{Y})_\alpha$ and $\widetilde{Y}^2_a$. It is natural to consider the relationship between them. The fibre products are related by the projection ϕ, first acting on the right then on the left to give the first row of:

$$\begin{array}{ccccc}
G^2 & \xrightarrow{\phi} & G\widetilde{Y} & \xrightarrow{\phi} & \widetilde{Y}^2 \\
\uparrow & & \uparrow & & \uparrow \\
G_a^2 & \xrightarrow{\phi_a} & (G\widetilde{Y})_a & \xrightarrow{\phi_a} & \widetilde{Y}_a^2 \\
\uparrow & & \uparrow & & \uparrow \\
G_\alpha^2 & & (G\widetilde{Y})_\alpha & \xrightarrow{\psi} & \widetilde{Y}_a^2
\end{array} \quad [18.6]$$

Under these projections the three adiabatic submanifolds are related in the strong sense that $B_a = \phi^{-1}(\widetilde{B}_a)$, and $\widetilde{B}_a = \phi^{-1}(\widehat{B}_a.)$ From this it follows readily that the maps ϕ in (12) lift to give the second row. On the other hand for the Θ-blow-up the relationships is quite different. Of course for the third space there is no Θ-blow-up and so ϕ_a can be lifted in that case to a $\mathcal{C}^\infty$ map, as shown. However this is essentially a blow-down map and it is not a b-fibration, because it is not a b-submersion. Furthermore $\phi_a^{-1}(\widetilde{B}_\Theta) \supsetneq B_\Theta$. Thus there is no $\mathcal{C}^\infty$ map from G_α^2 to $(G\widetilde{Y})_\alpha$ which would extend (12) to a full commutative diagram.

CHAPTER 13

The stretched product $(G^2\widetilde{Y})_\alpha$

Composition formulæ for the other classes of operators introduced in §10 are proved by constructing appropriate stretched 'triple products'. The first triple product of this type we consider is used to examine the composition of an operator from $\widetilde{Y}$ to G and one from G to itself. Thus it is a blown-up version of the fibre product $G^2\widetilde{Y}$. The blown-up version will be denoted $(G^2\widetilde{Y})_\alpha$; its construction is very similar to that G^3_α but is somewhat simplified by the absence of one of the Θ-partial diagonals.

Starting from the fibre product $(G^2\widetilde{Y})$ there are four stages, two adiabatic blow-ups and two Θ-blowups. To fix the notation, we label the six projections from $G^2\widetilde{Y}$ as follows:

$$\begin{aligned}\pi^3_f &: G^2\widetilde{Y} \longrightarrow G\widetilde{Y},\\ \pi^3_c &: G^2\widetilde{Y} \longrightarrow G\widetilde{Y},\\ \pi^3_s &: G^2\widetilde{Y} \longrightarrow G^2,\\ \pi^3_l &: G^2\widetilde{Y} \longrightarrow G,\\ \pi^3_m &: G^2\widetilde{Y} \longrightarrow G,\\ \pi^3_r &: G^2\widetilde{Y} \longrightarrow \widetilde{Y}.\end{aligned} \quad [19.1]$$

Here the first three projections are obtained by dropping the first, central and last factors of the product and the second three are projections onto the left, middle and right factors respectively. We use the obvious notation for the boundaries

$$M_1(G^2\widetilde{Y}) = \{\mathrm{ab}, \Theta\mathrm{bl}, \Theta\mathrm{bm}\}\,.[19.2]$$

Consider the three submanifolds of $\mathrm{ab}(G^2\widetilde{Y})$:

$$\widehat{B}^2_{a,o} = (\pi^3_o)^{-1}(\widetilde{B}_a), \quad o = f, c,\ \widehat{B}^2_{a,s} = (\pi^3_s)^{-1}(B_a).[19.3]$$

Here $\widetilde{B}_a$ is defined in (11) and B_a is defined in (7.2).

Let $\widehat{B}^3_a$ denoted the intersection

$$\widehat{B}^3_a = \widehat{B}^2_{a,o} \cap \widehat{B}^2_{a,o'} \text{ if } o \neq o'.[19.4]$$

Then set

$$(G^2\widetilde{Y})_a = \left[G^2\widetilde{Y}; \widehat{B}^3_a; \widehat{B}^2_a\right][19.5]$$

where, conventionally we let $\widehat{B}^2_a$ denote the union of the three submanifolds $\widehat{B}^2_{a,o}$ which are actually disjoint once $\widehat{B}^3_a$ has been blow up. The projections π^3_o for $o = f, c, s$ lift to maps $\pi^3_{o,a}$, from $(G^2\widetilde{Y})_a$ to G^2_a and $(G\widetilde{Y})_a$ respectively.

Similarly consider the partial Θ-diagonals:

$$\widehat{B}^2_{\Theta,s} = (\pi^3_s)^{-1}(B_\Theta),\ \widehat{B}^2_{\Theta,c} = (\pi^3_c)^{-1}(\widetilde{B}_\Theta)[19.6]$$

where B_Θ is defined in (7.2) and $\widetilde{B}_\Theta$ is defined in (11). As noted with their definitions both submanifolds have associated parabolic directions; we set

$$S_{2,s} = (\pi_c^3)^* S_2,\ S_{2,c} = (\pi_c^3)^* (\widetilde{S}_2).[19.7]$$

These are p-subbundles of the conormal bundles. The intersection of the two partial Θ-diagonals will be denoted

$$\begin{gathered} \widehat{B}^3_\Theta = \widehat{B}^2_{\Theta,c} \cap \widehat{B}^2_{\Theta,s} \\ \widehat{S}_3 = (\widehat{S}_{2,c})_{\restriction \widehat{B}^3_\Theta} \oplus (\widehat{S}_{2,s})_{\restriction \widehat{B}^3_\Theta} \subset N^* \widehat{B}^3_\Theta. \end{gathered} [19.8]$$

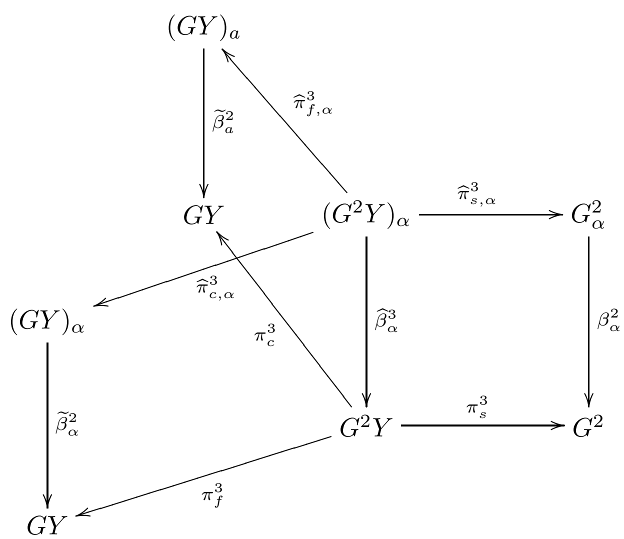
All these submanifolds are boundary p-submanifolds and the bundles are p-subbundles. Moreover every pair of these submanifolds meets parabolically cleanly. The lifts of the three Θ-submanifolds still meet parabolically cleanly in $(G^2\widetilde{Y})_a$. We then set

$$(G^2\widetilde{Y})_\alpha = \left[G^2\widetilde{Y}; \widehat{B}^3_a; \widehat{B}^2_a; \widehat{B}^3_\Theta, \widehat{S}_3; \widehat{B}^2_\Theta, \widehat{S}_2 \right] [19.9]$$

using the notation of §C4. This manifold with corners has the following boundary hypersurfaces:

$$M_1(G^2\widetilde{Y})_\alpha = \{\mathrm{at}, \mathrm{asf}, \mathrm{asc}, \mathrm{ass}, \mathrm{ab}, \Theta\mathrm{d}, \Theta\mathrm{sc}, \Theta\mathrm{ss}, \Theta\mathrm{l}, \Theta\mathrm{m}, \mathrm{eb}\}.[19.10]$$

PROPOSITION 13.1. *There are three b-maps from $(G^2\widetilde{Y})_\alpha$ giving rise to a commutative diagram:*



[19.12]

in which $\widehat{\pi}^3_{s,\alpha}$ and $\widehat{\pi}^3_{c,\alpha}$ are b-fibrations but $\widehat{\pi}^3_{f,\alpha}$ is not a b-submersion. The two lifted fibre diagonals $\Delta_{c,\alpha}$ and $\Delta_{f,\alpha}$ are transversal to $\widehat{\pi}^3_{s,\alpha}$; the lifted diagonal $\Delta_{s,\alpha}$ and fibre diagonal $\Delta_{f,\alpha}$ are transversal to $\widehat{\pi}^3_{c,\alpha}$.

PROOF. As already noted, after the first blow-up in (13) the adiabatic side faces are disjoint

$$\widehat{B}^2_{a,o} \cap \widehat{B}^2_{a,o'} = \emptyset,\ o \neq o' \text{ in } \left[G^2\widetilde{Y}; \widehat{B}^3_a\right] [19.13]$$

so we can blow-down and then exchange orders to get

$$(G^2\widetilde{Y})_\alpha \longrightarrow \left[G^2\widetilde{Y}; \widehat{B}^3_a; \widehat{B}^2_{a,f}\right] \simeq \left[G^2\widetilde{Y}; \widehat{B}^2_{a,f}; \widehat{B}^3_a\right]$$

since $\widehat{B}^3_a \subset \widehat{B}^2_{a,f}$, whence we can blow down to $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,f}\right] = G(G\widetilde{Y})_a$. Thus the b-map $\widehat{\pi}^3_{f,\alpha}$ exists. This map fails to be a b-submersion along Θd

To show the existence of $\widehat{\pi}^3_{c,\alpha}$ it suffices to show that

$$(G^2\widetilde{Y})_\alpha \simeq \left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_a; \widehat{B}^3_\Theta, \widehat{S}_3; \widehat{B}^2_{a,f}; \widehat{B}^2_{a,s}; \widehat{B}^2_{\Theta,s}, \widehat{S}_{2,s}\right] [19.14]$$

since $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}\right]$ is the fibre product $G(G\widetilde{Y})_\alpha$.

Starting from (13) we get directly

$$(G^2\widetilde{Y})_\alpha \simeq \left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^3_a; \widehat{B}^2_{a,f}; \widehat{B}^2_{a,s}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_\Theta, \widehat{S}_3; \widehat{B}^2_{\Theta,s}, \widehat{S}_{2,c}\right] [19.15]$$

using the disjointness of the partial diagonals after their intersection has been blown up and then applying Lemma C.8 twice. Then observe that, even in $\left[G^2\widetilde{Y}; \widehat{B}^3_a\right]$, $\widehat{B}^2_{\Theta,c}$ is disjoint from $\widehat{B}^2_{a,f}$ and $\widehat{B}^2_{a,s}$ (see Figure 4) so from (13) we get

$$(G^2\widetilde{Y})_\alpha \simeq \left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^3_a; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^2_{a,f}; \widehat{B}^2_{a,s}; \widehat{B}^3_\Theta, \widehat{S}_3; \widehat{B}^2_{\Theta,s}, \widehat{S}_{2,s}\right].[19.16]$$

The final step to get (13) is to show that

$$\widehat{B}^3_a \pitchfork B^2_{\Theta,c} \text{ in } \left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}\right]. \quad [19.17]$$

As in the proof of (7.3) first note that $\widehat{B}^3_a \cap \widehat{B}^2_{\Theta,c} \subset \widehat{B}^2_{a,c}$ in $G^2\widetilde{Y}$ so it suffices to examine the images in ${}_+N\widehat{B}^2_{a,c}$. There $\widehat{B}^3_a$ is the restriction of the bundle, so it is enough to show that $\widehat{B}^3_a$ intersects $\widehat{B}^2_{\Theta,c}$ transversally within $\widehat{B}^2_{a,c}$, i.e. in the zero section of the bundle. Now recall that $\widehat{B}^2_{a,c}$ is itself a bundle over Y_0^2, of which $\widehat{B}^3_a$ is the restriction to the diagonal in Y_0^2 and $\widehat{B}^2_{\Theta,c}$ is a subbundle, so they are transversal. Thus we have proved (13). The construction of $\widehat{\pi}^3_{s,\alpha}$ is similar, namely its existence follows from

$$(G^2\widetilde{Y})_\alpha \simeq \left[G^2\widetilde{Y}; \widehat{B}^2_{a,s}; \widehat{B}^2_{\Theta,s}, \widehat{S}_{2,s}; \widehat{B}^3_a; \widehat{B}^2_{a,f}; \widehat{B}^2_{a,c}; \widehat{B}^3_\Theta, \widehat{S}_3; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}\right]. \quad [19.18]$$

This can be proved much as (13).

Tables 5 and 6 show the exponents linking the adiabatic and Θ-boundary faces respectively. The exponents linking the a and Θ faces are all zero. Thus all three stretched projections are b-normal maps.

As in the proof of Proposition 7.5 we break the proof (B.4) into two steps. If

$$F_1 : Q \longrightarrow (G\widetilde{Y})_\alpha,\ F_2 : (G^2\widetilde{Y})_\alpha \longrightarrow Q$$
$$Q = \left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_a; \widehat{B}^2_{a,s}; \widehat{B}^2_{a,f}\right].$$

then $\widehat{\pi}^3_{c,\alpha} = F_1 \circ F_2$.

These two maps each involve only adiabatic or Θ blow-ups, so away from the intersection of these two type of front faces (B.4) follows readily from the fibration properties of blow-up. We therefore concentrate on the less obvious points. Relative to F_1, $\Theta\mathrm{d} \cap \{\epsilon = 0\} \subset Q$ is a half n-sphere bundle over $\Theta\mathrm{f} \cap \{\epsilon = 0\} \subset (GY)_\alpha$ and $\Theta\mathrm{ss} \cap \{\epsilon = 0\} \subset Q$ is a half n-sphere bundle over $\Theta\mathrm{b} \cap \{\epsilon = 0\} \subset (G\widetilde{Y})_\alpha$. Then if

$$\begin{gathered} p \in (G^2\widetilde{Y})_\alpha \text{ with } \widehat{\pi}^3_{c,\alpha}(p) = q \\ q \in \Theta\mathrm{d} \Longrightarrow \operatorname{range}(F_2)_* = T_q\,\Theta\mathrm{d}, \qquad [19.19] \\ q \in \Theta\mathrm{ss} \setminus \Theta\mathrm{d} \Longrightarrow \operatorname{range}(F_2)_* = {}^bT_q\,\Theta\mathrm{ss}. \end{gathered}$$

The geometry of the map F_1, (13) and the fibration properties of blow-down maps imply that $\widehat{\pi}^3_{c,\alpha}$ satisfies (B.4) and hence is a b-fibration.

The proof that $\widehat{\pi}^3_{s,\alpha}$ satisfies (B.4) follows the same lines with F_1, and F_2 becoming

$$F_1 : G^2_\alpha\widetilde{Y} \longrightarrow G^2_\alpha$$
$$F_2 : (G^2\widetilde{Y})_\alpha \longrightarrow \left[G^2\widetilde{Y}; \widehat{B}^2_{a,s}; \widehat{B}^2_{\Theta,s}, \widehat{S}_{2,s}\right] = G^2_\alpha\widetilde{Y}.$$

To establish the transversality at the partial diagonals we argue as before in the proof of Proposition 7.5. Unlike that case, $(G^2\widetilde{Y})_\alpha$ is not symmetric so each assertion of transversality has to be treated separately. First we consider $\widehat{\pi}^3_{c,\alpha}$ along the submanifold $\Delta_{s,\alpha}$. We use the description of $(G^2\widetilde{Y})_\alpha$ given in (13).

In the space

$$G(G\widetilde{Y})_\alpha = \left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}\right],$$

the submanifold, Δ_s lifts to be a graph over $(G\widetilde{Y})_\alpha$. Thus it suffices (and is necessary) to prove the injectivity of $\widehat{\pi}^3_{c,i\,*}\restriction_{T\Delta_{s,\alpha}}$, where

$$\widehat{\pi}^3_{c,i} : (G^2\widetilde{Y})_\alpha \longrightarrow G(G\widetilde{Y})_\alpha,$$

is the intermediate blow-down map.

In $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}\right]$, $\widehat{B}^3_a$ is an interior p-submanifold of a hypersurface boundary component and $\Delta_{s,\alpha}$ satisfies the hypotheses of Lemma C.5. Thus the projection at this step is injective. In $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_a\right]$, $\widehat{B}^3_\Theta$ lies in a corner of co-dimension 2 and $\Delta_{s,\alpha}$ satisfies the hypotheses of Lemma C.6. In

$$\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_\Theta, \widehat{S}_3\right]$$

$\Delta_{s,\alpha}$ does not intersect does not meet $\widehat{B}^2_{a,f}$ and satisfies the hypotheses of Lemma C.5 relative to its intersection with $\widehat{B}^2_{a,s}$. Once this submanifold is blown-up $\Delta_{s,\alpha}$ satisfies the hypotheses of Lemma C.5 relative to its intersection with $\widehat{B}^2_{\Theta,s}, \widehat{S}_{2,s}$, which completes the argument in this case.

In $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}\right]$ $\Delta_{f,\alpha}$ lifts to give a fibration over $(G\widetilde{Y})_\alpha$. Since the projection from $TG(G\widetilde{Y})_\alpha$ restricted to $T\Delta_{f,\alpha}$ has a kernel it is not necessary to show that $\widehat{\pi}^3_{c,i\,*}\restriction_{T\Delta_{f,\alpha}}$ is injective. This submanifold does not meet $\widehat{B}^2_{a,s}$ or $\widehat{B}^2_{\Theta,s}$ in $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_a; \widehat{B}^3_\Theta; \widehat{S}^3\right]$ and its intersection with and $\widehat{B}^2_{a,f}$ satisfies the hypotheses of Lemma C.5. The problem arises at the intersection with $\widehat{B}^3_\Theta$ in $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}; \widehat{B}^3_a\right]$.

This submanifold lies in a co-dimension 2 corner and the lift of Δ_f is not like a 'partial diagonal' in a neighborhood of it. The complement of the range of the differential restricted to $\Delta_{t,\alpha}$ under this blow-down lies in the kernel of the projection $G(G\widetilde{Y})_\alpha \longrightarrow (G\widetilde{Y})_\alpha$. This can be verified by a local coordinate calculation. *********local coordinate calculation****************************

The intersection of $\Delta_{f,\alpha}$ with $\widehat{B}^3_a$ in $\left[G^2\widetilde{Y}; \widehat{B}^2_{a,c}; \widehat{B}^2_{\Theta,c}, \widehat{S}_{2,c}\right]$ satisfies the hypotheses of Lemma C.5 completing the argument for $\Delta_{f,\alpha}$.

Easier arguments establish the transversality of $\widehat{\pi}^3_{s,\alpha}$ with respect to $\Delta_{f,\alpha}$ and $\Delta_{c,\alpha}$. We give a brief outline using the description of $(G^2\widetilde{Y})_\alpha$ given in (13). First observe that both partial fiber diagonals lift to $(G^2)_\alpha\widetilde{Y}$ as graphs over $(G^2)_\alpha$, thus we need to establish the injectivity of the differential of the intermediate projection,

$$\widehat{\pi}^3_{s,i} : (G^2\widetilde{Y})_\alpha \longrightarrow (G^2)_\alpha\widetilde{Y},$$

restricted to the tangent spaces of these submanifolds. This follows because in the intermediate space $(G^2)_\alpha\widetilde{Y}$ all the submanifolds, remaining to be blown-up, lift to be interior p-submanifolds of boundary hypersurfaces. One easily establishes, for either partial fiber diagonal, that the hypotheses of Lemma C.5 are verified at each stage of the construction. This completes the proof of the transversality statements. □

The triple product $(G^2\widetilde{Y})_\alpha$ is asymmetric, even as far as the factors of G are concerned. Thus there are actually six different spaces of the same type (i.e. diffeomorphic), in three pairs corresponding to whether the factor of $\widetilde{Y}$ is on the right, as above, in the centre or on the left with the pairs distinguished by the 'order' of the factors of G. Since we shall use these triple products to prove composition formulæ, only those four cases where the stretched projection corresponding to $\widehat{\pi}^3_{f,\alpha}$ does *not* occur as the 'central' projection are of interest. These occur in two pairs, corresponding to the adjoint operation, given by completely reversing the orders of the factors. Thus, apart from the space $(G^2\widetilde{Y})_\alpha$ just discussed, we need only consider the space we denote $(G\widetilde{Y}G)_\alpha$ which has the three projections

$$\begin{aligned}\pi^3_{f,\alpha} &: (G\widetilde{Y}G)_\alpha \longrightarrow (\widetilde{Y}G)_a\\ \pi^3_{c,\alpha} &: (G\widetilde{Y}G)_\alpha \longrightarrow G^2_\alpha \qquad [19.20]\\ \pi^3_{s,\alpha} &: (G\widetilde{Y}G)_\alpha \longrightarrow (G\widetilde{Y})_\alpha.\end{aligned}$$

The second two of these are therefore b-fibrations, but the first map is not. The labelling convention on these maps follows the order of the products, i.e. under the diffeomorphism

$$(G^2\widetilde{Y})_\alpha \longleftrightarrow (G\widetilde{Y}G)_\alpha [19.21]$$

$\pi^3_{f,\alpha}$ on the left corresponds to $\pi^3_{f,\alpha}$ on the right but the two b-fibrations, $\pi^3_{s,\alpha}$ and $\pi^3_{c,\alpha}$, are interchanged.

CHAPTER 14

The space $(\widetilde{Y}G\widetilde{Y})_\alpha$

The third triple product we construct is $(\widetilde{Y}G\widetilde{Y})_\alpha$, obtained by blowing up $\widetilde{Y}G\widetilde{Y}$, the triple ϵ-fibre product of $\widetilde{Y}$, G and $\widetilde{Y}$. As before we label the three projections:

$$\pi^3_f : \widetilde{Y}G\widetilde{Y} \longrightarrow G\widetilde{Y},\ \pi^3_c : \widetilde{Y}G\widetilde{Y} \longrightarrow \widetilde{Y}^2,\ \pi^3_s : \widetilde{Y}G\widetilde{Y} \longrightarrow \widetilde{Y}G.[20.1]$$

The p-submanifold, $\widetilde{B}_a$ of $G\widetilde{Y}$ (or $\widetilde{Y}G$) is defined in (11), for $\widetilde{Y}^2$ this p-submanifold is the diagonal in the fibre $\epsilon = 0$.

$$\widetilde{B}^2_{a,o} = (\pi^3_o)^{-1}(\widetilde{B}_a),\ \text{for } o = f, c, s.[20.2]$$

The triple adiabatic diagonal is, as usual, the common intersection

$$\widetilde{B}^3_a = \widetilde{B}^2_{f,a} \cap \widetilde{B}^2_{s,a} = \widetilde{B}^2_{f,a} \cap \widetilde{B}^2_{c,a} = \widetilde{B}^2_{s,a} \cap \widetilde{B}^2_{c,a}.[20.3]$$

Let $(\widetilde{Y}G\widetilde{Y})_a$ denote the space obtained by first blowing up $\widetilde{B}^3_a$ and then blowing up the lifts of $\widetilde{B}^2_{o,a}$, for $o = f, c, s$. Let

$$\widetilde{\beta}_a : (\widetilde{Y}G\widetilde{Y})_a \longrightarrow \widetilde{Y}G\widetilde{Y}$$

denote the blow-down map. The projections in (14) lift to give smooth maps

$$\pi^3_{f,a} : (\widetilde{Y}G\widetilde{Y})_a \longrightarrow (G\widetilde{Y})_a,\ \pi^3_{c,a} : (\widetilde{Y}G\widetilde{Y})_a \longrightarrow \widetilde{Y}^2_a,\ \pi^3_{s,a} : (\widetilde{Y}G\widetilde{Y})_a \longrightarrow (\widetilde{Y}G)_a.$$

They are easily seen to be b-fibrations. The space, $(\widetilde{Y}G\widetilde{Y})_a$ has seven boundary components, an adiabatic boundary ab, an adiabatic front face at, three adiabatic side faces asf, asc, ass, the Θ-boundary Θb and the free boundary eb.

To complete the construction of $(\widetilde{Y}G\widetilde{Y})_\alpha$ we need to blow up the appropriate Θ-diagonals. Consider

$$\widetilde{B}^2_{\Theta,o} = (\widetilde{\pi}^3_{o,a})^{-1}\widetilde{B}_\Theta,\ o = f, s.$$

Let $\widetilde{S}$ denote the 1-form defined at $\widetilde{B}_\Theta \subset (G\widetilde{Y})_a$ and used to define the parabolic directions in (11). Then $\widetilde{S}_{2,f} = (\widetilde{\pi}^3_{f,a})^*\widetilde{S}$ defines the parabolic bundle over $\widetilde{B}^2_{\Theta,f}$. Similarly if $\widetilde{S}$ is the subbundle of the normal bundle to $\widetilde{B}_\Theta$ in $(\widetilde{Y}G)_a$ let $\widetilde{S}_{2,s}$ denote the lift to a subbundle of the normal bundle to $\widetilde{B}^2_{\Theta,s}$. As usual the intersection is parabolically clean with

$$\widetilde{B}^3_\Theta = \widetilde{B}^2_{\Theta,f} \cap \widetilde{B}^2_{\Theta,s},\ \widetilde{S}_3 = \widetilde{S}_{2,f} \oplus \widetilde{S}_{2,s} \subset {}_+N^*\widetilde{B}^3_\Theta[20.4]$$

and the Θ diagonals lift to meet cleanly in $(\widetilde{Y}G\widetilde{Y})_a$. Then $(\widetilde{Y}G\widetilde{Y})_\alpha$ is the space obtained by first parabolically blowing-up $(\widetilde{Y}G\widetilde{Y})_a$ along $\widetilde{B}^3_\Theta$ and then the resultant space along the two (disjoint) lifts of the $\widetilde{B}^2_{\Theta,o}$, $o = f, s$. Using the notation given in §C4:

$$(\widetilde{Y}G\widetilde{Y})_\alpha = \left[\widetilde{Y}G\widetilde{Y}; \widetilde{B}^3_a; \widetilde{B}^2_a; \widetilde{B}^3_\Theta, \widetilde{S}_3; \widetilde{B}^2_\Theta, \widetilde{S}_2\right][20.5]$$

where, as usual, we let $\widetilde{B}^2_v$ denote the union of the corresponding $\widetilde{B}^2_{v,o}$ for $v = a, \Theta$. The boundary hypersurfaces of $(\widetilde{Y}G\widetilde{Y})_\alpha$ are

$$M_1(\widetilde{Y}G\widetilde{Y})_\alpha = \{\mathrm{at}, \mathrm{asf}, \mathrm{asc}, \mathrm{ass}, \mathrm{ab}, \Theta\mathrm{d}, \Theta\mathrm{sf}, \Theta\mathrm{ss}, \Theta\mathrm{b}\}.[20.6]$$

The important properties of this space are summarized as follows:

Proposition 14.1. *There are three smooth maps defined by the following commutative diagrams:*

$$\begin{array}{ccc} (\widetilde{Y}G\widetilde{Y})_\alpha & \xrightarrow{\widetilde{\pi}^3_{c,\alpha}} & \widetilde{Y}^2_a \\ \downarrow{\scriptstyle \widetilde{\beta}^3_\alpha} & & \downarrow{\scriptstyle \widetilde{\beta}^2_a} \\ \widetilde{Y}G\widetilde{Y} & \xrightarrow[\pi^3_c]{} & \widetilde{Y}^2 \end{array} \qquad \begin{array}{ccc} (\widetilde{Y}G\widetilde{Y})_\alpha & \xrightarrow{\widetilde{\pi}^3_{o,\alpha}} & (\widetilde{Y}G)_\alpha \\ \downarrow{\scriptstyle \widetilde{\beta}^3_\alpha} & & \downarrow{\scriptstyle \widetilde{\beta}^2_\alpha} \\ \widetilde{Y}G\widetilde{Y} & \xrightarrow[\pi^3_o]{} & \widetilde{Y}G \end{array}, \quad o = f, s.[20.8]$$

The maps $\widetilde{\pi}^3_{o,\alpha}$, $o = f, s$ are b-fibrations whereas $\widetilde{\pi}^3_{c,\alpha}$ is not a b-submersion. If $\Delta_{o,\alpha}$, for $o = f, s, c$ denote the lifted fibre diagonals ($o = f, s$) or diagonal ($o = c$) then $\widetilde{\pi}^3_{f,\alpha}$ is transversal to $\Delta_{o,\alpha}$, for $o = c, s$ and $\widetilde{\pi}^3_{s,\alpha}$ is transversal to $\Delta_{o,\alpha}$, for $o = f, c$.

Proof. The existence of the stretched projection $\widetilde{\pi}^3_{c,\alpha}$ follows easily as it does not require the commutation of an adiabatic blowup with a Θ-blowup. It is a b-map as it is a composition of projections and blow-down maps. The failure of $\widetilde{\pi}^3_{c,\alpha}$ to be a b-submersion occurs along the boundary component Θd where the b-differential has rank n.

The $\mathbb{Z}_2$-action on $\widetilde{Y}G\widetilde{Y}$ which interchanges the two $\widetilde{Y}$ factors lifts to a smooth diffeomorphism of $(\widetilde{Y}G\widetilde{Y})_\alpha$ thus it suffices to show that $\widetilde{\pi}^3_{f,\alpha}$ exists and is a b-fibration. The existence of the smooth b-map $\widetilde{\pi}^3_{f,\alpha}$ follows much as in the previous two cases. We use an alternate construction of $(\widetilde{Y}G\widetilde{Y})_\alpha$ starting from $\widetilde{Y}(G\widetilde{Y})_\alpha$. First blow up $\widetilde{B}^2_{a,f}$ and obtain the space $\widetilde{Y}(G\widetilde{Y})_a$. As in (13) we have

$$\widetilde{B}^3_a \pitchfork \widetilde{B}^2_{\Theta,f} \text{ in } \widetilde{Y}(G\widetilde{Y})_a.[20.9]$$

Thus we can blow these submanifolds up in either order.

We can now construct $(\widetilde{Y}G\widetilde{Y})_\alpha$ by a series of blowups starting from $\widetilde{Y}(G\widetilde{Y})_\alpha$. Namely

$$(\widetilde{Y}G\widetilde{Y})_\alpha \simeq \left[\widetilde{Y}(G\widetilde{Y})_\alpha; \widetilde{B}^3_a; \widetilde{B}^3_\Theta, \widetilde{S}_3; \widetilde{B}^2_{a,c}; \widetilde{B}^2_{a,s}; \widetilde{B}^2_{\Theta,s}, \widetilde{S}_{2,s}\right].[20.10]$$

This shows the existence of the stretched projection

$$\widetilde{\pi}^3_{f,\alpha} : (\widetilde{Y}G\widetilde{Y})_\alpha \longrightarrow (G\widetilde{Y})_\alpha.$$

This is a b-map, being a composition of diffeomorphisms, blow-down maps and projections and in view of Tables 7 and 8 which gives the exponents linking the adiabatic and Θ-boundary faces respectively it is b-normal. The exponents linking the a and Θ faces are all zero.

Arguing as in the previous two cases we show that $\widetilde{\pi}^3_{f,\alpha}$ is a b-fibration by using Proposition B.4, which leaves us to show (B.4). As before we split $\widetilde{\pi}^3_{f,\alpha}$ into two maps:

$$F_1 : Q \longrightarrow (GY)_\alpha,\ F_2 : (\widetilde{Y}G\widetilde{Y})_\alpha \longrightarrow Q$$
$$Q = \left[\widetilde{Y}(G\widetilde{Y})_\alpha; \widetilde{B}^3_a; \widetilde{B}^2_{a,s}; \widetilde{B}^2_{a,c}\right]$$

That F_1 satisfies (B.4) and that $\widetilde{\pi}^3_{f,\alpha}$ is a b-submersion away from $\{\epsilon = 0\}$ follow easily from the fibration properties of blow-down maps. Relative to F_1, $\Theta\mathrm{d} \cap \{\epsilon = 0\} \subset Q$ is a half n-sphere bundle over $\Theta\mathrm{f} \cap \{\epsilon = 0\} \subset (\widetilde{Y}G)_\alpha$ and $\Theta\mathrm{ss} \cap \{\epsilon = 0\} \subset Q$ is a half n-sphere bundle over $\Theta\mathrm{b} \cap \{\epsilon = 0\} \subset (\widetilde{Y}G)_\alpha$. As before these observations suffice to complete the proof that $\widetilde{\pi}^3_{f,\alpha}$ satisfies (B.4).

The transversality statements are simpler than in the previous two cases. All the submanifolds blown-up in the construction of $(\widetilde{Y}G\widetilde{Y})_\alpha$ are interior p-submanifolds of hypersurface boundary components. Furthermore the symmetry of the construction allows us to consider only $\widetilde{\pi}^3_{f,\alpha}$. The two submanifolds $\Delta_o; o = c, s$ lift to $\widetilde{Y}(G\widetilde{Y})_\alpha$ as graphs over $(G\widetilde{Y})_\alpha$. Thus we need to establish the injectivity of the differential of the intermediate projection,

$$\widetilde{\pi}^3_{f,i} : (\widetilde{Y}G\widetilde{Y})_\alpha \longrightarrow \widetilde{Y}(G\widetilde{Y})_\alpha,$$

restricted to the tangent spaces of $\Delta_{o,\alpha}, o = c, s$. This statement is verified by successive applications of Lemma C.5. The details in this case are left to the reader. □

CHAPTER 15

Adiabatic calculus

In order to compose kernels in $\Psi_a^m(\widetilde{Y};\Omega^{\frac{1}{2}})$ we need to construct one more very simple stretched triple product, $\widetilde{Y}_a^3$. Let

$$\pi_o^3 : \widetilde{Y}^3 \longrightarrow \widetilde{Y}^2,\ o = f, c, s,$$

denote the three projections. With $\widehat{B}_a$ defined in (12), set

$$\begin{aligned}\widehat{B}_{a,o}^2 &= (\pi_o^3)^{-1}(\widehat{B}_a), o = f, c, s \text{ and} \\ \widehat{B}_a^3 &= \widehat{B}_{a,o}^2 \cap \widehat{B}_{a,o'}^2,\ o \neq o'.\end{aligned} \quad [21.1]$$

These four manifolds meet cleanly and in $\left[\widetilde{Y}^3;\widehat{B}_a^3\right]$ the three lifts of the $\widehat{B}_{a,o}^2$ are disjoint. Denoting there union as $\widehat{B}_a^2$ set

$$\widetilde{Y}_a^3 = \left[\widetilde{Y}^3;\widehat{B}_a^3;\widehat{B}_a^2\right].[21.2]$$

Using Lemma C.7 to rearrange the blow-down there are clearly three b-maps $\pi_{o,a}^3$ which make the following diagrams commute

$$\begin{array}{ccc} \widetilde{Y}_a^3 & \xrightarrow{\pi_{o,a}^3} & \widetilde{Y}_a^2 \\ \downarrow{\scriptstyle \beta_a^3} & & \downarrow{\scriptstyle \beta_a^2} \\ \widetilde{Y}^3 & \xrightarrow[\pi_o^3]{} & \widetilde{Y}^2 \end{array} \qquad o = f, c, s.[21.3]$$

Since the following proposition, which summarizes the important properties of this space, is a simpler version of earlier results, such as Proposition 7.5, the proof is omitted.

PROPOSITION 15.1. *The three stretched projections $\pi_{o,a}^3$ in* (15) *are b-fibrations with $\pi_{o,a}^3$ transversal to the lifts of the three partial fibre diagonals $\Delta_{o',a} = (\pi_o^3)^*\Delta$ if $o' \neq o$ and such that each $\pi_{o,a}^3$ is a fibration near the lift of the triple diagonal $\Delta_{t,a}$ and maps it diffeomorphically onto $\Delta_a \subset \widetilde{Y}_a^2$.*

Table 9 shows the exponents linking the boundaries.

As we are only interested in the small $\widetilde{Y}$-calculus, where the kernels vanish to infinite order at ab, the index family has only one essential element, $\mathcal{E} = (E_{\mathrm{af}}, \emptyset)$. A more general calculus is considered in [**12**]. By what are now routine applications of the pull-back and push-forward theorems and the discussion in §18 of $\widetilde{Y}_a^2$ and that of $\widetilde{Y}_a^3$ above we obtain both conormal mapping properties and composition formulæ:

LEMMA 15.1. *For any* $m \in \mathbb{R}$, $\mathcal{E} = (E_{\mathrm{af}}, \emptyset)$, $m_a \in \mathbb{R}$ *and* $A \in \Psi_a^{m,\mathcal{E}}(\widetilde{Y}; \Omega^{\frac{1}{2}})$

$$A : \mathcal{A}^m_{\mathrm{phg}}(\widetilde{Y}; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}^{m'}_{\mathrm{phg}}(\widetilde{Y}; \Omega^{\frac{1}{2}}) \quad \textit{where } m'_a = m_a + E_{\mathrm{af}}.[21.6]$$

PROPOSITION 15.2. *Suppose* $A \in \Psi_a^{m,\mathcal{E}}(\widetilde{Y}; \Omega^{\frac{1}{2}})$ *with* $\mathcal{E} = (E_{\mathrm{af}}, \emptyset)$ *and* $B \in \Psi_a^{m',\mathcal{E}'}(\widetilde{Y}; \Omega^{\frac{1}{2}})$ *with* $\mathcal{E}' = (E'_{\mathrm{af}}, \emptyset)$ *then*

$$A \circ B \in \Psi_a^{m+m',\mathcal{F}}(\widetilde{Y}; \Omega^{\frac{1}{2}}) \textit{ where } \mathcal{F} = (F_{\mathrm{af}}, \emptyset) \textit{ and } F_{\mathrm{af}} = E_{\mathrm{af}} + E'_{\mathrm{af}}.[21.8]$$

We wish to invert an element of $\Psi_a^{m,\mathcal{E}}(\widetilde{Y}; \Omega^{\frac{1}{2}})$ in the simple case that

$$E_{\mathrm{af}} = p[21.9]$$

is just the index family $i(p + \mathbb{N}_0)$ for some real p. To do so we need to discuss the normal operator in this context. The front face of $\widetilde{Y}_a^2$ is just a half n-sphere bundle over Y with the interior being naturally a vector bundle, over Y, namely the normal bundle to the diagonal in Y^2 and hence canonically identified with TY. Thus the symbol of the kernel of an element $A \in \Psi_a^{m,p}(\widetilde{Y}; \Omega^{\frac{1}{2}})$ at $\mathrm{af}(\widetilde{Y}_a^2)$ is of the form $\epsilon^p N_{a,p}(A)$ where $N_{a,p}(A)$ acts on half-densities on TY through convolution with a kernel on each fibre:

$$N_{a,p}(A)u(y, v) = N_{a,p}(A, y) * u(y, \cdot).[21.10]$$

Here each of the kernels $N_{a,p}(A, y)$ is conormal at $0 \in N_yY$, rapidly decreasing at infinity, and depends smoothly on y. This means that the Fourier transform, normalized by the half-density constructed from the symplectic form ω :

$$\widehat{N}_{a,p}(A, y) = |\omega|^{-\frac{1}{2}} \int_{T_yY} e^{iY\cdot\eta} N_{a,p}(A, y)(Y)|dY d\eta| \in S^m(T_y^*Y)[21.11]$$

is a well-defined symbol on the cotangent bundle. It is important to notice that a complete symbol is really well-defined, not just modulo symbols of one order lower.

Notice that the transversality of the Lie algebra $\mathcal{V}_a(\widetilde{Y})$ to the lifted diagonal Δ_a gives natural isomorphisms

$$N\left\{\widetilde{Y}_a^2; \Delta_a\right\} \equiv TY \times [0, \epsilon_0][21.12]$$

so the symbol at the lifted diagonal can be interpreted, again normalizing with the symplectic density, as a symbol on the conormal bundle of Y lifted to $\widetilde{Y}$. Again we remove a factor of ϵ so that

$$\begin{gathered} \sigma_{m,p} : \Psi_a^{m,p}(\widetilde{Y}; \Omega^{\frac{1}{2}}) \twoheadrightarrow S^{[m]}(T^*Y \times [0, \epsilon_0]) \\ S^{[m]}(T^*Y \times [0, \epsilon_0]) = S^m(T^*Y \times [0, \epsilon_0])/S^{m-1}(T^*Y \times [0, \epsilon_0]). \end{gathered} [21.13]$$

From the definitions of each of these we see that

$$[\widehat{N}_{a,p}(A)] = \sigma_{m,p}(A)_{\restriction\epsilon=0} \text{ in } S^{[m]}(T^*Y).[21.14]$$

Proposition 15.3. *Together the adiabatic normal operator and diagonal symbol give a surjective homomorphism*

$$\Psi_a^{m,p}(\widetilde{Y};\Omega^{\frac{1}{2}}) \twoheadrightarrow \left\{(\widehat{N}_{a,p},\sigma_{m,p}) \in S^m(T^*Y) \times S^{[m]}(T^*Y \times [0,\epsilon_0]; [\widehat{N}] = \sigma_{\restriction \epsilon=0}\right\} [21.16]$$

which filters the adiabatic algebra in the sense that

$$\begin{gathered} A \in \Psi_a^{m,p}(\widetilde{Y};\Omega^{\frac{1}{2}}),\ B \in \Psi_a^{m',p'}(\widetilde{Y};\Omega^{\frac{1}{2}}) \Longrightarrow \\ \left(\widehat{N}_{a,p+p'}(A \cdot B)\right), \sigma_{m+m',p+p'}(A \cdot B) = \left(\widehat{N}_{a,p}(A)\widehat{N}_{a,p'}(B), \sigma_{m,p}(A)\sigma_{m',p'}(B)\right) \end{gathered} [21.17]$$

and

$$A \in \Psi_a^{m,p}(\widetilde{Y};\Omega^{\frac{1}{2}}),\ \widehat{N}_{a,p}(A) = 0 \Longleftrightarrow A \in \Psi_a^{m,p+1}(\widetilde{Y};\Omega^{\frac{1}{2}}).[21.18] \tag{15.1}$$

$$A \in \Psi_a^{m,p}(\widetilde{Y};\Omega^{\frac{1}{2}}),\ \sigma_{m,p}(A) = 0 \Longleftrightarrow A \in \Psi_a^{m-1,p}(\widetilde{Y};\Omega^{\frac{1}{2}}).[21.19] \tag{15.2}$$

An element $A \in \Psi_a^{m,p}(\widetilde{Y},\Omega^{\frac{1}{2}})$ *is invertible with inverse in* $\Psi_a^{-m,-p}(\widetilde{Y},\Omega^{\frac{1}{2}})$, *for* $\epsilon_0 > 0$ *sufficiently small, if and only if it is elliptic in the strong sense that*

$$\widehat{N}_{a,p}(A) \neq 0 \text{ and } \sigma(A) \neq 0.[21.20]$$

Proof. As far as the symbolic properties are concerned only (15.3) really requires any comment. To prove this it suffices to work with operators having no diagonal singularity and with kernels having support in $\widetilde{Y}_a^2$ not meeting ab and then to use continuity. Notice that an operator in $\Psi_a^*(\widetilde{Y};\Omega^{\frac{1}{2}})$ with kernel having support disjoint from the diagonal and $\mathrm{af}(\widetilde{Y}_a^2)$ is a smoothing operator on Y depending smoothly on ϵ and with kernel vanishing with all derivatives as $\epsilon \to 0$. Such operators form the residual ideal, just $\Psi_a^{-\infty,\emptyset}(\widetilde{Y};\Omega^{\frac{1}{2}})$, on which all symbol maps vanish. Thus we can actually assume that the kernel of each operator has compact support contained in the square of a coordinate patch. On the support of the kernel ϵ can be used as a defining function for $\mathrm{af}(\widetilde{Y}_a^2)$ and the projective local coordinates $Y = (y - y')/\epsilon$ and y can be used. In these coordinates, with the simplifying assumptions just made, the kernel is of the form

$$\kappa_A = K_A(\epsilon, y, Y)\epsilon^{-\frac{n}{2}}|dydY|^{\frac{1}{2}},\ K_A \in \mathcal{C}_c^\infty([0,\epsilon_0] \times \mathbb{R}^{2n}).$$

When the two kernels, of A and of B, are lifted to $\widetilde{Y}_a^3$ they are $\mathcal{C}^\infty$ and their product has support meeting the boundary only in the interior of the triple face $\mathrm{at}(\widetilde{Y}_a^3)$. Thus ϵ can again be used as a defining function for the boundary and the coordinates

$$y, Y', Y'',\ \text{where } y' = y - \epsilon Y',\ y'' = y' - \epsilon Y'' = y + \epsilon(Y' + Y'')[21.21]$$

can be used. The composite kernel also has support meeting the boundary in the interior of the front face and in terms of the coordinates $y, Z = (y'' - y)/\epsilon$ the projection π_c^3 can be written

$$\pi_c^3(y, Y', Y'') = (y, Y' + Y'') = (y, Z)$$

Thus the composite kernel can be written

$$\begin{aligned}\kappa_{A\cdot B} &= (\pi^3_c)_*\left[K_A\big(\epsilon, y, \frac{y-y'}{\epsilon}\big)|dydy'|^{\frac12}K_B\big(\epsilon, y', \frac{y'-y''}{\epsilon}\big)|dy'dy''|^{\frac12}\right]\\ \Longrightarrow K_{A\cdot B}(\epsilon,y,Z) &= \int\limits_{\mathbb{R}^n} K_A(\epsilon,y,Y')K_B(y-\epsilon Y', Z-Y')dY'.\end{aligned} \quad [21.22]$$

At $\epsilon = 0$ this shows that the restriction of the composite kernel to the front face arises as the convolution of the restrictions of factors. Taking the Fourier transform gives the homomorphism (15.3).

The non-vanishing of $\widehat{N}_{a,p}$ means precisely that it has a multiplicative inverse

$$b\cdot \widehat{N}_{a,p} = 1,\ b\in S^{-m}(T^*Y)[21.23]$$

where b can be extended from $\epsilon = 0$ in such a way that

$$\sigma_{m,p}(A)\cdot b = 1 \text{ in } S^{[0]}(T^*Y\times[0,\epsilon_0]).[21.24]$$

The surjectivity in (15.3) and the multiplicativity in (15.1) together show that we can find $B_0 \in \Psi_a^{-m,-p}(\widetilde{Y};\Omega^{\frac12})$ such that

$$A\cdot B_0 = \mathrm{Id} - R_1,\ R_1\in\Psi_a^{-1,1}(\widetilde{Y};\Omega^{\frac12}).[21.25]$$

The composition formula (15.2) now shows that

$$\begin{gathered} R_1^p \in \Psi_a^{-p,p}(\widetilde{Y};\Omega^{\frac12}) \Longrightarrow \\ \exists\ S\in\Psi_a^{-1,1}(\widetilde{Y};\Omega^{\frac12}) \text{ with } S - \sum_{0<p<N} R_1^p \in \Psi_a^{-N,N}(\widetilde{Y};\Omega^{\frac12})\ \forall\ N.\end{gathered} [21.26]$$

Here we use the definition, (11), to see that asymptotic summation is possible. Then, again from (15.2), $B' = B(\mathrm{Id} - S) \in \Psi_a^{-m,-p}(\widetilde{Y};\Omega^{\frac12})$ is such that (15) is improved to

$$A\cdot B' = \mathrm{Id} - R',\ R'\in\Psi_a^{-\infty,\emptyset}(\widetilde{Y};\Omega^{\frac12}).[21.27]$$

An element such as R' of the doubly residual space is simply a smoothing operator with kernel vanishing to all orders at $\epsilon = 0$. Thus, for small ϵ_0, $\mathrm{Id} - R'$ has inverse of the form $\mathrm{Id} + R''$ with R'' residual in the same sense. Thus $B = B' + B'\cdot R''$ is a right and left inverse, as claimed. This proves the sufficiency of (15.3) for invertibility with inverse in $\Psi_a^{-m,-p}(\widetilde{Y};\Omega^{\frac12})$. The necessity is obvious from the symbol homomorphism. □

In the sequel it is useful to understand the inverse of an 'adiabatic Laplacian' on $\widetilde{Y}$. To define such an operator choose a Riemannian metric, g on Y and set

$$\Delta^a_{\widetilde{Y}} = \epsilon^2\Delta_g.$$

This operator is obviously an element of $\mathrm{Diff}^2_a(\widetilde{Y};\Omega^{\frac12})$. An elementary calculation shows that the normal operator is simply the Euclidean Laplacian on $\mathbb{R}^n$. As a simple consequence of the Proposition 15.3 it follows that

$$(\Delta^a_{\widetilde{Y}} + \mathrm{Id})^{-1} \in \Psi_a^{-2;0,\emptyset}(\widetilde{Y};\Omega^{\frac12}).[21.271]$$

This is so because the symbol of the normal operator is simply $|\xi|^2 + 1$, where $|\cdot|$ is the norm defined on T^*Y by the metric g chosen above. For applications it is useful to have a more explicit description of the kernel of $(\Delta^a_{\widetilde{Y}} + \mathrm{Id})^{-m}$,

PROPOSITION 15.4. *In the coordinates, $Y, y; \epsilon$ introduced above, the kernel, $K_m(Y, y; \epsilon)$ of $(\Delta^a_{\widetilde{Y}} + \mathrm{Id})^{-m}, m > 0$ has the following decomposition*

$$K_m(Y, y; \epsilon) = |Y|^{2m-n} F_m(Y, y; \epsilon) + H_m(Y, y; \epsilon),$$

where

$$F_m \in \Psi^{0;0}_a(\widetilde{Y}; \Omega^{\frac{1}{2}}), H_m \in \Psi^{-\infty,0}_a(\widetilde{Y}; \Omega^{\frac{1}{2}}).$$

PROOF. For $\epsilon > 0$ this follows immediately from the fact that $(\Delta^a_{\widetilde{Y}} + \mathrm{Id})^{-m}$ is a polyhomogeneous pseudodifferential operator with principle symbol $(|\xi|^2 + 1)^{-m}$. As is evident from the proof of Theorem 15.3 this description extends smoothly to the front face in the scaled coordinates. The regularity at $\mathrm{af} \cap \mathrm{ab}$ follows easily from fact that the kernel of the normal operator vanishes to infinite order at this locus. □

We conclude this section with a discussion of the L^2-mapping properties of operators in $\Psi^{m;E_{\mathrm{ab}},E_{\mathrm{af}}}_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$. Like the calculus these are rather simple

THEOREM 15.1. *An operator $A \in \Psi^{m;E_{\mathrm{ab}},E_{\mathrm{af}}}_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$ defines a bounded operator from $L^2(\widetilde{Y}; \Omega^{\frac{1}{2}})$ to itself provided*

$$m \geq 0, E_{\mathrm{ab}} > 0, E_{\mathrm{af}} \geq 0.$$

PROOF. We apply Hörmander's technique to reduce to operators of order of $-\infty$ and then use Schur's theorem. ************This proof could probably be left****************************** to the reader. □

We can also define 'adiabatic' Sobolev spaces :

$$\mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}) = \{f \in \mathcal{C}^{-\infty}(\widetilde{Y}; \Omega^{\frac{1}{2}}); Af \in L^2(\widetilde{Y}; \Omega^{\frac{1}{2}}), A \in \Psi^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}})\}, s \geq 0,$$
$$\mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}) = \{u \in \mathcal{C}^{-\infty}(\widetilde{Y}; G); u = \sum_i P_i u_i; P_i \in \Psi^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}), u_i \in L^2(\widetilde{Y}; \Omega^{\frac{1}{2}})\}, s \leq 0;^{[21.29]}$$

and weighted spaces

$$\rho^t \mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}) = \{u \in \mathcal{C}^{-\infty}(\widetilde{Y}; \Omega^{\frac{1}{2}}); \rho^{-t}_{\mathrm{ab}} u \in \mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}).[21.30]$$

The mapping theorem for the weighted spaces is

PROPOSITION 15.5. *An operator $A \in \Psi^{m;E_{\mathrm{ab}},E_{\mathrm{af}}}_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$ defines a bounded operator between $\rho^t \mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$ and $\rho^{t'} \mathcal{H}^{s'}_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$ provided that*

$$t - m \geq t', s + E_{\mathrm{af}} \geq s', s + E_{\mathrm{ab}} > s'.[21.32]$$

If all inequalities hold strictly then the operator is compact.

PROOF. This follows easily from the composition properties of the calculus by considering the operator $\rho^{-s'}_{\mathrm{ab}} A \rho^s_{\mathrm{ab}}$. □

CHAPTER 16

α-projection calculus

Let $\mathcal{E}$ be an index family for $(G\widetilde{Y})_\alpha$ labelled according to

$$\mathcal{E} = (E_{\mathrm{af}}, E_{\mathrm{ab}}, E_{\Theta\mathrm{f}}, E_{\Theta\mathrm{b}}).[22.1]$$

We generalize (11) to include kernels which do not vanish rapidly along the Θ-boundary by setting

$$\Psi_\alpha^{m,\mathcal{E}}(\widetilde{Y}, G; \Omega^{\frac{1}{2}}) = I_{\mathrm{phg}}^{m-\frac{1}{4},\mathcal{E}}((G\widetilde{Y})_\alpha, \widetilde{\Delta}_\phi; \widetilde{\mathrm{KD}}_\alpha).[22.2]$$

The space of operators $\Psi_\alpha^{m,\mathcal{E}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}})$ is defined by transposition in (16). These operators define continuous linear maps

$$\begin{aligned} \Psi_\alpha^{m,\mathcal{E}}(\widetilde{Y}, G; \Omega^{\frac{1}{2}}) \ni A &: \dot{\mathcal{C}}^\infty(\widetilde{Y}; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}(G; \Omega^{\frac{1}{2}}) \\ \Psi_\alpha^{m,\mathcal{E}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}}) \ni B &: \dot{\mathcal{C}}^\infty(G; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}(\widetilde{Y}; \Omega^{\frac{1}{2}}) \end{aligned} [22.3]$$

respectively by the usual formulæ:

$$\begin{aligned} (A\phi)\nu &= (\widetilde{\pi}_{l,\alpha}^2)_* \left[(\widetilde{\pi}_{l,\alpha}^2)^* \nu \kappa_A (\widetilde{\pi}_{r,\alpha}^2)^* \phi |d\epsilon|^{-\frac{1}{2}} \right] \in \mathcal{C}^{-\infty}(G; \Omega), \\ (B\psi)\nu' &= (\widetilde{\pi}_{l,\alpha}^2)_* \left[(\widetilde{\pi}_{l,\alpha}^2)^* \nu' \kappa_B (\widetilde{\pi}_{r,\alpha}^2)^* \psi |d\epsilon|^{-\frac{1}{2}} \right] \in \mathcal{C}^{-\infty}(\widetilde{Y}; \Omega). \end{aligned} [22.4]$$

Here ν and ν' are non-vanishing half-densities on G and $\widetilde{Y}$ respectively and one should note that the maps $\widetilde{\pi}_{l,\alpha}^2$ and $\widetilde{\pi}_{r,\alpha}^2$ are interchanged by the transposition isomorphism $(G\widetilde{Y})_\alpha \longleftrightarrow (\widetilde{Y}G)_\alpha$.

In order to state composition formulæ involving these operators we need to show that the appropriate composite operators exist. For this purpose the mapping properties on spaces of conormal distributions suffice.

Lemma 16.1. *For any $m \in \mathbb{R}$, index set M for $\widetilde{Y}$ and any index family $\mathcal{E}$ for $(G\widetilde{Y})_\alpha$, $A \in \Psi_\alpha^{m,\mathcal{E}}(\widetilde{Y}, G; \Omega^{\frac{1}{2}})$ defines a linear operator*

$$A : \mathcal{A}_{\mathrm{phg}}^M(\widetilde{Y}; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}(G; \Omega^{\frac{1}{2}}) \text{ where } \mathcal{F} = ((M + E_{\mathrm{af}})\overline{\cup} E_{\mathrm{ab}}, E_{\Theta\mathrm{b}} \overline{\cup} E_{\Theta\mathrm{f}}).[22.6]$$

For any index family $\mathcal{G}$, for G each $A \in \Psi_\alpha^{m,\mathcal{E}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}})$ gives a continuous linear map

$$\begin{aligned} A : \mathcal{A}_{\mathrm{phg}}^{\mathcal{G}}(G; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}_{\mathrm{phg}}^L(\widetilde{Y}; \Omega^{\frac{1}{2}}) \text{ provided} \\ E_{\Theta\mathrm{b}} + G_{\Theta\mathrm{b}}, E_{\Theta\mathrm{f}} + G_{\Theta\mathrm{b}} > -1 \text{ and } L = (G_{\mathrm{ab}} + E_{\mathrm{af}})\overline{\cup} E_{\mathrm{ab}}. \end{aligned} [22.7]$$

PROOF. These results are obtained by applying Proposition B.1 or Proposition B.2 and Theorem B.3 to the formulæ (16). Note that the two stretched projections in (11) are b-fibrations transversal to the lifted fibre diagonal, see Lemma 11.1. □

If $\mathcal{E} = (E_{\text{af}}, E_{\text{ab}}, E_{\Theta\text{b}})$ is an index family for $(G\widetilde{Y})_a$ we can extend the definition (11) and consider more general 'adiabatic' operators

$$\Psi_a^{m,\mathcal{E}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}}) = \left\{ k \in I_{\text{phg}}^{m,\mathcal{E}}((\widetilde{Y}G)_a, \Delta_\phi; \widetilde{\text{KD}}_a) \right\}. [22.8]$$

A proof similar to that of Lemma 16.1 above gives:

$$\begin{gathered} A \in \Psi_a^{m,\mathcal{E}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}}) \Longrightarrow A : \mathcal{A}_{\text{phg}}^{\mathcal{F}}(G; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}_{\text{phg}}^{G}(\widetilde{Y}; \Omega^{\frac{1}{2}}) \\ \text{provided } F_{\Theta\text{b}} + E_{\Theta\text{b}} > -1 \text{ and } G = (F_{\text{ab}} + E_{\text{af}}) \overline{\cup} E_{\text{ab}}. \end{gathered} [22.9]$$

Operators in this class with $m = -\infty$ are a 'Θ-residual class' for $\Psi_\alpha(G, \widetilde{Y}; \Omega^{\frac{1}{2}})$ as they have kernels which are polyhomogeneous conormal on $(G\widetilde{Y})_a$. From (16) we see that for $A \in \Psi_\alpha^{m',\mathcal{E}'}(\widetilde{Y}, G; \Omega^{\frac{1}{2}})$ and $B \in \Psi_a^{m,\mathcal{E}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}})$ the composite operator $A \circ B$ is always well-defined on $\dot{\mathcal{C}}^\infty(G; \Omega^{\frac{1}{2}})$. We are really only interested in the case that A has diagonal order $-\infty$:

PROPOSITION 16.1. *If* $A \in \Psi_\alpha^{-\infty,\mathcal{E}}(\widetilde{Y}, G; \Omega^{\frac{1}{2}})$ *with* $E_{\text{ab}} = \emptyset$ *and* $B \in \Psi_a^{m,\mathcal{F}}(G, \widetilde{Y}; \Omega^{\frac{1}{2}})$ *with* $F_{\text{ab}} = \emptyset$ *then*

$$\begin{gathered} A \circ B \in \Psi_\alpha^{-\infty,\mathcal{G}}(G; \Omega^{\frac{1}{2}}) \textit{ with} \\ G_{\Theta\text{f}} = E_{\Theta\text{f}} + F_{\Theta\text{b}},\ G_{\Theta\text{r}} = 0,\ G_{\Theta\text{l}} = E_{\Theta\text{b}} \overline{\cup} E_{\Theta\text{f}} [22.11] \\ G_{\text{af}} = E_{\text{af}} + F_{\text{af}},\ G_{\text{ab}} = \emptyset. \end{gathered}$$

PROOF. To analyze the product we use the triple product $(G\widetilde{Y}G)_\alpha$, where the asymmetry between the factors of G is such that the b-fibration is the stretched version of the map $\widehat{\pi}_s^3 : G^2\widetilde{Y} \longrightarrow G\widetilde{Y}$ arises from projection off the right factor of G. Thus, from Proposition 13.1, the stretched projection which is not a b-fibration is $\widehat{\pi}_f^3$, arising from projection off the left factor of G. The kernel of the composite operator $C = A \circ B$ can be written

$$\kappa_C = (\widehat{\pi}_c^3)_* \left[(\widehat{\pi}_s^3)^* \kappa_A \cdot (\widehat{\pi}_f^3)^* \kappa_B \right] [22.12]$$

if due allowance is made for the behaviour of densities. Proposition B.1 applies to the pull-back of the kernel of A. The map $\widehat{\pi}_f^3$ is the composition of a b-fibration from $(G\widetilde{Y}G)_a$ to $(\widetilde{Y}G)_a$ and parabolic blow-down maps, see the discussion following (13). Since Proposition B.2 applies to the b-fibration the kernel κ_B lifts to $(G\widetilde{Y}G)_a$ to be polyhomogeneous conormal with respect to the p-submanifold $\widehat{B}_{a,f}^2$. Thus it remains to examine the lift under the two parabolic blow-up maps needed to construct $(G\widetilde{Y}G)_\alpha$ from $(G\widetilde{Y}G)_a$. Proposition C.3 applies to the blow-up of the Θ triple diagonal and gives a polyhomogeneous conormal distribution with respect to an intersection pair of p-submanifolds, namely the lifted $\widehat{B}_{a,f}^2$ and the additional boundary p-submanifold Q. The blow-up of the two partial Θ-diagonals is transversal to both these submanifolds so κ_B lifts to $(G\widetilde{Y}G)_\alpha$ to be polyhomogeneous to a pair of p-submanifolds. Thus the product in (16) is similarly polyhomogeneous. Now the b-fibration $\widehat{\pi}_c^3$ to G_α^2 is transversal to both these submanifolds and their intersection, so Theorem B.4 applies to show that the kernel of the composite operator is polyhomogeneous conormal on G_α^2.

To complete the proof of the proposition we need to check the index sets at the various boundary faces; this follows from the statements on orders in the various results used above. This information is summarized in the McDonald tables 5 and 6. $\square$

A similar formula arises from the composition of an operator of diagonal order $-\infty$ on G with an operator from $\widetilde{Y}$ to G :

PROPOSITION 16.2. *Let* $A \in \Psi_\alpha^{-\infty,\mathcal{E}}(G;\Omega^{\frac{1}{2}})$ *and* $B \in \Psi_a^{m,\mathcal{F}}(\widetilde{Y},G;\Omega^{\frac{1}{2}})$ *where*

$$E_{\Theta\mathrm{r}} + F_{\Theta\mathrm{b}} > -1,\ E_{\mathrm{ab}} = F_{\mathrm{ab}} = \emptyset[22.13]$$

then

$$A \circ B \in \Psi_\alpha^{-\infty,\mathcal{F}}(\widetilde{Y},G;\Omega^{\frac{1}{2}}) \text{ with}$$
$$F_{\Theta\mathrm{f}} = (E_{\Theta\mathrm{f}} + F_{\Theta\mathrm{b}})\overline{\cup}(E_{\Theta\mathrm{l}} + n + 1),\ F_{\Theta\mathrm{b}} = E_{\Theta\mathrm{l}}, [22.14]$$
$$F_{\mathrm{af}} = E_{\mathrm{af}},\ F_{\mathrm{ab}} = E_{\mathrm{ab}} + n.$$

PROOF. The proof is essentially the same as for Proposition 16.1. The same triple product space is involved, but in the order $(GG\widetilde{Y})_\alpha$. Again the map which is not a b-fibration is the one to $(G\widetilde{Y})_a$, corresponding to the lifting of the kernel of B, and the rôles of the other two maps have simply been reversed. Thus only the computation of the index family for the composite is different. $\square$

To study the mapping properties of operators in $\Psi_\alpha^{-\infty;E}(\widetilde{Y},G;\Omega^{\frac{1}{2}})$ it is useful to have the a formula for composition with $\mathrm{Diff}_\alpha^m(G;\Omega^{\frac{1}{2}})$.

PROPOSITION 16.3. *Let* $P \in \mathrm{Diff}_\alpha^m(G;\Omega^{\frac{1}{2}})$ *and* $A \in \Psi_\alpha^{-\infty;E}(\widetilde{Y},G;\Omega^{\frac{1}{2}})$ *then*

$$P \circ A \in \Psi_\alpha^{-\infty;E}(\widetilde{Y},G;\Omega^{\frac{1}{2}})$$

as well.

PROOF. This is because $\mathcal{V}_\alpha(G)$ lifts to $\mathcal{V}_b((G\widetilde{Y})_\alpha)$. $\square$

The last general composition we consider is between $\Psi_\alpha^{-\infty,\mathcal{E}}(\widetilde{Y},G;\Omega^{\frac{1}{2}})$ and $\Psi_a^{m,\mathcal{F}}(\widetilde{Y};\Omega^{\frac{1}{2}})$. It follows from Lemma 15.1 and Lemma 16.1 that the composite is well-defined for all index families.

PROPOSITION 16.4. *Suppose* $A \in \Psi_\alpha^{-\infty,\mathcal{E}}(\widetilde{Y}, G; \Omega^{\frac12})$ *with* $E_{\rm ab} = \emptyset$ *and* $B \in \Psi_a^{m,\mathcal{F}}(\widetilde{Y}; \Omega^{\frac12})$ *also with* $F_{\rm ab} = \emptyset$ *if* $m \ge 0$ *then*

$$A \circ B \in \Psi_\alpha^{-\infty,\mathcal{G}}(\widetilde{Y}, G; \Omega^{\frac12}) \text{ with}$$
$$G_{\Theta\rm f} = (E_{\Theta\rm f} - 2m)\overline{\cup}(E_{\Theta\rm b} + n + 1), G_{\Theta\rm b} = E_{\Theta\rm b}\overline{\cup}E_{\Theta\rm f}, [22.16]$$
$$G_{\rm af} = (E_{\rm af} + F_{\rm af}),\ G_{\rm ab} = \emptyset.$$

If $m < 0$ *then*

$$\begin{gathered} A \circ B \in \Psi_\alpha^{-\infty,\mathcal{G}}(\widetilde{Y}, G; \Omega^{\frac12}) + \Psi^{-\infty,\mathcal{G}'}(\widetilde{Y}, G; \Omega^{\frac12}) \text{ with} \\ G_{\Theta\rm f} = (E_{\Theta\rm f} - m)\overline{\cup}(E_{\Theta\rm b} + n + 1), G_{\Theta\rm b} = E_{\Theta\rm b}\overline{\cup}E_{\Theta\rm f}, \\ G_{\rm af} = (E_{\rm af} + F_{\rm af}), G_{\rm ab} = \emptyset; \\ G'_{\Theta\rm b} = E_{\Theta\rm b}\overline{\cup}E_{\Theta\rm f}, G'_{\rm af} = (E_{\rm af} + F_{\rm af})\, G'_{\rm ab} = \emptyset. \end{gathered} [22.161]$$

PROOF. Here we use the triple product $(G\widetilde{Y}^2)_\alpha$, which is diffeomorphic to $(\widetilde{Y}G\widetilde{Y})_\alpha$. Rearranging the discussion in §20 we see the stretched projection

$$\widetilde{\pi}_f^3 : (G\widetilde{Y}^2)_\alpha \longrightarrow \widetilde{Y}_\alpha^2 [22.17]$$

is not a b-fibration, whereas the other two stretched projections, to $(G\widetilde{Y})_\alpha$ and $(\widetilde{Y}G)_\alpha$ are b-fibrations. Thus the proof is again quite similar to that of Proposition 16.1.

The behavior at the Θ front face is obtained for positive even integral values of m by writing

$$B = (\mathrm{Id} + \varDelta^a_{\widetilde{Y}})^m (\mathrm{Id} + \varDelta^a_{\widetilde{Y}})^{-m} B [22.171]$$

. As a consequence of (15), (15.3), Proposition 15.4 it follows that

$$(\mathrm{Id} + \varDelta^a_{\widetilde{Y}})^{-m} B \in \Psi_a^{0;\mathcal{F}}(\widetilde{Y}; \Omega^{\frac12})$$

The action of the adiabatic $\widetilde{Y}$-Laplacian is easily determined in local coordinates. Thus we are reduced to order zero case.

To handle the case of negative even integers we use the decomposition of the Schwartz kernel of $(\varDelta^a_{\widetilde{Y}} + \mathrm{Id})^{-m}$ established in Proposition 15.4

$$K_m(y, y') = |y - y'|^{2m-n} F_m(y, y', \epsilon) + H_m(y, y'; \epsilon) [22.172]$$

Here $F_m(y, y'; \epsilon) \in \Psi_\alpha^{0;F'_{\rm af}}(\widetilde{Y}; \Omega^{\frac12})$ and H_m is smooth along the diagonal. Combining this with the expression for B given in(16) we easily obtain the residual term. The lift of $|y - y'|^{2m-n}$ combines with the densities to give a density conormal to the partial fiber diagonal multiplied by $\rho_{\Theta\rm t}^{2m}$. Since the second factor is again of order zero we can apply the order zero result to obtain the conclusion in this case as well. □

****************** This behavior at the front face is important in the study of the L^2-mapping properties for this calculus. Need to remove integral restrictions.

In this last case it is important to understand the behaviour of the adiabatic normal operators. *********************

As with the previous calculi we consider the L^2-mapping properties. It is essential to consider these operators as maps between $\mathcal{H}^s_a(\widetilde{Y};\Omega^{\frac12})$ and $\mathcal{H}^{s'}_\alpha(G;\Omega^{\frac12})$. This is because $E_{\Theta\mathrm{f}} < 0$, for the inverse of the push forward.

THEOREM 16.1. *An operator $A \in \Psi^{-\infty;\mathcal{E}}_\alpha(\widetilde{Y}, G;\Omega^{\frac12})$ defines a bounded operator $A : \mathcal{H}^s_a(\widetilde{Y};\Omega^{\frac12}) \longrightarrow \mathcal{H}^{s'}_\alpha(G;\Omega^{\frac12})$, provided*

$$E_{\mathrm{ab}} > 0, E_{\mathrm{af}} \geq 0, E_{\Theta\mathrm{b}} > -\frac12, \frac12 E_{\Theta\mathrm{f}} + s \geq 0.[22.19]$$

The transpose, A^t defines a bounded map $A^t : \mathcal{H}^{s'}_\alpha(G;\Omega^{\frac12}) \longrightarrow \mathcal{H}^s_a(\widetilde{Y};\Omega^{\frac12})$ provided

$$E_{\mathrm{ab}} > 0, E_{\mathrm{af}} \geq 0, E_{\Theta\mathrm{b}} > -\frac12, \frac12 E_{\Theta\mathrm{f}} \geq s.[22.20]$$

PROOF. We use the composition formula to write A as $\widetilde{A}(1+\Delta_{\widetilde{Y}^a})^{\frac{m}{2}} + A'$. The operator $\widetilde{A}$ has $\widetilde{E}_{\Theta\mathrm{f}} \geq 0$, and A' is a residual operator. We can then apply the Schwarz inequality to $< Af, g >$ with the weights $\rho_{\Theta\mathrm{b}}^{\pm\frac12}$ to reduce to an application of Schur's theorem as before. The fact that these kernels have range in $\mathcal{H}^\infty_\alpha(G;\Omega^{\frac12})$ follows from the fact that the kernel is smooth on the lifted diagonal and Proposition 16.3

Mapping the other direction goes essentially the same way, we need to observe that elements of negative, integral α-Sobolev spaces can be written as the image of elements of $L^2(G;\Omega^{\frac12})$ under α-differential operators. Then we use the fact that such operators lift to be tangent to the boundary of $(G\widetilde{Y})_\alpha$ and the uniform boundedness principle. ******need the uniform integrability lemma***************** □

We can also consider maps between weighted Sobolev spaces.

THEOREM 16.2. *An operator $A \in \Psi^{-\infty;\mathcal{E}}_\alpha(\widetilde{Y}, G;\Omega^{\frac12})$ defines a bounded operator $A : \rho^{t_1}_{\mathrm{ab}}\mathcal{H}^s_a(\widetilde{Y};\Omega^{\frac12}) \longrightarrow \rho^{t'_1}_{\mathrm{ab}}\rho^{t'_2}_{\Theta\mathrm{b}}\mathcal{H}^{s'}_\alpha(G;\Omega^{\frac12})$, provided*

$$E_{\mathrm{ab}} + t_1 > t'_1, E_{\mathrm{af}} + t_1 \geq t'_1, E_{\Theta\mathrm{b}} > t'_2 - \frac12, \frac12 E_{\Theta\mathrm{f}} + s \geq \frac12 t'_2.[22.31]$$

The transpose, A^t defines a bounded map provided $A^t : \rho^{t'_1}_{\mathrm{ab}}\rho^{t'_2}_{\Theta\mathrm{b}}\mathcal{H}^{s'}_\alpha(G;\Omega^{\frac12}) \longrightarrow \rho^{t_1}_{\mathrm{ab}}\mathcal{H}^s_a(\widetilde{Y};\Omega^{\frac12})$ provided

$$E_{\mathrm{ab}} + t'_1 > t_1, E_{\mathrm{af}} + t'_1 \geq t_1, E_{\Theta\mathrm{b}} + t'_2 > -\frac12, \frac12 (E_{\Theta\mathrm{f}} + t'_2) \geq s.[22.120]$$

PROOF. We simply consider $\rho^{-t'_1}_{\mathrm{ab}}\rho^{-t'_2}_{\Theta\mathrm{b}}A\rho^{t_1}_{\mathrm{ab}}$ for the first case and $\rho^{-t_1}_{\mathrm{ab}}A\rho^{t'_1}_{\mathrm{ab}}\rho^{t'_2}_{\Theta\mathrm{b}}$ for the second and apply Theorem 16.1. □

In this case we require a more powerful mapping result, which considers the mappings between adiabatic Sobolev spaces on $\widetilde{Y}$ and Heisenberg-adiabatic Sobolev spaces on G. The latter spaces were defined at the end of §12.

It is useful to have a lifting result for the adiabatic Laplacian on $\widetilde{Y}$.

PROPOSITION 16.5. *The the lift of $\Delta^a_{\widetilde{Y}}$ to $(G\widetilde{Y})_\alpha$ can be expressed as*

$$\begin{gathered}\widetilde{\pi}2r^*_\alpha(\Delta^a_{\widetilde{Y}}) = P_2 + V_1P_1 + V_2, \text{ where}\\ P_2, P_1 \text{ are lifts from the left with } \mathfrak{w}(P_2) = 4, \mathfrak{w}(P_1) = 2,\\ \text{of orders } 2, 1 \text{ respectively, } V_i \in \operatorname{Diff}^i_b(G\widetilde{Y})_\alpha.\end{gathered}$$

PROOF. This is a local coordinate calculation. □

The mapping result is

THEOREM 16.3. *An operator $A \in \Psi^{-\infty,\mathcal{E}}_\alpha(\widetilde{Y}, G; \Omega^{\frac{1}{2}})$ with $E_{\Theta\mathrm{b}} \subset \mathbb{N}_0$. defines a bounded map $A : \mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}) \longrightarrow \mathfrak{H}^t_a(G; \Omega^{\frac{1}{2}})$ provided $s \geq 0$ and*

$$E_{\Theta\mathrm{f}} \geq t - s, E_{\Theta\mathrm{b}} > -\frac{1}{2}, E_{\mathrm{ab}} > 0, E_{\mathrm{af}} \geq 0.[22.393]$$

For $s \leq 0$ the operator is a bounded map $A : \mathcal{H}^s_a(\widetilde{Y}; \Omega^{\frac{1}{2}}) \longrightarrow \mathfrak{H}^t_a(G; \Omega^{\frac{1}{2}})$ provided

$$E_{\Theta\mathrm{f}} \geq t - 2s, E_{\Theta\mathrm{b}} > -\frac{1}{2}, E_{\mathrm{ab}} > 0, E_{\mathrm{af}} \geq 0.[22.395]$$

PROOF. We present the proof for integral values of s, t. First we suppose that $s \geq 0$. Let $P \in \operatorname{Diff}^*_a(G; \Omega^{\frac{1}{2}})$ have weight t, then it follows from (**??**) and Proposition 16.3 and Proposition 15.4 that $P \circ A(\mathrm{Id} + \Delta^a_{\widetilde{Y}})^{-\frac{1}{2}s}$ belongs to $\Psi^{-\infty;???}_\alpha(G; \Omega^{\frac{1}{2}}) + \Psi^{-\infty;???}_a(G; \widetilde{Y})$. The index set of the non-residual term is non-negative at Θf. Therefore Theorem 16.1 implies that this composition defines a bounded map from $L^2(\widetilde{Y}; \Omega^{\frac{1}{2}})$ to $L^2(G; \Omega^{\frac{1}{2}})$. The assertion of the theorem follows easily from this.

To obtain the mapping result for $s < 0$ we use Proposition 16.5. First we consider $s = -2m$, an even integer. If $u \in \mathcal{H}^{-2m}_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$ then $u = (\Delta^a_{\widetilde{Y}} + \mathrm{Id})^m f$ for some $f \in L^2(\widetilde{Y}; \Omega^{\frac{1}{2}})$. Applying the lifting result we see that Au is a some of terms with the most singular belonging to $\mathfrak{H}^{-4m}_a(G; \Omega^{\frac{1}{2}})$. This completes the proof of the theorem in this case.

The result for general real values of s, t follows from an interpolation argument. *********I hope*************** □

The failure of the map $\widetilde{\pi}^3_{c,\alpha}$ in (14.1) to be a b-fibration implies that even the simplest composite of an operator from $\widetilde{Y}$ to G with one from G to $\widetilde{Y}$, namely the push-forward of the kernel of an operator in $A \in \Psi^{-\infty,\mathcal{E}}_\alpha(\widetilde{Y}, G; \Omega^{\frac{1}{2}})$, is not obviously an element of $\Psi^{m,\mathcal{F}}_a(\widetilde{Y}; \Omega^{\frac{1}{2}})$. Even away from the adiabatic boundary components such a push-forward may fail to be a classical pseudodifferential operator. However if the kernel of A is holomorphic or anti-holomorphic, that is

$$^{\alpha}\overline{\partial} \cdot \ A = 0 \text{ or } ^{\alpha}\partial \cdot A = 0, [22.40]$$

then κ_A has additional regularity at Θf . We analyze this regularity and then, in §24, give a composition result for an operator with holomorphic kernel and the push-forward. The regularity result is a generalization of Proposition 5.2 and allows us to recapture the precise form of the asymptotic expansion for the Bergman kernel proved in [**?**], uniformly in ϵ.

CHAPTER 17

Holomorphic kernels

In [**?**] the almost-analytic extension of a defining function $\rho(x)$ for a domain $\Omega \subset \mathbb{C}^n$ is defined as a $\mathcal{C}^\infty$ function, $R(x,y)$, defined on Ω^2 such that

$$R(x,x) = \rho(x),\ \overline{\partial}_x R \sim 0, \partial_y R \sim 0 \text{ along } \Delta.[23.1]$$

More generally a smooth function defined on a totally real submanifold of a complex manifold has an almost analytic extension, see [**?**]. In Lemma 6.1 we slightly extend this concept to families of functions that depend smoothly on a parameter with a degenerating complex structure. We now need to define an almost analytic extension for functions defined on M_a to a neighbourhood of the diagonal in M_α^2 and also an extension to a neighborhood of the fiber diagonal in $(M\widetilde{Y})_\alpha$.

These extensions are very similar to that obtained in Lemma 6.1. Even though the complex structure degenerates at $\{\epsilon = 0\}$, a neighbourhood of the lifted diagonal in M_a^2 and the lifted fiber diagonal in $(M\widetilde{Y})_a$ intersect only af, Thus, in so far as the extension process is concerned, no degeneration is actually apparent. In this section we assume that the defining function for the tube, $\widetilde{\phi}$ and the complex structure on T^*Y satisfy (10.1) and (10.2).

Definition 17.1. *Let $f(\epsilon,x)$ be a smooth real valued function on M. The almost holomorphic-antiholomorphic extension of f is a function $F \in \mathcal{C}^\infty(M_a^2)$ such that*

$$\begin{gathered} F \restriction_{\Delta_a} = \pi_{l,a}^{2\,*} f \restriction_{\Delta_a} \\ {}^a\overline{\partial}_l F \sim 0,\ {}^a\partial_r F \sim 0 \text{ along } \Delta_a[23.3] \\ F(\epsilon,x,y) = -\overline{F(\epsilon,y,x)}. \end{gathered}$$

The proof that such an extension exists and is unique in Taylor series along Δ_a is essentially the same as in [**?**]; the only point is to check that the Borel summation argument can be done smoothly in the parameter. Henceforth we assume that such an almost holomorphic-antiholomorphic extension for the defining function, $\widetilde{\phi}$ of the Θ-boundary is fixed and denote it by ζ. Starting with a smooth real valued function on the Θ-boundary of M, we can extend it smoothly into M and then as an almost holomorphic-antiholomorphic function on M_a^2. In [**?**] it is established that ζ can be chosen to take values in the right half plane. This implies that a smooth branch of $\log(\zeta)$ can be chosen in M_a^2 and thus a smooth branch of $\zeta^\mu, \forall \mu \in \mathbb{C}$.

Because the fibres of $M \longrightarrow \widetilde{Y}$ are totally real, for ϵ sufficiently small, there is a second notion of almost holomorphic extension of a function defined on M. We simply identify M with the fibre diagonal, $\Delta_{\phi\, a}$, in $(M\widetilde{Y})_a$ and apply the method used in [**?**]. We will call this almost holomorphic **fiber** extension

DEFINITION 17.2. *Let $f(\epsilon, x)$ be a smooth function on M. The almost holomorphic fiber extension of f is a function $F \in \mathcal{C}^\infty((M\widetilde{Y})_a)$ such that*

$$\begin{aligned} F \restriction_{\Delta_{\phi\, a}} &= \widetilde{\pi}^{2\,*}_{l,a}(f) \restriction_{\Delta_{\phi\, a}} \\ {}^a\overline{\partial}_l F &\sim 0, \quad \text{along } \Delta_{\phi\, a}. \end{aligned} \quad [23.5]$$

As before the existence and uniqueness in Taylor series is a simple modification of the construction in [**?**]. We assume that such an almost holomorphic fiber extension of $\widetilde{\phi}$ is fixed and denote it by $\widetilde{\zeta}$. Again, if f is only defined on ∂M we can extend it smoothly into M and then fiber extend it almost holomorphically as a function on $(M\widetilde{Y})_a$.

To analyze the push-forward of holomorphic kernels it is important to have some control on the form of the function $\widetilde{\zeta}$. The almost holomorphic fiber extension of $\widetilde{\phi}$ in Definition 17.2 can be performed a fibre at a time. That is if we choose coordinates $(\epsilon; y, x; y')$ in a fibre neighbourhood of $(\epsilon, q_0, q_0) \in \widetilde{Y}^2_a$ then the almost holomorphic extension of $\widetilde{\phi}$ can be computed in ZM_{q_0} with y' simply a parameter.

Choose q to be a geodesic normal coordinate centered at q_0 In these coordinates

$$\begin{aligned} \phi(\epsilon; q, p) &= \epsilon^{-2}\psi(q, p) - 1 \\ &= \epsilon^{-2}[a_{ij}p^i p^j + O(|p|^3)] - 1 \end{aligned}$$

with a_{ij} a positive definite matrix. The space ZM_{q_0} is constructed by introducing the coordinates

$$\widetilde{q} = \frac{q}{\epsilon}, \widetilde{p} = \frac{p}{\epsilon}.$$

To do the almost analytic continuation it is better to use a coordinate system adapted to the complex structure. To that end we introduce a holomorphic coordinate system $x_i + iy_i, i = 1, \ldots, n$, defined by the conditions

$$\begin{gathered} (1)(0,0) \longleftrightarrow (0,0), \\ (2) \text{ The one jet of } \operatorname{Im} \overline{\partial}\widetilde{\phi} \text{ at } (0,0) \text{ agrees with that of } y \cdot dx [23.61] \\ (3) \text{ The totally real submanifold } x = 0 \text{ maps to } q = 0. \end{gathered}$$

The existence of complex coordinates satisfying (17),1,2 follows from (10.1) and (10.2). Condition (3) is obtained in the real analytic category by 'almost analytically' extending the fiber variables $ip_j, j = 1, \ldots, n$ from $q = 0$ and then applying a linear transformation. If the complex structure and the fibration are analytic then the formal series defining the almost analytic extensions actually converge in some neighborhood of $q = 0$. In the smooth category one solves to infinite order along $q = 0$ and then employs Hörmander's solution of the $\overline{\partial}$-problem with a weight, singular along $q = 0$, to obtain the desired holomorphic change of variables. Coordinates for ZM_{q_0} are also given by

$$\widetilde{x} = \frac{x}{\epsilon}, \widetilde{y} = \frac{y}{\epsilon}.[23.6]$$

We call these scaled holomorphic coordinates.

Using a formal series we almost analytically extend

$$\psi(x, y) = |y|^2 + O(|y|^3 + |x|^2|y|^2)[23.62]$$

from $x = 0$. Denote the almost analytic extension by $\Psi(x, y)$:

$$\Psi(x, y) = \psi(0, y) - i\partial_y \psi(0, y) \cdot x - \frac{1}{2}\left\langle \partial^2_{y_i y_j} \psi(0, y) x, x \right\rangle + O(|x|^3).[23.63]$$

Observe that in the scaled coordinates we have

$$\Psi(\widetilde{x}, \widetilde{y}; \epsilon) = \epsilon^{-2}\psi(0, \epsilon\widetilde{y}) - i\epsilon\partial_y \psi(0, \epsilon\widetilde{y}) \cdot \widetilde{x} - \frac{1}{2}\left\langle \partial^2_{y_i y_j} \psi(0, \epsilon\widetilde{y})\widetilde{x}, \widetilde{x} \right\rangle + O(\epsilon|\widetilde{x}|^3).[23.64]$$

From (17) we conclude that $\Psi(\widetilde{x}, \widetilde{y}; \epsilon)$ is smooth on ZM_{q_0}.

The almost analytic fiber extension of $\widetilde{\phi}$ is accomplished by recalling the dependence of the coordinates x, y on the 'base fiber' $q = q_0$. The holomorphic coordinates defined in (17) can be constructed with smooth dependence of the base point, q. The variables $(q, \widetilde{x}, \widetilde{y}; \epsilon)$ define coordinates on a neighborhood of the fiber diagonal in $(M\widetilde{Y})_a$; the fiber diagonal is given in them by $\widetilde{x} = 0$. Evidently we have:

Lemma 17.1. *If $(\epsilon; \widetilde{x}, \widetilde{y}, q)$ are scaled holomorphic coordinates with $\widetilde{x}(q) = 0$ then the almost holomorphic fibre extension of $\widetilde{\phi}$ into a neighbourhood of the fiber diagonal in $(M\widetilde{Y})_a$ is given by*

$$\widetilde{\zeta}(\epsilon; \widetilde{x}, \widetilde{y}; q_0) = 1 - \Psi(\widetilde{x}, \widetilde{y}; \epsilon)[23.8]$$

The linear term is nondegenerate and the quadratic term is positive definite for sufficiently small ϵ.

Proof. All the statements but the last follow from the discussion preceding the statement of the proposition. The positive definiteness of the quadratic term follows from (17) and (17). □

Before we apply the almost analytic continuation we need one further fact, its relationship to the parabolic blow-up of $(M\widetilde{Y})_a$. From the discussion preceding (11) the following is clear:

Proposition 17.1. *The constant and linear terms in the almost analytic continuation of $\widetilde{\phi}$ define the parabolic subbundle of the boundary of the fiber diagonal in $(M\widetilde{Y})_a$ in that*

$$\widetilde{S}_2 = \mathrm{sp}\{d\epsilon^{-2}\psi(0, \epsilon\widetilde{y}) \restriction_{\partial\Delta_a}, d\epsilon\partial_y \psi(0, \epsilon\widetilde{y}) \cdot \widetilde{x} \restriction_{\partial\Delta_a}\}.[23.66]$$

Thus $|\Psi - 1|^{\frac{1}{2}}$ lifts to $(G\widetilde{Y})_\alpha$ as a defining function for Θf .

In §5 we showed that the Bergman kernel lifts to be polyhomogeneous conormal on $\mho^2_\Theta$; this allows a much more serious singularity at the $\partial\Delta$ than obtained in [**?**]. The additional smoothness of the kernel was deduced in Proposition 5.2 as a simple consequence of analyticity and conormality. Here we extend this to the adiabatic context.

Proposition 17.2. *Let $K \in I^{\mathcal{E}}_{\mathrm{phg}}(G^2_\alpha)$ satisfy*

$$^{\alpha}\overline{\partial}_l K = 0,\ {}^{\alpha}\partial_r K = 0.[23.10]$$

Assume that $E_{\Theta\mathrm{f}} = (m, n+1), m \in \mathbb{Z}$ *then*

$$K = \sum_{i=0}^{n} \zeta^{\frac{m}{2}} (\log \zeta)^i k_i \ \textit{where}\ k_i \in \mathcal{C}^\infty(M^2_a).[23.11]$$

Proof. The different logarithmic terms do not interact; we will give the complete proof only in the simplest case, $n = 0$. Since the Θ-boundary of G is strictly pseudoconvex, $\overline{\partial}$ is non-characteristic along Θl and similarly ∂ is non-characteristic along Θr . From this we conclude that a holomorphic-antiholomorphic, polyhomogeneous conormal kernel on G^2_α is actually smooth at $\Theta\mathrm{l} \cup \Theta\mathrm{r}$ in the standard $\mathcal{C}^\infty$-structure. Thus we can consider K as a kernel on M^2_α.

From Proposition 17.1 it follows that $|\zeta|^{\frac{1}{2}}$ lifts to M^2_α as a defining function for Θf . Therefore $F_0 = \zeta^{\frac{-m}{2}} K$ satisfies (17.1) with index set $E_{\Theta\mathrm{f}} = (0, 1)$ and so has an expansion at Θf :

$$F_0 \sim \sum_{j \geq 0} a_i \rho^i_{\Theta\mathrm{f}}.[23.12]$$

The front face is fibred and (17.1) implies that the function a_0 in (17) is actually constant on each fibre. Thus

$$a_0 \restriction_{\Theta\mathrm{f}} = \left(\beta^2_\alpha\right)^* b_0$$

where $b_0 \in \mathcal{C}^\infty(\partial\Delta_a)$. Let b'_0 denote the almost holomorphic-antiholomorphic extension of b_0 to a neighbourhood of $\partial\Delta_a$ in M^2_a and set $a'_0 = (\beta^2_\alpha)^* b'_0 \in (M^2_\alpha)$. The difference $F_0 - a'_0$ satisfies (17.1) with $E_{\Theta\mathrm{f}} = (1, 1)$, thus we can apply the above argument to $F_1 = (F - a'_0)\zeta^{-\frac{1}{2}}$. Continuing inductively in this fashion we obtain the desired expansion. □

Using a similar argument one can prove

Proposition 17.3. *Let $K \in I^{\mathcal{E}}_{\mathrm{phg}}((G\widetilde{Y})_\alpha)$ satisfy ${}^{\alpha}\overline{\partial}_l K = 0$ and assume that* $E_{\Theta\mathrm{f}} = (m, n+1), m \in \mathbb{Z}$ *then*

$$K \sim \sum_{i=0}^{n} \widetilde{\zeta}^{\frac{m}{2}} (\log \widetilde{\zeta})^i k_i \ \textit{with}\ k_i \in \mathcal{C}^\infty((M\widetilde{Y})_a).[23.14]$$

As is clear from the proof of Proposition 17.2 it suffices to assume that the kernel of the operator is almost holomorphic-antiholomorphic (or almost holomorphic) at Θf and smooth in the standard $\mathcal{C}^\infty$-structure along the remainder of the Θ-boundary.

CHAPTER 18

Holomorphic calculus

In §22 certain composition results which one might expect to be valid are not discussed. This applies in particular to the composition

$$\Psi_\alpha^{-\infty,\mathcal{E}}(\widetilde{Y},G;\Omega^{\frac12})\cdot\Psi_\alpha^{-\infty,\mathcal{F}}(G,\widetilde{Y};\Omega^{\frac12}).[24.1]$$

Even when $\mathcal{E}$ and $\mathcal{F}$ are such that the composition exists we believe that it is not necessarily in the polyhomogeneous adiabatic calculus. However it can be shown that the composite operator is in the 'very big' adiabatic calculus, meaning that it has kernel which is conormal at the boundary and the lifted diagonal. Let $\mathfrak{e}=(e_{\mathrm{af}},e_{\mathrm{ab}},e_\Delta)$ be a 'power' set for $\widetilde{Y}_a^2$, with Δ_a as a distinguished submanifold. The elements of the 'big' adiabatic calculus, with multi-order $\mathfrak{e}$ are those elements of $I^*(\widetilde{Y}_a^2,\Delta_a;\widehat{\mathrm{KD}})$ which have orders at most e_{af} at af, e_{ab} at ab and e_Δ at the lifted diagonal. The 'order' here is defined in [**12**]; for conormal distributions with respect to the boundary of a manifold with corners it is fixed by the iterated weighted L^∞ estimates satisfied. This is extended to define the order at an interior p-submanifold by transferring the distribution to the normal bundle, taking the Fourier transform and considering the order on the conormal bundle compactified by stereographic projection. Of course, as in [**12**], one needs to show that the order is independent of choices made. For our purposes here the precise order is not essential. Let us denote by $\Psi_{a,\mathrm{big}}^{\mathfrak{e}}(\widetilde{Y};\Omega^{\frac12})$ the 'big' space adiabatic pseudo-differential operators of order $\mathfrak{e}$ in this sense.

We are actually most interested in the composition of such an operator from $\widetilde{Y}$ to G with the push-forward operator

$$P:\mathcal{C}^\infty(G;\Omega^{\frac12})\longrightarrow\mathcal{C}^\infty(\widetilde{Y};\Omega^{\frac12})[24.2]$$

defined by the choice of a section of the fibre half-density bundle for $\widetilde{\pi}$ in (10).

Proposition 18.1. *If* $K\in\Psi_\alpha^{-\infty,\mathcal{E}}(\widetilde{Y},G;\Omega^{\frac12})$ *where*

$$E_{\Theta\mathrm{b}}>-1 \text{ and } E_{\Theta\mathrm{f}}>-1[24.4]$$

then $P\circ K\in\Psi_{a,\mathrm{big}}^{*}(\widetilde{Y};\Omega^{\frac12})$.

Proof. By Lemma 16.1 the composite operator $P\circ K$ is defined on $\dot{\mathcal{C}}^\infty(\widetilde{Y};\Omega^{\frac12})$ provided $E_{\Theta\mathrm{f}}\overline{\cup}E_{\Theta\mathrm{b}}>-1$, which is just the condition (18.1). If ν' is a non-vanishing half-density on $\widetilde{Y}_a^2$ then the kernel of the composite, κ, is given as a distribution on $\widetilde{Y}_a^2$ by

$$\kappa\nu' = \psi_* \left[\kappa_K \psi^* \nu'\right] [24.5]$$

where $\psi : (G\widetilde{Y})_\alpha \longrightarrow \widetilde{Y}_a^2$ is the lift of the projection from G to $\widetilde{Y}$, see (12).

Let $\mathcal{V}_b(\widetilde{Y}_a^2; \Delta_a)$ denote the Lie algebra of vector fields on $\widetilde{Y}_a^2$ which are tangent to the boundary and to Δ_a. The space of conormal distributions, with respect to the boundary and to Δ_a is defined by iterated regularity with respect to this Lie algebra, i.e. we need to show that

$$V_1 \cdots V_m \kappa \text{ has fixed regularity } \forall\ V_1, \dots, V_m \in \mathcal{V}_b(\widetilde{Y}_a^2; \Delta_a).[24.6]$$

Here the vector fields are really acting by Lie derivation on the half-densities. The natural way to show this iterative regularity is to show that every element of $\mathcal{V}_b(\widetilde{Y}_a^2; \Delta_a)$ is ψ-related to a $\mathcal{C}^\infty$ vector field on $(G\widetilde{Y})_\alpha$ which is tangent to all boundaries, since the kernel of K is conormal with respect to the boundary.

Lemma 18.1. *For every $V \in \mathcal{V}_b(\widetilde{Y}_a^2; \Delta_a)$ there exists $W \in \mathcal{V}_b((G\widetilde{Y})_\alpha)$ with $\psi_*(W) = V$.*

Proof. To find a vector field W which is ψ-related to a given V it suffices to work locally on $(G\widetilde{Y})_\alpha$. Notice, from (12), that ψ is the product of a fibration and a blow-down map:

$$\psi = \phi_a \cdot \beta,\ \beta = \beta\left[(G\widetilde{Y})_a; \widetilde{B}_\Theta, \widetilde{S}_2\right], [24.8]$$

see (11). Thus any vector field on $\widetilde{Y}_a^2$ has a lift to $(G\widetilde{Y})_a$ which is tangent to the boundary. To have a lift to $(G\widetilde{Y})_\alpha$ the intermediate lift to $(G\widetilde{Y})_a$ should be $\widetilde{S}_2$-parabolically tangent to $\widetilde{B}_\Theta$; see Proposition C.2. Thus we are reduced to showing that each

$$V \in \mathcal{V}_b(\widetilde{Y}_a^2; \Delta_a) \text{ has a lift under } \phi_a \text{ in } \mathcal{V}_b((G\widetilde{Y})_a; \widetilde{B}_\Theta, \widetilde{S}_2).[24.9]$$

Since $\phi_a(\widetilde{B}_\Theta) = \Delta_a$ there is nothing to prove away from $\Delta_a \subset \widetilde{Y}_a^2$, and even above Δ_a nothing to prove away from $\widetilde{B}_\Theta$. We can always introduce local coordinates $(\epsilon; y, x; y')$ in $(G\widetilde{Y})_a$ near $\widetilde{B}_\Theta \subset (G\widetilde{Y})_a$ so that

$$G = \{(y, x); |x| \le 1\} \text{ and } \theta = x \cdot d(y - y').[24.10]$$

The Lie algebra $\mathcal{V}_b(\widetilde{Y}_a^2; \Delta_a)$ is locally generated by

$$\epsilon\partial_\epsilon,\ R = \sum_{i=1}^n y_i(\partial_{y_i} - \partial_{y_i'}),\ Y_i = \partial_{y_i} + \partial_{y_i'}, i = 1, \dots, n[24.11]$$

and the infinitesimal generators of the invertible linear transformation about $\Delta_a = \{y = y'\}$. The trivial lifts of $\epsilon\partial_\epsilon$ and the Y_i each annihilates θ identically so these lifts are as required for (18). So consider a linear transformation, A, in the variables $y-$

y', i.e. rotation around Δ_a. In the fibre variables consider the related homogeneous transformation:

$$x \longmapsto \frac{{}^tAx}{|{}^tAx|}[24.12]$$

near $|x| = 1$. The combined map, $\tilde{A}$, is clearly a diffeomorphism of G near the boundary which covers A as a local diffeomorphism of $\widetilde{Y}^2_a$. The submanifold $\widetilde{B}_\Theta$, being $y = y'$, is invariant and from (18) the line bundle, $\widetilde{S}_2$, spanned by θ in the conormal bundle to $\widetilde{B}_\theta$ is mapped into itself. Thus if $A = \exp(rV)$ then the lift $\tilde{A}$ is generated by a vector field W ϕ_a-related to V. This shows that the linear vector fields, i.e. the infinitesimal generators of linear transformations around Δ_a, have lifts as required in (18). This completes the proof of (18) and hence of the lemma. □

Using the lifted vector fields provided by Lemma 18.1 we easily complete the proof of the proposition. □

Now we consider holomorphic, or almost holomorphic, kernels. As a consequence of the regularity of holomorphic kernels shown in §23 we shall show that if K in Proposition 18.1 has kernel which is almost holomorphic at Θf then $P \circ K$ is polyhomogeneous.

Proposition 18.2. *Suppose $K \in \Psi^{-\infty,\mathcal{E}}_\alpha(\widetilde{Y}, G; \Omega^{\frac12})$ has kernel which is almost holomorphic at $\Theta\mathrm{f}(G\widetilde{Y})_\alpha$ then, assuming* (18.1)*, composition with push-forward gives*

$$P \circ K \in \Psi^{\mathcal{F}}_a(\widetilde{Y}_a; \Omega^{\frac12})[24.14]$$

where the index set $\mathcal{F} = (F_\Delta, F_{\mathrm{af}}, F_{\mathrm{ab}})$ is given by

$$F_\Delta = ????,\ F_{\mathrm{af}} = E_{\mathrm{af}},\ F_{\mathrm{ab}} = E_{\mathrm{ab}}.[24.15]$$

Proof. Since this is an important result, quite basic to the result of §25, we give two proofs; these are related but rather different in style. The first proof consists of a refinement of the proof of Proposition 18.1 whilst the second relies on direct asymptotic evaluation of the integrals involved in the push-forward.

To prove (18.2) with (18.2) we need to show that the kernel of the composite operator, which is shown to be conormal at Δ_a by Proposition 18.1, has a symbol with a complete asymptotic expansion in powers determined by F_Δ which has an expansion at af as indicated and vanishes to all orders at ab. The behaviour at the adiabatic faces is essentially immediate, indeed it is not necessary to assume the holomorphy condition to ensure this. Under ψ defining functions ρ_{af} and ρ_{ab} for the adiabatic front face and adiabatic boundary lift to be defining functions for the similarly denoted boundary hypersurfaces of $(G\widetilde{Y})_\alpha$. Thus from (18) the meromorphy of $\rho_v^{iz}\kappa$, where κ is the kernel of $P \circ K$ follows from that of $(\psi^*\rho_v)^{iz}\kappa$

for $v = \text{ab}$ or $v = \text{af}$. Thus at these boundary hypersurfaces κ is polyhomogeneous with the same index set as assumed for the kernel of K.

Thus it only remains to show that κ is polyhomogeneous as a conormal distribution with respect to Δ_a and to show that the index set is given by (18.2). To do this we make substantial use of Proposition 17.3 which decomposes K in terms of powers of the one function $\widetilde{\zeta}$ with $\mathcal{C}^\infty$ coefficients on $(G\widetilde{Y})_a$. Notice that the map, ϕ_a, from $(G\widetilde{Y})_a$ to $\widetilde{Y}_a^2$ is a fibration so under it smooth kernels map to smooth kernels. As shown in [**12**] a conormal distribution, κ, at an interior p-submanifold such as Δ_a of a manifold with corners is polyhomogeneous, with index family $E = \{(z_j, m_j)\}$, if the distribution

$$\prod_{j<J}(R - z_j)^{m_j+1}\kappa \in I^{M,p(J)}(\widetilde{Y}_a^2, \Delta_a),\ \text{where } p(J) \longrightarrow \infty \text{ as } J \longrightarrow \infty.[24.16]$$

Here R is a radial vector field for the submanifold, i.e. R vanishes at Δ_a and its linear part is the identity as a bundle map on $N^*\Delta_a$. Of course in local coordinates we can take

$$R = \frac{1}{2}(y - y') \cdot (\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}).[24.17]$$

To complete the (first) proof of the proposition we shall use:

LEMMA 18.2. *There is a radial vector field $R \in \mathcal{V}_b(\widetilde{Y}_a^2)$ for $\Delta_a \subset \widetilde{Y}_a^2$ which has a lift to a radial vector field $\widetilde{R} \in \mathcal{V}_b((G\widetilde{Y})_a)$ for $\widetilde{B}_\Theta$ such that*

$$\widetilde{R}\widetilde{\zeta} = c\widetilde{\zeta},\ c \in \mathcal{C}^\infty((G\widetilde{Y})_a),\ c = 1 \text{ on } \widetilde{B}_\Theta[24.19]$$

where $\widetilde{\zeta}$ is the almost holomorphic function in Proposition 17.3.

PROOF. Since $\phi_a : (G\widetilde{Y})_a \longrightarrow \widetilde{Y}_a^2$ is a fibration one can certainly lift R and (18.2) is a restriction only near $\widetilde{B}_\Theta$. In fact we can work locally near a point in Δ_a. Then (17.1) gives the form of $\widetilde{\zeta}$ in local coordinates. For simplicity suppose first that the higher order terms are absent, i.e.

$$\widetilde{\zeta} = 1 - |x|^2 + 2ix \cdot (y - y') + |y - y'|^2.[24.20]$$

Let R_1 be the product lift of R in (18), so

$$R_1\widetilde{\zeta} = 2ix \cdot (y - y') + 2|y - y'|^2.[24.21]$$

We look for a lift of R in the form

$$\begin{aligned}\widetilde{R} &= R_1 + a(1 - |x|^2)x \cdot \frac{\partial}{\partial x} + b(y - y')^\perp \cdot \frac{\partial}{\partial x} \\ y - y' &= [x \cdot (y - y')]\frac{x}{|x|^2} + (y - y')^\perp\end{aligned}[24.22]$$

near $|x| = 1$. Applying (18) to (18) the identity (18.2) becomes

$$2ix\cdot(y-y')+2|y-y'|^2-2a(1-|x|^2)+a(1-|x|^2)x\cdot(y-y')+2ib|y-y'|^2 = c\zeta.[24.23]$$

This certainly can be solved for the coefficients, say in the form

$$c\left[1+\frac{(y-y')\cdot x}{2i|x|^2}\right]=\left[1+\frac{2(y-y')\cdot x}{2i|x|^2}\right],\ c=a[-2+x\cdot(y-y')],\ c=(2+2ib).[24.24]$$

This gives the lift $\widetilde{R}$. On $\widetilde{B}_\Theta$ the solution of (18) satisfies $a=-1/2$, $b=i/2$ and $c=1$. It follows that $\widetilde{R}$ is indeed a radial vector field for $\widetilde{B}_\Theta$.

The general case, in which (18) is replaced by (17.1) is similar except that to handle the higher order terms (18) should be generalized to

$$\widetilde{R}=R_1+a(1-|x|^2)x\cdot\frac{\partial}{\partial x}+(y-y')^\perp\cdot b\cdot\frac{\partial}{\partial x}[24.25]$$

where b is now a smooth linear transformation on the orthocomplement of x, depending smoothly on x,y,y' and ϵ as parameters. The conditions in (18) are just the equality of coefficients of two of the defining functions for $\widetilde{B}_\Theta$, namely $(1-|x|^2)$ and $x\cdot(y-y')$, and of a quadratic form in the remaining functions, $(y-y')^\perp$. The more general form (18) clearly gives the freedom to remove a general quadratic form so the perturbed analogue of (18) can also be solved, still with $a=-1/2$, $b=\mathrm{Id}$ and $c=1$ at $\widetilde{B}_\Theta$. This completes the proof of the lemma. □

Returning to the proof of the proposition observe that by Proposition 17.3 the kernel K has an expansion (17.3), with remainder vanishing to increasingly high order at $\Theta\mathrm{f}$. Since the coefficients are smooth functions on $(M\widetilde{Y})_a$ and $\widetilde{R}$ is a radial vector field for $\widetilde{B}_\Theta$ satisfying (18.2)

$$\kappa_J=\prod_{j<J}(\widetilde{R}-z_j)^{m_j+1}\kappa_K$$

is the Schwartz kernel of

$$K_J\in\Psi_\alpha^{-\infty,\mathcal{E}(J)}(\widetilde{Y},G;\Omega^{\frac12})[24.26]$$

where $\mathcal{E}(J)=(E_{\mathrm{af}},E_{\mathrm{ab}},E_{\Theta\mathrm{f}}(J),0)$ and the index set $E_{\Theta\mathrm{f}}(J)\longrightarrow\infty$ as $J\longrightarrow\infty$. Thus we see from Proposition 18.1 that as $J\longrightarrow\infty$ the kernels of the operators $P\circ K_J$ are conormal with fixed order at the adiabatic boundaries but order tending to $-\infty$ on Δ_a. Since, R and $\widetilde{R}$ are ψ-related $P\circ K_J$ has kernel the distribution in (18), which we have therefore proved. Thus we have completed the first proof of the proposition.

We now give a second proof. As already noted the statement is local in a neighbourhood of a point $P\in\Delta_a$. We use the scaled holomorphic coordinates $(q,\widetilde{x},\widetilde{y};\epsilon)$ defined before the statement Lemma 17.1. This lemma implies that

$$\widetilde{\zeta}(\epsilon;\widetilde{x},\widetilde{y},y')=1-\epsilon^{-2}\psi(0,\epsilon\widetilde{y})+i\epsilon^{-1}a(\epsilon\widetilde{y})\cdot\widetilde{x}+a_{ij}(\epsilon\widetilde{y})\widetilde{x}_i\widetilde{x}_j+O(\epsilon|\widetilde{x}|^3).[24.27]$$

Where a_{ij} is positive definite for small enough epsilon and the error terms are smooth functions on $(M\widetilde{Y})_a$.

We assume that K is either of the form

$$K=\frac{\mathcal{F}}{\widetilde{\zeta}^m}\text{ or }\mathcal{F}\log\widetilde{\zeta},\ \mathcal{F}\in\mathcal{C}^\infty((M\widetilde{Y})_a;\Omega^{\frac12}).$$

The general case follows easily from these cases. The operator determined by $\widehat{K} = \pi_{a*}^2 K$ is defined as the principle value

$$\widehat{K} \cdot f(y) = \lim_{\delta \to 0} \int\limits_{d(y,y')>\delta} \widehat{K}(y, y') f(y') dV. [24.28]$$

We will show that $\widehat{K}$ has an asymptotic expansion of the form

$$\widehat{K} \sim \sum_{i=-(m-1)}^{\infty} [a_i(\epsilon; \omega, y') + \log|R| R^{m-1} b_{i+m-1}(\epsilon; \omega, y')] R^i \text{ where}$$
$$\omega_j = \frac{\widetilde{x}_j}{|\widetilde{x}|}, R = |\widetilde{x}|, a_i, b_i \in \mathcal{C}^\infty([0, \mu] \times \mathbb{S}^{n-1} \times \mathbb{B}^n). \quad [24.29]$$

The proposition follows easily from (18). Proposition 17.3 implies that $\widehat{K}$ is $\mathcal{C}^\infty$ in Δ_a^c.

Without loss of generality we can assume the $\mathcal{F}$ is supported in an arbitrarily small neighbourhood of $\partial\Delta_a$. It follows from the positive definiteness of the quadratic term in (18) that, for small enough ϵ, all other contributions to the integral are smooth. For simplicity we assume that $m \in \mathbb{N}$, the general case follows by essentially the same argument. We argue inductively, first considering the case $m = 1$. Since the leading order part of $\widetilde{\zeta}$ vanishes only on $\partial\Delta_a$ the integrals for $d(y, y') > 0$ are absolutely convergent for small enough ϵ. In the remainder of the proof F, F', F'', etc. and G, G', G'', etc. will denote smooth functions of their arguments without any identification between different occurrences of the same symbol.

Let

$$r = 1 - \epsilon^{-2}\psi(0, \epsilon\widetilde{x}); \xi_i = \frac{\partial_{y_i}\psi(0, \epsilon\widetilde{y})}{\psi(0, \epsilon\widetilde{y})^{\frac{1}{2}}}.$$

From (17) it follows that this defines a smooth coordinate transformation in a neighborhood of the boundary of the fiber diagonal. In these coordinates we have

$$\widehat{K}(\epsilon; R, \omega, y') = \int\limits_{\mathbb{S}^{n-1}} \int\limits_0^1 \frac{F dr \wedge \mathrm{dVol}_{n-1}}{(r + 2i(1-r)^{\frac{1}{2}} R\xi \cdot \omega + R^2 + e)}. [24.30]$$

The volume form on the unit k-sphere is denoted by dVol_k. The error term $e = O(\epsilon R^3)$ is a smooth function of $\epsilon, r, R, \xi, \omega, y'$. Integrating by parts in r we obtain

$$\widehat{K} = \int\limits_{\mathbb{S}^{n-1}} \int\limits_0^1 F \log(r + 2i(1-r)^{\frac{1}{2}} R\xi \cdot \omega + R^2 + e) dr \wedge \mathrm{dVol}_{n-1} +$$
$$\int\limits_{\mathbb{S}^{n-1}} F' \log(2iR\xi \cdot \omega + R^2 + e)\, \mathrm{dVol}_{n-1} = \quad [24.31]$$
$$I_1 + \int\limits_{\mathbb{S}^{n-1}} F' \log(\xi \cdot \omega + R + e')\, \mathrm{dVol}_{n-1} + \log R \int\limits_{\mathbb{S}^{n-1}} F'\, \mathrm{dVol}_{n-1}.$$

Here $e' = O(\epsilon R^2)$ is a smooth function of $\epsilon, r, R, \xi, \omega, y'$.

The last term in (18) is exactly of the type described in the conclusion of the proposition. A simple induction shows that the volumetric term, I_1 in (18) has an expansion of the desired type modulo a smoother boundary term. We leave it to the reader to verify:

$$\int_{\mathbb{S}^{n-1}} \int_0^1 F \log(\widetilde{\zeta})\widetilde{\zeta}^k dr \wedge \mathrm{dVol}_{n-1} = \int_{\mathbb{S}^{n-1}} \int_0^1 [F' \log(\widetilde{\zeta})\widetilde{\zeta}^{k+1} + F''\widetilde{\zeta}^k] dr \wedge \mathrm{dVol}_{n-1} + \int_{\mathbb{S}^{n-1}} [G \log(\widetilde{\zeta})\widetilde{\zeta}^{k+1} + G'\widetilde{\zeta}^k] \,\mathrm{dVol}_{n-1} \text{ for } k \in \mathbb{N}_0. \quad [24.32]$$

The terms without a log-factor have the desired type of expansion.

The only remaining terms are of the form

$$R^k \int_{\mathbb{S}^{n-1}} F \log(2i\xi \cdot \omega + R + e')(2i\xi \cdot \omega + R + e')^k \,\mathrm{dVol}_{n-1} . [24.33]$$

As a function of $(\epsilon; R, \omega, y')$ the integral in (18) has a finite number of derivatives, $N(k)$. Clearly $N(k) \longrightarrow \infty$ as $k \longrightarrow \infty$.

To handle such terms we introduce the coordinates

$$u = \xi \cdot \omega \text{ and } \eta \in \mathbb{S}^{n-2} = \{\omega^{\perp} \cap \mathbb{S}^{n-1}\}.$$

The only singular contribution to the integral arises where $u = 0$, thus we can assume that F is supported near this locus. The integral in (18) becomes

$$I_2 = \int_{\mathbb{S}^{n-2}} \int_{-1}^1 F \log(2iu + R + e')(2iu + R + e')^k du \wedge \mathrm{dVol}_{n-2} . [24.34]$$

Integrating by parts with respect to u in (18) and using the support properties of F we obtain

$$I_2 = \int_{\mathbb{S}^{n-2}} \int_{-1}^1 [F \log(2iu+R+e')(2iu+R+e')^{k+1} + F'(2iu+R+e')^k] du \wedge \mathrm{dVol}_{n-2} . [24.35]$$

Applying (18) recursively verifies the assertion of the proposition for $m = 1$ and also handles the log-term.

For $m > 1$ we integrate by parts in r to obtain

$$\int_{\mathbb{S}^{n-1}} \int_0^1 \frac{F}{\widetilde{\zeta}^m} dr \wedge \mathrm{dVol}_{n-1} = \int_{\mathbb{S}^{n-1}} \int_0^1 \frac{F'}{\widetilde{\zeta}^{m-1}} dr \wedge \mathrm{dVol}_{n-1} + \int_{\mathbb{S}^{n-1}} \frac{F''}{\widetilde{\zeta}^{m-1}} \,\mathrm{dVol}_{n-1} . \quad [24.36]$$

The inductive hypothesis implies that the volumetric term on the right hand side of (18) has the desired expansion. To complete the proof we only need to analyze the boundary term; denote it by I_3. Using the coordinates (u, η) for $\mathbb{S}^{n-1}$ introduced above we obtain

$$I_3 = \int_{\mathbb{S}^{n-2}} \int_{-1}^1 \frac{F}{[R(2iu + R + e')]^{m-1}} du \wedge \mathrm{dVol}_{n-2} . [24.37]$$

As before we can assume that F is supported near $u = 0$; integrating by parts with respect to u, $m - 1$ times we see that

$$I_3 = R^{1-m} \int_{\mathbb{S}^{n-2}} \int_{-1}^{1} F' \log(2iu + R + e')du \wedge \mathrm{dVol}_{n-2} \,.[24.38]$$

As the integral in (18) is of the type treated in (18) this completes the second proof of the proposition. □

The second proof gives a direct method for computing the coefficients in the expansion (18). It is clear that these coefficients are smooth functions of the parameter ϵ. If $m > 0$ then the leading term is given by

$$2(m-1)a_0(\epsilon; \omega, y') = \lim_{R \to 0} \int_{\mathbb{S}^{n-1}} \frac{\mathcal{F}(\epsilon; 0, \xi, \omega, y')}{(2i\omega \cdot \xi + R)^{m-1}} \,\mathrm{dVol}\,.[24.39]$$

CHAPTER 19

The Töplitz isomorphism

By the Töplitz isomorphism we mean the map (0.1) and its continuous extensions. Thus we proceed to the proof of Theorem 10.1. More precisely we first construct a left inverse for the push-forward map (10.1).

Proposition 19.1. *If $\epsilon_0 > 0$ is sufficiently small then there is an operator*

$$T \in \Psi_{\alpha}^{-\infty;\mathcal{E}}(\widetilde{Y}, G; {}^{\alpha}\Lambda^{n,0}, \mathbb{C})$$
$$\mathcal{E} = (E_{\mathrm{af}}, E_{\mathrm{ab}}, E_{\Theta\mathrm{f}}, E_{\Theta\mathrm{b}}) = (\frac{n}{2}, \emptyset, ?, ?))^{[25.2]}$$

such that

$$K \circ T = T,\ P \circ T = \mathrm{Id},\ T \circ P \circ K = K[25.3]$$

where K is the Bergman projector.

Proof. To construct T we first investigate the operator $P \circ K \circ {}^tP$. Applying Proposition 16.2 to the composition $K \circ {}^tP$ and then Proposition 18.2 to $P \circ (K \circ {}^tP)$ we conclude that (18.2) holds with the index family given by (18.2). Moreover using (18) we can compute the symbol of $P \circ K \circ {}^tP$ at the adiabatic front face. We find that

$$\sigma_{\mathrm{af}}(P \circ K \circ {}^tP) = [25.4]$$

is an invertible convolution operator, i.e. has non-vanishing Fourier transform which is an elliptic symbol. Thus Proposition 15.3 shows that $P \circ K \circ {}^tP$ is invertible for small ϵ_0 with inverse

$$(P \circ K \circ {}^tP)^{-1} \in \Psi_a^{\mathcal{G}}(\widetilde{Y}).[25.5]$$

We take

$$T = K \circ {}^tP \circ (P \circ K \circ {}^tP)^{-1}.[25.6]$$

By Proposition 16.2 and Proposition 16.4 we conclude that

$$T \in \Psi_{\alpha}^{-\infty,\mathcal{H}},\ \mathcal{H} = .[25.7]$$

By Proposition 8.1 we can compose on the left with K and since this is a projection, from (19) we see that the first condition in (19.1) holds. The second follows from the construction of $(P \circ K \circ {}^tP)^{-1}$:

$$P \circ T = P \circ K \circ {}^tP \circ (P \circ K \circ {}^tP)^{-1} = P \circ K \circ {}^tP(P \circ K \circ {}^tP)^{-1} = \mathrm{Id}[25.8]$$

by construction.

Thus it only remains to show that T is also a left inverse of $P \circ K$, in the sense of the third identity in (19.1). To show this we note from Proposition 16.2 and Proposition 16.4 that

$$K' = T \circ P \circ K = (K \circ {}^tP) \circ (P \circ K \circ {}^tP)^{-1} \circ (P \circ K) \in \Psi_\alpha^{-\infty,\mathcal{H}'}.[25.9]$$

Moreover, $K' \circ K' = K'$ by the first property in (19.1) and

$$K' \circ K = K', \; K \circ K' = K[25.10]$$

as follows from (19) and (19) respectively. Thus K' is projection onto a subspace of the Bergman space. Moreover we can compute the adiabatic symbol of K' :

$$\sigma_{\mathrm{af}}(K') = \sigma_{\mathrm{af}}(T) \circ \sigma_{\mathrm{af}}(P) \circ \Sigma_{\mathrm{af}}(K).[25.11]$$

Since this is the just the computation of these operators in the adiabatic model case we see that $\sigma_{\mathrm{af}}(K') = \sigma_{\mathrm{af}}(K)$. Thus we can write

$$K' = K' \circ K = [K + (K' - K)] \circ K = F \circ K, \; F = \mathrm{Id} + K' - K[25.12]$$

and conclude that $\sigma_{\mathrm{af}} = \mathrm{Id}$ so using (8) that F is invertible for small ϵ_0. It follows that the range of K' is the same as that of K, i.e. that $K' = K$ since both are self-adjoint projections. This proves the last of the identities (19.1). □

Of course it is an immediate consequence of (19.1) that T is an isomorphism onto the Bergman space, for small $\epsilon > 0$ which gives a right and left inverse of P in (10.1). This completes the proof of Theorem 10.1.

Part 4

Appendices

APPENDIX A

Elementary examples

A.1. Model Problems

We briefly describe the application of blow-ups to the study of the solution operators for two ordinary differential equations. The model for a degenerate operator on a manifold with boundary is

$$-(x\partial_x)^2 + \lambda^2,\ \lambda \in \mathbb{R}_+,\ \text{ on } \dot{\mathcal{C}}^\infty(\mathbb{R}_+; |dx|^{\frac{1}{2}}).[A1.1]$$

The model for a degenerating family is

$$-(\epsilon\partial_x)^2 + 1 \text{ on } \dot{\mathcal{C}}^\infty(\mathbb{R}\times[0,1]; |dxd\epsilon|^{\frac{1}{2}}).[A1.2]$$

Following the normalization adopted throughout this monograph, we consider all operators to act on non-singular half densities. We show that the kernels of the inverses to these two operators lift to be polyhomogeneous conormal distributions on appropriately blown-up spaces. These examples are in one minor way not representative of the operators we study in the sequel, in that they act on non-compact spaces. As we confine our attention to points in the L^2-resolvent set the relevant kernels are actually exponentially decreasing 'at ∞.' We could therefore compactify the spaces. However this would only detract from the main point of these exercises.

The Schwarz kernel for $[-(x\partial_x)^2 + \lambda^2]^{-1}$ is given by

$$k(x,y) = \begin{cases} \left(\frac{x}{y}\right)^\lambda \frac{|dxdy|^{\frac{1}{2}}}{y}, & x \leq y \\ \left(\frac{y}{x}\right)^\lambda \frac{|dxdy|^{\frac{1}{2}}}{y}, & x \geq y. \end{cases} [A1.3]$$

This kernel has polyhomogeneous conormal singularities along $\mathrm{lb} = \{x = 0\}, \mathrm{rb} = \{y = 0\}$ and along the diagonal Δ. Note however that the kernel is not polyhomogeneous conormal in the corner $x = y = 0$. This is because the diagonal does not meet the corner cleanly. To remedy this we blow-up the corner. This is what would normally be called the introduction of polar coordinates:

$$r = (x^2+y^2)^{\frac{1}{2}}, \theta = \tan^{-1}(\frac{y}{x}).[A1.4]$$

We denote this space by $[\mathbb{R}^2; 0]$. Let $\beta_0^2 : [\mathbb{R}^2; 0] \longrightarrow \mathbb{R}^2$ denote the obvious blow-down map

$$\beta_0^2(r,\theta) = (r\cos(\theta), r\sin(\theta)).[A1.5]$$

The blown-up space has a new boundary component $(\beta_0^2)^{-1}(0,0)$, which we call the front face. We denote the boundary components of $[\mathbb{R}^2; 0]$ by $\mathrm{lb}, \mathrm{rb}, \mathrm{ff}$ these are the lifts of $x = 0, y = 0$ and $(0,0)$ respectively.

As a manifold with corner

$$[\mathbb{R}^2; 0] \simeq [0,1] \times \mathbb{R}_+.$$

The lift of the diagonal, which we define to be

$$\Delta_0 = \mathrm{cl}(\beta_0^2)^{-1}(\Delta \setminus (0,0)),$$

intersects the $\partial[\mathbb{R}^2;0]$ only in the interior of the front face. In polar coordinates

$$\Delta_0 = \{(r,\theta); \theta = \frac{1}{4}\pi\}.$$

Taking into account the lift of the density bundle we see that $k(x,y)$ lifts, under β_0^2 to give

$$k_0 = \beta_0^{2*}(k) = \begin{cases} \left(\frac{\sin\theta}{\cos\theta}\right)^\lambda \frac{|drd\theta|^{\frac{1}{2}}}{r^{\frac{1}{2}}\sin\theta}, & \theta \in [0, \frac{1}{4}\pi] \\ \left(\frac{\cos\theta}{\sin\theta}\right)^\lambda \frac{|drd\theta|^{\frac{1}{2}}}{r^{\frac{1}{2}}\sin\theta}, & \theta \in [\frac{1}{4}\pi, \frac{1}{2}\pi]. \end{cases} \quad [A1.6]$$

This is a polyhomogeneous conormal distribution section of the singular density bundle $\rho_{\mathrm{ff}}^{-\frac{1}{2}}\Omega^{\frac{1}{2}}([\mathbb{R}^2;0])$. Note that the singularity in the density bundle does not depend on λ. Near the lb

$$k_0 = \rho_{lb}^{\lambda}\widetilde{k},$$

where $\widetilde{k}$ is $\mathcal{C}^\infty$ in a neighborhood of lb. Similarly near rb

$$k_0 = \rho_{\mathrm{rb}}^{\lambda-1}\widehat{k},$$

where $\widehat{k}$ in $\mathcal{C}^\infty$ in a neighborhood of rb. We express this by saying that the index set of k_0 is λ at lb and $\lambda - 1$ at rb. The index sets are different because $(x\partial_x)^2$ is not self adjoint acting on ordinary half densities, $f(x)|dx|^{\frac{1}{2}}$. A neighborhood of the front face is fibred by intervals. The fibers are simply the sets r =constant. We can think of k_0 as a smooth function of r with values in the space of distributions on $[0, \frac{1}{2}\pi]$, polyhomogeneous conormal with respect to $0, \frac{1}{4}\pi, \frac{1}{2}\pi$. When thought of in this way it is evident that the kernel has a well defined restriction to ff which is also a distribution of this type. This is called the normal operator of k_0. The half density factor is canonically identified with the 'fiber' density on the front face.

To summarize, the kernel, k, lifts to $[\mathbb{R}^2;0]$ to give a polyhomogeneous conormal distribution with respect to $\mathrm{lb}, \mathrm{rb}, \Delta_0$. It is a section of a **fixed** singular half density bundle. The lifted diagonal does not intersect $\mathrm{lb}\cup\mathrm{rb}$ and therefore the singularities along these loci do not interact. Thus a considerable simplification in the description of the singularities of the kernel, k is afforded by lifting to the blow-up space. As is also apparent from the formula, the analytic continuation of k_0 to the 'non-physical' half plane $\mathrm{Re}\,\lambda < 0$ shares these properties as well.

Now we consider the degenerating family given in (A.1). The kernel for $[-(\epsilon\partial_x)^2 + \mathrm{Id}]^{-1}$ acting on half densities is given by

$$h(x,y;\epsilon) = \frac{e^{-\frac{|x-y|}{\epsilon}}}{2\epsilon}\frac{|dxdyd\epsilon|^{\frac{1}{2}}}{|d\epsilon|^{\frac{1}{2}}}.[A1.7]$$

We represent the fiber half density bundle as $\frac{|dxdyd\epsilon|^{\frac{1}{2}}}{|d\epsilon|^{\frac{1}{2}}}$. Since the operator is tangent to the ϵ-fibers it is simpler to retain ϵ as a parameter for the kernel of the inverse. Away from $\epsilon = 0$ this is an analytic family of pseudodifferential operators of -2. It converges to the identity operator as $\epsilon \to 0$, a rather singular limit. The singularity of the kernel is contained in the 'diagonal', $\Delta \times [0,1]$. The kernel vanishes to infinite order along $\epsilon = 0$ as long as we avoid the intersection with diagonal. The nasty singularity along $\Delta \times \{0\}$ is resolved by blowing this submanifold up.

We introduce polar coordinates about the 'boundary of the diagonal:'

$$R^2 = \epsilon^2 + (x-y)^2, \omega = \frac{\epsilon}{R}, \xi = \frac{x-y}{R}, y.[A1.8]$$

Using R, ω, y or R, ξ, y we obtain coordinates in a neighborhood of the front face of the blown-up space $[\mathbb{R}^2 \times [0,1]; \Delta \times \{0\}]$. Let

$$\beta_a : [\mathbb{R}^2 \times [0,1]; \Delta \times \{0\}] \longrightarrow \mathbb{R}^2 \times [0,1]$$

denote the blow-down map. This space has two boundary components, an adiabatic boundary, $\mathrm{ab} = \beta_a^{-1}(\mathbb{R}^2 \times \{0\} \setminus \Delta \times \{0\})$, and the front face $\mathrm{af} = \beta_a^{-1}(\Delta \times \{0\})$.

The front face is a fibration over $\Delta \times \{0\}$ with fiber isomorphic to the compactified normal bundle of $\Delta \times \{0\} \subset \mathbb{R}^2 \times \{0\}$. The lift of the diagonal is the submanifold $\xi = 0 \equiv \omega = 1$. It intersects the boundary only in the interior of af. The lift of the kernel, h is given by

$$h_a(R, \omega, \xi, y) = \beta_a^*(h) = \frac{e^{-\frac{|\xi|}{\omega}}}{\omega} \frac{|dR(\xi d\omega - \omega d\xi)dy|^{\frac{1}{2}}}{|Rd\epsilon|^{\frac{1}{2}}}.[A1.9]$$

The lifted kernel is a polyhomogeneous conormal section of the singular fiber density bundle $\rho_{\mathrm{af}}^{-\frac{1}{2}}\Omega([\mathbb{R}^2\times[0,1]; \Delta\times\{0\})^{\frac{1}{2}}\otimes|d\epsilon|^{-\frac{1}{2}}$. We use this slightly involved description of the density bundle as it simplifies the discussion of composition and mapping properties.

The lifted kernel vanishes to infinite order at ab and has a well defined restriction to af as a fiber density. This again requires the identification of the singular density bundle with the density bundle of the fibers of the front face. This is the 'adiabatic' normal operator.

Though as a family of operators the behavior at $\epsilon = 0$ is rather singular, the kernel lifts to be quite simple on blow-up space. The kernel h_a is polyhomogeneous conormal with respect to $\mathrm{af}, \mathrm{ab}, \Delta_a$ and has a well defined normal operator. The constructions we use throughout this monograph are similar in spirit to these very simple models.

A.2. An elementary composition formula

In this section we generalize the first construction in the previous section to define a calculus of pseudodifferential operators and explain the ideas underlying the 'geometric' proof of the composition formula in this calculus. Our purpose here is not to give a rigorous proof but rather to expose the fundamental **geometric** concepts in a simple, but entirely representative example. The interested reader is strongly urged to fill in the details of the construction in this case.

The underlying space is $\mathbb{R}_+$, as above we could compactify by adding a point at ∞ but this would only rob this example of its inherent simplicity. Let $\mathcal{A}^{\{z\}}(\mathbb{R}_+)$ denote functions in $\mathcal{C}^\infty(0,\infty)$ which are rapidly decreasing at infinity and have an expansion at $x = 0$ of the form:

$$u \sim \sum_{j=0}^{\infty} u_j x^{iz+j}.$$

These are simple examples of polyhomogeneous conormal distributions. More generally if $E = \{z_j ik, j \in \mathbb{N}, k \in \mathbb{N}_0\} \subset \mathbb{C}$ with

$$\lim_{j\to\infty} \operatorname{Im} z_j = -\infty, [A2.1]$$

then we define $\mathcal{A}^E(\mathbb{R}_+)$ to be functions $u \in \mathcal{C}^\infty(0,\infty)$, rapidly decreasing at infinity with an asymptotic expansion at 0:

$$u \sim \sum_j \sum_{k=0}^{\infty} u_{jk} x^{iz_j+k}.[A2.100]$$

A set satisfying the condition (A.2) is called an index set for $\mathbb{R}_+$. Let $\Omega^{\frac{1}{2}}$ denote a half density bundle; we denote by $\mathcal{A}^E(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ the space of polyhomogeneous conormal half densities with index set E.

The Mellin transform plays a very important role in the study of conormal distributions. In fact the spaces $\mathcal{A}^E(\mathbb{R}_+)$ can be characterized in terms of the Mellin transform. Since these distributions are smooth in $(0,\infty)$ and rapidly decreasing at ∞ we are only interested in the behavior near to $x=0$. Choose a function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ which equals 1 in a neighborhood of 0. We define the local Mellin transform for a distribution $u \in \mathcal{C}^{-\infty}(\mathbb{R}_+)$ by

$$\widehat{u}(s) =< \phi(x)x^{-(is+1)}, u > .[A2.101]$$

It is an elementary fact that this pairing is well defined for $s \in \mathbb{C}$ with sufficiently large imaginary part. In fact $\widehat{u}(s)$ is a holomorphic function for s in some half plane $\operatorname{Im} s > M$. For $u \in \mathcal{A}^E(\mathbb{R}_+)$, integration by parts along with the asymptotic expansion given in (A.2) allow one to show that $\widehat{u}(s)$ has a meromorphic extension to the entire complex plane with poles at

$$\widehat{E} = \{z_j - ik;\, z_j \in E,\, k \in \mathbb{N}_0\}.[A2.202]$$

In fact this condition is necessary and sufficient for a rapidly decreasing element of $\mathcal{C}^{-\infty}(\mathbb{R}_+)$ to belong to $\mathcal{A}^E(\mathbb{R}_+)$.

Suppose that A is an operator which carries $\dot{\mathcal{C}}^\infty(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ into $\mathcal{C}^{-\infty}(\mathbb{R}_+;\Omega^{\frac{1}{2}})$, then A has a Schwarz kernel, k_A which is an element of $\mathcal{C}^{-\infty}(\mathbb{R}_+^2;\Omega^{\frac{1}{2}})$. This distribution can be pulled back via the map β_0^2 defined in (A.1) to give a distribution on the blown-up space $[\mathbb{R}_+^2;0]$. We denote this distribution by κ_A. Let us consider a very simple example, the Schwarz kernel of the identity map:

$$\beta_0^{2*}(\delta(x-y)|dxdy|^{\frac{1}{2}}) = \delta(r(\cos\theta - \sin\theta))|rdrd\theta|^{\frac{1}{2}}.[A2.2]$$

The δ-function is homogeneous of degree -1 and therefore

$$\kappa_{\mathrm{Id}} = \delta(\cos\theta - \sin\theta)\left[\frac{|drd\theta|}{r}\right]^{\frac{1}{2}}[A2.3]$$

Observe that κ_{Id} is a delta distribution supported along the lifted diagonal Δ_0 and is rapidly vanishing along the left and right boundary faces of $[\mathbb{R}_+^2;0]$. As in (A.1) we express it as a multiple of the singular half density,

$$\nu_0 = \left[\frac{drd\theta}{r}\right]^{\frac{1}{2}}.$$

More generally we define $\Psi_b^m(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ as those operators A such that κ_A is rapidly vanishing at lb and rb and can be represented in a neighborhood of Δ_0 by the Fourier integral:

$$\kappa_A(r,\theta) = \int_{-\infty}^{\infty} a(r,\xi)e^{i\theta\xi}d\xi\nu_0.[A2.4]$$

Here $a(r,\xi) \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R})$ has an expansion

$$a(r,\xi) \sim \sum_{j=0}^{\infty} a_j(r)\xi^{m+j}.[A2.5]$$

The lifted kernel has a singularity along the lifted diagonal of exactly the same sort as a classical pseudodifferential operator. The 'symbol' of the operator is simply $a(r,\xi)$, we denote it by ${}^b\sigma_m(A)$. Is is invariantly a function on the conormal bundle of Δ_0 in $[\mathbb{R}^2_+;0]$. As a simple computation verifies, differential operators on $\mathbb{R}_+$ which can be expressed as polynomials of degree m in $x\partial_x$ with coefficients in $\mathcal{C}^\infty(\mathbb{R}_+)$ belong to $\Psi^m_b(\mathbb{R}_+;\Omega^{\frac{1}{2}})$. These operators are usually denoted by $\mathrm{Diff}^m_b(\mathbb{R}_+;\Omega^{\frac{1}{2}})$. The calculus, Ψ^m_b is intended as a 'microlocalization' of this algebra differential operators. If

$$P = \sum_{j=0}^{m} a_j(x)(x\partial_x)^j$$

then the symbol is

$${}^b\sigma_m(P) = a_m(r)(i\xi)^m \mod \xi^{m-1}.[A2.7]$$

An operator $P \in \mathrm{Diff}^m_b(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ is 'elliptic' provided $a_m(0) \neq 0$.

The operator considered in (**??**) is elliptic. From the construction there it is clear that the inverse to $-(x\partial_x)^2 + 1$ is not in Ψ^m_b for any m as the kernel does not vanish to infinite order at the left and right boundaries. This motivates us to augment this algebra by adding boundary terms. For $m \in \mathbb{R}$, α, $\beta \in \mathbb{C}$ we say that $A \in \Psi^{m;\alpha,\beta}_b(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ if the lifted Schwarz kernel satisfies the following conditions:

1. In a neighborhood of Δ_0 it has a representation as in (A.2), (A.2),
2. In a neigborhood of lb it has an asymptotic expansion

$$\kappa_A(r,\theta) = \sum_{j=0}^{\infty} b_j(r)(\cos\theta)^{i\alpha+j}, b_j(r) \in \mathcal{C}^\infty(\mathbb{R}_+),\ j = 0,1,\ldots,$$

3. In a neighborhood of rb it has an asymptotic expansion

$$\kappa_A(r,\theta) = \sum_{j=0}^{\infty} c_j(r)(\sin\theta)^{i\beta+j}, c_j(r) \in \mathcal{C}^\infty(\mathbb{R}_+),\ j = 0,1,\ldots.$$

In the blown-up space the submanifolds $\Delta_0, \mathrm{lb}, \mathrm{rb}$ are disjoint so the conditions in the above definition are well defined and independent of one another. This is the primary motivation for introducing the blown-up space. We can generalize this a little by allowing index sets $E_{\mathrm{lb}}, E_{\mathrm{rb}}$ instead of single exponents in the expansions at lb and rb . These operators are denoted are denoted by

$$\Psi^{m;\mathcal{E}}_b(\mathbb{R}_+;\Omega^{\frac{1}{2}}),\ \text{where } \mathcal{E} = \{E_{\mathrm{lb}}, E_{\mathrm{rb}}\}.$$

The collection of index sets, $\mathcal{E}$ is called an index family for $[\mathbb{R}^2_+;0]$. In fact we will need something slightly more complicated below. We have implicitly assumed that the kernel is smooth up to the front face away from Δ_0, however we could also have a non-trivial index set there. That is $\mathcal{E} = \{E_{\mathrm{lb}}, E_{\mathrm{rb}}, E_{\mathrm{ff}}\}$. When $E_{\mathrm{ff}} = \mathbb{N}_0$ it is omitted. If each E_o is of the form $\alpha_o - ik$, $k \in \mathbb{N}_0$ then we use the simplified notation $\mathcal{E} = (\alpha_{lb}, \alpha_{\mathrm{rb}}, \alpha_{\mathrm{ff}})$. Note that $m = -\infty$ corresponds to a kernel which is smooth in the interior but has polyhomogeneous conormal singularities along the boundary of $[\mathbb{R}^2_+;0]$. Such operators are not in general 'residual' as they do act compactly on weighted L^2-spaces.

There is a characterization of the kernels $\kappa_A \in \Psi_b^{-\infty;\{\alpha_1,\alpha_2,\alpha_3\}}$ in terms of their 'Mellin transform.' Let $\phi \in \mathcal{C}^\infty_c([\mathbb{R}^2_+;0];\Omega^{\frac{1}{2}})$ and let $\rho_{\mathrm{lb}}, \rho_{\mathrm{rb}}, \rho_{\mathrm{ff}}$ be defining functions for the left boundary, right boundary and front face respectively. We define a local Mellin transform of κ_A by

$$\widehat{\kappa}_A(s_1,s_2,s_3;\phi) =< \phi\rho_{\mathrm{lb}}^{-(is_1+1)}\rho_{\mathrm{rb}}^{-(is_2+1)}\rho_{\mathrm{ff}}^{-(is_3+1)}, \kappa_A > .[A2.103]$$

As with (A.2) this is defined and holomorphic in an 'orthant':

$$\operatorname{Im} s_i > M_i, i = 1,2,3.$$

One can show that $A \in \Psi_b^{-\infty;\{\alpha_1,\alpha_2,\alpha_3\}}$ if and only if $\widehat{\kappa}(s_1,s_2,s_3;\phi)$ has a meromorphic continuation to $\mathbb{C}^3$ for any choice of $\phi \in \mathcal{C}^\infty_c([\mathbb{R}^2_+;0])$ with poles on varieties of the form

$$s_i = \alpha_i + j; j \in \mathbb{N}_0,\ i = 1,2,3$$

There is also a characterization of this sort for more general index sets and for $m \neq -\infty$.

In order for $\Psi_b^{m;\mathcal{E}}$ to be a 'calculus' of operators we need to have conditions under which

$$A \in \Psi_b^{m_1;\mathcal{E}_1} \text{ and } B \in \Psi_b^{m_2;\mathcal{E}_2}$$

can be composed to give an operator

$$A \circ B = C \in \Psi_b^{m_3;\mathcal{E}_3}.$$

As a first step in this direction we consider the action of operators in our calculus on the spaces of polyhomogeneous conormal distributions defined above. By composing the blow-down map β_0^2 with the left and right projections to $\mathbb{R}_+$ we obtain maps π_{0l} and π_{0r}:

$$\pi_{0o} : [\mathbb{R}^2_+;0] \longrightarrow \mathbb{R}_+,\ o = l,r.$$

The action of A on $\dot{\mathcal{C}}^\infty(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ is defined by

$$Au = \pi_{0l*}(\kappa_A\pi_{0r}^*(u)).[A2.8]$$

Using (A.2), the Mellin transform and the results of appendix B one can show that

Proposition A.1. *If* $A \in \Psi_b^{m;\alpha,\beta}(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ *and* $u \in \mathcal{A}^\gamma(\mathbb{R}_+;\Omega^{\frac{1}{2}})$ *then* $A\cdot u$ *is defined provided* $\operatorname{Im}(\beta+\gamma) < 1$ *and satisfies*

$$A\cdot u \in \mathcal{A}^{\{\alpha,\gamma\}}(\mathbb{R}_+;\Omega^{\frac{1}{2}}).[A2.10]$$

Remark A.1. *This mapping result is easily extended to general index sets and families by linearity.*

Since $\dot{\mathcal{C}}^\infty(\mathbb{R}_+) = \mathcal{A}^{-i\infty}(\mathbb{R}_+)$, it follows from Proposition A.1 that C is defined if

$$A \in \Psi_b^{m_1;\alpha_1,\beta_1},\ B \in \Psi_b^{m_2;\alpha_2,\beta_2} \text{ provided } \operatorname{Im}(\beta_1+\alpha_2) < 1.[A2.11]$$

For simplicity we restrict ourselves to the case $m_1 = m_2 = -\infty$ We would like to show that $C \in \Psi_b^{-\infty;\mathcal{E}}$ for some index family $\mathcal{E}$. To do this we need to obtain some expression for $\beta_0^{2*}(k_C)$ in terms of κ_A, κ_B. Suppose that $k_A(x,y)$, $k_B(x,y)$

are smooth kernels on a compact manifold X. From Fubini's theorem it follows that the kernel of the composite operator is given by:

$$k_C(x,y) = \int\limits_X k_A(x,z)k_B(z,y)dz.[A2.12]$$

If we think of the kernels as half densities on X^2 then there is a functorial expression in terms of the triple product space X^3 which is equivalent to (A.2). Let π_f, π_c, π_s denote the projections of X^3 onto X^2 obtain by omitting the first, second and third factors respectively. The kernel for the composite is given by

$$k_C = \pi_{c*}(\pi_s^* k_A \otimes \pi_f^* k_B).[A2.13]$$

This expression extends by continuity to give the kernels of composites of pseudo-differential operators on a compact manifold.

The case at hand is slightly more complicated because we need to compute the pullback of the kernel of C to $[\mathbb{R}^2_+;0]$ and we need to preserve the structure of the singularities. This leads to the following geometric problem: construct a space $\widetilde{\mathbb{R}}^3_+$ to replace $\mathbb{R}^3_+$ in the way that $[\mathbb{R}^2_+;0]$ replaces $\mathbb{R}^2_+$. This space has to have certain properties:

1. $\widetilde{\mathbb{R}}^3_+$ should be a blow-up of $\mathbb{R}^3_+$ with blow-down map β^3_0,
2. there should be maps $\pi_{0o} : \widetilde{\mathbb{R}}^3_+ \longrightarrow [\mathbb{R}^2_+;0]$,
3. these maps must satisfy

$$\pi_{0o} \circ \beta^2_0 = \beta^3_0 \circ \pi_o,\ o = f,c,s.$$

4. The maps π_{0o}, $o = f,c,s$ should be b-fibrations.

Most of these conditions are fairly evident, the third condition implies that

$$\beta^{2*}_0(k_{A\circ B}) = \pi_{0c*}(\pi^*_{0s}\kappa_A \otimes \pi^*_{0f}\kappa_B)[A2.14]$$

when it makes sense. The fourth is a technical condition described in §B5. It is needed to ensure that the pullback and pushforward operations preserve polyhomogeneity.

A moments thought shows that in order for the maps π_{0o} to exist we must in some sense blow-up all the submanifolds in $\mathbb{R}^3_+$ which project under π_o to $(0,0)$. That is we must blow-up

$$B_f = \{(0,0,x); x \in \mathbb{R}_+\},\ B_c = \{(0,x,o); x \in \mathbb{R}_+\},\ B_s = \{(x,0,0); x \in \mathbb{R}_+\}.[A2.15]$$

This leads to a small difficulty: we can obtain a space by blowing up these submanifolds one at a time and lifting the remaining ones to the resultant space and then blowing them up. However the space one obtains depends on the order in which B_f, B_c, B_s are blown-up. The resultant spaces also fail to be b-fibrations. The reason is that the submanifolds intersect in a non-clean (in the sense of Bott) fashion. We remedy this situation by first blowing up their common intersection, $B_t = B_f \cap B_c = B_f \cap B_s = B_c \cap B_s$. We denote this space by $[\mathbb{R}^3_+; B_t]$. Generally we denote a manifold X blown-up along a submanifold Y by $[X;Y]$, see §C4 for more details about this notation.

The lifts of the submanifolds in (A.2) to $[\mathbb{R}^3_+; B_t]$ are disjoint; again we denote them by B_f, B_c, B_s. Clearly we can blow them up in any order we please. We denote the the resultant space by $[\mathbb{R}^3_+; B_t; B_f \cup B_c \cup B_s]$. This is a space with all the necessary blowups to define the maps π_{0o}. Let β_0^3 denote the canonical blow-down:

$$\beta_0^3 : [\mathbb{R}_+^3; B_t; B_f \cup B_c + \cup B_s] \longrightarrow \mathbb{R}_+^3.[A2.16]$$

Now we encounter another problem: how should the projections, π_{0o} be defined, from the construction we have given for $[\mathbb{R}_+^3; B_t; B_f \cup B_c \cup B_s]$ it is not clear that they exist. To remedy this we need to give alternate constructions for this space which entail **commuting** the order of the blow-ups. Observe that

$$[\mathbb{R}_+^3; B_o] = [\mathbb{R}_+^2; 0] \times \mathbb{R}_+.[A2.17]$$

From (A.2) it is clear that $[\mathbb{R}_+^3; B_o]$ projects to $[\mathbb{R}_+^2; 0]$, thus our strategy is to construct canonically isomorphic to $[\mathbb{R}_+^3; B_t; B_f \cup B_c \cup B_s]$ starting with $[\mathbb{R}_+^3; B_o]$. One can easily verify that

$$[\mathbb{R}_+^3; B_t; B_f \cup B_c \cup B_s] \simeq [\mathbb{R}_+^3; B_f; B_t; B_c \cup B_s].[A2.18]$$

This isomorphism persists if f, c, s are cyclically permuted. Instead of giving a formal proof of this statement we give a graphical demonstration, see fig. ??. Note that the alternate construction is obtained by changing the order in which the manifolds are blown-up. From (A.2) the existence of the map π_{0f} is immediate. There is a blow-down map:

$$F : [\mathbb{R}_+^3; B_f; B_t; B_c \cup B_s] \longrightarrow [\mathbb{R}_+^3; B_f],$$

and a projection

$$G : [\mathbb{R}_+^3; B_f] \longrightarrow [\mathbb{R}^2; 0].$$

The map π_{0f} is the composition

$$\pi_{0f} = G \circ F.[A2.19]$$

All of the properties of the maps π_{0o}, $o = f, c, s$ follow easily from this construction. See §C4 for a general discussion of when it is permissible to interchange the order of blow-ups.

With the construction of the 'stretched triple product' complete we can use (A.2) to analyze the kernel κ_C. To complete this step we use the Mellin transform characterization of polyhomogeneous conormal distributions. That is for $\phi \in \mathcal{C}_c^\infty([\mathbb{R}_+^2; 0]; \Omega^{\frac{1}{2}})$ we compute:

$$\begin{gathered}\widehat{\kappa}_C(s_1, s_2, s_3; \phi) = \\ < \phi \rho_{\mathrm{lb}}^{-(is_1+1)} \rho_{\mathrm{ff}}^{-(is_2+1)} \rho_{\mathrm{rb}}^{-(is_3+1)} \nu_0, \kappa_C > .\end{gathered}[A2.20]$$

Using the formula (1.1) we obtain that

$$\begin{gathered}\widehat{\kappa}_C(s_1, s_2, s_3; \phi) = \\ < \pi_{0c}^*(\phi \rho_{\mathrm{lb}}^{-(is_1+1)} \rho_{\mathrm{ff}}^{-(is_2+1)} \rho_{\mathrm{rb}}^{-(is_3+1)} \nu_0), \pi_{0s}^* \kappa_A \otimes \pi_{0f}^* \kappa_B > .\end{gathered}[A2.21]$$

From (A.2) and the charaterization of polyhomogeneous conormal distributions in terms of their local Mellin transforms, it is essentially an elementary calculation to prove the composition formula:

Proposition A.2. *Composition formula If*

$$A \in \Psi_b^{-\infty;\alpha_1,\beta_1},\ B \in \Psi_b^{-\infty;\alpha_2,\beta_2} \text{ with } \operatorname{Im}(\beta_1 + \alpha_2) < 1.[A2.23]$$

then

$$A \circ B \in \Psi_b^{-\infty;\{\alpha_1,\alpha_2\},\{\beta_1,\beta_2\},\{0,\alpha_2+\beta_1+???\}}.$$

REMARK A.2. *This formula is only strictly correct if none of the pairs* $\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}, \{0, \alpha_2 + \beta_1 + ???\}$ *differs by an integer.*

This shows that when the composition of two operators in Ψ_b^* is defined the resultant operator is also in Ψ_b^*. The "orders" of $A \circ B$ are simply related to those of A and B.

APPENDIX B

Conormal distributions

Since heavy use is made in this monograph of polyhomogeneous conormal distributions we shall describe their definition, in general on a manifold with corners, and main properties. Full proofs of these facts for general conormal distributions can be found in [**12**]. We also refer to [**12**] for the precise notion of a manifold with corners but recall that these are spaces locally smoothly modeled on the products $\mathbb{R}^n_k = [0, \infty)^k \times \mathbb{R}^{n-k}$ of half-lines and lines with the additional property that all boundary hypersurfaces are embedded. This includes products of manifolds with boundary, which is how many of the spaces here originate. If X is a manifold with corners then by local coordinates based at $p \in X$ we always mean coordinates x_j with respect to which p is the origin and X is locally given by $x_j \geq 0$, for $j = 1, \dots, k$.

B.1. Conormality at the boundary

On a compact topological manifold without boundary, X a smooth structure is defined by the choice of a subring of $\mathcal{C}^0(X)$ which satisfies certain seperation and consistency properties. This subring is denoted by $\mathcal{C}^\infty(X)$. Homeomorphisms of X which preserve the structure ring under pullback are called diffeomorphisms. Infinitesimal diffeomorphisms are called vector fields. The vector fields can also be described as the smooth sections of a vector bundle TX over X. Or algebraically they can be defined as the derivations of the structure ring.

If X is a compact manifold with corners then the space $\mathcal{V}_b(X)$ is defined to be the smooth vector fields on X tangent to the boundary hypersurfaces, and hence to all boundary faces. This is the space of infinitesimal diffeomorphisms of X. As is shown in [**12**] these vector fields can also be characterized as the smooth sections of a vector bundle over X, ${}^bTX$. The space $\mathcal{V}_b(X)$ is a $\mathcal{C}^\infty(X)$-Lie algebra, its universal enveloping algebra is denoted by $\mathrm{Diff}^*_b(X)$. These are called 'totally characteristic' differential operators. The space $L^\infty(X)$ is defined as in the boundaryless case; the space $\mathcal{C}^{-\infty}(X) = (\dot{\mathcal{C}}^\infty(X))'$ of extendible distributions on X is defined in detail in [**12**]. Moreover $L^\infty(X) \subset \mathcal{C}^{-\infty}(X)$ is a well-defined subspace. An action of $\mathcal{V}_b(X)$ on $\mathcal{C}^{-\infty}(X)$ is defined by using the standard integration by parts formalism, we then define

$$\mathcal{A}^0(X) = \{u \in L^\infty(X); V_1 \cdot \dots \cdot V_m u \in L^\infty(X) \ \forall \ V_i \in \mathcal{V}_b(X)\} .[B1.1]$$

This is the space of boundedly-conormal functions. The definition shows that in a natural way it is one possible replacement for $\mathcal{C}^\infty(X)$ (which is itself well-defined as the structure algebra). In the interior of X the vector fields in $\mathcal{V}_b(X)$ are unconstrained so certainly these conormal functions are smooth there:

$$\mathcal{A}^0(X) \subset \mathcal{C}^\infty(\overset{\circ}{\to} X).[B1.2]$$

Since each boundary hypersurface $H \subset X$, where X is always a compact manifold with corners, is embedded there is a defining function $\rho_H \in \mathcal{C}^\infty(X)$ with the properties

$$\rho_H \geq 0,\ d\rho_H \neq 0 \text{ on } H,\ H = \{x \in X; \rho_H(x) = 0\}.[B1.3]$$

Any two such defining functions are smooth positive multiples of each other. Since ρ_H is positive on the interior of X we can define

$$\rho_H^z = \exp(z \log \rho_H) \in \mathcal{C}^\infty(\overset{\circ}{\to} X) \quad \forall\ z \in \mathbb{C}.[B1.4]$$

These powers are always extendible distributions, in fact multiplication of an extendible distribution by any such power is always well-defined and gives an isomorphism

$$\rho_H^z : \mathcal{C}^{-\infty}(X) \longleftrightarrow \mathcal{C}^{-\infty}(X).[B1.5]$$

For simplicity we can always assume that the defining function ρ_H also satisfies

$$\rho_H(X) < 1 \quad \forall\ x \in X.[B1.6]$$

This means that $\log \rho_H \neq 0$ on the interior of X. Then for any complex number z and any integer j multiplication

$$\rho_H^z (\log \rho_H)^j : \mathcal{C}^{-\infty}(X) \longleftrightarrow \mathcal{C}^{-\infty}(X)[B1.7]$$

is again an isomorphism.

In general we let $M_k(X)$ be the set of codimension k boundary components of X and $M(X)$ be the set of all boundary faces; both are finite sets. By a power set on X we mean a map

$$\mathfrak{m} : M_1(X) \longrightarrow \mathbb{R} \times \mathbb{N}_0,\ \mathbb{N}_0 = \{0, 1, 2 \dots\}.[B1.8]$$

If we choose a defining function for each $H \in M_1(X)$ satisfying (B.1) then we can consider the products of powers:

$$\rho^{\mathfrak{m}} = \prod_{H \in M_1(X)} \rho_H^{m(H)} (\log \rho_H)^{j(H)}[B1.9]$$

where $\mathfrak{m}(H) = (m(H), j(H))$. The general L^∞ weighted conormal space on X is simply

$$\mathcal{A}^{\mathfrak{m}}(X) = \rho^{\mathfrak{m}} \mathcal{A}^0(X).[B1.10]$$

This is equivalent to a definition as in (B.1), i.e.

$$\begin{aligned} \mathcal{A}^{\mathfrak{m}}(X) = \{u \in \rho^{\mathfrak{m}} L^\infty(X); V_1 \cdot \dots \cdot V_m u \in \rho^{\mathfrak{m}} L^\infty(X)\ \forall\ V_i \in \mathcal{V}_b(X)\} \\ \rho^{\mathfrak{m}} L^\infty(X) = \{u \in \mathcal{C}^{-\infty}(X); u = \rho^{\mathfrak{m}} v,\ v \in L^\infty(X)\}. \end{aligned} [B1.11]$$

As well as these L^∞-based conormal spaces we consider the L^2-based spaces. These are similar to standard Sobolev spaces. Thus if we let $L^2_b(X)$ denote the space of (equivalence classes of) measurable functions which are square integrable with respect to a non-vanishing b-density (i.e. one that becomes $a dx_1 dx_2 \dots dx_n / x_1 \dots x_k$ with $a > 0$ in local coordinates) then for positive integers

$$H^m_b(X) = \left\{u \in L^2_b(X); \mathrm{Diff}^m_b(X) u \subset L^2_b(X)\right\}.[B1.49]$$

This definition extends directly to sections of any vector bundle. These have Hilbert space topologies and the duals can also be regarded as spaces of extendible distributions, so one defines

$$H^m_b(X) = (H^{-m}_b(X; \Omega_b))',\ m \in -\mathbb{N}[B1.50]$$

where the pairing is the continuous extension of the pairing of $L^2_b(X)$ and $L^2_b(X;\Omega_b)$. One can define these spaces for all real exponents by interpolation or using b-pseudodifferential operators (totally characteristic pseudodifferential operators, see [**?**], [**?**] or [**12**]). Even more generally one can define weighted versions of these spaces, which we denote:

$$\rho^{\mathfrak{m}} H^m_b(X) \ni u \Longleftrightarrow \rho^{-\mathfrak{m}} u \in H^m_b(X).[B1.51]$$

One then has

$$\begin{aligned} \bigcup_{\mathfrak{m}} \left[\bigcap_m \rho^{\mathfrak{m}} H^m_b(X) \right] &= \mathcal{A}(X) \\ \bigcap_{\mathfrak{m}} \left[\bigcap_m \rho^{\mathfrak{m}} H^m_b(X) \right] &= \dot{\mathcal{C}}^\infty(X) \;\; [B1.52] \\ \bigcup_{\mathfrak{m}} \left[\bigcup_m \rho^{\mathfrak{m}} H^m_b(X) \right] &= \mathcal{C}^{-\infty}(X). \end{aligned}$$

Although we sometimes need the full generality of these 'logarithmically weighted' spaces it is often sufficient to use simpler spaces and notation. First we often choose an order for $M_1(X)$:

$$M_1(H) = \{H_1, H_2, \dots, H_N\}[B1.12]$$

so that the power sets are just ordered sequences of pairs:

$$\mathfrak{m} = \{(m_1, j_1), (m_1, j_2), \dots, (m_N, j_N)\}.[B1.13]$$

If $j_k = 0$ for some hypersurface we simply denote the corresponding pair as m_k instead of $(m_k, 0)$. If the map $\mathfrak{m}$ is constant it is denoted by its value, either (m, j) or m as the case may be. Power sets have a natural partial order, namely if

$$\begin{gathered} \mathfrak{m}(H) = (m(H), j(H)), \mathfrak{n}(H) = (n(h), \ell(H)) \text{ then } \mathfrak{m} \le \mathfrak{n} \Longleftrightarrow \\ \forall\ H \in M_1(X),\ m(H) \le n(H) \text{ and } j(H) \le \ell(H) \text{ if } m(H) = n(H). \end{gathered} [B1.14]$$

With this partial order

$$\mathcal{A}^{\mathfrak{m}}(X) \subset \mathcal{A}^{\mathfrak{n}}(X) \Longleftrightarrow \mathfrak{m} \ge \mathfrak{n}.[B1.15]$$

This reversal of the obvious order is one reason (the main one being historical) for introducing the 'product-type symbol spaces' on X simply by setting

$$S^{\mathfrak{m}}(X) = \mathcal{A}^{-\mathfrak{m}}(X).[B1.16]$$

We also extend the definition of a power set by adjoining ∞ and $-\infty$ to the range of the map (B.1). If $\mathfrak{m}$ is a power set with all entries finite or $+\infty$ we let $\mathfrak{m}(j)$ be the power set obtained by replacing $+\infty$ by j and then set

$$\mathcal{A}^{\mathfrak{m}}(X) = \bigcap_j \mathcal{A}^{\mathfrak{m}(j)}(X).[B1.17]$$

If some of the entries are $-\infty$ we let $\mathfrak{m}(j)$ be the power set obtain by replacing $-\infty$ by $-j$ (so that some entries may be $+\infty$) and then define

$$\mathcal{A}^{\mathfrak{m}}(X) = \bigcup_j \mathcal{A}^{\mathfrak{m}(j)}(X), [B1.18]$$

using (B.1) if necessary.

Leibniz' formula for the action of vector fields shows that these spaces are multiplicative

$$\mathcal{A}^{\mathfrak{m}}(X)\cdot\mathcal{A}^{\mathfrak{n}}(X)=\mathcal{A}^{\mathfrak{m}+\mathfrak{n}}(X)[B1.19]$$

with the obvious notion of additivity of maps into $\mathbb{R}\times\mathbb{N}_0$ and with the conventions

$$\begin{aligned}(m,j)+\infty=+\infty,\ (m,j)-\infty=-\infty,\ (m,j)\text{ finite}\\ +\infty+\infty=+\infty,\ -\infty-\infty=-\infty,\ +\infty-\infty=+\infty.\end{aligned}[B1.20]$$

For power sets $\mathfrak{m}$ not taking the value $-\infty$ the spaces $\mathcal{A}^{\mathfrak{m}}(X)$ are Frechet spaces; in the general case (B.1) gives an inductive limit topology. The smallest space amongst the $\mathcal{A}^{\mathfrak{m}}$ is where $\mathfrak{m}$ is constant with value $+\infty$; then

$$\mathcal{A}^{+\infty}(X)=\dot{\mathcal{C}}^{\infty}(X)[B1.21]$$

is the space of $\mathcal{C}^{\infty}$ functions on X vanishing with all derivatives at the boundary. If $\epsilon>0$ and the compressed notation for power sets described above is used then for any power set $\mathfrak{m}$

$$\dot{\mathcal{C}}^{\infty}(X)\subset\mathcal{A}^{\mathfrak{m}}(X)\text{ is dense in the topology of }\mathcal{A}^{\mathfrak{m}-\epsilon}(X).[B1.22]$$

The largest of the conormal spaces corresponds to a power set with constant value $-\infty$; we use the simpler notation

$$\mathcal{A}(X)=\mathcal{A}^{-\infty}(X).[B1.23]$$

One of the most important properties of these conormal spaces is their asymptotic completeness. Suppose that $\mathfrak{m}_j$ is a sequence of power sets which is such that

$$\text{for each } H\in M_1(X)\text{ either }\mathfrak{m}_j(H)\text{ is constant or }\mathfrak{m}_j(H)\longrightarrow\infty.[B1.24]$$

Consider a sequence $a_j\in\mathcal{A}^{\mathfrak{m}_j}(X)$. Asymptotic completeness is the condition that there exist some element

$$\begin{aligned}a\in\mathcal{A}^{\mathfrak{m}}(X),\ \mathfrak{m}=\inf_j\mathfrak{m}_j\text{ such that}\\ \forall\ J,\ a-\sum_{j<J}a_j\in\mathcal{A}^{\mathfrak{n}_J},\ \mathfrak{n}_J=\inf_{j\geq J}m_j.\end{aligned}[B1.25]$$

This asymptotic completeness is proved by the summation method of Borel's lemma. The asymptotic sum, a, is unique up to the addition of any element of $\mathcal{A}^{\mathfrak{m}_\infty}(X)$ where $\mathfrak{m}_\infty=\sup\mathfrak{m}_j$, meaning it is equal to the power sets $\mathfrak{m}_j$ on the hypersurfaces on which the sequence is constant and equal to $+\infty$ on the others. As usual we write

$$a\sim\sum_j a_j.[B1.26]$$

Another useful property of these spaces to bear in mind is interpolation.

LEMMA B.1. *Suppose* $\mathfrak{m}'$, $\mathfrak{m}''$ *and* $\mathfrak{m}$ *are power sets for a compact manifold with corners and* $t\in[0,1]$ *is such that for each* $H\in M_1(X)$

$$\begin{aligned}\mathfrak{m}'(H)=(m'(H),j'(H)),\ \mathfrak{m}''(H)=(m''(H),j''(H)),\ \mathfrak{m}(H)=(m(H),j(H))\\ \text{with } m(H)\leq tm'(H)+(1-t)m''(H)\end{aligned}[B1.28]$$

and where

$$j(J)\leq tj'(H)+(1-t)j''(H)[B1.29]$$

whenever equality occurs in (B.1) *then*

$$\mathcal{A}^{\mathfrak{m}'}(X) \cap \mathcal{A}^{\mathfrak{m}''}(X) \subset \mathcal{A}^{\mathfrak{m}}(X).[B1.30]$$

Using the filtration defined by (B.1) one can define the notion of a symbol for general conormal distributions as a residue class. This symbol has many of the familiar properties of the symbol of a pseudodifferential operator. In this monograph, very little use is made of general conormal distributions. Instead we work with the much simpler subalgebra of polyhomogeneous conormal distributions. These bear the same relationship to general conormal distirbutions as Kohn-Nirenberg pseudodifferential operators bear to operators of type $(1, 0)$.

B.2. Polyhomogeneous conormal distributions

A very simple type of conormal distribution at the boundary of a manifold with corners is a $\mathcal{C}^\infty$ multiple of some power of a defining function ρ_H for $H \in M_1(X)$:

$$\rho_H^{iz} a(x) \in \mathcal{A}^{\mathfrak{m}}(X),\ \mathfrak{m}(H) = \operatorname{Im} z,\ \mathfrak{m}(G) = 0, G \neq H \text{ if } a \in \mathcal{C}^\infty(X).[B3.1]$$

More generally we can consider logarithmic terms of the same general type:

$$\rho_H^{iz}(\log \rho_H)^j a(x) \in \mathcal{A}^{\mathfrak{m}}(X),\ \mathfrak{m}(H) = (\operatorname{Im} z, j)\ \mathfrak{m}(G) = 0, G \neq H \text{ if } a \in \mathcal{C}^\infty(X).[B3.2]$$

A polyhomogeneous conormal distribution is one which is, in an appropriate sense, an asymptotic sum of such 'simple' distributions. An *index set* is a discrete subset $E \subset \mathbb{C} \times \mathbb{N}_0$, $N_0 = \{0, 1, \dots\}$ with the following two properties:

(B.1) $$(z_\ell, j_\ell) \in E \text{ and } |z_\ell| + |j_\ell| \longrightarrow \infty \Longrightarrow \operatorname{Im} z_\ell \longrightarrow -\infty.[B3.3]$$

(B.2) $$(z, j) \in E \Longrightarrow (z - ip, k) \in E\ \forall\ p \in \mathbb{N} \text{ and } k \in \{0, 1, \dots, j\}.[B3.4]$$

The first condition ensures that any sequence of distinct powers corresponds to a sequence or terms as in (B.2) which is rapidly decreasing at the boundary hypersurface in question and the second ensures independence of the choice of ρ_H.

An *index family* $\mathcal{E}$ assigns an index set to each boundary hypersurface, thus we can write $\mathcal{E} = (E_1, \dots, E_N)$, with the E_j index sets, if $M_1(X) = \{H_1, \dots, H_N\}$ is an enumeration of the boundary hypersurfaces. We wish to define the space $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ of polyhomogeneous conormal distributions corresponding to the index family $\mathcal{E}$. An inductive definition in terms of expansions is given in [**12**]. This is a very 'constructive' definition which shows the effective computability of the terms in the expansion. Here we shall simply discuss one of the more useful characterizations.

Any boundary face H has a 'radial' vector field, that is a vector field ${}^b\nu_H \in \mathcal{V}_b(X)$ which vanishes (in the sense of $\mathcal{C}^\infty$ vector fields) at H and is such that

$${}^b\nu_H \rho_H = \rho_H + O(\rho_H^2).[B3.5]$$

Any two such vector fields differ by a term in $\rho_H \mathcal{V}_b(X)$. If E is an index set associated to H and $N \subset \mathbb{N}$ set

$$P_{H,N,E} = \prod_{\substack{(z,j) \in E \\ \operatorname{Im} z \geq -N}} ({}^b\nu_H - iz).[B3.6]$$

Notice that a given factor ${}^b\nu_H - iz$ appears exactly $j+1$ times if $j \in \mathbb{N}_0$ is the largest integer such that $(z, j) \in E_H$. The assumption (B.1) on the index sets ensures that the product in (B.2) is finite for each N. A conormal distribution $u \in \mathcal{A}(X)$ is said to be polyhomogeneous at $H \in M_1(X)$ with index set E if there exists a fixed power set $\mathfrak{m}$ such that

$$P_{H,N,E}u \in \mathcal{A}^{\mathfrak{m}-N1(H)}(X) \quad \forall\ N \in \mathbb{N}.[B3.7]$$

That is the application of $P_{H,N,E}$ lowers the order at H by N. If $\mathcal{E}$ is an index family for X then set

$$P_{N,\mathcal{E}} = \prod_{H \in M_1(X)} P_{H,N,E_H}[B3.8]$$

for some chosen ordering of $M_1(H)$ (since the normal vector fields of different boundary hypersurfaces need not commute). Then we set

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) = \left\{u \in \mathcal{A}(X); P_{N,\mathcal{E}}u \in \mathcal{A}^N(X)\ \forall\ N\right\}.[B3.9]$$

Of course it needs to be shown that this definition is independent of the choices made, see [**12**].

With any index family $\mathcal{E}$ we can associate the power set $\mathfrak{e}$ where

$$\begin{gathered}\mathfrak{e}(H) = (m, j) \iff \\ m = -\sup\{\operatorname{Im} z; (z,j) \in E_H\},\ j = \max\{k, \exists\ (z,k) \in E_H \text{ with } \operatorname{Im} z = -m\}.\end{gathered}[B3.10]$$

This power set has the property that $\mathcal{A}^{\mathfrak{e}}(X)$ is the smallest such space with

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X) \subset \mathcal{A}^{\mathfrak{e}}(X).[B3.11]$$

Recall from [**12**] the form of the collar neighbourhood, or normal fibration, theorem for boundary hypersurfaces. This states that there is a distinguished (and non-empty) class of local diffeomorphisms

$$F : {}_+N\{X; H\} \supset O \longrightarrow O' \subset X[B1.33]$$

from neighbourhoods O of H as the zero section of its inward-pointing normal bundle to neighbourhoods O' of H in X. The distinguishing feature is that these local diffeomorphisms fix each point of H and induce the identity transformation on $N\{X; H\}$. The normal bundle $N\{X; H\}$ has a natural compactification (by stereographic projection) which we denote X_H. It is diffeomorphic to $H \times [0,1]$, the boundary hypersurface $H \times \{0\}$ is often denoted again as H. If $\mathfrak{m}$ is a power set on X then there is a corresponding power set $\mathfrak{m}^H$ on X_H. All the boundary hypersurfaces of X_H, except $H \times \{1\}$, correspond uniquely (under a normal fibration) with boundary hypersurfaces of X so $\mathfrak{m}^H$ is just defined to be $+\infty$ on $H \times \{1\}$ and to equal $\mathfrak{m}$ on the other boundary hypersurfaces.

For polyhomogeneous spaces it is a simple matter to define symbol maps. We first define the symbol at a boundary hypersurface H. If $\mathcal{E}$ is an index family for X it fixes an index family $\widetilde{\mathcal{E}}^H$ for X_H by setting $\widetilde{E}^H_{H\times\{0\}} = E_H,\ \widetilde{E}^H_{H\times\{1\}} = \emptyset$ and taking the index sets on the other boundary hypersurfaces of X_H to be those of the boundary hypersurfaces of X to which they map. One easily verifies that there is a natural identifiaction:

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)/\mathcal{A}^{\mathcal{E}-i}_{\text{phg}}(X) \cong \mathcal{A}^{\widetilde{\mathcal{E}}^H}_{\text{phg}}(X_H)/\mathcal{A}^{\widetilde{\mathcal{E}}^H-i}_{\text{phg}}(X_H).[B3.19]$$

The compactified normal bundle X_H has an $\mathbb{R}^+$-action which fixes the notion of homogeneity up to the boundary hypersurface $H \times \{0\}$. In particular on X_H there are homogeneous defining functions for $H \times \{0\}$, namely the differentials $d\rho_H$ of defining functions for H on X. Two such defining functions are multiples of each other with the factor in $\mathcal{C}^\infty(H)$. Thus we can fix the symbol of $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ at H as a finite sum of quasi-homogeneous terms identified via (B.2) with a polyhomogeneous conormal distribution on X_H :

$$\sigma_H(u) = \sum_{\{(z,j)\in E_H;(-\operatorname{Im} z,j)=\mathfrak{e}(H)\}} a(z,j)(d\rho_H)^{iz}(\log(d\rho_H))^j,\ a(z,j)\in\mathcal{A}^{\mathcal{E}^H}_{\text{phg}}(H).[B3.20]$$

The notation $\mathcal{E}^H$ is used for the index family on H induced by $\mathcal{E}$. We let $\mathcal{E}^H \ni E^H_K = E_G$ if $G \in M_1(X)$, $K \in M_1(H)$ and $K \in G \cap H$. The symbol is the 'most singular' term in the expansion of u at H.

The range of the symbol map at H on $\mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ is precisely the space of sums as in (B.2) but there are consistency conditions between the symbols at intersecting boundary hypersurfaces. If $G \in M_d(X)$ is any boundary face of codimension $d > 0$ then there is an analogously defined symbol map. The compactified normal bundle X_G is a trivial $[0,1]^d$-bundle over G with d individual $\mathbb{R}^+$-actions on each fibre. If $P = \{H \in M_1(X); G \cap H \neq \emptyset\}$ then

$$G \subset \bigcap_{H\in P} H,[B3.21]$$

and the symbol at G can be written

$$\sigma_G(u) = \sum_{(Z,J)}\prod_{H\in P} A(Z,J)(d\rho_H)^{iZ(H)}(\log(d\rho_H))^{J(H)},\ a(Z,J)\in\mathcal{A}^{\mathcal{E}^G}_{\text{phg}}(G)[B3.22]$$

where the finite sum is over the index sets

$$\begin{gathered}(Z,J) : P \longrightarrow \mathbb{C}\times\mathbb{N}_0 \text{ such that}\\ (Z(H),J(H)) \in E_H \text{ and } (-\operatorname{Im} Z(H),J(H)) = \mathfrak{e}(H) \quad \forall\ H \in P.\end{gathered}[B3.23]$$

The consistency conditions on the symbol at hypersurface boundary faces can then be written

$$\sigma_K\sigma_H(u) = \sigma_K\sigma_G(u) \quad \forall\ K \in M(H)\cap M(G).[B3.24]$$

Here we use the convention of [**12**] of regarding $M(H)$ as a subset of $M(X)$ if $H \in M(X)$. These are the only consistency conditions:

LEMMA B.2. *Let $\mathcal{E}$ be an index family for X and suppose $P \subset M_1(X)$ is any set of boundary hypersurfaces. Then writing $\mathcal{E}^H$ for the index family induced on H by $\mathcal{E}$ the elements*

$$\begin{gathered}v_H = \sum_{\{(z,j)\in E_H;(-\operatorname{Im} z,j)=\mathfrak{e}(H)\}} a_H(z,j)(d\rho_H)^{iz}(\log(d\rho_H))^j,\\ \text{where } a_H(z,j)\in\mathcal{A}^{\mathcal{E}^H}_{\text{phg}}(H),\ \forall\ H\in P\end{gathered}[B3.26]$$

are such that there exists $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ with

$$\sigma_H(u) = v_H \quad \forall\ H \in P[B3.27]$$

if and only if

$$\sigma_K(v_H) = \sigma_K(v_G) \quad \forall\ K \in M(H)\cap M(G),\ H,G\in P.[B3.28]$$

The compactified normal bundles to the boundary hypersurfaces, X_H, also act as carriers of the leading parts of the tangent vector fields $\mathcal{V}_b(X)$. Thus if a normal fibration, F, around H is used to transfer $V \in \mathcal{V}_b(X)$ to X_H the result is well-defined up to a term $\rho_H W$, with $W \in \mathcal{V}_b(X_H)$. Since X_H has an action of $[0,\infty]$, arising from its structure as a line bundle, this image fixes a unique vector field

$I_H(V) \in \mathcal{V}_b(X_H)$ which is $(0,\infty)$-invariant. This is called the indicial map and corresponds to a short exact sequence of Lie algebras:

$$0 \longrightarrow \rho_H \mathcal{V}_b(X) \longrightarrow \mathcal{V}_b(X) \longrightarrow \mathcal{V}_{b,I}(X_H) \longrightarrow 0.[B1.46]$$

Here $\mathcal{V}_{b,I}(X_H)$ is the subalgebra of $\mathcal{V}_b(X_H)$ consisting of the $(0,\infty)$-invariant elements. The map (B.2) extends to a map of the enveloping algebras:

$$0 \longrightarrow \rho_H \operatorname{Diff}_b^m(X) \longrightarrow \operatorname{Diff}_b^m(X) \xrightarrow{I_H} \operatorname{Diff}_{b,I}^m(X_H) \longrightarrow 0[B1.47]$$

where again the subscript 'I' refers to the $(0,\infty)$-invariance. If $P \in \operatorname{Diff}_b^m(X)$ then $I_H(P) \in \operatorname{Diff}_{b,I}^m(X_H)$ is its indicial operator at H. One of the most important properties of symbol map is:

$$\sigma_H(Pu) = I_H(P)\sigma_H(u) \quad \forall\ P \in \operatorname{Diff}_b^m(X),\ u \in \mathcal{A}_{\text{phg}}^{\mathcal{E}}(X).[B1.48]$$

Here the right side is in $\mathcal{A}_{\text{phg}}^{\mathcal{E}^H}(X_H)$.

An important tool in the analysis of polyhomogeneous conormal distributions is the local Mellin transform. Let $(x_1,\dots,x_k,y_1,\dots,y_{n-k})$ be linear coordinates for $\mathbb{R}_k^n$; for a k-tuple α define

$$x^\alpha = x_1^{\alpha_1}\cdot\dots\cdot x_k^{\alpha_k}.$$

Let $u \in \mathcal{C}^{-\infty}(\mathbb{R}_k^n)$, $\psi \in \mathcal{C}_c^\infty(\mathbb{R}_k^n)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n-k})$, there exists an M such that if

$$\operatorname{Im} s_i > M, i = 1,\dots,k, [B1.99]$$

then the pairing

$$\phi \longrightarrow < x^{-(is-1)}\phi(y), \psi u >, [B1.100]$$

defines an element of $\mathcal{C}^{-\infty}(\mathbb{R}^{n-k})$. Denote this distribution by $\widehat{u}(s;\psi)$. In fact it is easy to show that

$$s \longrightarrow \widehat{u}(s;\psi)$$

is a holomorphic function from the orthant defined in (B.2) to $\mathcal{C}^{-\infty}(\mathbb{R}^{n-k})$. This is the local Mellin transform of u in a neighborhood of the point $(0,\dots,0)$.

If $u \in \mathcal{A}^{\mathfrak{m}}(\mathbb{R}_k^n)$ then it is a simple consequence of (B.1) that $\widehat{u}(s;\psi)$ is represented by a function in $\mathcal{C}^\infty(\mathbb{R}^{n-k})$ for s lying in some orthant. If $u \in \mathcal{A}_{\text{phg}}^{\mathcal{E}}(\mathbb{R}_k^n)$ then by using the asymptotic expansion and integrating by parts one easily establishes that $\widehat{u}(s;\psi)$ has a meromorphic extension to $\mathbb{C}^k$ with values in $\mathcal{C}^\infty(\mathbb{R}^{n-k})$. The poles of the continuation lie along varieties of the form

$$\operatorname{Im} s_i = \alpha \text{ for } \alpha \in E_i,\ \mathcal{E} = (E_1,\dots,E_k).[B2.104]$$

In fact this property characterizes polyhomogeneous conormal distributions on $\mathbb{R}_k^n$.

By introducing admissible coordinates we can extend this construction to an arbitrary manifold with corners. Let X be such a manifold with corners and let $p \in \partial X$ define $k(p)$ to be the maximal codimension of a boundary component containing p. For $u \in \mathcal{C}^{-\infty}(X)$ we proceed as above: for $p \in \partial X$ introduce local coordinates $(x_1,\dots,x_{k(p)},y_1,\dots,y_{n-k(p)})$ centered at p. For s a k-tuple satisfying (B.2) for some sufficiently large M and ψ a smooth function with compact support in the coordinate chart centered at p we define $\widehat{u}_p(s;\psi)$ by

$$< \phi, \widehat{u}_p(s;\psi) > = < \phi x^{-(is-1)}, \psi u >, \text{ for all } \phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n-k(p)}).[B1.101]$$

If $u \in \mathcal{A}_{\text{phg}}^{\mathcal{E}}$ then for every $p \in \partial X$ and admissible ψ the map

$$s \longrightarrow \widehat{u}_p(s;\psi)$$

takes values in $\mathcal{C}^\infty(\mathbb{R}^{n-k(p)})$ and has a meromorphic continuation to $\mathbb{C}^{k(p)}$. The poles of $\widehat{u}_p$ occur along linear subspaces of the form in (B.2). These conditions are easily seen to be independent of the choice of local coordinates or cut off function ψ and characterize polyhomogeneous conormal distributions on X.

Theorem B.1. *Let X be a manifold with corners and $u \in \mathcal{C}^{-\infty}(X)$ then $u \in \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X)$ if and only if the local Mellin transforms of u, $\widehat{u}_p(s;\psi)$ take values in $\mathcal{C}^\infty(\mathbb{R}^{n-k(p)})$ and continue meromorphically to $\mathbb{C}^{k(p)}$ with polar varieties of the form in* (B.2) *for all $p \in \partial X$ and admissible ψ.*

The converse statement in the theorem is proved by using the Mellin inversion formula to prove that u has the required asymptotic expansions. From this it is clear that one only needs to check the local Mellin transforms for a finite number of points $p \in \partial X$ and corresponding cut off functions.

B.3. p-submanifolds

Some of the operators in which we are interested have kernels which are singular along a submanifold other than a boundary face. The singularities are still conormal and can be reduced by blow-up to boundary singularities of the type just discussed, but this destroys some of the structure. We therefore treat conormal distributions at p-submanifolds directly. For the proofs of certain composition formulæ we need to discuss some cases of conormal distributions at two p-submanifolds which meet cleanly (i.e. form a clean p-submanifold).

To begin with we very briefly recall the theory of conormal distributions at an embedded submanifold in a manifold, X, without boundary, see [**?**];§18.2, [**9**]. If $\mathcal{V}(X;Y)$ is the space of $\mathcal{C}^\infty$ vector fields on X tangent to Y then the space of conormal distributions at Y is fixed by iterative regularity, as in (B.1):

$$I^*(X,Y) = \left\{u \in \mathcal{C}^{-\infty}(X); \exists\ m = m(u) \text{ with } \mathcal{V}^k(X;Y)u \subset H^m(X)\ \forall\ k\right\}.[B2.1]$$

The use of a Sobolev space here is not very significant. Indeed if one wishes to capture the order of the conormal distribution precisely in such an iterative definition one should use Besov spaces (see in particular [**?**]). If $x_1,\dots,x_n = (x',x'')$, with $x' = (x_1,\dots,x_{n-p})$, $x'' = (x_{n-p+1},\dots,x_n)$, are local coordinates in which

$$Y = \{x_{n-p+1} = \cdots = x_n = 0\}\,[B2.2]$$

then the convention for orders is:

$$\begin{gathered} u \in I^m(X,Y) \Longleftrightarrow u(x) = (2\pi)^{-p}\int_{\mathbb{R}^p} e^{ix''\cdot\xi''}a(x',\xi'')d\xi'' \\ \text{with } a\left(x', \frac{\eta}{(1-|\eta|^2)}\right) \in S^{m+\frac{n}{4}-\frac{p}{2}}(\mathbb{R}^{n-p}\times\mathbb{B}^p_\eta). \end{gathered} \quad [B2.3]$$

The symbol space $S^m(X)$ on a manifold with corners (in this case with boundary) is defined in (B.1). If the symbol is a polyhomogeneous conormal distribution, i.e.

$$a\left(x, \frac{\eta}{1-|\eta|^2}\right) \in S^{m+\frac{n}{4}-\frac{p}{2}}_{\text{phg}}(\mathbb{R}^{n-p}\times\mathbb{B}^p_\eta), [B2.3']$$

then we say that $u \in I^m_{\text{phg}}(X;Y)$. As with conormal distributions at the boundary the leading part of the symbol a in (B.3) has simpler transformation properties than the lower order terms. Consider the space $I^m(X,Y;\Omega^{\frac{1}{2}})$ of conormal sections of the half-density bundle; this still reduces to (B.3) in local coordinates. Let X^*_Y

denote the compactification, by stereographic projection, of the conormal bundle $N^*\{X;Y\}$ to Y in X. Then the principal symbol is invariantly defined as an element of the residue space, giving the symbol exact sequence:

$$0 \hookrightarrow I^{m-1}(X,Y;\Omega^{\frac{1}{2}}) \hookrightarrow I^m(X,Y;\Omega^{\frac{1}{2}}) \xrightarrow{\sigma_m} S^{m+\frac{n}{4}}(X^*_Y;\Omega_b^{\frac{1}{2}})/S^{m+\frac{n}{4}+1}(X^*_Y;\Omega_b^{\frac{1}{2}}) \longrightarrow 0.[B2.4]$$

The change in order comes from the homogeneity of the half-density factor.

If we set $\rho_Y = (x^2_{n-p+1} + \cdots + x^2_n)^{\frac{1}{2}}$ then $u \in I^m_{\mathrm{phg}}(X,Y;\Omega^{\frac{1}{2}})$ has an asymptotic expansion in terms of the powers $\rho^j_Y \log \rho^k_Y$ with coefficients that lift to smooth functions on the spherical conormal bundle of Y. If one sums the formal asymptotic expansion one obtains an element $\widetilde{u} \in I^m_{\mathrm{phg}}(X,Y;\Omega^{\frac{1}{2}})$ such that

$$u - \widetilde{u} \in \mathcal{C}^\infty(X).$$

In other words the residual space for distributions conormal to a submanifold consists of smooth functions in a neighborhood of the submanifold not functions which vanish to infinite order as in the case of boundary hypersurfaces considered above.

We wish to extend the definition and the symbol map to polyhomogeneous conormal distributions singular at a p-submanifold of a compact manifold with corners. By definition such a submanifold is a (by default closed) subset $Y \subset X$ with the property that each point of it has a neighbourhood with a product decomposition, $[0,1)^k \times \mathbb{R}^{n-k}$, which the submanifold meets in a partial product, $[0,1)^{k-d'} \times \{0\} \times \mathbb{R}^{n-k-d''}$. It is a boundary p-submanifold, as for example B_a in (7.2), if $d' > 0$, otherwise it is an interior p-submanifold. Thus a p-submanifold has the property that in some local coordinates, in the precise sense described above, it is given by

$$Y = \{x_j = 0; k - d' + 1 \le j \le k, n - d'' + 1 \le j \le n\}.[B2.5]$$

Here $p = d' + d''$ is the codimension of Y in X. Clearly Y is itself a manifold with corners.

In the case of an interior submanifold the extension is straightforward. The global definition is obtained simply by replacing $\mathcal{V}(X;Y)$ in (B.3) with $\mathcal{V}_b(X;Y)$, consisting of the vector fields tangent to both Y and to the boundary hypersurfaces of X :

$$I^*(X,Y) = \left\{u \in \mathcal{C}^{-\infty}(X); \exists\ m = m(u) \text{ with } \mathcal{V}^k_b(X;Y)u \subset \rho^{-m} H^m_b(X)\ \forall\ k\right\}.[B2.6]$$

We use the weighted Sobolev spaces here only because they have already been defined above. The local form for Y, (B.3), is essentially the same as (B.3) so the 'boundary variables' $x_1, \dots, x_k$ are in essence parameters. Let $\mathcal{E}$ be an index family for X and suppose $m \in \mathbb{R}$. Then we define the space of polyhomogeneous conormal distributions of order m at Y and polyhomogeneous at the boundary with index family $\mathcal{E}$ by localization. First we require

$$\phi u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X) \text{ if } \operatorname{supp}\phi \cap Y = \emptyset.[B2.7]$$

Suppose ϕ has support in a coordinate neighbourhood in which (B.3) holds. Then ϕu has a representation (B.3) where a is obtained by partial Fourier transformation in the variable x''. As before we can compactify the dual space and demand

$$a\left(x', \frac{\eta}{(1-|\eta|^2)}\right) \in \mathcal{A}^{\mathcal{E},-(m+\frac{n}{4}-\frac{p}{2})}_{\mathrm{phg}}(\mathbb{R}^{n-p}_k \times \mathbb{B}^p_\eta).[B2.8]$$

This is consistent with (B.3) if $k = 0$. There is a partition of unity on X where each element ϕ has support either disjoint from Y or in a coordinate neighbourhood in which (B.3) holds. Then we say $u \in I^{m,\mathcal{E}}_{\mathrm{phg}}(X,Y)$ if the appropriate one of (B.3) or (B.3) always holds. Of course it is important to show that this definition is independent of choices, see [**12**].

As in the boundaryless case there is a well-defined principal symbol at Y. Let X^*_Y be the conormal bundle to Y compactified by stereographic projection, as in (B.3). The symbol map at Y, defined for simplicity of notation on b-half-density sections is

$$\sigma_Y : I^{m,\mathcal{E}}_{\mathrm{phg}}(X,Y;\Omega^{\frac{1}{2}}_b) \twoheadrightarrow \mathcal{A}^{\mathcal{E}^Y,-m-\frac{n}{4}}_{\mathrm{phg}}(X^*_Y,\Omega^{\frac{1}{2}}_b)/\mathcal{A}^{\mathcal{E}^Y,-m-\frac{n}{4}+1}_{\mathrm{phg}}(X^*_Y,\Omega^{\frac{1}{2}}_b).[B2.9]$$

Here the boundary hypersurfaces of X^*_Y, other than that corresponding to ∞ on the fibres of $N^*\{X;Y\}$, arise from boundary hypersurfaces of X (namely those meeting Y) and $\mathcal{E}^Y$ is the index set obtained from $\mathcal{E}$ by this association. As for the spaces discussed in §B2 there are also symbol maps at each boundary hypersurface. For boundary hypersurfaces not meeting Y these are exactly as in §B2, since $I^{m,\mathcal{E}}_{\mathrm{phg}}(X,Y;\Omega^{\frac{1}{2}}_b) = \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X;\Omega^{\frac{1}{2}}_b)$ locally. At $H \in M_1(X)$ with $H \cap Y \neq \emptyset$ the symbol map becomes

$$\sigma_H(u) = \sum_{\{(z,j)\in E_H;(-\operatorname{Im} z,j)=\mathfrak{e}(H)\}} a(z,j)(d\rho_H)^i z(\log(d\rho_H))^j,\ a(z,j) \in I^{m,\mathcal{E}^H}_{\mathrm{phg}}(H, Y\cap H).[B2.10]$$

This discussion extends to define a symbol at each boundary face and again there are consistency conditions between the various symbols corresponding to boundary faces which meet, or to boundary faces and Y if they intersect. If $H,G \in M(X)$ and $H \cap G \neq \emptyset$ but $H \cap G \cap Y = \emptyset$ then the consistency condition just reduces to (B.2). In fact there is a similar condition in all cases; see [**12**] for the details. These consistency conditions are the only restrictions on the values of the symbol maps. We also remark that asymptotic completeness extends directly to these spaces too. If the sequence corresponds to decreasing order at Y, i.e. $m(j) \longrightarrow -\infty$ then summation corresponds to increasing smoothness across Y, not to vanishing at Y. Thus

$$\bigcap_m I^{m,\mathcal{E}}_{\mathrm{phg}}(X,Y) = \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X).[B2.11]$$

The foregoing constructions are almost as simple for general conormal distributions. We also need to consider boundary p-submanifolds and pairs of submanifolds meeting in various ways. The definition of $I^*(X,Y)$ given in (B.3) generalizes immediately to these situations, however the definition of order is rather complicated in the general case. For polyhomogeneous conormal distributions the simple definition given in (B.2) also generalizes easily.

For boundary p-submanifolds we can use precisely the same definition, (B.3). Next consider two interior p-submanifolds, Y_1 and Y_2, which intersect transversally in an interior p-submanifold

$$Y = Y_1 \pitchfork Y_2.[B2.14]$$

In this case (see [**12**]) near each point of Y there are local coordinates in which

$$\begin{gathered} Y_1 = \{x_{n-p_1+1} = \cdots = x_n = 0\} \\ Y_2 = \{x_{k+1} = \cdots = x_{k+p_2} = 0\} \\ Y = \{x_{k+1} = \cdots = x_{k+p_2} = x_{n-p_2+1} = \cdots = x_n = 0\}, \\ p = p_1 + p_2 \leq n - k. \end{gathered} [B2.15]$$

Notice that we *assume* here that Y is an interior p-submanifold. Then we define $I^*(X, Y_1 \cup Y_2)$ by (B.3) with $\mathcal{V}_b(X;Y)$ replaced by

$$\mathcal{V}_b(X;Y_1 \cup Y_2) = \mathcal{V}_b(X;Y_1) \cap \mathcal{V}_b(X;Y_2).[B2.16]$$

The other case of intersecting p-submanifolds which we shall consider here correspond to an interior p-submanifold, Y_1, meeting a boundary p-submanifold, Y_2. Let $G \in M_{(1)}(X)$ be the smallest boundary face containing Y_2. We require the intersection, Y, be an embedded submanifold and that

$$Y = Y_1 \cap Y_2 = Y \cap G.[B2.26]$$

Certainly Y is then a boundary p-submanifold. Near each point of Y there are local coordinates in terms of which

$$\begin{aligned} Y_1 &= \{x_{n-p_1+1} = \cdots = x_n = 0\} \\ Y_2 &= \{x_{k-d'+1} = \cdots = x_k = 0 = x_{n-d''+1} = \cdots = x_n\} \\ G &= \{x_{k-d'+1} = \cdots = x_k = 0\} \qquad [B2.27] \\ Y &= \{x_{k-d'+1} = \cdots = x_k = 0 = x_{n-p_1+1} = \cdots = x_n\} \\ p_2 &= d' + d'',\ p_1 \geq d''. \end{aligned}$$

Even though the intersection of Y_1 and Y_2 is not transversal it is clean in the sense of Bott. The definition (B.3) is still appropriate if $\mathcal{V}_b(X, Y)$ is replaced by

$$\mathcal{V}_b(X, Y_1 \cup Y_2) = \mathcal{V}_b(X, Y_1) \cap \mathcal{V}_b(X, Y_2).[B2.28]$$

In each of these cases we can define subspaces of polyhomogeneous conormal distributions. If Y is a p-submanifold it always has a radial vector field ${}^b\nu_Y \in \mathcal{V}_b(X;Y)$. That is, ${}^b\nu_Y$ vanishes at Y and furthermore

$${}^b\nu_Y f = f + \mathcal{I}_Y^2 \quad \forall\ f \in \mathcal{I}_Y [B3.12]$$

where $\mathcal{I}_Y \subset \mathcal{C}^\infty(X)$ is the ideal of functions vanishing at Y. Another way of expressing (B.3) is to say that the linear transformation induced on N^*Y by ${}^b\nu_Y$ is the identity. Any two radial vector fields differ by an element of $\mathcal{I}_Y \cdot \mathcal{V}_b(X;Y)$. If $\mathcal{E}$ is an index family for the manifold X and m is an order for Y then we can give a definition of polyhomogeneity quite analogous to (B.2). Set $\mathcal{F} = (m, \mathcal{E})$ and put

$$\begin{aligned} P_{N,\mathcal{F},Y} &= P_{N,\mathcal{E}} \cdot P_{N,m,Y} \\ P_{N,m,Y} &= \prod_{\{j \in \mathbb{N}_0; j \leq d''+m+N\}} ({}^b\nu_Y + m + \frac{n}{4} - \frac{p}{2} + d'' - j)[B3.13] \end{aligned}$$

where $P_{N,\mathcal{E}}$ is given by (B.2), p is the codimension of Y and d'' its boundary codimension (meaning $p - k$ if Y is the smallest boundary face containing Y has codimension k). Then define

$$I^{\mathcal{F}}_{\text{phg}}(X, Y) = \left\{ u \in I^*(X, Y); \mathcal{V}_b^k(X;Y) P_{N,\mathcal{F},Y} u \in \rho^N H^{-m(u,N)}(X)\ \forall\ N \text{ and } k \in \mathbb{N}_0 \right\}, [B3.14]$$

here $m(u, N)$ is a sequence of real numbers tending to infinity as N tends to infinity, and

$$\rho^N = \prod_{H \in M_1(X)} \rho_H^N.$$

One can show directly that this definition is independent of choices. Again one can give equivalent definitions in terms of expansions.

For the cases of two intersecting p-submanifolds considered in §B1 we can still use a similar definition. It is particularly convenient in these cases to choose the radial vector fields carefully, so that

$$^b\nu_Y \text{ is tangent to both } Y_1 \text{ and } Y_2 [B3.16]$$

and similarly so that

$$^b\nu_{Y_i} \text{ is tangent to } Y_{i'},\ i \neq i',\ i, i' = 1, 2. [B3.17]$$

That this is possible can be seen from the local form (B.3) or (B.3). Suppose that m_1, m_2 and m are index sets for the submanifolds Y_1, Y_2 and $Y = Y_1 \cap Y_2$. Then we can use the second part of (B.3) to define a 'radial operator' for each of the three submanifolds and set

$$P_N = P_{N,\mathcal{E}} \cdot P_{N,m_1,Y_1} \cdot P_{N,m_2,Y_2} \cdot P_{N,m,Y}$$
$$I^{m_1,m_2,m,\mathcal{E}}_{\text{phg}}(X, Y) = \left\{ u \in I^*(X, Y); \mathcal{V}^k_b(X; Y_1 \cup Y_2) P_N u \in \rho^N H^{-m(u,N)}(X)\ \forall\ N \text{ and } k \in \mathbb{N}_0 \right\}, [B3.18]$$

with ρ^N and $m(u, N)$ as above.

For Y_1, Y_2 transversal interior p-submanifolds there are symbol maps at each of the boundary faces and Y_1, Y_2 and at Y. The symbols at $Y_i \setminus Y$, $i = 1, 2$ are computed as above for a single submanifold. The precise definition of the symbol at Y can be found in [**12**], it satisfies:

$$\begin{aligned} \sigma_Y(u) \in \mathcal{A}^{[-m-\frac{n}{4}],\mathcal{E},M_1,M_2}_{\text{phg}}(X^*_Y, Z; \Omega^{\frac12}_b) \text{ if } u \in I^{m_1,m_2,m,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2; \Omega^{\frac12}_b) \\ \sigma_Y(u) = 0 \Longleftrightarrow u \in I^{m_1,m_2,m-1,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2; \Omega^{\frac12}_b), \end{aligned} [B2.23]$$

where

$$M_1 = -m_1 - \frac{n}{4} - \frac{p_1}{2} - p_2,\ M_2 = -m_2 - \frac{n}{4} - p_1 - \frac{p_2}{2}. [B2.20]$$

Here Z is a submanifold of the boundary component of X^*_Y introduced by stereographic projection on the fiber. It corresponds to the union of the 'fiber boundaries' of the inclusions of N^*Y_1 and N^*Y_2 into N^*Y.

There is an important multiplicative property:

$$\begin{aligned} I^{m_1,\mathcal{E}_1}_{\text{phg}}(X, Y_1) \times I^{m_2,\mathcal{E}_2}_{\text{phg}}(X, Y_2) \ni (u_1, u_2) \longmapsto \\ u_1 u_2 \in I^{m_1,m_2,m_1+m_2+\frac{n}{4},\mathcal{E}_1+\mathcal{E}_2}_{\text{phg}}(X, Y_1 \cup Y_2) \end{aligned} [B2.24]$$

and the symbol map behaves well:

$$\sigma_Y(u_1 u_2) = \sigma_{Y_1}(u_1)_{\restriction Y} \cdot \sigma_{Y_2}(u_2)_{\restriction Y}. [B2.25]$$

Of course (B.3) needs a little interpretation which can be found in [**12**].

These notations are consistent in the sense that

$$\begin{aligned} I^{m,\mathcal{E}}_{\text{phg}}(X, Y) \subset I^{m_1,m_2,m,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2), \\ I^{m,\mathcal{E}}_{\text{phg}}(X, Y_i) \subset I^{m_1,m_2,m,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2), i = 1, 2 \end{aligned} [B2.21]$$

and in fact

$$\begin{aligned} I^{m_1,\mathcal{E}}_{\text{phg}}(X, Y_1) = I^{m_1,-\infty,-\infty,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2) \\ I^{m_2,\mathcal{E}}_{\text{phg}}(X, Y_2) = I^{-\infty,m_2,-\infty,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2) [B2.34] \\ I^{m,\mathcal{E}}_{\text{phg}}(X, Y) = I^{-\infty,-\infty,m,\mathcal{E}}_{\text{phg}}(X, Y_1 \cup Y_2). \end{aligned}$$

B.4. Pull-back

If X, X' are compact manifolds with corners and $H_j = \{\rho_j = 0\}$, for $j = 1, \dots, N$ $H'_j = \{\rho'_j = 0\}$ for $j = 1, \dots, N'$ are their respective boundary hypersurfaces then a $\mathcal{C}^\infty$ map $f : X \longrightarrow X'$ is a b-map if

$$f^*\rho'_j = a_j \prod_{k=1}^{N} \rho_k^{e(j,k)},\ j = 1, \dots, N',\ 0 < a_j \in \mathcal{C}^\infty(X).[B4.1]$$

Geometrically this means that the inverse image of a boundary hypersurface is a union of boundary hypersurfaces, e.g. the map from $[-1,1]$ to $[0,1]$ given by $x \longrightarrow x^2$ is not a b-map. The numbers $e(j,k)$ are necessarily non-negative integers. With a b-map there is an associated map of index families. Let $\mathcal{E}' = \{E'_1, \dots, E'_{N'}\}$ be an index family for X', i.e. E'_i is the index set associated to $H_i \in M_1(X)$. Then set

$$\begin{aligned} f^\flat(\mathcal{E}') = \mathcal{E} = \{E_1, \dots, E_N\} \text{ where} \\ E_j = \big\{(z,k) \in \mathbb{C} \times \mathbb{N}_0; \exists\ (z_i, k_i) \in E'_i\ \forall\ i \text{ s.t.} \\ e(i,j) \neq 0 \text{ with } z = \sum_{e(i,j)\neq 0} e(i,j) z_i,\ k = \sum_{e(i,j)\neq 0} k_i\big\}. \end{aligned} \quad [B4.2]$$

Proposition B.1. *Let $f : X \longrightarrow X'$ be a b-map between compact manifolds with corners then for any index family $\mathcal{E}'$ on X' the pull-back of $\mathcal{C}^\infty$ functions in the interior restricts to give a linear map:*

$$F^* : \mathcal{A}^{\mathcal{E}'}_{\text{phg}}(X') \longrightarrow \mathcal{A}^{f^\flat \mathcal{E}'}_{\text{phg}}(X).[B4.4]$$

We also need to consider the pull-back of distributions conormal to a p-submanifold. In this case we add an extra condition so that the pull-back is well-defined. Any b-map induces a linear map of the b-tangent spaces

$$^b f_* : {}^bT_pX \longrightarrow {}^bT_{f(p)}X' \quad \forall\ p \in X[B4.5]$$

which is called the b-differential. Now if Y' is an interior p-submanifold of X' we shall require the transversality condition:

$$\text{range}_x({}^b f_*) + {}^bT_{y'}Y' = {}^bT_{y'}X' \quad \forall\ y' \in Y' \text{ and } x \in X \text{ with } f(x) = y'.[B4.6]$$

Proposition B.2. *If $Y' \subset X'$ is an interior p-submanifold and $f : X \longrightarrow X'$ is a b-map satisfying* (B.4) *then $Y = f^{-1}Y'$ is a p-submanifold and by continuous extension from $\dot{\mathcal{C}}^\infty(X')$,*

$$f^* : I^{m',\mathcal{E}'}_{\text{phg}}(X',Y') \longrightarrow I^{m,\mathcal{E}}_{\text{phg}}(X,Y),\ m = m' + \frac{1}{4}(\dim X' - \dim X),\ \mathcal{E} = f^\flat \mathcal{E}'.[B4.8]$$

B.5. b-fibrations

The notion of a b-fibration is fundamental for the push-forward operation and hence in the proof of composition formulæ. We recall the definition, from [**12**], and give a useful procedure for verifying that a map satisfies these conditions.

We shall say that a b-map, f, as in (B.4) is a b-normal map if:

$$\text{For each } k \in \{1, \dots, N,\}\ e(j,k) \neq 0 \text{ for at most one } j \in \{1, \dots, N'\}.[B5.1]$$

Geometrically, the image of a codimension one boundary component is never contained in a codimension two boundary component. This condition is important in ensuring that polyhomogeneity is preserved under push-forwards. The b-map is a b-submersion if (B.4) is surjective for every p.

At each point $p \in X$ the b-tangent space has a well-defined subspace, ${}^bN_pX \subset {}^bT_pX$, the b-normal space to the boundary face of smallest dimension containing p. Of course this is just zero in the interior and in general has dimension exactly the codimension of this smallest boundary face containing p. A b-submersion is said to be a b-fibration if in addition

$$ {}^bf_* : {}^bN_pX \twoheadrightarrow {}^bN_{f(p)}X' \quad \forall\ p \in X.[B5.2] $$

The following two results are taken from [**12**].

PROPOSITION B.3. *The b-differential of a b-map restricts to a surjective map* (B.5) *if and only if the map is b-normal, i.e.* (B.5) *holds.*

COROLLARY B.1. *A b-submersion is a b-fibration if and only if* (B.5) *holds.*

We shall make frequent use of the following, closely related, criterion for a b-fibration.

PROPOSITION B.4. *A b-normal map $f : X \longrightarrow X'$ between compact manifolds with corners is a b-fibration if and only if for every $p \in \partial X$ the ordinary differential is surjective as a map*

$$ f_* : T_pF_p \longrightarrow T_{f(p)}F'_{f(p)}[B5.6] $$

where $F_p \in M(X)$ and $F'_{f(p)} \in M(X')$ are the smallest boundary faces containing the points p and $f(p)$ respectively.

PROOF. Since f is a b-map one certainly has $f(F_p) \subset F'_{f(p)}$ so (B.4) is a well-defined map. At every point $p \in X$ the tangent space T_pF_p can be canonically identified with the quotient

$$ T_pF_p \equiv {}^bT_pX/{}^bN_pX.[B5.7] $$

Thus the surjectivity of (B.4) is equivalent to the surjectivity of the quotient of the b-differential

$$ {}^bf_* : {}^bT_pX/{}^bN_pX \longrightarrow {}^bT_{f(p)}X'/{}^bN_{f(p)}X'.[B5.8] $$

Proposition B.3 shows that ${}^bf_*$ is surjective as a map (B.5) on the b-normal spaces, so f is a b-submersion if and only if (B.4)is surjective at every point. □

B.6. Push-forward

For a $\mathcal{C}^\infty$ map between compact manifolds without boundary, $f : X \longrightarrow X'$, the push-forward of distributional densities is always well-defined as the dual to the pull-back operator on $\mathcal{C}^\infty$ functions:

$$ f_* : \mathcal{C}^{-\infty}(X;\Omega) \longrightarrow \mathcal{C}^{-\infty}(X';\Omega).[B6.1] $$

On compact manifolds with corners some integrability conditions is generally required.

Restricting attention to b-fibrations, $f : X \longrightarrow X'$, consider the subset

$$ P_f = \left\{H \in M_1(X); H \not\subseteq f^{-1}(H') \text{ for any } H' \in M_1(X')\right\}.[B6.2] $$

For any of the conormal spaces defined above the order at H is well-defined, we impose the condition:

$$ u \in I^*(X, \mathcal{Y}; \Omega_b) \text{ has order } \mathfrak{m} > 0 \text{ at each } H \in P_f.[B6.3] $$

Here $\mathcal{Y}$ could be empty, a p-submanifold or two intersecting p-submanifolds as in §B1.

For b-fibrations we define an operation on index sets, and hence index families. Namely if $\mathcal{E}$ is an index family for X then

$$\mathcal{F} = f_{\#}\mathcal{E} = \{F_1, \dots, F_{N'}\} \, [B6.4]$$

is the index family for X' defined by

$$F_\ell = \big\{(z,j) \in \mathbb{C} \times \mathbb{N}_0; \exists \, (z_r, j_r) \in F_r \text{ for those } r \text{ s.t. } e(\ell, r) \neq 0, \\ z = \sum_r e(\ell, r) z_r, \ j+1 = \sum_r (j_r + 1)\big\}. \qquad [B6.5]$$

THEOREM B.2. *Suppose $f : X \longrightarrow X'$ is a b-fibration of compact manifolds with corners and $\mathcal{E}$ is an index family for X such that*

$$\mathfrak{e}(H) > 0 \ \forall \ H \in P_f, [B6.7]$$

where $\mathfrak{e}$ is the power set defined by $\mathcal{E}$, then push-forward defines a linear map

$$f_* : \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X; \Omega_b) \longrightarrow \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X'; \Omega_b), \ \mathcal{F} = f_{\#}\mathcal{E}.[B6.8]$$

PROOF. Sketch of Proof This result is proved by using the characterization of polyhomogeneous conormal distributions in terms of their local Mellin transforms, Theorem B.1. The formula

$$< x^{-(is-1)}\phi, \psi f_* u > = < f^*(x^{-(is-1)}\phi\psi), u > .[B6.81]$$

implies that local Mellin transforms of f_*u are expressible in terms of the local Mellin transforms of u. Applying the b-fibration condition allows one to deduce the structure of the meromorphic continuation of $\widehat{f_* u}$ from that of $\widehat{u}$. □

If $Y \subset X$ is an interior p-submanifold we say that a b-fibration f is b-transversal to Y if its restriction to $Y$is also a b-fibration onto X'; this is just the same as requiring it to be transversal to Y, i.e.

$$\mathrm{null}(f_*) + T_y Y = T_y X \quad \forall \ y \in Y.[B6.9]$$

If Y is a boundary p-submanifold and $g \in M(X)$ is the smallest boundary face containing Y then f is b-transversal to Y if it is a b-fibration from Y to $f(G)$ which is necessarily a boundary face of X'.

THEOREM B.3. *Under the hypotheses of Theorem B.2 suppose additionally that Y is an interior p-submanifold to which f is transversal, then*

$$f_* : I^{m,\mathcal{E}}_{\mathrm{phg}}(X, Y; \Omega_b) \longrightarrow \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X'; \Omega_b), \ \mathcal{F} = f_{\#}\mathcal{E}.[B6.11]$$

In fact we need a more complicated case where the singularities on submanifolds are erased by push-forward.

THEOREM B.4. *Under the hypotheses of Theorem B.2 suppose that Y_1 is an interior p-submanifold, Y_2 is a boundary p-submanifold satisfying* (B.3) *and such that f is b-transversal to Y_1, Y_2 and Y then*

$$f_* : I^{m_1, m_2, m, \mathcal{E}}_{\mathrm{phg}}(X, Y_1 \cup Y_2; \Omega_b) \longrightarrow \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X'; \Omega_b), \ \mathcal{F} = f_{\#}\mathcal{E}.[B6.13]$$

Suppose that instead of being b-transversal f is actually a fibration in a neighbourhood of an interior p-submanifold Y and embeds Y as an interior p-submanifold $Y' \subset X'$. Then the differential of f fixes an embedding of $N^*Y' = N^*\{X'; Y'\}$ as a subbundle of $N^*Y = N^*\{X; Y\}$

$$f^* : N^*Y' \longrightarrow N^*Y. [B6.14]$$

THEOREM B.5. *Under the hypotheses of Theorem B.2 suppose that Y_1 and Y_2 are interior p-submanifolds intersecting transversally in an interior p-submanifold Y and that f is transversal to Y_1 and Y_2, is a fibration in a neighbourhood of Y and embeds it as an interior p-submanifold $Y' \in X'$ then*

$$\begin{gathered} f_* : I^{m_1,m_2,m,\mathcal{E}}_{\mathrm{phg}}(X, Y_1 \cup Y_2; \Omega_b) \longrightarrow I^{m',\mathcal{F}}_{\mathrm{phg}}(X', Y'; \Omega_b), \\ \mathcal{F} = f_\# \mathcal{E},\ m' = m - \frac{1}{4}(\dim X - \dim X') \end{gathered} \quad [B6.16]$$

and

$$\sigma_{Y'}(f_* u) = (f^*)^* \cdot \sigma_Y(u). [B6.17]$$

APPENDIX C

Parabolic blow-up

C.1. Normal blow-up

The tangent bundle to a manifold with corners TX just consists of the derivations of the space $\mathcal{C}^\infty(X)$; i.e. for each $p \in X$, T_pX consists of the linear maps

$$\begin{gathered} v : \mathcal{C}^\infty(X) \longrightarrow \mathbb{R} \text{ s.t.} \\ v_p(fg) = f(p)v_p(g) + g(p)v_p(f) \quad \forall\ f, g \in \mathcal{C}^\infty(X). \end{gathered} [C1.1]$$

It is a vector bundle over X. If $B \subset X$ is a p-submanifold the tangent bundle to B is naturally identified with a subbundle of TX restricted to B :

$$TB \subset T_BX = TX \restriction_B .[C1.2]$$

The quotient bundle is the normal bundle to B :

$$NB = T_BX/TB.[C1.3]$$

The inward-pointing part, ${}_+T_pX$ of T_pX consists of those $v \in T_pX$ which are non-negative on non-negative functions which vanish at p; if p is in the interior then ${}_+T_pX = T_pX$; in general it is linearly isomorphic to $\mathbb{R}^n_k$, with k the number of boundary hypersurfaces containing p (i.e. the maximal codimension of a boundary face containing p.) The inward-pointing part of NB, ${}_+NB$, is a manifold with corners which can be used as a local model for X near B.

Consider the space of all vector fields on X which are tangent both to the boundary and to a given p-submanifold B :

$$\mathcal{V}_b(X;B) = \{V \in \mathcal{V}_b(X); V \text{ is tangent to } B\}.[C1.4]$$

Since NB is a vector bundle over B the space $\mathcal{C}^\infty(B)$ can be considered as the space of fibre-constant functions on ${}_+NB$. Moreover the sections of the dual bundle N^*B define all the $\mathcal{C}^\infty$ functions on ${}_+NB$ which are linear on the fibres; together they fix the $\mathcal{C}^\infty$ structure of ${}_+NB$, i.e. give local coordinates everywhere. Notice that there is a natural identification

$$\mathcal{C}^\infty(B; N^*B) = \mathcal{I}_B/\mathcal{I}_B^2, [C1.5]$$

where $\mathcal{I}_B \subset \mathcal{C}^\infty(X)$ is the ideal of functions vanishing on B. Thus if $V \in \mathcal{V}_b(X;B)$ it defines linear maps

$$L_V : \mathcal{C}^\infty(B) \longrightarrow \mathcal{C}^\infty(B),\ L_V : \mathcal{C}^\infty(B; N^*B) \longrightarrow \mathcal{C}^\infty(B; N^*B); [C1.6]$$

the first by restriction to B and the second by action on $\mathcal{I}_B$. The maps are consistent, in the sense that

$$L_V[fg] = gL_V(f) + fL_V(g),\ f \in \mathcal{C}^\infty(B),\ g \in \mathcal{C}^\infty(B) \text{ or } \mathcal{C}^\infty(B; N^*B).[C1.7]$$

Thus it extends uniquely to a $\mathcal{C}^\infty$ vector field, L_V on NB; clearly L_V is tangent to all boundaries of ${}_+NB$. The following result is proved in [**12**].

PROPOSITION C.1. *The 'linearization' map*

$$L : \mathcal{V}_b(X;B) \longrightarrow \mathcal{C}^\infty({}_+NB;{}^bTNB)[C1.9]$$

has range consisting of all the $\mathcal{C}^\infty$ vector fields tangent to the boundary faces of ${}_+NB$ and invariant under the $\mathbb{R}^+$-action on the fibres, its null space is $\mathcal{I}_B\mathcal{V}_b(X;B)$.

Since the vector fields L_V are $\mathbb{R}^+$-invariant they project to $\mathcal{C}^\infty$ vector fields on the inward-pointing spherical normal bundle to B :

$$L_V \in \mathcal{C}^\infty(S_+NB;{}^bTS_+NB),\ S_+NB = {}_+NB \setminus 0/\mathbb{R}^+.[C1.10]$$

The blow-up of X along B :

$$[X;B] = (X)_B[C1.11]$$

is a well-defined manifold with corners. The new notation, on the left, is helpful in simplifying the discussion of iterated blow-up. As an abstract set the blown-up space is the disjoint union of the complement of B in X and the inward-pointing part of the spherical normal bundle to B in X :

$$[X;B] = [X \setminus B] \sqcup [S_+NB\smallsetminus 0/\mathbb{R}^+].[C1.12]$$

The second part, $\operatorname{ff}[X;B] = [S_+NB\smallsetminus 0/\mathbb{R}^+]$, is called the front face. There is a natural 'blow-down' map,

$$\beta = \beta\,[X;B] : [X;B] \longrightarrow X[C1.13]$$

which is the identity on the first term in (C.1) and projection onto B from the second. In fact $[X;B]$ has a unique (hence natural) $\mathcal{C}^\infty$ structure, given by polar coordinates, which is such that $\beta\,[X;B]$ is $\mathcal{C}^\infty$, has rank equal to $\dim B$ over B and under it every element $V \in \mathcal{V}_b(X;B)$ lifts to an element of $\mathcal{V}_b([X;B])$ which restricts to the front face to give L_V in (C.1).

C.2. Parabolic blow-up defined

The notion of the parabolic blow up of a manifold with corners along a submanifold was introduced in [**5**] to analyze the boundary behaviour of the Bergman Laplacian. Here we give a slightly more abstract definition and also describe the invariance properties a little further (although all proofs are to be found in [**5**]). To streamline the handling of iterated blow-ups in §C4 we introduce a slightly different notation analogous to (C.1).

Since the inward-pointing part of the normal bundle to a p-submanifold of a manifold with corners, ${}_+N^*B$, is itself a manifold with corners the notion of a p-submanifold of ${}_+N^*B$ is well-defined. We say that a subset $S \subset {}_+N^*B$ is a p-subbundle if it is a p-submanifold and is the intersection of ${}_+N^*B$ with a subbundle of N^*B. For a p-submanifold B and a p-subbundle $S \subset {}_+N^*B$ we shall define a new manifold with corners which is X blown up S-parabolically along B :

$$[X;B,S] = (X)_{B,S}\,,\ \beta = \beta\,[X;B,S] : [X;B,S] \longrightarrow X[C2.1]$$

where the new notation on the left, $[X;B,S]$, differs from that on the right which was used in [**5**].

To define (C.2) we identify the space of (inward-pointing) S-parabolic normal vectors to B. Let $\mathcal{I}_B \subset \mathcal{C}^\infty(X)$ be the ideal of functions vanishing on B and consider the smaller ideal of those functions satisfying

$$\mathcal{I}_{B,S} = \{f \in \mathcal{I}_B;\ df \restriction B \in \mathcal{C}^\infty(B;S)\};[C2.2]$$

just the functions with differentials in S. For $p \in B$ consider the S-parabolic curves in X through p, *viz* those having the properties

$$\chi : [0,1] \longrightarrow X,\ \chi(0) = p, [C2.3] \tag{C.1}$$

$$\chi^* f(t) = O(t^2) \text{ as } t \downarrow 0 \quad \forall\ f \in \mathcal{I}_{B,S}.[C2.4] \tag{C.2}$$

Note that the first condition implies that

$$\chi^* f(t) = O(t)\ \forall\ f \in \mathcal{I}_B.$$

The second condition just means that the image of the tangent vector at 0, $[d\chi(0)/dt] \in S_p^\circ \subset {}_+NB$, the annihilator of S_p. Denote by ${}_+N_p\{X;B,S\}$ the quotient space obtained by identifying curves under the equivalence relation

$$\begin{aligned}\chi_1 \sim_p \chi_2 \Longleftrightarrow [\chi_1^* - \chi_2^*]f = O(t^2) \quad \forall\ f \in \mathcal{I}_B \\ [\chi_1^* - \chi_2^*]f = O(t^3) \quad \forall\ f \in \mathcal{I}_{B,S}.\end{aligned}[C2.5]$$

This is the space of inward-pointing normal vectors S-parabolic to B.

As with the usual normal bundle ${}_+N_p\{X;B,S\}$ has a natural $\mathbb{R}^+$-action and additive structure:

$$a[\chi] = [\chi_a],\ \chi_a(t) = \chi(at), \quad a \in [0,\infty][C2.6] \tag{C.3}$$

$$\chi \sim_p \chi_1 + \chi_2 \Longleftrightarrow \begin{aligned} f = O(t^2) \quad \forall\ f \in \mathcal{I}_B \\ [\chi^* - \chi_1^* - \chi_2^*]f = O(t^3) \quad \forall\ f \in \mathcal{I}_{B,S}.\end{aligned}[C2.7] \tag{C.4}$$

By reference to local coordinates it is easy to see that

$${}_+N_p\{X;B,S\} \equiv \mathbb{R}^{n-d'-d''}_{k-d'} \equiv [0,\infty)^{k-d'} \times \mathbb{R}^{n-k-d''}$$

with this structure. In fact these point wise spaces vary smoothly in the sense that

$${}_+N\{X;B,S\} = \bigsqcup_{p\in B} {}_+N_p\{X;B,S\} \text{ is a smooth } \mathbb{R}^{n-d'-d''}_{k-d'}\text{-bundle over } B.[C2.8]$$

However it is important to note that (C.3) and (C.4) *do not* constitute a linear structure, unless $S = \{0\}$. This is because the multiplication by scalars does not distribute over addition. As we shall see there are (non-unique) linear structures which are consistent with (C.4) and in terms of which the $\mathbb{R}^+$-action (C.3) reduces to the standard $\mathbb{R}^+$-action on one summand and the square of the standard $\mathbb{R}^+$-action on the other.

Notice that the inward-pointing part of the usual normal bundle is the $\{0\}$-parabolic normal bundle. That is, it can be obtained from the space of vector fields satisfying (C.1) (since (C.2) is trivial if $S = \{0\}$) and the first condition in (C.2). Thus there is a natural projection

$${}_+N\{X;B,S\} \longrightarrow {}_+NB.[C2.9]$$

From (C.1) it is clear that the range of (C.2), S°, is the annihilator of S. The null space is denoted

$$S^\# = \{v = [\chi] \in {}_+N\{X;B,S\}; \chi^* f = O(t^2) \quad \forall\ f \in \mathcal{I}_B\}.[C2.10]$$

It is a p-subbundle of ${}_+N\{X;B,S\}$. The map (C.2) therefore projects to a natural isomorphism

$${}_+N\left\{X;B,S\right\}/S^\# \cong S^\circ.[C2.11]$$

There is also a natural map

$${}_+NB \ni [\chi(t)] \longmapsto [\chi(t^2)] \in {}_+N\left\{X;B,S\right\}.[C2.12]$$

This has null space S° and range $S^{\#}$. Thus

$$S' \cong {}_+NB/S^\circ \cong S^{\#}[C2.13]$$

where S' is the dual of S as a p-subbundle.

As in the normal case there is a 'linearization' map for vector fields. If $B \subset X$ is a p-submanifold and $S \subset {}_+N^*B$ is a p-subbundle consider the space of vector fields

$$\mathcal{V}_b\{X;B,S\} = \big\{V \in \mathcal{V}_b(X); V \text{ is tangent to } B \\ \text{and if } \eta \text{ is a section of } S \text{ then } \mathcal{L}_V\eta \text{ is as well}\big\}. [C2.14]$$

This space of vector fields consists of 'infinitesimal diffeomorphisms' of X which fix B and $S \subset N^*B$.

Lemma C.1. *The action of $V \in \mathcal{V}_b\{X;B,S\}$ on $\mathcal{C}^\infty(B)$, on $\mathcal{I}_B$ and on $\mathcal{I}_{B,S}$ defines a vector field L_V on ${}_+N\{X;B,S\}$ which is invariant under the $\mathbb{R}^+$-action* (B.3)*; the map*

$$\mathcal{V}_b\{X;B,S\} \ni V \longmapsto L_V \in \mathcal{V}_b({}_+N\{X;B,S\})[C2.16]$$

has as range all the $\mathcal{C}^\infty$ vector fields invariant under (C.3)*, these span ${}^bT_+B\{B,S\}$ at each point; the null space of* (C.1) *is $\mathcal{I}_B\mathcal{V}_b\{X;B,S\}$*

Now as an abstract set we define the space obtained by blowing X up S-parabolically along B to be

$$[X;B,S] = (X \setminus B) \sqcup ({}_+N\{X;B,S\}\diagdown 0/\mathbb{R}^+)[C2.17]$$

i.e. just the disjoint union of the complement of B in X and the inward-pointing part of the spherical S-parabolic normal bundle to B in X. The blow-down map

$$\beta\,[X;B,S] : [X;B,S] \longrightarrow X[C2.18]$$

is defined to be the identity on the first summand in (C.2) and the natural projection, to B, on the second. Of course we wish to show that this blown-up space has a natural structure as a manifold with corners with respect to which (C.2) is smooth. We can refer to [**5**] for the proof of this once we discuss the identification (C.2).

Since $S^\circ \subset {}_+NB$, the annihilator of S in the normal bundle to B, is a p-subbundle we can choose another p-subbundle, H, which is complementary to it:

$${}_+NB = H \oplus S^\circ.[C2.19]$$

Similarly we can choose a p-subbundle G of ${}_+N\,\{X;B,S\}$ which is a complement to $S^{\#}$. Such splittings induce an isomorphism

$${}_+NB = \begin{matrix} H \longleftrightarrow S^{\#} \\ \\ \\ S^\circ \longleftrightarrow G \end{matrix} = {}_+N\{B,S\}.[C2.20]$$

where the two maps are obtained from (C.2) and (C.2). Consider the $\mathbb{R}^+$-action on ${}_+NB$ induced by the choice of H :

$$M_\delta(v+w) = \delta^2 v + \delta w,\ v \in H, w \in S^\circ.[C2.21]$$

Using (C.2) we can identify

$$({}_+NB \setminus B)/M_\delta \equiv ({}_+N\{X;B,S\} \setminus B)/\mathbb{R}^+[C2.22]$$

where B is the zero section in each case and the $\mathbb{R}^+$ action on the right is from (C.3). We can identify a neighborhood of B in X with a neighborhood of the zero section in ${}_+NB$, this in turn gives an identification

$$(X \setminus B) \sqcup ({}_+NB \setminus B)/M_\delta \equiv [X; B, S] \,.[C2.23]$$

The front face of $[X; B, S]$ is $({}_+NB \setminus B)/M_\delta$, this is the space of trajectories of the action of $\mathbb{R}_+$ on ${}_+NB$. In [**5**] a $\mathcal{C}^\infty$ structure was given to the space on the left, which we transfer to the space on the right. Of course we need to show that it does not depend on the choice of H. It is generated by the lift of $\mathcal{C}^\infty({}_+NB)$ and $\mathcal{C}^\infty$ functions on ${}_+NB \setminus B$ homogeneous of degrees 0 and 1 under the action of M_δ. However in [**5**] it is shown that if the space on the left in (C.2) is defined, with its $\mathcal{C}^\infty$ structure, by another choice of H then the identity transformation on $X \setminus B$ extends to a diffeomorphism of the two blown-up spaces. This shows that the $\mathcal{C}^\infty$ structure on $[X; B, S]$ is independent of the choice of H.

PROPOSITION C.2. *Under S-parabolic blow-up of $B \subset X$ the space of vector fields in* (C.2) *lifts into $\mathcal{V}_b([X; B, S])$ and spans $\mathcal{V}_b([X; B, S])$ over $\mathcal{C}^\infty([X; B, S])$; moreover the lift of $V \in \mathcal{V}_b\{X; B, S\}$ is such that its restriction to the front face gives the projection of the vector field L_V in Lemma C.1.*

PROOF. See [**5**]. □

C.3. Lifting under blow-up

When a manifold is blown up along a submanifold it is important to note how various objects lift to the new space. Indeed one of the main uses of blowing up is to 'correct' the intersection properties of various submanifolds with the boundary. In particular although we only wish to blow up the very well-behaved p-submanifolds we need to lift more general 'b-submanifolds'. We recall from [**12**] that $Y \subset B$ is a b-submanifold (by default closed) if near each point of $p \in Y$ there is a local coordinate neighborhood in which Y is the intersection of X with a linear subspace. It is crucial here that the coordinates be admissible in the sense described above, i.e. the boundary hypersurfaces of X be given by $\{x_j = 0\}$ for $j = 1 \dots, k$ since this strongly affects the intersection properties of Y with X. If X is a manifold with boundary then the diagonal in X^2 is a typical example of a b-submanifold which fails to be a p-submanifold.

For a b-submanifold of $Y \subset X$ we define the lift to $[X; B, S]$ of Y, denoted $\beta^*(Y)$, in two cases. First

$$Y \subset B \Longrightarrow \beta^*(Y) = \beta^{-1}(Y) \subset [X; B, S] \, [C3.1]$$

is always a submanifold, just the restriction to Y of the S-parabolic normal bundle to B. If Y meets $X \setminus B$ we only define the lift under the following "cleanness" condition:

$$Y \cap B \text{ is a submanifold with } N_p^*(Y \cap B) = N_p^*Y + N_p^*B \quad \forall\, p \in Y \cap B[C3.2]$$

and we further demand that there be a splitting $N^*_{Y \cap B}B = S + H$ with H a p-subbundle such that

$$\text{if } \tilde{S} = N^*_{Y\cap B}Y \cap N^*_{Y\cap B}B \text{ then } \tilde{S} = (\tilde{S} \cap S) \oplus (\tilde{S} \cap H).[C3.3]$$

The first condition, (C.3), is just the cleanness condition of Bott. Of course if $Y \subset B$ then both conditions are satisfied! For submanifolds Y meeting the complement of

B we define

$$\beta^*(Y) = \mathrm{cl}\left(\beta^{-1}(Y \setminus B)\right) \subset [X; B, S] \,.[C3.4]$$

Lemma C.2. *Suppose $B \subset X$ is a p-submanifold of a manifold with corners and $S \subset N^*B$ is a p-subbundle then if $Y \subset X$ is a b-submanifold not contained in B and meeting it S-cleanly in the sense of* (C.3) *and* (C.3) *the lift of Y to $[X; B, S]$, defined by* (C.3)*, is a b-submanifold; the lift is naturally isomorphic to $[Y; Y \cap B, S \cap N^*Y]$ with the restriction of $\beta\,[X; B, S]$ being the blow-down map for $Y \cap B$ in Y; if additionally Y is a p-submanifold then so is its lift.*

Proof. This follows by introducing coordinates in which everything is linear, notice in particular that $Y \cap B$ is necessarily a p-submanifold of Y. □

Frequently we need to blowup several intersecting submanifolds parabolically. Thus we need to consider how subbundles of a conormal bundle lift under scuh blowups. Suppose that $U \subset N^*Y$ is a subbundle. Then we define the lift, β^*U of this bundle to the blown-up space to be just the image of β^* restricted to U if $Y \subset B$. If Y satisfies the hypotheses of Lemma C.2 then the lift of $U \subset N^*Y$ is defined to be the closure in $T^*[X; B, S]$ of the image of β^* restricted to U over $Y \setminus B$ in case Y meets the complement of B.

More generally if $U \subset {}_+N^*Y$ is a p-subbundle we say that $\{B, S\}$ and $\{Y, U\}$ meet parabolically cleanly if B and Y meet cleanly in the sense of (C.3) and in addition

(C.5) $$S \cap U_{\restriction Y\cap B} \subset {}_+N^*_{Y\cap B}B \cap {}_+N^*_{Y\cap B}Y \text{ is a } p\text{-subbundle}[C3.6]$$

(C.6) $$S \cap {}_+N^*_{Y\cap B}Y \subset {}_+N^*_{Y\cap B}Y \text{ is a } p\text{-subbundle and}[C3.7]$$

(C.7) $$U \cap {}_+N^*_{Y\cap B}B \subset {}_+N^*_{Y\cap B}B \text{ is a } p\text{-subbundle.}[C3.8]$$

Since a p-subbundle always has a p-subbundle complement this is equivalent to (C.3) if $U = \{0\}$. These conditions imply the existence of a splitting into p-subbundles

$$ {}_+N^*(Y \cap B) = S_1' \oplus S_1 \oplus U_1 \oplus (S \cap U) \oplus I \oplus S_2 \oplus U_2 \oplus S_2'[C3.9]$$

where the sum of the first six terms is ${}_+N^*_{Y\cap B}B$, the sum of the last six terms is ${}_+N^*_{Y\cap B}Y$, the sum of the middle two is ${}_+N^*_{Y\cap B}B \cap {}_+N^*_{Y\cap B}Y$, $S = S_1 \oplus (S \cap U) \oplus S_2$ and $U = U_1 \oplus (S \cap U) \oplus U_2$.

Lemma C.3. *Under the conditions of Lemma C.2 if Y is a p-submanifold and $U \subset {}_+N^*Y$ is a p-subbundle such that $\{B, S\}$ and $\{Y, U\}$ meet parabolically cleanly in the sense of* (C.3), (C.5), (C.6) *and* (C.7) *it follows that the lift $\beta^*U \subset {}_+N^*\beta^*(Y)$ is a p-subbundle.*

We also recall (and slightly extend) the following simple lemma from [**12**]; it is just the formula for the Lebesgue measure in polar coordinates.

Lemma C.4. *Under the blow-down map $\beta : [X; B] \longrightarrow X$ for a p-submanifold the density bundle lifts to a bundle which maps smoothly into the density bundle of $[X; B]$, and*

$$\beta^* : \mathcal{C}^\infty(X; \Omega) \longrightarrow \mathcal{C}^\infty([X, B]\,; \rho_{\mathrm{ff}}^q \Omega),\ q = \operatorname{codim} B - 1[C3.12]$$

where ρ_{ff} is a defining function for the new front face and the range of (C.4) *spans the target space over $\mathcal{C}^\infty([X; B])$. For an S-parabolic blow-up, where $S \subset {}_+N^*B$ is a p-subbundle,* (C.4) *is replaced by*

$$\beta^* : \mathcal{C}^\infty(X; \Omega) \longrightarrow \mathcal{C}^\infty([X; B, S]\,; \rho_{\mathrm{ff}}^q \Omega),\ q = \operatorname{codim} B - 1 + \dim S.[C3.13]$$

For the study of the stretched projections used in the composition formulæ the following tansversality results are quite useful.

LEMMA C.5. *Let X be a manifold with corners, $\{B, S\}$ a clean interior p-submanifold of the hypersurface boundary component $M \subset X$. Suppose that Z is an interior p-submanifold of X such that*

$$\begin{aligned} Z \cap M &= Z \cap B, \\ S \restriction_{TZ} &= 0. \end{aligned} \quad [C3.15]$$

If $\beta : [X; B, S] \longrightarrow X$ denotes the blow-down map then $\beta_* \restriction_{T\beta^* Z}$ is an isomorphism.

PROOF. The hypotheses of the lemma imply that in a neighborhood of any point $p \in Z \cap B$ there exist coordinates $x, y_1, \dots, y_l, z_1, \dots, z_m$ such that locally

$$B = \{x = 0, y_1 = \dots = y_l = 0\} \quad Z = \{y_1 = \dots = y_l = 0, z_{r+1} = \dots = z_m = 0\}, 0 < r.[C3.16]$$

The y-variables can be chosen so that projective coordinates on $[X; B, S]$ are given by

$$x, Y_i = \frac{y_i}{x}, i = 1, \dots, s; Y_i = \frac{y_i}{x^2} i = s+1, \dots, l; z_j, j = 1, \dots, m.[C3.17]$$

From (C.3) it is evident that in this coordinate system $\beta^* Z$ is given by the system of equations

$$Y_i = 0, i = 1, \dots, l; z_j = 0, j = r+1, \dots, m.[C3.18]$$

From (C.3) we conclude that $\beta^* Z$ is compactly contained inside of sets with projective coordinates as in (C.3). The assertion of the lemma follows immediately this. □

We also need a transversality result for 'partial diagonals.'

LEMMA C.6. *Let X be a manifold with corners, $\{B, S\}$ a clean interior p-submanifold of the co-dimension k boundary component $M \subset X$. Suppose that Z is an interior b-submanifold of X and that there are defining functions for M, $x_1, \dots, x_k$ such that*

$$\begin{aligned} Z &\subset \{x_1 = \dots = x_k\} \\ Z \cap M &= Z \cap B, \quad [C3.20] \\ S \restriction_{TZ} &= 0. \end{aligned}$$

If $\beta : [X; B, S] \longrightarrow X$ denotes the blow-down map then $\beta_* \restriction_{T\beta^* Z}$ is an isomorphism.

PROOF. The proof is very similar to that of Lemma C.5. We observe that coordinates can be introduced

$$x_1, \dots, x_k; y_1, \dots, y_l; z_1, \dots, z_m$$

so that

$$B = \{x_1 = \dots = x_k = y_1 = \dots = y_l = 0\} \text{ and}$$
$$Z = \{x_1 = \dots = x_k; y_1 = \dots = y_l = 0; z_{r+1} = \dots = z_m = 0\}.$$

The cleanness condition and the hypothesis that S annihilates the tangent space to Z implies that $\mathrm{sp}\{dx_1, \dots, dx_k\} \cap S = \{0\}$. Thus we can choose these coordinates so that projective coordinates on the blow-up are given by

$$X_1 = x_1, X_i = \frac{x_i}{x_1}, i = 2, \dots, k; Y_i = \frac{y_i}{x_1}, i = 1, \dots, sY_j = \frac{y_j}{x_1^2}, j = s+1, \dots, l, z_i, i = 1, \dots, m.$$

The lift of Z in these coordinates is

$$\beta^* Z = \{X_i = 1, i = 2, \dots, k;\ Y_i = 0, i = 1, \dots, l;\ z_{r+1} = \cdots = z_m = 0\}.[C3.21]$$

From (C.3) it is clear that a neighborhood of $\beta^* Z \cap \mathrm{ff}$ is covered by such projective coordinate systems. An elementary calculation in these coordinate systems proves the assertion of the lemma. □

One can allow the boundary differentials to belong to S however Z must then be assumed to lie in an appropriate 'parabolic' diagonal. For example, if we have a codimension three corner and $dx_3 \in S$ then the correct hypothesis would be $Z \subset \{x_1 = x_2 = x_3^2\}$.

There is a further lifting result which we state here and use in §22. Suppose that X is a compact manifold with corners and $Y \subset X$ is an interior p-submanifold. Suppose $H \in M_1(X)$ and $B = H \cap Y$ is an interior p-submanifold of H. Let $S \subset N_B^* Y$ be a subbundle, it is therefore an interior p-subbundle of $N_B^* X$, and consider the S-parabolic blow-up of X along B, $X' = [X; B, S]$. Recall that $S^\#$ is a well-defined submanifold of the S-parabolic normal bundle to B, let $Q \in \mathrm{ff}(X')$ be its image in the front face. Let us denote by Y' the lift of Y to X' then

$$Y' \cap \mathrm{ff}(X') = Y' \cap Q.[C3.22]$$

This is the arrangement of submanifolds considered in (B.3) and §B3, so the spaces $I_{\mathrm{phg}}^{m_1, m_2, m, \mathcal{F}}(X', Y' \cup Q)$ have been discussed above. Let $\mathcal{E}$ be an index family for X with

$$E_H = 0.[C3.23]$$

Then let $\mathcal{E}'$ be the index family on X' given by lifting index sets from boundary hypersurfaces of X to the corresponding hypersurface of X' and taking

$$E'_{\mathrm{ff}(X')} = .[C3.24]$$

Proposition C.3. *Let X, Y, H, B, S, X', Y', Q, $\mathcal{E}$ and $\mathcal{E}'$ be as above, in particular with* (C.3) *and* (C.3) *valid, then the blow-down map $\beta = \beta[X; B, S]$ gives a lifting map*

$$\beta^* : I_{\mathrm{phg}}^{m, \mathcal{E}}(X, Y) \longrightarrow I_{\mathrm{phg}}^{m, m', m'', \mathcal{E}'}(X', Y \cup Q).[C3.26]$$

C.4. Iterated blow-ups

In the body of the paper we construct several spaces by a process of iterated blow-ups. The most complex of these spaces are constructed for the purpose of proving composition formulæ between algebras of operators. To prove these theorems one must show that the final (large) space maps smoothly onto other blown-up spaces. The existence of these smooth maps is shown by making alternate constructions of the final space through different intermediate spaces. Thus we are led to consider conditions under which an interchange in the order of two blow-ups is permissible. For a simple example see §A2.

The iterated blow-up constructions which actually occur here are of a fairly simple type. Suppose that X is a manifold with corners X and that $\{Y_j, S_j\}$ are for $j = 1, \dots, N$ pairs consisting of a p-submanifold and a p-subbundle of its conormal bundle, $N^* Y_j$. We can therefore define the space $[X; Y_1, S_1]$. Suppose that for every $j > 1$ the Y_j meet Y_1 S_1-cleanly in the sense of (C.3) or (C.3). Thus the lifts

$$\beta_1^*(Y_j),\ \beta_1^*(S_j) \subset N^*(\beta_1^*(S_j))[C4.1]$$

are all defined for $j > 1$. Then provided $\beta_1^*(S_2)$ is a p-subbundle we can define the second blow up

$$[[X; Y_1, S_1]\,; \beta_1^*(Y_2), \beta_1^*(S_2)]\,.[C4.2]$$

The notation quickly becomes tedious; since there is no ambiguity we denote the space (C.4) simply as

$$[X; Y_1, S_1; Y_2, S_2]\,.[C4.3]$$

Here of course we assume (or by writing the space *assert*) that the appropriate conditions hold for the lifts. Then proceeding iteratively we denote the general space

$$[X; Y_1, S_1; Y_2, S_2, \dots; Y_N, S_N]\,[C4.4]$$

assuming all the successive lifts to be p-submanifolds and the lifted bundles to be p-subbundles (in practice this is usually easy to check).

With this notation we have the successive blow-down maps

$$\beta_j : [X; Y_1, S_1, \dots; Y_{j-1}, S_{j-1}; Y_j, S_j] \longrightarrow [X; Y_1, S_1; \dots; Y_{j-1}, S_{j-1}]$$

and iterated blow-down maps

$$\begin{gathered}\beta_{ji} : [X; Y_1, S_1, \dots;; Y_i, S_i] \longrightarrow [X; Y_1, S_1; \dots; Y_j, S_j] \quad \forall\; i > j \\ \beta_{ji} = \beta_{j+1} \circ \beta_{j+2} \dots \beta_i.\end{gathered} \; [C4.5]$$

If $\{Y'_j, S'_j\}$ for $j = 1, \dots N'$ is another system of submanifolds and parabolic bundles in X then certainly, assuming all the intersection conditions needed to ensure that they are both defined, the spaces $[X; Y_1, S_1; \dots; Y_N, S_N]$ and $[X; Y'_1, S'_1; \dots; Y'_{N'}, S'_{N'}]$ are naturally diffeomorphic on $X \setminus [Y_1 \cup \dots \cup Y_N \cup Y'_1 \cup \dots \cup Y'_{N'}]$ which lifts into both spaces as an open dense set. We write

$$[X; Y_1, S_1; \dots; Y_N, S_N] \simeq [X; Y'_1, S'_1; \dots; Y'_{N'}, S'_{N'}]\,[C4.6]$$

if this densely defined map extends to a diffeomorphism of the two spaces; they are then naturally isomorphic.

There are two elementary situations in which one can interchange the order of two blow-ups. First if

$$Y_1 \pitchfork Y_2 \Longleftrightarrow N_p^* Y_1 \cap N_p^* Y_2 = \{0\}\ \forall\ p \in Y_1 \cap Y_2.[C4.7]$$

Lemma C.7. *If $Y_i \subset X$, $i = 1, 2$ are two p-submanifolds meeting transversally in the sense of* (C.4) *and $S_i \in N^* Y_i$ are any p-subbundles then*

$$[X; Y_1, S_1; Y_2, S_2] \simeq [X; Y_2, S_2; Y_1, S_1]\,.[C4.9]$$

Proof. The transversality assumption means that locally, near each point $p \in Y_1 \cap Y_2$, there is a decomposition of X as a product of manifolds with corners $X = X_1 \times X_2$ and p-submanifolds $Y'_i \subset X_i$ such that

$$Y_i = Y'_i \times X_{i+1},\ i = 1, 2 \mod 2.[C4.10]$$

The blow-up of X along Y_i is equivalent to the blow-up of X_i along Y'_i and the result follows directly. □

One important special case of transversal intersection is the trivial case $Y_1 \cap Y_2 = \emptyset$. Then we can write (C.7) in the form

$$Y_1 \cap Y_2 = \emptyset \Longrightarrow [X; Y_1, S_1; Y_2, S_2] \simeq [X; Y_2, S_2; Y_1, S_1] \simeq [X; Y_1 \cup Y_2, S_1 \cup S_2]\,[C4.11]$$

where by the union of the bundles we mean the union of their total spaces.

The other important case where the commutativity of blow-up can be readily analyzed is when

$$Y_2 \subset Y_1 \text{ and } S_2 \supset (S_1)_{\restriction Y_2}.[C4.12]$$

LEMMA C.8. *If Y_i for $i = 1, 2$ are two p-submanifolds of a manifold with corners X and $S_i \subset N^*Y_i$ are p-subbundles and* (C.4) *holds then*

$$[X; Y_1, S_1; Y_2, S_2] \simeq [X; Y_2, S_2; Y_1, S_1]\,.[C4.14]$$

PROOF. The case of normal blow-up is discussed in detail in [**12**] and the parabolic case is not essentially different. □

APPENDIX D

Vector bundle coefficients

D.1. Differential operators on bundles

This appendix deals with the extension of the Θ-calculus developed in [**5**] to operators acting on vector bundles. A vector bundle over a manifold with corners, X is defined as in the compact case with the sole modofication that the base space is locally modelled on $\mathbb{R}^n_k$. If B is a vector bundle then we denote the dual bundle by B'. Suppose that B_1 and B_2 are smooth vector bundles over X, a manifold with boundary and that P is a differential operator which carries smooth sections of B_1 to smooth sections of B_2.

Definition D.1. *We will say that P is a Θ-differential operator from B_1 to B_2 if upon introduction of smooth local bases for B_1 and B_2, P is represented by a matrix with entries in* $\operatorname{Diff}^*_\Theta$. *The set of such operators of order m will be denoted by* $\operatorname{Diff}^m_\Theta(X; B_1, B_2)$.

Since $\operatorname{Diff}^*_\Theta$ is a right and left $\mathcal{C}^\infty(X)$-module, the property described in (D.1) does not depend on the choice of bases. Essentially all the local facts proved for scalar operators in [**5**] generalize to the bundle case. The proofs follow by working in local bases. For example:

Lemma D.1. *If*

$$P \in \operatorname{Diff}^m_\Theta(X; B_1, B_2), Q \in \operatorname{Diff}^{m'}_\Theta(X; B_2, B_3)$$

then

$$Q \cdot P \in \operatorname{Diff}^{m+m'}_\Theta(X; B_1, B_3),$$

and the operator

$${}^tP \in \operatorname{Diff}^m_\Theta(X; B'_2, B'_1).$$

Before proceeding with our analysis we establish a simple but important geometric fact. Let M and N be two manifolds, possibly with corners and ψ a smooth map from M to N; if $\pi : B \longrightarrow N$ is a vector bundle over N then we can define a bundle, $\psi^*(B)$ over M as a subset of the product $M \times B$:

$$\psi^*(B) = \{(m, b) \in M \times B : \psi(m) = \pi(b)\}.[D1.3]$$

The projection onto M is simply the restriction of the projection $M \times B \longrightarrow M$. The other projection, $M \times B \longrightarrow B$ restricts to define a map $\psi_* : \psi^*(B) \longrightarrow B$. We have a commutative diagram:

$$\begin{array}{ccc} \psi^*(B) & \xrightarrow{\psi_*} & B \quad [D1.4] \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\psi} & N \end{array}$$

The following lemma is immediate from this construction.

LEMMA D.2. *With M, N, B, and ψ as above, let L be a subset of M such that $\psi(L) = \{n\}$, a single point, then $\psi^*(B) \restriction_L$ is canonically trivial with fibre $\pi^{-1}(n)$.*

Let $Z = X \times [0,1]$, $p \in X$, S_p the line in T_p^*X defined by Θ_p. Set $Z_p = [Z;(p,0),S]$. If B is a vector bundle over X we let $\widetilde{B}$ denote the pullback to Z via the canonical projection and $\widetilde{B_p}$ denote the pullback to Z_p. As a simple application of (D.2) we have

LEMMA D.3. *If B is a vector bundle over X then*

$$\widetilde{B_p} \restriction_{\text{ff}(Z_p)} = \widehat{B_p}[D1.7]$$

is canonically trivial with fibre isomorphic to the fibre of B over p.

The lie group G_p, defined by the lie algebra ${}^{\Theta}T_pX$, acts on the $\text{ff}(Z_p)$. We can define a left action of G_p on sections of $\widehat{B_p}$ by declaring a section s to be left invariant if $\psi_*(s)$ is constant, a right action is defined analogously

If $Q \in \text{Diff}^m_\Theta(X;B_1,B_2)$ then Q lifts to define an element of $\text{Diff}^m_\Theta(Z;\widetilde{B_1},\widetilde{B_2})$ which we will also denote by Q. It follows from the fact that a matrix representation of Q has entries in $\text{Diff}^m_\Theta(X)$ that we can lift Q to an element of $\text{Diff}^m_\Theta(Z_p;\widetilde{B_{1p}},\widetilde{B_{2p}})$ which is tangent to the front face. The local bases of B_1 and B_2 lift at p to define left invariant framings of $\widehat{B_{1p}}$ and $\widehat{B_{2p}}$ moreover the restriction of this operator to the front face is a left invariant operator between sections of $\widehat{B_{1p}}$ and sections of $\widehat{B_{2p}}$; this is the normal operator, $N_p(Q)$. Abstractly $N_p(Q) \in \mathcal{D}^m({}^{\Theta}T_pX) \otimes \hom(\widehat{B_{1p}},\widehat{B_{2p}})$. The coefficient bundle can be identified with the vector space $\hom(B_{1p},B_{2p})$. That $N_p(Q)$ is well defined follows from

LEMMA D.4. *If $V_1,\dots,V_m$ are vector fields in $\mathcal{V}_\Theta$ and $a,b \in \mathcal{C}^\infty(X)$ then*

$$V_1 \dots V_m(ab) = (V_1 \dots V_m a)b \mod \rho\,\text{Diff}^{m-1}_\Theta .[D1.9]$$

PROOF. The proof is a simple induction. If $m = 1$ then

$$V(ab) = (Va)b + a(Vb) = (Va)b \mod O(\rho).$$

If we assume the result for $m-1$ then we have

$$V_1 \dots V_m(ab) = V_1 \dots V_{m-1}((V_m a)b + a(V_m b)),$$

from which the claim of the lemma is clear. □

The following corollary follows easily from (D.4):

COROLLARY D.1. *Suppose that A and B are change of basis matrices for B_1 and B_2 respectively and N_p and N_p' are the matrices of the normal operators relative to the two pairs of bases then*

$$N_p' = {}^tB_p^{-1} N_p A_p.$$

It follows from Corollary D.1 that by introducing smooth local bases for B_1 and B_2 near to p the construction of the normal operator reduces to the componentwise application of (7.1) to the matrix of Q. The following properties of the normal operator are immediate consequences:

PROPOSITION D.1. *If* $P \in \mathrm{Diff}^m_\Theta(X; B_1, B_2)$ *and* $Q \in \mathrm{Diff}^{m'}_\Theta(X; B_2, B_3)$ *then* $N_p(Q \cdot P) \in \mathcal{D}^{m+m'}({}^\Theta T_p X) \otimes \hom(\widehat{B_1}_p, \widehat{B_3}_p)$ *and*

$$N_p(Q \cdot P) = N_p(Q) \cdot N_p(P).[D1.12]$$

The normal operator of the transpose is the transpose of the normal operator, that is, if p_{ij} *is the matrix representation of* $N_p(P)$ *then* ${}^t p_{ji}$ *is the matrix representation for* $N_p({}^t P)$ *relative to the dual bases.*

In addition to the normal operator we also need to define an indicial operator. Recall that

$$K_{1,p} = [{}^\Theta T_p X, {}^\Theta T_p X]$$

The abstract definition is very simple: $\mathcal{D}^m({}^\Theta T_p X) \otimes \hom(\widehat{B_1}_p, \widehat{B_2}_p)$ has a subspace

$$K_{1,p} \otimes \mathcal{D}^{m-1}({}^\Theta T_p X) \otimes \hom(\widehat{B_1}_p, \widehat{B_2}_p).$$

For $p \in \partial X$ the indicial operator, $I_p(Q)$ is simply the normal operator modulo this subspace. In a local basis one takes the residue of each coefficient modulo $K_{1,p} \otimes \mathcal{D}^{m-1}({}^\Theta T_p X)$. It follows from the short exact sequence,

$$0 \hookrightarrow K_{1,p} \hookrightarrow {}^\Theta T_p X \longrightarrow {}^b N_p X = [{}^b T_p X / T_p \partial X] \longrightarrow 0,$$

that these residue classes are canonically identified with elements of $\mathcal{D}^m({}^b N_p^+ X)$, which can in turn be identified with $\mathbb{R}^+$- invariant operators on the fibres of $N^+(X)$. Thus the indicial operator is an element of $\mathcal{D}^m({}^b N_p^+ X) \otimes \hom(B_{1p}, B_{2p})$. The coefficient bundle being the pullback of $\hom(B_1, B_2) \restriction_{\partial X}$ to $N^+ X$, via projection onto the zero section. The indicial operator, $I_p(Q)$ acts as an $\mathbb{R}^+$-invariant operator between sections of $N_p^+(X) \times B_{1p}$ and $N_p^+(X) \times B_{2p}$.

The indicial operator at a point p is a system of $\mathbb{R}^+$-invariant ordinary differential operators; a complex number z is an indicial root for $I_p(Q)$ if the linear map of B_{1p} to B_{2p} defined by

$$I_p(Q; z)u = (\rho^{-z} I_p(Q)(\rho^z u)) \restriction_{\rho=0}, [D1.13]$$

has a nontrivial null space. Here ρ denote a defining function for ∂X. A moments consideration shows that this is independent of the choice of defining function. If $\dim B_1 = \dim B_2$ then it follows from Corollary D.1 that we can restate this as

$$\det I_p(Q; z) = 0.$$

D.2. Vector bundle coefficients for the full calculus

In this section we extend the considerations of §D1 to the full calculus of Θ–pseudodifferential operators. As is evident from the discussion of differential operators, little substance is added by considering operators acting on sections of vector bundles. The definitions all entail an introduction of local bases, a description of the matrix entries and an argument to show that the definition is independent of the choice of basis. The proofs of the generalizations of the theorems in sections 11 and 12 of [**5**] also follow these lines. Consequently, this and the succeeding section contain mostly definitions, and statements of theorems with the detailed justification left to the reader.

To be consistent with the development in [**5**] we define all operators to act between half-density valued sections of bundles. A linear operator that carries elements of $\dot{\mathcal{C}}^{\infty}(X; B_1 \otimes \Omega^{\frac{1}{2}})$ to elements of $\mathcal{C}^{-\infty}(X; B_2 \otimes \Omega^{\frac{1}{2}})$ has a Schwarz kernel in $\mathcal{C}^{-\infty}(X^2; \widetilde{\hom}(B_1, B_2) \otimes \Omega^{\frac{1}{2}})$. The fibre of the bundle $\widetilde{\hom}(B_1, B_2)$ at (x, y) is

$$\hom(B_{1y}, B_{2x}).$$

In order to pull these kernels up to X^2_Θ we first need to consider the pull-back of the homomorphism bundles.

Let

$$\hom_\Theta(B_1, B_2) = {\beta^{(2)}_\Theta}^*(\widetilde{\hom}(B_1, B_2)).[D2.1]$$

If k_A is the Schwartz kernel of A as a distribution on X^2 then $\widetilde{\kappa}_A = {\beta^{(2)}_\Theta}^*(k_A)$ is a distributional section of $\hom_\Theta(B_1, B_2) \otimes {\beta^{(2)}_\Theta}^* \Omega^{\frac{1}{2}}$. As in the scalar case we shift a factor of $\rho^N_{\Theta\mathrm{f}}$, $N = \dim X + 1$, onto the kernel to obtain κ_A as a section of $\hom_\Theta(B_1, B_2) \otimes \rho_{\mathrm{ff}}^{-\frac{1}{2}N} \Omega^{\frac{1}{2}}$.

Since $\beta^{(2)}_\Theta$ is a diffeomorphism away from the front face it follows that a fibre of the $\hom_\Theta(B_1, B_2)$ is canonically identified with the corresponding fibre of $\widetilde{\hom}(B_1, B_2)$. If $\mathcal{F}_p$ denotes the fibre of $\mathrm{ff}(X^2_\Theta)$ lying over $p \in \partial X$ then $\beta^{(2)}_\Theta(\mathcal{F}_p) = (p, p)$ and therefore it follow from (D.2) that $\hom_\Theta(B_1, B_2) \restriction_{\mathcal{F}_p}$ is canonically trivial with fibre $\hom(B_{1p}, B_{2p})$.

For computations it is useful to use an alternate description of the homomorphism bundles that follows from the fact that if V and W are two vector spaces then $\hom(V, W) \simeq W \otimes V'$. Therefore

$$\widetilde{\hom}(B_1, B_2) \simeq \pi_L^*(B_2) \otimes \pi_R^*(B_1')[D2.2]$$

and hence

$$\hom_\Theta(B_1, B_2) \simeq {}^\Theta\pi_L^*(B_2) \otimes {}^\Theta\pi_R^*(B_1').[D2.3]$$

If there is a Hermitian inner product $\langle , \rangle_{B_1}$ defined on B_1 then we can further identify B_1' with $\overline{B_1}$ and thus

$$\hom_\Theta(B_1, B_2) \simeq {}^\Theta\pi_L^*(B_2) \otimes {}^\Theta\pi_R^*(\overline{B_1}).[D2.4]$$

If A is an operator acting between half density sections as above then we think of its kernel, κ_A as defined on X^2_Θ as a section of $\mathcal{C}^{-\infty}(X^2_\Theta; \hom_\Theta(B_1, B_2) \otimes \rho_{\mathrm{ff}}^{-\frac{1}{2}N} \Omega^{\frac{1}{2}})$ with the action defined by

$$A \cdot f = ({}^\Theta\pi_L)_*(\kappa_A {}^\Theta\pi_R^*(f)).[D2.5]$$

It follows easily from (D.2) and (D.1) that (D.2) defines a half density section of B_2.

As in the scalar case we define our spaces of pseudodifferential operators in terms of the conormal regularity properties of the kernel κ_A along the lifted diagonal, Δ_Θ and boundary hypersurfaces of X^2_Θ. Let E denote an index for ∂X, see (B.1)-(B.2).

Definition D.2. *A half density section, s of a vector bundle $B \longrightarrow X$ is in $\mathcal{A}^E_{\mathrm{phg}}(X; B \otimes \Omega^{\frac{1}{2}})$ if the coefficients of s relative to a smooth local basis for B lie in $\mathcal{A}^E_{\mathrm{phg}}(X; \Omega^{\frac{1}{2}})$.*

That these spaces are well defined follows immediately from the fact that hte space of polyhomogeneous conormal distributions with a given index set is a $\mathcal{C}^\infty$-module. An index family $\mathcal{E}$ for X^2_Θ is a collection of three index sets $E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$.

DEFINITION D.3. *An element of* $\mathcal{C}^{-\infty}(X^2_\Theta; \hom_\Theta(B_1, B_2) \otimes \rho_{\mathrm{ff}}^{-\frac{1}{2}N}\Omega^{\frac{1}{2}})$ *belongs to*

$$\mathcal{I}^{m,\mathcal{E}}_{\mathrm{phg}}(X^2_\Theta, \Delta_\Theta; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$$

if the entries in a matrix representation relative to smooth local bases for B_1 *and* B_2 *lie in* $\mathcal{I}^{m,\mathcal{E}}_{\mathrm{phg}}(X^2_\Theta, \Delta_\Theta; \rho_{\mathrm{ff}}^{-\frac{1}{2}N}\Omega^{\frac{1}{2}})$.

From the standard theory of conormal distributions, as found for example in [**?**] it follows that these classes of distributions are well defined and do not depend on the choices of local bases. Now that we have defined these spaces of distributions we can define spaces of operators.

DEFINITION D.4. *An operator is in the 'small calculus' of order* m,

$$\Psi^m_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$$

if its Schwartz kernel,κ_A *lies in* $\mathcal{I}^{m,\mathcal{E}}_{\mathrm{phg}}(X^2_\Theta, \Delta_\Theta; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ *with* $\mathcal{E} = \emptyset, \emptyset, (0, 0)$.

DEFINITION D.5. *An operator is a 'boundary term' of order* $\mathcal{E}$,

$$\Psi^{-\infty;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$$

if its kernel belongs to $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X^2_\Theta; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$

The full calculus is defined as

$$\Psi^{m;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) = \Psi^{-\infty;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) + \Psi^m_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}). \quad [D2.10]$$

For operators in the small calculus we can define a principle or diagonal symbol by using a matrix representation of the operator and defining the symbol of the matrix to be the matrix of symbols. It follows from [**5**] equation (7.5) that the symbolic matrix coefficients define elements of

$$S^{\{m\}}({}^\Theta T^*(X)) = S^m({}^\Theta T^*(X))/S^{m+1}({}^\Theta T^*(X)).$$

An invariant symbol is obtained by considering the symbol as taking values in $\hom_\Theta(B_1, B_2) \restriction_{\Delta_\Theta}$ pulled back to ${}^\Theta T^*(X)$. From (D.2) it follows that this bundle can be canonically identified with $\hom(B_1, B_2)$ pulled back to ${}^\Theta T^*(X)$. Thus we define the symbol ${}^\Theta\sigma_m(A)$ to be the element of $S^{\{m\}}({}^\Theta T^*(X); \hom(B_1, B_2))$ obtained by the construction outlined above.

We say that an operator $P \in \Psi^m_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ is elliptic if its principle symbol ${}^\Theta\sigma_m(P)$ defines an isomorphism between the fibres of B_1 and B_2 for every nonzero element of ${}^\Theta T^*(X)$.

As in the scalar case the intersection of the small calculus and the boundary terms consists of operators with kernels in $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X^2_\Theta; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ with $\mathcal{E} = \emptyset, \emptyset, \{(0, 0)\}$. Therefore we can extend the diagonal symbol to the full calculus by setting

$$\begin{gathered} {}^\Theta\sigma_m(A) = {}^\Theta\sigma_m(A') \text{ where } A = A' + A''; \\ A' \in \Psi^m_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}); A'' \in \Psi^{-\infty;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) \end{gathered} \quad [D2.11]$$

As usual we have a short exact sequence:

$$\begin{gathered} 0 \hookrightarrow \Psi^{m-1;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) \hookrightarrow \Psi^{m;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) \xrightarrow{{}^\Theta\sigma_m} \\ S^{\{m\}}({}^\Theta T^*(X); \hom(B_1, B_2)) \longrightarrow 0. \end{gathered} \quad [D2.12]$$

In addition to the principle symbol we also can define a normal operator at least for operators with a kernel smooth up to the front face. For $A \in \Psi_{\Theta}^{m;\mathcal{E}}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ with $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, \{(0,0)\}$ we define the normal operator, $N(A)$ to be $\kappa_A \restriction_{\Theta\mathrm{f}}$. By working with local bases and applying [**5**],(7.7) one easily sees that $N(A)$ takes values in $\mathcal{I}^{m,\mathcal{E}^{\mathrm{ff}}}(\mathrm{ff}(X^2_{\Theta}), \partial\Delta_{\Theta}; \hom_{\mathrm{ff}}(B_1, B_2)) \otimes \Omega^{\frac{1}{2}}_{\mathrm{fibre}} \otimes \Omega^{\frac{1}{2}}_{\mathrm{fibre}}$. Here

$$\hom_{\mathrm{ff}}(B_1, B_2) = \hom_{\Theta}(B_1, B_2) \restriction_{\mathrm{ff}(X^2_{\Theta})}.$$

It follows from (D.2) and (D.2) that this bundle is trivial when restricted to a fibre of the front face, $\mathcal{F}_p$ with fibre $\hom(B_{1p}, B_{2p})$. As usual there is a short exact sequence:

$$\begin{aligned} 0 \hookrightarrow \Psi_{\Theta}^{m;E_{\Theta\mathrm{l}},E_{\Theta\mathrm{r}},1}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) \hookrightarrow \Psi_{\Theta}^{m;E_{\Theta\mathrm{l}},E_{\Theta\mathrm{r}}}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) \xrightarrow{N} \\ \mathcal{I}^{m,E_{\Theta\mathrm{l}},E_{\Theta\mathrm{r}}}_{\mathrm{phg}}(\mathrm{ff}(X^2_{\Theta}), \partial\Delta_{\Theta}; \hom_{\mathrm{ff}}(B_1, B_2)) \otimes \Omega^{\frac{1}{2}}_{\mathrm{fibre}} \otimes \Omega^{\frac{1}{2}}_{\mathrm{fibre}} \longrightarrow 0. \end{aligned} \quad [D2.13]$$

To prove exactness at the third factor one needs to use the 'collar neighbourhood with coefficients' lemma:

Lemma D.5. *If Y is a manifold with corners, B a vector bundle over Y and Z is an embedded hypersurface boundary component of Y then there is a neighbourhood U of Z in Y and a bundle isomorphism, ϕ taking*

$$\phi : \widehat{B \restriction_Z} \longrightarrow B \restriction U.$$

Here $\widehat{B \restriction_Z}$ is the pullback of $B \restriction Z$ to a neighbourhood of the zero section in $N^+(Z)$ by the canonical projection. The bundle map ϕ reduces to the identity on Z.

Proof. The proof is a small modification of the usual argument. □

Finally we can define left and right boundary symbols. If we fix smooth local bases for B_1 and B_2 then the kernel of an operator in $\Psi_{\Theta}^{m;\mathcal{E}}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ is represented near $\Theta\mathrm{l}$ or $\Theta\mathrm{r}$ by a matrix with entries in $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X^2_{\Theta}; \Omega^{\frac{1}{2}})$, here $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$. The boundary symbols ${}^{\Theta}\sigma_{\Theta\mathrm{l}}(A)$ and ${}^{\Theta}\sigma_{\Theta\mathrm{r}}(A)$ are defined by applying the construction of §B2 to to the matrix entries. One obtains a matrix with terms as in (B.2).

Invariantly the coefficient bundle for the left symbol is

$$\hom_l(B_1, B_2) = \Pi_0^*(\hom_{\Theta}(B_1, B_2) \restriction_{\Theta\mathrm{l}}),$$

where Π_0 is the projection of $N^+(\Theta\mathrm{l})$ onto its zero section which is canonically isomorphic to $\Theta\mathrm{l}(X^2_{\Theta})$. It is important in applications to be able to apply the indicial operator of a differential operator to the left boundary symbol thus it is more convenient to identify

$$N^+(\Theta\mathrm{l}(X^2_{\Theta})) \simeq {}^{\Theta}\pi_L^*(N^+(\partial X)).[D2.15]$$

Using (D.2) we can also identify the homomorphism bundle as

$$\hom_l(B_1, B_2) \simeq {}^{\Theta}\pi_L^*(\widehat{B_2}) \otimes {}^{\Theta}\pi_R^*(B_1) \restriction_{\Theta\mathrm{l}}, [D2.16]$$

here $\widehat{B_2} = \pi_0^*(B_2 \restriction_{\partial X})$ where π_0 is the projection onto the zero section in $N^+(\partial X)$. Therefore the left boundary symbol is of the form (**??**) with

$$a(z,j) \in \mathcal{A}^{\mathcal{E}^{\Theta\mathrm{l}}}_{\mathrm{phg}}({}^{\Theta}\pi_l^*(N^+(\partial X)); {}^{\Theta}\pi_L^*(\widehat{B_2})) \otimes {}^{\Theta}\pi_R^*(B_1) \restriction_{\Theta\mathrm{l}} \otimes \rho_{\mathrm{ff}}^{-\frac{1}{2}N} \Omega^{\frac{1}{2}}.[D2.17]$$

For a boundary face o we let $\{E_o\}$ denote the most singular terms in the index set E_o and $\mathcal{A}_{\text{phg}}^{\{E_o\},\mathcal{E}^o}$ denote sums as in (B.2). We have a short exact sequence

$$0 \hookrightarrow \Psi_\Theta^{m;E_{\Theta\text{l}}+\delta,E_{\Theta\text{r}},E_{\Theta\text{f}}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12}) \hookrightarrow \Psi_\Theta^{m;E_{\Theta\text{l}},E_{\Theta\text{r}},E_{\Theta\text{f}}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12}) \quad [D2.18]$$

$$\xrightarrow{{}^\Theta\sigma_L} \mathcal{A}_{\text{phg}}^{\{E_{\Theta\text{l}}\},\mathcal{E}^{\Theta\text{l}}}({}^\Theta\pi_l^*(N^+(\partial X));{}^\Theta\pi_L^*(\widehat{B_2}))\otimes{}^\Theta\pi_R^*(B_1)\restriction_{\Theta\text{l}}\otimes\rho_{\text{ff}}^{-\frac12 N}\Omega^{\frac12}\longrightarrow 0.$$

****Here δ needs to be appropriately defined to be the 'gap'.***** A similar discussion applies to ${}^\Theta\sigma_R(A)$.

To summarize we have

Theorem D.1. *For a compact manifold with boundary, X a pair of vector bundles B_1, B_2 and a nonvanishing projective class of 1-forms, $[\Theta]$ defined on ∂X there is a algebra of operators, $\Psi_\Theta^{m;\mathcal{E}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12})$ with four symbol maps $N(A)$, ${}^\Theta\sigma_m(A)$, ${}^\Theta\sigma_{\Theta\text{l}}(A)$, and ${}^\Theta\sigma_{\Theta\text{r}}(A)$ and exact sequences,* (D.2), (D.2) *and* (D.2).

D.3. Composition and mapping properties

Next we need to discuss the mapping and composition properties of these classes of operators. The simplest such theorem regards the composition between Θ-differential and Θ–pseudodifferential operators:

Theorem D.2. *Suppose that B_1, B_2, and B_3 are vector bundles over X and that $A\in\Psi_\Theta^{m;\mathcal{E}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12})$ and $P\in\text{Diff}^{m'}(X;B_2\otimes\Omega^{\frac12},B_3\otimes\Omega^{\frac12})$ then*

(D.1) $$P\cdot A\in\Psi_\Theta^{m+m';\mathcal{E}}(X;B_1\otimes\Omega^{\frac12},B_3\otimes\Omega^{\frac12})[D3.2]$$

(D.2) $${}^\Theta\sigma_{m+m'}(P\cdot A)={}^\Theta\sigma_{m'}(P)\cdot{}^\Theta\sigma_m(A).[D3.3]$$

The composition on the right hand side of (D.2) *is as bundle maps from $B_1\longrightarrow B_2$ and $B_2\longrightarrow B_3$ respectively. The normal operators satisfy:*

$$N_p(P\cdot A)=N_p(P)\cdot N_p(A),[D3.4]$$

here $N_p(P)$ acts as a differential operator tangent to the fibres of the front face. The left boundary symbol satisfies:

$${}^\Theta\sigma_{\Theta\text{l}}(P\cdot A)=I(P)\cdot{}^\Theta\sigma_{\Theta\text{l}}(A),[D3.5]$$

here $I(P)$ acts as a differential operator on the pull-back of bundles under ${}^\Theta\pi_L$.

Proof. Introducing local bases for the bundles reduces the proof to the scalar case. □

As corollaries of the symbolic composition formulae we obtain

Corollary D.2. *Suppose that $P\in\text{Diff}^{m'}(X;B_2\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12})$ is elliptic then for any index family $\mathcal{E}=E_{\Theta\text{l}},E_{\Theta\text{r}},E_{\Theta\text{f}}$ and any $R\in\Psi_\Theta^{m;\mathcal{E}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12})$ there is an $F\in\Psi_\Theta^{m-m';\mathcal{E}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12})$ such that*

$$P\cdot F-R\in\Psi_\Theta^{-\infty;\mathcal{E}}(X;B_1\otimes\Omega^{\frac12},B_2\otimes\Omega^{\frac12});[D3.7]$$

moreover the construction of F can be carried out smoothly in parameters in any compact manifold.

Proof. This just uses the standard iteration associated with the diagonal symbol map. □

Corollary D.3. *Suppose that* $P \in \mathrm{Diff}^{m'}(X; B_2 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ *and the indicial operator,* $I(P)$, *on the fibres of* $N^+(\partial X)$ *has constant indicial roots (that is independent of the basepoint) and* $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$ *are index sets such that there is no point* $(z, m) \in E_{\Theta\mathrm{l}}$ *with* z *an indicial root of* $I(P)$ *then for any* $R \in \Psi^{m;\mathcal{E}}_{\Theta}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ *there is an* $F \in \Psi^{-\infty;\mathcal{E}}_{\Theta}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ *with*

$$P \cdot F - R \in \Psi^{m;E'}_{\Theta}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}}) \text{ and } \mathcal{E}' = \emptyset, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}.[D3.9]$$

Proof. This is simply a matter of solving the indicial system iteratively. □

The next order of business is to generalize the mapping and composition theorems. The following lemmas allows us to introduce local bases and repeat the scalar proofs essentially verbatim:

Lemma D.6. *Let* B_1 *and* B_2 *be vector bundles over* X *the canonical pairing*

$$\widetilde{\mathrm{hom}}(B_1, B_2) \otimes \pi_R^* B_1 \longrightarrow \pi_L^* B_2 [D3.11]$$

lifts to a natural pairing

$$\mathrm{hom}_\Theta(B_1, B_2) \otimes {}^{\Theta}\pi_R^* B_1 \longrightarrow {}^{\Theta}\pi_L^*(B_2).[D3.12]$$

Proof. We use the identification (D.2) and the functorial properties of the pullback operation to obtain

$$\begin{aligned} &\mathrm{hom}_\Theta(B_1, B_2) \otimes {}^{\Theta}\pi_R^*(B_1) \simeq \\ &{}^{\Theta}\pi_L^*(B_2) \otimes {}^{\Theta}\pi_R^*(B_1') \otimes {}^{\Theta}\pi_R^*(B_1) \simeq {}^{\Theta}\pi_L^*(B_2) \otimes {}^{\Theta}\pi_R^*(B_1' \otimes B_1). \end{aligned}[D3.13]$$

The lemma is clear from (D.3) □

Lemma D.7. *Suppose that* B_1, B_2, *and* B_3 *are bundles over* X *then the canonical pairing*

$$\pi_S^*(\widetilde{\mathrm{hom}}(B_2, B_3)) \otimes \pi_F^*(\widetilde{\mathrm{hom}}(B_1, B_2)) \longrightarrow \pi_C^*(\widetilde{\mathrm{hom}}(B_1, B_3))[D3.15]$$

lifts to a natural pairing

$${}^{\Theta}\pi_S^*(\mathrm{hom}_\Theta(B_2, B_3)) \otimes {}^{\Theta}\pi_F^*(\mathrm{hom}_\Theta(B_1, B_2)) \longrightarrow {}^{\Theta}\pi_C^*(\mathrm{hom}_\Theta(B_1, B_3)).[D3.16]$$

Proof. We use (D.2) to obtain that the left hand side of (D.7) is isomorphic to

$${}^{\Theta}\pi_S^*({}^{\Theta}\pi_L^* B_3 \otimes {}^{\Theta}\pi_R^* B_2') \otimes {}^{\Theta}\pi_F^*({}^{\Theta}\pi_L^* B_2 \otimes {}^{\Theta}\pi_R^* B_1')[D3.17]$$

and the right hand side is isomorphic to:

$${}^{\Theta}\pi_C^*({}^{\Theta}\pi_L^* B_3 \otimes {}^{\Theta}\pi_R^* B_1').[D3.18]$$

The fact that $\beta^{(2)}_\Theta \cdot {}^{\Theta}\pi_O = \pi_O \cdot \beta^{(3)}_\Theta$ for $O = F$, S, or C implies that

$$\begin{aligned} {}^{\Theta}\pi_R \cdot {}^{\Theta}\pi_F &= {}^{\Theta}\pi_R \cdot {}^{\Theta}\pi_C \\ {}^{\Theta}\pi_L \cdot {}^{\Theta}\pi_F &= {}^{\Theta}\pi_R \cdot {}^{\Theta}\pi_S \, [D3.19] \\ {}^{\Theta}\pi_L \cdot {}^{\Theta}\pi_S &= {}^{\Theta}\pi_L \cdot {}^{\Theta}\pi_C. \end{aligned}$$

From (D.3) it follows that (D.3) can be replaced by

$${}^{\Theta}\pi_C^*({}^{\Theta}\pi_L^* B_3 \otimes {}^{\Theta}\pi_R^* B_1') \otimes {}^{\Theta}\pi_S^*({}^{\Theta}\pi_R^*(B_2' \otimes B_2)).[D3.20]$$

The lemma follows easily from (D.3). □

The mapping and composition results are

PROPOSITION D.2. *For any index family $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$ for X^2_Θ and index set E_I for X*

$$A \in \Psi^{m;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac12}, B_2 \otimes \Omega^{\frac12}) \Longrightarrow A : \mathcal{A}^{E_I}_{\mathrm{phg}}(X; B_1 \otimes \Omega^{\frac12}) \longrightarrow \mathcal{A}^{E_O}_{\mathrm{phg}}(X; B_2 \otimes \Omega^{\frac12}), \quad \text{provided } E_I + E_{\Theta\mathrm{r}} > -1 \text{ and } E_O = E_{\Theta\mathrm{l}}\overline{\overline{\cup}}(E_{\Theta\mathrm{f}} + E_I). \qquad [D3.22]$$

The general composition result is

THEOREM D.3. *If $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$ and $\mathcal{E}' = E'_{\Theta\mathrm{l}}, E'_{\Theta\mathrm{r}}, E'_{\Theta\mathrm{f}}$ are index families for X^2_Θ such that*

$$E_{\Theta\mathrm{r}} + E'_{\Theta\mathrm{l}} > -1 [D3.24]$$

then for any $A \in \Psi^{m;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac12}, B_2 \otimes \Omega^{\frac12})$ and $B \in \Psi^{m';\mathcal{E}'}_\Theta(X; B_2 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12})$ the composite operator $A \cdot B$ is well defined and

$$A \cdot B \in \Psi^{m+m';\mathcal{E}''}_\Theta(X; B_1 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12}) \text{ where } \mathcal{E}'' = (E''_{\Theta\mathrm{l}}, E''_{\Theta\mathrm{r}}, E''_{\Theta\mathrm{f}}),$$
$$E''_{\Theta\mathrm{l}} = E_{\Theta\mathrm{l}}\overline{\overline{\cup}}[E_{\Theta\mathrm{f}} + E'_{\Theta\mathrm{l}}], E''_{\Theta\mathrm{r}} = [E'_{\Theta\mathrm{f}} + E_{\Theta\mathrm{r}}]\overline{\overline{\cup}}E'_{\Theta\mathrm{r}}, E''_{\Theta\mathrm{f}} = [E'_{\Theta\mathrm{f}} + E_{\Theta\mathrm{f}}]\overline{\overline{\cup}}[E'_{\Theta\mathrm{r}} + E_{\Theta\mathrm{l}} + N]. \qquad [D3.25]$$

The diagonal symbol of the composition is given by

$$^{\Theta}\sigma_{m+m'}(A \cdot B) = {}^{\Theta}\sigma_m(A) \cdot {}^{\Theta}\sigma_{m'}(B); [D3.26]$$

the composition on the right hand side is as linear maps.

In addition to the symbol classes $\Psi^{m;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac12}, B_2 \otimes \Omega^{\frac12})$ introduced above we also have certain 'residual' terms whose kernels are not smooth at the boundary but are nonetheless polyhomogeneous conormal distributions on X^2 itself.

DEFINITION D.6. *We will say that $A \in \Psi^{-\infty,\mathcal{E}}(X; B_1 \otimes \Omega^{\frac12}, B_2 \otimes \Omega^{\frac12})$ where $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}$ is an index family for X^2 if in terms of local bases the Schwarz kernel of A belongs to $\beta^{(2)*}_\Theta(\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X^2) \otimes \widetilde{\hom}(B_1, B_2)) \otimes \rho_{\mathrm{ff}}^{-\frac12 N}\Omega^{\frac12}$.*

We have the following composition results for these operators

THEOREM D.4. *For index sets $\mathcal{E} = E_{\Theta\mathrm{l}}, E_{\Theta\mathrm{r}}, E_{\Theta\mathrm{f}}$, an index family for X^2_Θ and $\mathcal{E}' = E'_{\Theta\mathrm{l}}, E'_{\Theta\mathrm{r}}$, an index family for X^2 satisfying* (D.3) *we have*

$$\Psi^{m;\mathcal{E}}_\Theta(X; B_1 \otimes \Omega^{\frac12}, B_2 \otimes \Omega^{\frac12}) \circ \Psi^{-\infty,\mathcal{E}'}(X; B_2 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12}) \subset \Psi^{-\infty,\mathcal{E}''}(X; B_1 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12}) \qquad [D3.29]$$

where $E''_{\Theta\mathrm{l}} = E_{\Theta\mathrm{l}}\overline{\overline{\cup}}[E_{\Theta\mathrm{f}} + E'_{\Theta\mathrm{r}} + E'_{\Theta\mathrm{l}}]$ and $E''_{\Theta\mathrm{r}} = E'_{\Theta\mathrm{r}}$. Similarly if $\mathcal{G}$ is a second index family for X^2 such that (D.3) *holds then*

$$\Psi^{-\infty,\mathcal{G}}(X; B_1 \otimes \Omega^{\frac12}, B_2 \otimes \Omega^{\frac12}) \circ \Psi^{-\infty,\mathcal{E}'}(X; B_2 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12}) \subset \Psi^{-\infty,\mathcal{E}''}(X; B_1 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12}) \qquad [D3.30]$$

where $E''_{\Theta\mathrm{l}} = E_{\Theta\mathrm{l}}, \quad E''_{\Theta\mathrm{r}} = G_{\Theta\mathrm{r}}$. If F is an index set for X with $F + E_{\Theta\mathrm{r}} > -1$ then

$$\Psi^{-\infty,\mathcal{E}}(X; B_2 \otimes \Omega^{\frac12}, B_3 \otimes \Omega^{\frac12}) \cdot \mathcal{A}^F_{\mathrm{phg}}(X; B_1 \otimes \Omega^{\frac12}) \subset \mathcal{A}^{E_{\Theta\mathrm{l}}}_{\mathrm{phg}}(X; B_2 \otimes \Omega^{\frac12}). [D3.31]$$

As a final result we restate the L^2-mapping properties; we assume that B_1 and B_2 are Hermitian bundles such that smooth sections in compact subsets of X have bounded length.

PROPOSITION D.3. *If $m \leq 0$ and the exponent sets satisfy $E_{\Theta\mathrm{l}} > -\frac{1}{2}, E_{\Theta\mathrm{r}} > -\frac{1}{2}$ and $E_{\Theta\mathrm{f}} > 0$ then each $A \in \Psi_{\Theta}^{m;\mathcal{E}}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ defines a bounded operator from $L^2(X; B_1 \otimes \Omega^{\frac{1}{2}})$ to $L^2(X; B_2 \otimes \Omega^{\frac{1}{2}})$; the operators in $\Psi^{-\infty,\mathcal{E}'}(X; B_1 \otimes \Omega^{\frac{1}{2}}, B_2 \otimes \Omega^{\frac{1}{2}})$ are compact.*

APPENDIX E

The tube around $\mathbb{R}^n$

In this appendix we give a more classical description of the kernel which provides the inverse to the map from holomorphic $(n,0)$–forms on the fiber of the adiabatic front face of G^2_α, ${}^aF \simeq \mathbb{R}^n \times \mathbb{B}^n$ to distributions on $\mathbb{R}^n$ defined by the push-forward operation. The fiber of the adiabatic front face should be thought of as the unit neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ relative to the standard flat metric. The existence of this kernel proves the Boutet de Monvel-Guillemin conjecture for the space ${}^aF$ and serves as the leading order term in the Taylor expansion in the scaling parameter ϵ for the inverse to the push forward on the ϵ-tube in T^*Y. We construct it using the partial Fourier transform and show that it lifts to be polyhomogeneous conormal on the parabolic blow-up of of the fibre diagonal in ${}^aF \times \mathbb{R}^n$. The description of this kernel afforded by the partial Fourier transform is as a 'Fourier integral operator with complex phase'.

E.1. The Fourier multiplier

Let $F(x)$ be a function in $\mathcal{S}(\mathbb{R}^n)$ which has an holomorphic extension to ${}^aF$. As a consequence of the Paley-Wiener theorem this is equivalent to the statement that for any $\eta > 0$ there is a constant C_η such that the Fourier transform of F satisfies the estimate.

$$\hat{F}(\xi) \leq C_\eta e^{-(1-\eta)|\xi|}.[E1.1]$$

If $F(z)$ defines a extendible tempered distribution on $|\operatorname{Im} z| \leq 1$ then we can replace (E.1) by an estimate of the form

$$\hat{F}(\xi) \leq K_N e^{-|\xi|}(1+|\xi|)^N; \quad \text{for some } N.[E1.2]$$

Let $F_y(x) = F(x+iy)$ then a simple application of Cauchy's theorem shows that

$$\hat{F}_y(\xi) = e^{-y\cdot\xi}\hat{F}(\xi).[E1.100]$$

Let

$$\pi_a : {}^aF \longrightarrow \mathbb{R}^n$$

denote the projection. For an holomorphic $(n,0)$–form $F(z)dz$ such that

$$F(x+iy) \leq M(1+|x|)^{-(n+1)}, \text{ for } |y| \leq 1,$$

we use Fubini's theorem and(E.1) to compute the action of the push forward in the Fourier representation:

$$\begin{aligned} \hat{}(\xi) &= \int_{\mathbb{R}^n}\int_{\mathbb{B}^n} F(x+iy)dy e^{-ix\cdot\xi}dx \\ &= \int_{\mathbb{B}^n} \hat{F}(\xi)e^{-y\cdot\xi}dy \cdot m_n(\xi) \end{aligned}.$$

This gives

$$\hat{}(\xi) = \widehat{F}(\xi) m_n(\xi),$$

$$\text{where } m_n(\xi) = \int\limits_{|y|\leq 1} e^{-y\cdot\xi}\,dy = \frac{c_n}{|\xi|^n}\int\limits_0^{|\xi|} I_{\frac{n-2}{2}}(r) r^{\frac{n}{2}}\,dr, [E1.3]$$

$I_\nu(z)$ is the I-Bessel function of order ν. The Schwarz kernel of the push forward is a delta distribution concentrated on the 'fibre diagonal'

$$^aF \times \mathbb{R}^n \supset \Delta_f = \{(x, x, y); \quad x \in \mathbb{R}^n, |y| \leq 1\}.$$

The important properties of $m_n(\xi)$ are summarized in the following proposition.

Proposition E.1. *The function $m_n(\xi)$ is an entire function of $\xi \cdot \xi$. For real ξ,*

$$m_n(\xi) > 0.$$

As ξ tends to infinity in $\mathbb{R}^n$, m_n, has an asymptotic development:

$$m_n(\xi) \sim \frac{e^{|\xi|}}{|\xi|^{\frac{n+1}{2}}}\sum_{k=0}^{\infty} a_k |\xi|^{-k}. [E1.5]$$

Proof. The positivity for real arguments is obvious from the first integral representation in (E.1). The asymptotic expansion follows from the standard expansion for the I-Bessel function, see [**?**]. □

E.2. Inverse to the push forward

The kernel for the inverse to the push forward is given as an oscillatory integral by

$$\kappa_{P^{-1}}(x - x', y) = \int\limits_{\mathbb{R}^n} \frac{e^{\imath\xi\cdot(x-x'+\imath y)}}{m_n(\xi)}\,d\xi \otimes dz. [E2.1]$$

If $|y| < 1$ then it is a simple consequence of Proposition E.1 that $\kappa_{P^{-1}}(x - x', y)$ is smooth and rapidly decreasing as $|x - x'|$ tends to infinity. It is in fact holomorphic as a function of $z = x - x' + iy$. Using (E.1) and an elementary stationary phase argument one easily establishes:

Lemma E.1. *The kernel $\kappa_{P^{-1}}(x - x', y)$ is in $\mathcal{C}^\infty(^aF \setminus \partial\Delta_f)$, where*

$$\partial\Delta_f = \{x, x', y; \quad |y| = 1, |x - x'| = 0\}.$$

For $|y| \leq 1$ it is uniformly rapidly deceasing as $|x - x'|$ tends to infinity.

The interesting part of this kernel lies in the 'boundary of the fibre diagonal,' $\partial\Delta_f$. To resolve the singularity we need to parabolically blow-up this submanifold. This is possible because the Θ-structure on ^aF is defined by the 'basic' 1-form $y\cdot dx$. Thus the 1-form on $^aF \times \mathbb{R}^n$ given by

$$\Theta = y \cdot d(x - x'), [E2.3]$$

along with $dr_{\partial\Delta_f}$, define a subbundle, S of $N^*\partial\Delta_f$. Let $[^aF \times \mathbb{R}^n; \partial_D iag_f, S]$ denote the parabolic blowup of $^aF \times \mathbb{R}^n$ along Δ_f. Let

$$\beta_\Theta : [^aF \times \mathbb{R}^n; \partial_D iag_f, S] \longrightarrow {}^aF \times \mathbb{R}^n$$

be the blow-down and set $\kappa_{P^{-1}\Theta} = \beta^*_\Theta(\kappa_{P^{-1}})$. We think of the blown-up space as being non-compact with two boundary components, a 'side face,' sf and a 'front face,' ff .

THEOREM E.1. *The kernel* $\kappa_{P^{-1}\Theta}$ *is a polyhomogeneous conormal distribution on* $[{}^aF \times \mathbb{R}^n; \partial_D iag_f, S]$, *rapidly decreasing at infinity with*

$$\kappa_{P^{-1}\Theta} \in \mathcal{A}^{0,E_{\Theta f}}_{\text{phg}}([{}^aF \times \mathbb{R}^n; \partial_D iag_f, S]) otimes \beta^*_\Theta(K_l).[E2.5]$$

The index set is given by

$$E_{\Theta f} = \{(-2n, 1), (0, 1)\}.[E2.6]$$

K_l *is the pullback of the canonical bundle on* ${}^aF$ *to* ${}^aF \times \mathbb{R}^n$.

PROOF. We use the asymptotic expansion (E.1) and the spherical symmetry of $m_n(\xi)$ to reduce the computation to the evaluation of one dimensional integrals. Let $t = x - x'$ we can rewrite (E.2)

$$\kappa_{P^{-1}}(t, y) = \int\limits_{\mathbb{S}^{n-1}} \int\limits_0^\infty \frac{e^{\imath\sigma\omega\cdot(t+\imath y)}}{m_n(\sigma)} \sigma^{n-1}\, d\sigma.[E2.7]$$

From (E.1) it is evident that all that is required are asymptotic expansions as $|t| \to 0$ and $|y| \to 1$ for the integrals

$$I_p(t, y) = \int\limits_{\mathbb{S}^{n-1}} \int\limits_1^\infty e^{-\sigma(1-\omega\cdot(t+\imath y))} \sigma^{\frac{3n-1}{2}-p}\, d\sigma, \quad p \in \mathbb{N}_0,$$

which show that they lift to be polyhomogeneous conormal on $[{}^aF \times \mathbb{R}^n; \partial_D iag_f, S]$.

Integrating in σ and ignoring terms which are smooth near $\partial\Delta_f$ leaves terms of the form:

$$\int\limits_{\mathbb{S}^{n-1}} (1 - \imath\omega\cdot z)^l\, d\omega \qquad \begin{aligned} l \in &\{-\frac{3n+1}{2}, -\frac{3n-1}{2}, \dots\} \text{ for } n \text{ even,} \\ l \in &\{-\frac{3n+1}{2}, \dots, -2, -1\} \text{ for } n \text{ odd} \end{aligned} \quad [E2.12]$$

and if n is odd:

$$\int\limits_{\mathbb{S}^{n-1}} (1 - \imath\omega\cdot z)^k \log(1 - \imath\omega\cdot z)\, d\omega, \quad k \in \mathbb{N}_0.[E2.13]$$

To evaluate these integrals we use the fact that they are holomorphic for $|\operatorname{Im} z| < 1$ and thus we only need to evaluate them where $\operatorname{Re} z = 0$ and use analytic continuation. Using spherical symmetry we reduce to one dimensional integrals:

$$\begin{aligned} \mathcal{I}_{l,n}(|y|) &= \int\limits_0^\pi (1 + |y| \cos\phi)^l (\sin\phi)^{n-2}\, d\phi \text{ and} \\ \mathcal{J}_{k,n}(|y|) &= \int\limits_0^\pi (1 + |y| \cos\phi)^k \log(1 + |y|\cos\phi)(\sin\phi)^{n-2}\, d\phi. \end{aligned} \quad [E2.14]$$

Applying the standard asymptotic expansions for these integrals and the fact that

$$-|y|^2 = z \cdot z \restriction_{\operatorname{Re} z = 0} [E2.22]$$

we obtain

$$\kappa_{P^{-1}}(z) = \frac{G_n(1 + z \cdot z)}{(1 + z \cdot z)^{n+1}} + \log(1 + z \cdot z)H_n(1 + z \cdot z)[E2.23]$$

where G_n and H_n are smooth functions. The statement of the theorem follows immediately from (E.2) and the fact that

$$\beta_\Theta^*(1 + (x - x' + \imath y) \cdot (x - x' + \imath y)) = \rho_{\Theta\mathrm{f}}^2 F, [E2.24]$$

where F is smooth and non-vanishing in a neighbourhood of ff . □

Let Γ be a lattice in $\mathbb{R}^n$. Since $\kappa_{P^{-1}}(t + iy)$ is rapidly decreasing as t tends to infinity we can sum the kernel over the lattice to obtain a kernel defined on the quotient ${}^aF/\Gamma \times \mathbb{R}^n/\Gamma$:

$$\kappa_{P^{-1}}^\Gamma(x - x', y) = \sum_{n \in \Gamma} \kappa_{P^{-1}}(x + n - x' + iy).$$

This kernel is easily seen to define the inverse to the pushforward from holomorphic $(n, 0)$–forms on ${}^aF/\Gamma$ to functions on $\mathbb{R}^n/\Gamma$. The conormal regularity follows easily from Theorem E.1. In this case the adiabatic limit can be computed explicitly. The ϵ neighborhood of $\mathbb{R}^n/\Gamma$ in ${}^aF/\Gamma$ is biholomorphically equivalent to ${}^aF/\Gamma_\epsilon$ where

$$\Gamma_\epsilon = \{\frac{1}{\epsilon}n; \quad n \in \Gamma\}.$$

The kernel on the shrinking tubes is given by

$$\kappa_{P^{-1}}^\epsilon(x - x' + iy) = \sum_{n \in \Gamma} \kappa_{P^{-1}}(x + \frac{n}{\epsilon} - x' + iy).$$

APPENDIX F

The Laplacian on (p, q)–forms

F.1. The model problems

The discussion at the beginning of §4 establishes that, on the unit ball with the Bergman metric, the equation

$$(\Delta_{p,q} - \mu)e_{p,q} = \delta_{p,q}(0)[F1.1]$$

can be reduced to a system of ordinary differential equations. This system has analytic coefficients and regular singular points at 0 and 1. The standard theory for such equations implies that infinite order, formal solutions at 1 are actually solutions, analytic in the disk of radius 1 about 1. The singularities on the boundary of this disk occur only at 0 and are completely described by the indicial roots at 0.

In this section we compute the indicial systems for the different possible values of p, q. For a given p, q, the indicial roots of this system are naturally parameterized as meromorphic functions on a compact Riemann surface, $\Sigma^{p,q}$. This surface has a finite group conformal automorphisms. A fundamental domain for this action is given by the closure of the 'physical' or L^2-resolvent set. There is a finite set of points on the surface corresponding to $\mu = \infty$. Denote this set by $B^{p,q}$. As in the case of functions, there is a discrete subset of $\Sigma^{p,q} \setminus B^{p,q}$ where two indicial roots differ by a positive integer denote this set by $A^{p,q}$.

In the complement of $A^{p,q} \cup B^{p,q}$ we can construct a holomorphic family of 'unnormalized' fundamental solutions. These are solutions of (F.1) with $\delta_{p,q}(0)$ replaced by some holomorphic function times $\delta_{p,q}(0)$. This function does not vanish in a open subset of the 'physical' resolvent set and thus is not identically zero. We can therefore obtain a meromorphic continuation of the fundamental solution to $\Sigma^{p,q} \setminus A^{p,q} \cup B^{p,q}$. Via (3), this defines a meromorphic continuation of the resolvent kernel for $\Delta_{p,q}$. Thus we see that the parametrization of the eigenvalue, $\mu = s(n-s)$, used for the Laplacian on functions has a reasonable generalization to (p, q)–forms: the eigenvalue is a meromorphic function on $\Sigma^{p,q}$. As this surface is not of genus zero, the eigenvalue does not have a global expression as a rational function.

To begin the calculation we observe that the indicial roots are intrinsically defined by the operator as in (D.1). Thus we are free to compute them relative to any smooth basis of ${}^{\Theta}\Lambda^{p,q}$. The hyperquadric model leads to the simplest calculations.

The hyperquadric is defined by

$$\rho = \frac{\operatorname{Im} w}{2} - \frac{1}{2}|z|^2 > 0.[F1.2]$$

We define a unit basis of $1, 0$ forms by

$$\nu = \frac{\partial\rho}{\rho}, \omega_i = \frac{dz_i}{\sqrt{2\rho}}, i = 1, \ldots, n-1.$$

The Hermitian metric is given by

$$ds^2_{\mathrm{B}} = \nu \circ \bar\nu + \sum_{i=1}^{n-1} \omega_i \circ \bar\omega_i.$$

Let I, J denote multi-indices, denote the $(p,0)$–forms and $(0,q)$–forms by

$$I = (i_1, \dots, i_p), \quad J = (j_1, \dots, j_q);\ 1 \le i_k, j_l \le n-1,$$

let

$$\omega^I = \omega_{i_1} \wedge \dots \wedge \omega_{i_p}, \quad \bar\omega^J = \bar\omega_{j_1} \wedge \dots \wedge \bar\omega_{j_q},$$

and the 'tangential Kähler form'

$$\eta = \sum_{i=1}^{n-1} \omega_i \wedge \bar\omega_i.$$

The volume form, which defines the orientation, is given by

$$\mathrm{dVol} = \frac{i(-1)^{n-1}}{2} \nu \wedge \bar\nu \wedge \mathbf{\Omega} \text{ where } \mathbf{\Omega} = C_n \omega_1 \wedge \dots \omega_{n-1} \wedge \bar\omega_1 \wedge \dots \wedge \bar\omega_{n-1},$$

C_n is a dimensional constant. As usual we denote by $|I|$ the length of a multi-index.

The exterior algebra splits into four disjoint pieces:

$$\begin{aligned} {}^{\Theta}\Lambda^{p,q}_{00} &= \mathrm{sp}\{\omega^I \wedge \bar\omega^J; |I| = p, |J| = q\}, \\ {}^{\Theta}\Lambda^{p,q}_{10} &= \mathrm{sp}\{\nu \wedge \omega^I \wedge \bar\omega^J; |I| = p-1, |J| = q\}, \\ {}^{\Theta}\Lambda^{p,q}_{01} &= \mathrm{sp}\{\omega^I \wedge \bar\omega^J \wedge \bar\nu; |I| = p, |J| = q-1\}, \\ {}^{\Theta}\Lambda^{p,q}_{11} &= \mathrm{sp}\{\nu \wedge \bar\nu \wedge \omega^I \wedge \bar\omega^J; |I| = p-1, |J| = q-1\}. \end{aligned} \quad .[F1.3]$$

For particular values of p, q, certain of these spaces may be empty.

As we will see, the indicial operator preserves this decomposition. To compute the indicial operator for $\Delta_{p,q}$ it suffices to determine the action of this operator on sections of ${}^{\Theta}\Lambda^{p,q}$ whose coefficients, with respect to the basis (F.1), depend only on ρ. To that end, we collect some formulæ for the action of $\overline\partial$ and the Hodge star operator in this basis. We denote by * the Hodge operator on ${}^{\Theta}\Lambda$ with respect to dVol and by * the Hodge operator on the subalgebra,

$${}^{\Theta}\Lambda_\omega = \mathrm{sp}\{\omega^I \wedge \bar\omega^J, |I| \le n-1, |J| \le n-1\},$$

with respect to $\mathbf{\Omega}$. We have

$$\begin{gathered} \overline\partial(\omega^I \wedge \bar\omega^J) = -\left(\frac{|I|+|J|}{2}\right) \bar\nu \wedge \omega^I \wedge \bar\omega^J, \quad {}^*(\omega^I \wedge \bar\omega^J) = \frac{\nu \wedge \bar\nu}{2} \wedge {}^\star(\omega^I \wedge \omega^J); \\ \overline\partial(\nu \wedge \omega^I \wedge \bar\omega^J) = (\eta + \nu \wedge \bar\nu) \wedge \omega^I \wedge \bar\omega^J + \left(\frac{|I|+|J|}{2}\right) \nu \wedge \bar\nu \wedge \omega^I \wedge \bar\omega^J, \\ {}^*(\nu \wedge \omega^I \wedge \bar\omega^J) = {}^\star(\omega^I \wedge \omega^J) \wedge \bar\nu; \\ \overline\partial(\omega^I \wedge \bar\omega^J \wedge \bar\nu) = 0, \quad {}^*(\omega^I \wedge \bar\omega^J \wedge \bar\nu) = -\nu \wedge {}^\star(\omega^I \wedge \omega^J); \\ \overline\partial(\nu \wedge \bar\nu \wedge \omega^I \wedge \bar\omega^J) = \eta \wedge \bar\nu \wedge \omega^I \wedge \bar\omega^J, \quad {}^*(\nu \wedge \bar\nu \wedge \omega^I \wedge \bar\omega^J) = 2^\star(\omega^I \wedge \omega^J). \end{gathered} \quad [F1.4]$$

In order to put the expressions for the action of the indicial operator in a convenient form, we need to define two further algebraic operators. These are defined on ${}^{\Theta}\Lambda_\omega$ in terms of the 'tangential Kähler' form, η:

$$U\xi = \frac{i\eta \wedge \xi}{2}, \quad \Upsilon\xi = (-1)^{p+q\star}[\frac{i\eta}{2} \wedge {}^\star\xi], \quad \xi \in {}^{\Theta}\Lambda^{p,q}_\omega$$

The operators, U, Υ, are adjoint relative to the induced metric on ${}^{\Theta}\Lambda_{\omega}$. To complete the discussion we let Π_r denote the projection of ${}^{\Theta}\Lambda_{\omega}$ onto $\oplus_{p+q=r}{}^{\Theta}\Lambda^{p,q}_{\omega}$. Define

$$H = \sum_{i=0}^{2(n-1)} (n-1-r)\Pi_r.$$

As is shown in [?], the operators U, Υ, H satisfy the commutation relations

$$[\Upsilon, U] = H,\ [H, U] = -2U,\ [H, \Upsilon] = -2\Upsilon.[F1.5]$$

It follows from (F.1) that ${}^{\Theta}\Lambda_{\omega}$ is an $\mathfrak{sl}(2, \mathbb{C})$-module. We discuss this further in a moment.

The Laplace operator on (p, q)–forms is normalized by

$$\Delta_{p,q}\psi = \left({}^{\Theta}\overline{\partial}^{*}\,{}^{\Theta}\overline{\partial} + {}^{\Theta}\overline{\partial}\,{}^{\Theta}\overline{\partial}^{*}\right)\psi.$$

Using the formulæ in (F.1) one easily derives, for $f = f(\rho)$, that

$$\begin{gathered}
-\Delta_{p,q} f\omega^I \wedge \bar{\omega}^J = \\
2\Big[\left((\rho\partial_\rho)^2 - n\rho\partial_\rho + \left(\frac{2n-p-q}{2}\right)\left(\frac{p+q}{2}\right)\right) f\omega^I \wedge \bar{\omega}^J [F1.6] \\
-fU\Upsilon(\omega^I \wedge \bar{\omega}^J)\Big],\ |I| = p, |J| = q, \\
-\Delta_{p,q} f\nu \wedge \omega^I \wedge \bar{\omega}^J = \\
2\Big[\left((\rho\partial_\rho)^2 - n\rho\partial_\rho + \left(\frac{2n-1-p-q}{2}\right)\left(\frac{p+q+1}{2}\right)\right) f\nu \wedge \omega^I \wedge \bar{\omega}^J [F1.7] \\
-f\nu \wedge \Upsilon U(\omega^I \wedge \bar{\omega}^J)\Big],\ |I| = p-1, |J| = q, \\
-\Delta_{p,q} f\omega^I \wedge \bar{\omega}^J \wedge \bar{\nu} = \\
2\Big[\left((\rho\partial_\rho)^2 - n\rho\partial_\rho + \left(\frac{2n+1-p-q}{2}\right)\left(\frac{p+q-1}{2}\right)\right) f\omega^I \wedge \bar{\omega}^J \wedge \bar{\nu} [F1.8] \\
-fU\Upsilon(\omega^I \wedge \bar{\omega}^J) \wedge \bar{\nu}\Big],\ |I| = p, |J| = q-1, \\
-\Delta_{p,q} f\nu \wedge \bar{\nu} \wedge \omega^I \wedge \bar{\omega}^J = \\
2\Big[\left((\rho\partial_\rho)^2 - n\rho\partial_\rho + \left(\frac{2n-p-q}{2}\right)\left(\frac{p+q}{2}\right)\right) f\omega^I \wedge \bar{\omega}^J \wedge \nu \wedge \bar{\nu} [F1.9] \\
-f\Upsilon U(\omega^I \wedge \bar{\omega}^J) \wedge \nu \wedge \bar{\nu}\Big],\ |I| = p-1, |J| = q-1.
\end{gathered}$$

As asserted above, the indicial operator respects the splitting in (F.1). Using the Lie algebra structure of ${}^{\Theta}\Lambda_{\omega}$ we can completely diagonalize the indicial operators. Following [?], we define the subspaces $P^{p,q}_{\omega} \subset {}^{\Theta}\Lambda^{p,q}_{\omega}$ by

$$P^{p,q}_{\omega} = \{\xi \in {}^{\Theta}\Lambda^{p,q}_{\omega};\ \Upsilon\xi = 0\}.[F1.10]$$

Elements in these subspaces are called primitive. The basic properties of the primitive subspaces are summarized in the following proposition. The proofs can be found in [?] or [?].

Proposition F.1. *The primitive subspace, $P^{p,q}_{\omega}$, is non-zero if and only if $p + q \leq n - 1$. The operator $U^{n-1-p-q}$ defines an isomorphism of $P^{p,q}_{\omega}$ onto $P^{2(n-1)-p-q}_{\omega}$. The primitive subspaces have the alternate characterization*

$$P^{p,q}_{\omega} = \operatorname{null} U^{n-p-q} \restriction_{{}^{\Theta}\Lambda^{p,q}_{\omega}} .[F1.12]$$

If $[n]^+$ *denotes the map from* $\mathbb{Z}$ *to* $\mathbb{N}_0$ *defined by*

$$[n]^+ = \begin{matrix} n \text{ if } n \geq 0 \\ 0 \text{ if } n < 0, \end{matrix}$$

then

$$^{\Theta}\Lambda^{p,q}_{\omega} = \bigoplus_{k=[p+q+1-n]^+}^{\min\{p,q\}} U^k P^{p-k,q-k}_{\omega}.[F1.13]$$

To diagonalize the indicial system we use the decomposition given in (F.1). To that end we prove the following lemma

LEMMA F.1. *Let* $k \in \mathbb{N}$ *then*

$$\begin{aligned}(U\Upsilon)U^k &= U^{k+1}\Upsilon + kHU^k + k(k+1)U^k, \\ (\Upsilon U)U^k &= U^{k+1}\Upsilon + (k+1)HU^k + k(k+1)U^k.\end{aligned}[F1.15]$$

PROOF. The second formula follows easily form the first using the commutation relation $[\Upsilon, U] = H$. The first formula we prove inductively. For $k = 1$, use the commutation relations to obtain

$$(U\Upsilon)U = U^2 + UH = U^2 + HU + 2U.$$

Assuming, by induction, that (F.1) holds for k, we have

$$\begin{aligned}(U\Upsilon)U^{k+1} &= U^2\Upsilon U^k + U[\Upsilon, U]U^k \\ &= U^2\Upsilon U^k + UHU^k\end{aligned}.[F1.16]$$

The result follows easily from (F.1), the inductive hypothesis and the commutation formula $[U, H] = 2U$. □

Since the primitive subspaces are annihilated by Υ, it follows easily from (F.1)-(F.1) and Lemma F.1 that the indicial operator is diagonal on elements of $^{\Theta}\Lambda^{p,q}_{\omega}$ of the following types:

$$fU^k\xi,\ f\nu \wedge U^k\xi,\ fU^k\xi \wedge \bar{\nu},\ f\nu \wedge \bar{\nu} \wedge U^k\xi \text{ where } \xi \text{ is primitive.}[F1.17]$$

Proposition F.1 implies that there is a basis composed of such elements. This allows us to determine the indicial roots. In this basis the indicial operator is a decoupled system of ODEs. Note that this system is actually the full Laplace operator acting on (p,q)–forms of the simple type $\xi = f(\rho)\psi$, here ψ is a basis element as in (F.1). Thus we see that 'fundamental' eigenfunctions of $\Delta_{p,q}$ are of the form $\rho^z\psi$. Here z is the indicial root corresponding to the characteristic direction ψ. One could construct a theory of Eisenstein series formally identical to the theory for functions. The resolvent kernel and its meromorphic continuation could also be constructed through a process of spectral synthesis. We prefer a more direct route.

The Laplace operator satisfies the following relations

$$^*\Delta_{p,q} = \Delta_{n-p,n-q}{}^*, \quad \overline{\Delta_{p,q}\psi} = \Delta_{q,p}\overline{\psi}.[F1.18]$$

In light of (F.1), it clearly suffices to determine the indicial roots for p, q satisfying $p + q \leq n$, $p \leq q$.

THEOREM F.1. *For p, q such that $0 < p+q \leq n$, $p \leq q$, the indicial roots of $\Delta_{p,q} - \mu$ are the solutions of the equations:*

$$
\begin{aligned}
&\text{on } {}^{\Theta}\Lambda_{00} : (z_{00}^k - \frac{n}{2})^2 - \left(\frac{n-p-q}{2}\right)^2 - k(n+k-p-q) = -\mu,\ k = 0\ldots,p,\\
&\text{on } {}^{\Theta}\Lambda_{10} : (z_{10}^k - \frac{n}{2})^2 - \left(\frac{n-1-p-q}{2}\right)^2 - (k+1)(n+k-p-q) = -\mu,\ k = 1\ldots,p,\\
&\text{on } {}^{\Theta}\Lambda_{01} : (z_{01}^k - \frac{n}{2})^2 - \left(\frac{n-1-p-q}{2}\right)^2 - (k+1)(n+k-p-q) = -\mu,\ k = 0\ldots,p,\\
&\text{on } {}^{\Theta}\Lambda_{11} : (z_{11}^k - \frac{n}{2})^2 - \left(\frac{n-p-q}{2}\right)^2 - k(n+k-p-q) = -\mu,\ k = 1\ldots,p.
\end{aligned}
\quad [F1.20]
$$

PROOF. The equations are simple algebraic consequences of (F.1)-(F.1) and Lemma F.1. It follows from Proposition F.1 that each of the equations corresponds to a non-trivial subspace of ${}^{\Theta}\Lambda^{p,q}$. □

The variables, $z_{ij}^k, i, j \in \{0, 1\}$ can be thought of as affine coordinates on $\mathbb{P}^N$, where $N = 4p + 2$. Eliminating μ from the equations in (F.1) leads to a system of $4p + 1$ independent equations. Let $\Sigma^{p,q}$ denote the curve in $\mathbb{P}^N$ defined by this system. The curve is smooth, μ is obviously a meromorphic function on it. The curve is immersed in $\mathbb{P}^N$ having a multiple normal intersection along the polar divisor of μ.

From (F.1) it is clear that all the functions z_{ij}^k have the same polar divisor. A simple calculation shows that the image of $\Sigma^{p,q}$ in $\mathbb{P}^N$ has a 2^{4p}-fold normal crossing along this divisor. We denote this divisor by $B^{p,q}$. The affine coordinate z_{00}^0 defines a mapping of $\Sigma^{p,q}$ to $\mathbb{P}^1$. It is a 2^{4p+1}-sheeted cover, with a branch locus of degree $2^{4p+1}(4p + 1)$. The branch locus is disjoint from $B^{p,q}$. Using the Riemann-Hurwitz formula we compute the genus of $\Sigma^{p,q}$:

$$g(\Sigma^{p,q}) = 1 + 2^{4p}(4p - 1).[F1.21]$$

The linear maps of $\mathbb{A}^N$ defined by

$$(z_{ij}^k - \frac{n}{2}) \longrightarrow (-1)^{\epsilon_{ij}^k}(z_{ij}^k - \frac{n}{2}), \quad \epsilon_{ij}^k \in \{0, 1\}[F1.22]$$

preserve the curve $\Sigma^{p,q}$ and thus define a action of $\mathbb{Z}/2^{4p+2}$ on this curve. A fundamental domain in $\mathbb{A}^N$ is given by

$$\mathcal{D} = \{\operatorname{Re} z_{ij}^k \geq \frac{n}{2}\}.[F1.23]$$

The intersection, $\mathcal{D}^{p,q} = \mathcal{D} \cap \Sigma^{p,q}$, is a fundamental domain for this group action on $\Sigma^{p,q}$. In the interior of $\mathcal{D}^{p,q}$ the indicial roots are:

$$\begin{aligned} z_{00}^k &= \frac{n}{2} + \sqrt{\left(\frac{n-p-q}{2}\right)^2 + k(n+k-p-q) - \mu}, \quad k = 0,\dots,p, \\ z_{11}^k &= \frac{n}{2} + \sqrt{\left(\frac{n-p-q}{2}\right)^2 + k(n+k-p-q) - \mu}, \quad k = 1,\dots,p, \\ z_{10}^k &= \frac{n}{2} + \sqrt{\left(\frac{n-1-p-q}{2}\right)^2 + (k+1)(n+k-p-q) - \mu}, \quad k = 1,\dots,p, \\ z_{01}^k &= \frac{n}{2} + \sqrt{\left(\frac{n-1-p-q}{2}\right)^2 + (k+1)(n+k-p-q) - \mu}, \quad k = 0,\dots,p. \end{aligned} \qquad [F1.24]$$

The parameter $\mu \in \mathbb{C} \setminus [\frac{(n-p-q)^2}{4}, \infty)$; the branch of the square root is taken to have positive real part for real, negative μ.

We define the set of accidental multiplicities, $A^{p,q}$ as the set of points on $\Sigma^{p,q}$ where $z_{ij}^k - z_{lm}^n \in \mathbb{N}$ for some pair of triples (i,j,k), (l,m,n) in the allowable range. We let

$$A_+^{p,q} = \mathcal{D}^{p,q} \cap A^{p,q}, \quad A_-^{p,q} = A^{p,q} \setminus A_+^{p,q}.[F1.25]$$

It is elementary to show that $A_+^{p,q}$ is a finite set. Using energy estimates, see [?], one can show that

$$\operatorname{spec}(\Delta_{p,q}) = \begin{cases} \left[\left(\frac{n-p-q}{2}\right)^2, \infty\right), \text{ for } p+q \neq n, \\ \left[\left(\frac{n+1-p-q}{2}\right)^2, \infty\right) \cup \{0\} \text{ for } p+q = n \end{cases} \qquad [F1.26]$$

For $p+q=n$ the isolated eigenvalue at zero is of infinite multiplicity. For $p+q \neq n$, the resolvent is an analytic family of operators in $\overset{\circ}{\mathcal{D}}{}^{p,q}$, however as a family of supported distributions it may not be analytic in $A_+^{p,q}$. As in the case of $(n,1)$–forms, this manifests itself by the appearance of log-terms in the asymptotic expansion at the left and right boundaries.

In order to construct a fundamental solution we first construct analytic families of radial solutions to the ODE system analogous to (3). This system is the Laplace operator on radial, vector valued (p,q)–forms, as in (3). We denote this operator by $\Delta_{p,q}^{\text{rad}}$. Its indicial roots are those computed in Theorem F.1. Identify the point $(i,0)$ in the hyperquadric model with $(0,\dots,0)$ in the unit ball model and the geodesic ray $\gamma_h = \{(i\rho, 0), \rho \in [0,1]\}$ with the geodesic ray $\gamma_b = \{(r,0), r \in [0,1]\}$. The splitting of ${}^{\Theta}\Lambda^{p,q}$ given in (F.1) restricts, along γ_h, to give an analogous splitting of ${}^{\Theta}\Lambda^{p,q}$ relative to the spheres centered at $(i,0)$. Thus we can use the bases given in (F.1) to define the characteristic directions on γ_b at $r=1$ in the ball model.

As above we fix p,q such that $p+q \leq n$, $p \leq q$. The general case is handled by applying (F.1). Let $D_{rs} = \dim P_\omega^{r,s}$. If $\{\xi_i^k, i = 1,\dots,D_{rs}\}$ denotes a unit basis for $P_\omega^{r,s}$ then define

$$\Xi^{r,s} = (\xi_1, \dots, \xi_{D_{rs}})^t.$$

Choose a local uniformizing parameter, τ for an open set on $\Sigma^{p,q}$. The radial (p,q)–forms, $u^k_{ij}(r;\tau)$, sections of ${}^{\Theta}\Lambda^{p,q}_{ij}$; $i,j\in\{0,1\}$, are solutions to

$$(\Delta^{\mathrm{rad}}_{p,q}-\mu(\tau))u=0,\ r\in(0,1),[F1.27]$$

normalized by

$$\begin{aligned} u^k_{00}(r;\tau)&=(1-r)^{z^k_{00}(\tau)}\Xi^{p-k,q-k}+O\left((1-r)^{1+z^k_{00}(\tau)}\right),k=0,\dots,p;\\ u^k_{10}(r;\tau)&=(1-r)^{z^k_{10}(\tau)}\nu\wedge\Xi^{p-k,q+1-k}+O\left((1-r)^{1+z^k_{10}(\tau)}\right),k=1,\dots,p;\\ u^k_{01}(r;\tau)&=(1-r)^{z^k_{01}(\tau)}\Xi^{p-k,q-1-k}\wedge\bar\nu+O\left((1-r)^{1+z^k_{01}(\tau)}\right),k=0,\dots,p;\\ u^k_{11}(r;\tau)&=(1-r)^{z^k_{11}(\tau)}\Xi^{p-k,q-k}\wedge\nu\wedge\bar\nu+O\left((1-r)^{1+z^k_{11}(\tau)}\right),k=1,\dots,p.\end{aligned}\quad[F1.28]$$

In virtue of the formal nature of their construction, the solutions defined in (F.1) are holomorphic in $\Sigma^{p,q}_a=\Sigma^{p,q}\setminus A^{p,q}\cup B^{p,q}$.

There is an analytic combination of the form

$$\widetilde{e}^{\,p,q}(r,\tau)=\sum_{k=0}^{p}(a_k(\tau)u^k_{00}+b_k(\tau)u^k_{01})+\sum_{k=1}^{p}(c_k(\tau)u^k_{10}+d_k(\tau)u^k_{11})[F1.30]$$

which satisfies the distributional equation

$$\Delta_{p,q}\widetilde{e}^{\,p,q}=m(\tau)\delta_{p,q}.$$

Here $m(\tau)$ is an analytic function on $\Sigma^{p,q}_a$. As the characteristic directions are independent on the domain of analyticity, it follows that $m(\tau)$ can be chosen not to vanish on $\overset{\circ}{\mathcal{D}}{}^{p,q}\setminus A^{p,q}_+$. Otherwise we could construct nontrivial L^2 solutions for energies in the resolvent set. Hence

$$e^{p,q}(r;\tau)=\frac{\widetilde{e}^{\,p,q}(r;\tau)}{m(\tau)}$$

is analytic in $\overset{\circ}{\mathcal{D}}{}^{p,q}\setminus A^{p,q}_+$ and meromorphic on $\Sigma^{p,q}_a$. It satisfies the distributional equation

$$\Delta^{\mathrm{B}}_{p,q}e^{p,q}=\delta_{p,q}$$

and is in L^2 near $r=1$ for $\tau\in\overset{\circ}{\mathcal{D}}{}^{p,q}\setminus A^{p,q}_+$. By applying (3) we obtain the resolvent kernel, $E^{p,q}(\tau)$ for $\Delta^{\mathrm{B}}_{p,q}$.

Using the blow-down map $\beta^{(2)}_{\Theta}$, we can lift the kernel up to $(\mathbb{CB}^n_{\frac12})^2_{\Theta}$. Set

$$E^{p,q}_{\Theta}=\beta^{(2)*}_{\Theta}E^{p,q}.$$

For $\tau\in\Sigma^{p,q}$ define the index set:

$$\begin{aligned}\mathcal{I}^{p,q}_{\tau}=\{2z^0_{00}(\tau)\}\overline{\cup}\dots\overline{\cup}\{2z^p_{00}(\tau)\}\overline{\cup}\{2z^0_{01}(\tau)\}\overline{\cup}\dots\overline{\cup}\{2z^p_{01}(\tau)\}\overline{\cup}\\ \{2z^1_{10}(\tau)\}\overline{\cup}\dots\overline{\cup}\{2z^p_{10}(\tau)\}\overline{\cup}\{2z^1_{11}(\tau)\}\overline{\cup}\dots\overline{\cup}\{2z^p_{11}(\tau)\}.\end{aligned}\quad[F1.130]$$

Recall that the indicial roots are meromorphic functions on $\Sigma^{p,q}$. On $\Sigma^{p,q}_a$ there are no log-terms in $\mathcal{I}^{p,q}_{\tau}$.

THEOREM F.2. *The lifted resolvent kernel for the Laplace operator on (p,q)–forms, defined by the Bergman metric on the unit ball, has a meromorphic extension to the Riemann surface $\Sigma^{p,q}_a$, as a supported distribution on the space $(\mathbb{CB}^n_{\frac12})^2_{\Theta}$.*

In its domain of analyticity the lifted kernel satisfies:

$$E^{p,q}_{\Theta}(\tau) \in \Psi^{-2;\mathcal{I}^{p,q}_{\tau},\mathcal{I}^{p,q}_{\tau}}_{\Theta}(\mathbb{CB}^n_{\frac{1}{2}};{}^{\Theta}\Lambda^{p,q},{}^{\Theta}\Lambda^{p,q}).[F1.32]$$

If $p+q\neq n$ then it is analytic as an extendible distribution on $\overset{\circ}{\mathcal{D}}{}^{p,q}$. If $p+q=n$ then it is analytic as an extendible distribution on $\overset{\circ}{\mathcal{D}}{}^{p,q}\setminus\{\mu=0\}$; it has a pole at $\{\mu=0\}\cap\overset{\circ}{\mathcal{D}}{}^{p,q}$. The residue is a projector onto an infinite dimensional null space. The extended kernel also satisfies (F.2).

For the sake of consistency with the previous development we restate (F.2) for the action of $\Delta_{p,q}$ on half density valued forms. Proceeding as in (3)-(3) we obtain the lifted resolvent kernel for this action $\widetilde{E}^{p,q}_{\Theta}$. The index set at the right boundary is shifted by $N=2n+1$:

$$\widetilde{E}^{p,q}_{\Theta} \in \Psi^{-2;\mathcal{I}^{p,q}_{\tau},\mathcal{I}^{p,q}_{\tau}-N}_{\Theta}(\mathbb{CB}^n_{\frac{1}{2}};{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}},{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}}).[F1.33]$$

The solution of the 'model' problem constitutes the main step in constructing the resolvent kernel and its meromorphic continuation for $\Delta_{p,q}$ on a strictly pseudoconvex domain with metric of the form (0.2). As in the case of $(n,1)$–forms, the general case has a 'Hodge' decomposition. For most values of p,q there is also a decomposition analogous to the Hard Lefschetz theorem for the Dolbeault cohomology of a compact Kähler manifold. These topics are addressed in the next section.

F.2. The resolvent kernel on (p,q)–forms

In this section Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$. The metric g_r is of the form given (0.2). The construction of the resolvent kernel for the Laplace operator on (p,q)–forms closely follows the pattern of the previous constructions, see §5. We will simply state the result, sketch the proof and discuss some of the finer structure of these kernels.

Theorem F.3. *The resolvent kernel $R^{p,q}(\tau)$ for $\Delta_{p,q}$ on half-density valued forms has a meromorphic continuation to $\tau\in\Sigma^{p,q}_a$. In its domain of holomorphy it satisfies*

$$\begin{gathered}R^{p,q}(\tau)\in\Psi^{-2;\mathcal{I}^{p,q}_{\tau},\mathcal{I}^{p,q}_{\tau}-N}_{\Theta}(\mho;{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}},{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}})+\\ \Psi^{-\infty;\mathcal{I}^{p,q}_{\tau},\mathcal{I}^{p,q}_{\tau}-N}(\mathbb{CB}^n_{\frac{1}{2}};{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}},{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}}).\end{gathered}\quad[F2.2]$$

If $p+q\neq n$ then $\Delta_{p,q}$ is invertible; $R^{p,q}(\tau)$ is meromorphic as an extendible distribution on $\overset{\circ}{\mathcal{D}}{}^{p,q}$. If $p+q=n$ then it is meromorphic as an extendible distribution on $\overset{\circ}{\mathcal{D}}{}^{p,q}\setminus\{\mu=0\}$; it has a pole at $\{\mu=0\}\cap\overset{\circ}{\mathcal{D}}{}^{p,q}$. The residue is a projector onto an infinite dimensional null space. The extended kernel also satisfies (F.3). *The residues at the poles in $\overset{\circ}{\mathcal{D}}{}^{p,q}\setminus\{\mu=0\}$ are finite dimensional projectors onto L^2 eigenspaces of $\Delta_{p,q}$.*

Proof. The argument is essentially identical to the proof of Theorem 4.1. We construct a kernel $Q_0(\tau)$ for $\tau\in\Sigma^{p,q}_a$ which satisfies

$$\begin{gathered}(\Delta_{p,q}-\mu(\tau))Q_0(\tau)-\mathrm{Id}=R_0(\tau)\in\Psi^{-\infty;\emptyset,\mathcal{I}^{p,q}_{\tau}-N,1}_{\Theta}(\mho;{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}},{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}}),\\ Q_0(\tau)\in\Psi^{-2;\mathcal{I}^{p,q}_{\tau},\mathcal{I}^{p,q}_{\tau}-N}_{\Theta}(\mho;{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}},{}^{\Theta}\Lambda^{p,q}\otimes\Omega^{\frac{1}{2}}).\end{gathered}\quad[F2.3]$$

This entails usage of the diagonal symbol, Corollary D.2, the normal operator, Theorem D.2 and the indicial operator at the left boundary, Corollary D.3. As all these operations are formal, the analyticity of Q_0 in τ is identical to that of $E_\Theta^{p,q}$. The error term R_0 is a compact operator on a weighted L^2-space, but can easily be improved.

Apply Theorem D.4 to obtain a Borel summable Neumann series for

$$(\mathrm{Id} + R_0)^{-1}.$$

This operator is of the form $(\mathrm{Id} + R_0')$, where

$$R_0' \in \Psi_\Theta^{-\infty;\emptyset,\mathcal{J}_\tau,1}(\mho; {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}, {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}).$$

Here $\mathcal{J}_\tau$ is a complicated index set, whose exact nature need not concern us, see §5. Setting $Q_1 = Q_0(\mathrm{Id} + R_0')$ we obtain an new parametrix, this time with a residual error:

$$\begin{aligned}(\Delta_{p,q} - \mu(\tau))Q_1\tau - \mathrm{Id} = R_1(\tau) \in \Psi^{-\infty;\emptyset,\mathcal{J}_\tau}(\mho; {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}, {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12})\\ Q_1 \in \Psi_\Theta^{-2;\mathcal{I}_\tau^{p,q},\mathcal{I}_\tau^{p,q}-N}(\mho; {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}, {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}) + \qquad [F2.4]\\ \Psi_\Theta^{-\infty;\mathcal{I}_\tau^{p,q},\mathcal{J}_\tau',\mathcal{K}_\tau}(\mho; {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}, {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}),\end{aligned}$$

where, again, $\mathcal{J}_\tau', \mathcal{K}_\tau$ are complicated index sets of no lasting importance.

Arguing as in §5, we can remove the log terms from the front face in Q_1 at the expense of adding log terms to the right boundary. We will retain the same notation. The error term $(\mathrm{Id} + R_1(\tau))$ is a analytic Fredholm family between weighted L^2 spaces. By a standard argument, it can be assumed invertible for some $\tau \in \overset{\circ}{\mathcal{D}}{}^{p,q} \setminus A_+^{p,q}$. In light of its analyticity it is invertible in the complement of a discrete subset of $\Sigma_a^{p,q}$. Let $Q_2(\tau) = Q_1(\tau)(\mathrm{Id} + R_1(\tau))^{-1}$. This kernel satisfies

$$\begin{aligned}(\Delta_{p,q} - \mu(\tau))Q_2(\tau) = \mathrm{Id}\\ Q_2(\tau) \in \Psi_\Theta^{-2;\mathcal{I}_\tau^{p,q},\mathcal{J}_\tau''}(\mho; {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}, {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}) + [F2.5]\\ \Psi^{-\infty;\mathcal{I}_\tau^{p,q},\mathcal{J}_\tau''}(\mho; {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}, {}^\Theta\Lambda^{p,q} \otimes \Omega^{\frac12}).\end{aligned}$$

The argument for generic τ is completed by using the self adjointness of $\Delta_{p,q}$ to remove the complicated index set from the right boundary. These symmetry considerations imply that Q_2 belongs to the symbol class given in (F.3), at least for τ in the complement of a discrete subset of $\Sigma_a^{p,q}$.

All that remains is to complete the discussion for τ in the resolvent set. The invertibility for $p + q \neq n$ follows from results in [?]. If $\tau \in \overset{\circ}{\mathcal{D}}{}^{p,q} \setminus A_+^{p,q}$ then the argument just concluded and the invertibility of the operator in the resolvent set suffice to complete the argument. In case $p + q \neq n$ then an argument similar to that used at the end of the proof of Theorem 4.1 is needed to obtain the correct index set for $\tau \in A_+^{p,q}$. The precise description of the continuity near $A_+^{p,q}$ follows from Lemma 3.1 as in the proof of Theorem 4.1.

For $p+q = n$ the only difficulty arises at $\mu(\tau) = 0, \tau \in \overset{\circ}{\mathcal{D}}{}^{p,q}$. We can circumvent this by expressing the resolvent kernel on the orthocomplement of the null space in terms of $(\Delta_{p,q-1} - \mu)^{-1}$ and $(\Delta_{p,q+1} - \mu)^{-1}$. This is discussed below. □

The Hodge decomposition is

$$L^2(\Omega, {}^\Theta\Lambda^{p,q}) = {}^\Theta\overline{\partial}\,\mathrm{Dom}_{p,q-1}({}^\Theta\overline{\partial}) \oplus {}^\Theta\overline{\partial}^*\,\mathrm{Dom}_{p,q+1}({}^\Theta\overline{\partial}^*) \oplus \mathrm{null}\,\Delta_{p,q}.[F2.6]$$

Here the domains of the operators ${}^{\Theta}\overline{\partial}$ and ${}^{\Theta}\overline{\partial}^*$ are simply the L^2 closures of these operators restricted to $\dot{\mathcal{C}}^\infty(\Omega;{}^{\Theta}\Lambda^{p,q})$. From (F.2) it follows easily that for $p+q\neq n$ the projectors onto the first two factors in (F.2) are

$$\Pi_{\Theta_{\overline{\partial}}} = {}^{\Theta}\overline{\partial}{}^{\Theta}\overline{\partial}^*\Delta_{p,q}^{-1} \text{ and } \Pi_{\Theta_{\overline{\partial}^*}} = {}^{\Theta}\overline{\partial}^*{}^{\Theta}\overline{\partial}\Delta_{p,q}^{-1}.$$

Thus for p,q in this range the 'Hodge' decomposition of the resolvent kernel is:

$$\begin{aligned} R_+^{p,q} &= (\Delta_{p,q}-\mu)^{-1}\restriction_{\Theta_{\overline{\partial}^*\Theta\Lambda^{p,q+1}}} = {}^{\Theta}\overline{\partial}^*{}^{\Theta}\overline{\partial}(\Delta_{p,q}-\mu)^{-1}\Delta_{p,q}^{-1},\\ R_-^{p,q} &= (\Delta_{p,q}-\mu)^{-1}\restriction_{\Theta_{\overline{\partial}\Theta\Lambda^{p,q-1}}} = {}^{\Theta}\overline{\partial}{}^{\Theta}\overline{\partial}^*(\Delta_{p,q}-\mu)^{-1}\Delta_{p,q}^{-1}.\end{aligned} \quad [F2.7]$$

The decompositions of the the form bundles defined in (F.1) are defined by the Hermitian metric and therefore generalize in an obvious way to a strictly pseudoconvex manifold with metric of the form (0.2). It follows from (F.1) that ${}^{\Theta}\overline{\partial}$ annihilates the leading order terms at the boundary in $(\Delta_{p,q}-\mu)^{-1}$ arising from ${}^{\Theta}\Lambda_{01}^{p,q}$ and ${}^{\Theta}\overline{\partial}^*$ annihilates those arising from ${}^{\Theta}\Lambda_{10}^{p,q}$. On the other hand the equations

$$\begin{aligned} (\Delta_{p,q+1}-\mu){}^{\Theta}\overline{\partial}(\Delta_{p,q}-\mu)^{-1} &= {}^{\Theta}\overline{\partial}\,\mathrm{Id},\\ (\Delta_{p,q-1}-\mu){}^{\Theta}\overline{\partial}^*(\Delta_{p,q}-\mu)^{-1} &= {}^{\Theta}\overline{\partial}^*\,\mathrm{Id}\end{aligned} \quad [F2.8]$$

and the formulæ for the indicial roots imply that the entire asymptotic series corresponding to these characteristic directions must be absent.

Let r be a defining function for $\partial\Omega$ and ψ a tangentially primitive form at $\partial\Omega$. The equation,

$$\overline{\partial}(\nu\wedge\bar{\nu}\wedge\eta^k\wedge\psi) = \bar{\nu}\wedge\eta^{k+1}\wedge\psi + O(r),$$

implies that the leading order part of the asymptotic expansion arising from this term will be killed off if $\eta^{k+1}\wedge\psi = 0$. As above (F.2) then implies that the entire asymptotic series corresponding to these characteristic directions are absent. Similar remarks apply to the action of ${}^{\Theta}\overline{\partial}^*$ on terms coming from ${}^{\Theta}\Lambda_{00}^{p,q}$.

From this discussion it is apparent that, in general, the index set of each Hodge component of $(\Delta_{p,q}-\mu)^{-1}$ is a proper subset of $\mathcal{I}_\tau^{p,q}$. Denote these subsets by $\mathcal{I}_{\tau+}^{p,q}$ and $\mathcal{I}_{\tau-}^{p,q}$ respectively. Each of these index sets corresponds to a subset of the equations in (F.1), thus they are uniformized by Riemann surfaces of lower genus $\Sigma_\pm^{p,q}$. There are covering maps

$$\pi_\pm : \Sigma^{p,q} \longrightarrow \Sigma_\pm^{p,q}.$$

Let $\Sigma_{a\pm}^{p,q}$ denote the surface $\Sigma_\pm^{p,q}$ with the divisor corresponding to accidental multiplicities among $\mathcal{I}_{\tau\pm}^{p,q}$ removed. Let $B_\pm^{p,q} = \pi_\pm(B^{p,q})$. The factors $R_\pm^{p,q}$ are actually meromorphic on the complement of discrete subsets of $\Sigma_{a\pm}^{p,q}\setminus B_\pm^{p,q}$

To conclude this discussion we consider $p+q=n$. We use the identities

$$\Delta_{p,q+1}{}^{\Theta}\overline{\partial} = {}^{\Theta}\overline{\partial}\Delta_{p,q}, \quad \Delta_{p,q-1}{}^{\Theta}\overline{\partial}^* = {}^{\Theta}\overline{\partial}^*\Delta_{p,q} [F2.9]$$

to express the resolvent in terms of the resolvent in adjacent grades. It follow easily from (F.2) that for $p+q=n$ the solutions to the problems:

$$\begin{aligned} {}^{\Theta}\overline{\partial}u &= f, \text{ for } f\in\mathrm{Dom}_{p,q}({}^{\Theta}\overline{\partial}), {}^{\Theta}\overline{\partial}f = 0,\ f\perp\mathrm{null}\,\Delta_{p,q}\\ {}^{\Theta}\overline{\partial}^*v &= g, \text{ for } g\in\mathrm{Dom}_{p,q}({}^{\Theta}\overline{\partial}^*), {}^{\Theta}\overline{\partial}^*g = 0,\ g\perp\mathrm{null}\,\Delta_{p,q}.\end{aligned} \quad [F2.10]$$

are given by

$$u = \Delta_{p,q-1}^{-1}{}^{\Theta}\overline{\partial}^*f, \quad v = \Delta_{p,q+1}^{-1}{}^{\Theta}\overline{\partial}g.[F2.11]$$

Combining (F.2) and (F.2) for $p+q=n$ we obtain

$$\begin{aligned} R_+^{p,q} &= (\Delta_{p,q}-\mu)^{-1}\restriction_{{}^\Theta\overline{\partial}^*\,{}^\Theta\Lambda^{p,q+1}} = {}^\Theta\overline{\partial}^*(\Delta_{p,q+1}-\mu)^{-1}\Delta_{p,q+1}^{-1}\,{}^\Theta\overline{\partial}, \\ R_-^{p,q} &= (\Delta_{p,q}-\mu)^{-1}\restriction_{{}^\Theta\overline{\partial}\,{}^\Theta\Lambda^{p,q-1}} = {}^\Theta\overline{\partial}(\Delta_{p,q-1}-\mu)^{-1}\Delta_{p,q-1}^{-1}\,{}^\Theta\overline{\partial}^*. \end{aligned} [F2.12]$$

The p,q-Bergman projector onto the null space is given by

$$K_0^{p,q} = (\mathrm{Id} - {}^\Theta\overline{\partial}\Delta_{p,q-1}^{-1}\,{}^\Theta\overline{\partial}^* - {}^\Theta\overline{\partial}^*\Delta_{p,q+1}^{-1}\,{}^\Theta\overline{\partial}).[F2.13]$$

and the resolvent kernel is

$$R^{p,q}(\tau) = R_+^{p,q}(\tau) + R_-^{p,q}(\tau) + \frac{K_0^{p,q}}{\mu(\tau)}.[F2.14]$$

The full resolvent kernel is analytic in the complement of a discrete subset of $\Sigma_a^{p,q}$ whereas each of the terms $R_\pm^{p,q}$ is analytic in the complement of a discrete subset of a surface which is covered by $\Sigma_a^{p,q}$. We leave the details to the interested reader. This substantiates the claim made in the proof of Theorem F.3

F.3. The hard Lefschetz theorem

The hard Lefschetz theorem of algebraic geometry, see [**?**] has a very simple analogue in the context of a metric of the form (0.2) on a strictly pseudoconvex domain. We recall the general set up. The Kähler form,

$$\omega = \partial\overline{\partial}\log r,$$

defines a operator of bidegree $(1,1)$ on the Dolbeault complex by the rule

$$U\xi = \frac{i}{2}\omega\wedge\xi.$$

We denote its adjoint by Υ. These operators define a decomposition of the exterior algebra into 'primitive' components. A (p,q)–form, ξ is called primitive if $\Upsilon\xi=0$ or equivalently $U^{n+1-p-q}\xi=0$. Let ${}^\Theta\mathrm{P}^{p,q}$ denote sections of ${}^\Theta\Lambda^{p,q}$ which are primitive. We let Π_r denote the projection onto $\oplus_{p+q=r}{}^\Theta\Lambda^{p,q}$ and define

$$H = \sum_{r=0}^{n}(n-r)\Pi_r.$$

As in §F1 we have the commutation relations

$$[\Upsilon,U]=H,\ [H,U]=-2U,\ [H,\Upsilon]=-2\Upsilon.[F3.1]$$

We summarize the important properties of these operators, the proofs can be found in [**?**].

Proposition F.2. *The sections of the bundles* ${}^\Theta\Lambda^{p,q}$ *split as orthogonal direct sums*

$$\mathcal{C}^\infty(\Omega;{}^\Theta\Lambda^{p,q}) = \bigoplus_{k=[p+q-n]^+}^{\min\{p,q\}} U^k\,{}^\Theta\mathrm{P}^{p-k,q-k}.[F3.3]$$

The operators U *and* Υ *are of norm one or zero on each grade of* ${}^\Theta\Lambda^{p,q}$ *and they satisfy the commutation relations*

$$\begin{aligned} &= [U,{}^\Theta\overline{\partial}] = [\Upsilon,{}^\Theta\overline{\partial}^*] = [\Upsilon,{}^\Theta\partial] = 0, \\ [U,{}^\Theta\overline{\partial}^*] = -i\partial,\ [U,{}^\Theta\partial^*] &= i\overline{\partial},\ [\Upsilon,{}^\Theta\overline{\partial}] = -i\,{}^\Theta\partial^*,\ [\Upsilon,{}^\Theta\partial] = i\,{}^\Theta\overline{\partial}^* \end{aligned} [F3.4]$$

and

$$[U,\Delta]=[\Upsilon,\Delta]=0.[F3.5]$$

PROOF. Most of these statements are proved in [?]. The only statements which requires comment are (F.2) and (F.2). In [?] it is shown that these formulæ hold in a formal sense. This of course implies that they hold on $\dot{\mathcal{C}}^\infty(\Omega;{}^\Theta\Lambda^{p,q})$. Since $\Delta_{p,q}$ is essentially self adjoint on this domain for every p,q, it follows easily from (F.2) and the fact that ∂ and ${}^\Theta\partial^*$ belong to Diff^1_Θ, that U and Υ carry the domain of Δ into itself. From this we conclude that (F.2) holds as well for the L^2 closure of Δ. □

The hard Lefschetz theorem describes the decomposition of the Dolbeault cohomology into primitive components. The result of Donnelly and Fefferman is that the L^2-cohomology is zero unless $p+q=n$.

COROLLARY F.1. *The summands in* (F.2) *are invariant subspaces for* $\Delta_{p,q}$. *If* $p+q=n$ *then*

$$\mathcal{H}^{p,q}_{\mathrm{Dol}}(\Omega)=\{\psi\in\mathrm{Dom}(\Delta_{p,q}):{}^\Theta\overline{\partial}\psi={}^\Theta\overline{\partial}^*\psi=0\}\subset{}^\Theta\,\mathrm{P}^{p,q}\,.[F3.7]$$

PROOF. That the primitive summands are invariant subspaces is an immediate consequence of (F.2), if ξ is primitive then

$$\Upsilon\Delta_{p,q}\xi=\Delta_{p,q}\Upsilon\xi=0,$$

hence $\Delta_{p,q}\xi$ is primitive as well. Moreover (F.2) easily implies that $[U^k,\Delta]=0$. This completes the proof of the first statement.

Suppose that $\psi\in\mathcal{H}^{p,q}_{\mathrm{Dol}}$, observe that

$$0=\Upsilon\Delta_{p,q}\psi=\Delta_{p-1,q-1}\Upsilon\psi.[F3.8]$$

On the other hand the result of Donnelly and Fefferman states that null $\Delta_{p-1,q-1}=\{0\}$ if $p+q=n$. Thus $\Upsilon\psi=0$. □

Beyond (F.1) there does not seem to be a strong connection between the Hodge and the Lefschetz decomposition. This is because (F.2) implies that ${}^\Theta\overline{\partial}$ and ${}^\Theta\overline{\partial}^*$ need not preserve the primitive subspaces.

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