THE THEOREM OF A. SCHUR IN HYPERBOLIC SPACE

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ABSTRACT

We prove a hyperbolic analogue of the Theorem of A. Schur that states:

Schur's Theorem: Let C and C' be two curves in $\mathbb{R}^3$ of the same length with curvatures k and k' respectively, such that

$$|k| \leq k'.$$

Suppose C' is a planar curve which, along with its chord bounds a convex region. Let d and d' be the chordal lengths of C and C' respectively, then:

$$d \leq d'.$$

Our proof is an adaptation of Schur's method of polygonal approximation.

Acknowledgement: I would like to thank Robert Connelly for pointing out an error in an earlier version of this paper and telling me of Stoker's work on the problem.

Introduction.

The geodesic curvature of a space curve is a measure of how far the curve is from being a geodesic. In light of this, it is reasonable to expect that an upper bound on the curvature will give a lower bound on the distance between points on the curve. In Euclidean space, there is such a result known as A. Schur's Theorem:

Theorem [Sc] - Let $\gamma_1(t)$ and $\gamma_2(t)$ be two curves in $\mathbb{R}^3$ of length L^2 parametrized by arclength. Let $\kappa_1(t)$ and $\kappa_2(t)$ be the geodesic curvatures of γ_1 and γ_2 respectively.
Suppose that $\gamma_1(t)$ is a planar curve without self intersection such that γ_1 along with the chord from $\gamma_1(0)$ to $\gamma_1(L)$ bounds a convex region. Furthermore suppose:

$$|\kappa_2(t)| \leq \kappa_1(t) \quad t \in [0,L].$$

Then

$$|\gamma_2(0)\gamma_2(L)| \geq |\gamma_1(0)\gamma_1(L)|.$$

(If A, B are two points in $\mathbb{R}^3$ then $|A\,B|$ is the distance between them.)

In this paper, we will show that a version of this theorem is true in hyperbolic space. We require an additional assumption:

Let $\tilde{\gamma}_2(t)$ be the planar curve with curvature $|\kappa_2(t)|$, we suppose that $\tilde{\gamma}_2(t)$ along with the chord from $\tilde{\gamma}_2(0)$ to $\tilde{\gamma}_2(L)$ also bounds a convex region. The theorem is very likely true without this assumption but our argument requires it or something like it.

Schur proved the theorem by demonstrating an analogous result for polygonal curves. We will use the same strategy. Our argument is quite similar to Schur's but is considerably more complicated.

For any curve $\gamma(t)$ in $\mathbb{H}^3$, parametrized on $[0,L]$, the geodesic from $\gamma(0)$ to $\gamma(L)$ is called the chord of γ; its length is called the chordal length of γ. If $\gamma(t)$ is a planar curve (in $\mathbb{H}^2$) which, along with its chord bounds a convex region then we will say that γ is chord convex.

Let A, B and C denote three points in $\mathbb{H}^3$; $\overline{AB}$ denotes the directed segment from A to B and $\angle ABC$ denotes the angle measured counterclockwise at B from $\overline{AB}$ to $\overline{BC}$; it lies in $[0,2\pi]$. See figure 1.

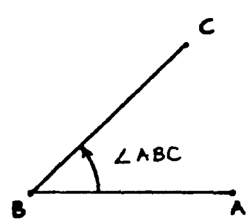


Figure 1

Let $\gamma = AP_1 \cdots P_n B$ be a polygonal curve in $\mathbb{H}^3$. We define an admissible deformation of γ to be a rigid rotation of the subcurve $AP_1 \cdots P_k P_{k+1}$ about the axis through $P_k P_{k+1}$ keeping the complimentary subcurve $P_k P_{k+1} \cdots P_n B$ fixed. The result is a polygonal curve $A'P_1' \cdots P_{k-1}' P_k \cdots P_n B$ with the same side lengths and exterior angles.

Let the exterior angles of γ be:

$\theta_1 = \pi - \angle AP_1P_2, \ldots, \theta_n = \pi - \angle P_{n-1}P_nB$, and let $\tilde{\gamma} = \tilde{A}\tilde{P}_1 \cdots \tilde{P}_n \tilde{B}$ be the planar polygonal curve with the same side lengths as γ and exterior angles:

$$|\theta_1|, \ldots, |\theta_n|.$$

Because $\mathbb{H}^3$ is homogeneous and isotropic, it follows that γ can be obtained from $\tilde{\gamma}$ by a sequence of admissible deformations. The hyperbolic analogue of Schur's theorem for polygons is:

<u>Theorem A</u>: Let γ and $\tilde{\gamma}$ be as above. If $\tilde{\gamma}$ is chord convex then:

$$|AB| \geq |\tilde{A}\tilde{B}|$$

(Heretofor $|XY|$ denotes the hyperbolic distance from X to Y.)

The theorem for C^2 - curves follows from Theorem A by a simple approximation argument and a comparison theorem for chord convex curves (§ 3-4).

Schur's proof of the polygonal theorem relies on two facts from Euclidean plane geometry and two facts from geometry in $\mathbb{R}^3$.

1) Every pair of lines in $\mathbb{R}^2$ intersects in a unique point or is parallel.

2) If $\gamma = A P_1 \cdots P_n B$ is a chord convex polygonal curve in $\mathbb{R}^2$ then the intersection points of the lines through P_i and P_{i+1} with the line through A and B are in the same cyclic ordering as the edges of γ.

3) The triangle inequality.

4) Every polygon in $\mathbb{R}^3$ can be obtained via admissible deformations from the planar polygon with the same side lengths and absolute exterior angles.

Facts 3) and 4) remain true in hyperbolic space but 1) fails. To fill the breach we have:

1') Every pair of lines in $\mathbb{H}^2$ intersects at a unique finite point, is asymptotic at ∞ or has a common perpindicular line.

The three possibilities are illustrated in the Poincaré disk model in figure 2.

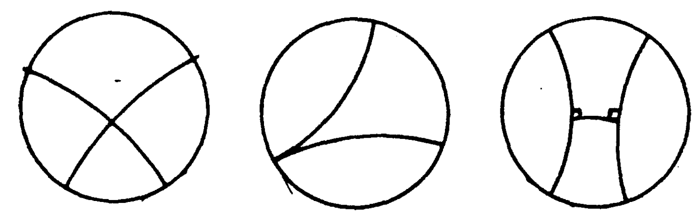


Figure 2

In the arguments which follow, we will generally assume that no two edges of a polygonal arc are asymptotic at infinity. As this is generically the case, continuity considerations show that this does not diminish the generality of our result.

The incidence pattern of the edges on AB and the intersection points of the common perpindicular lines display order properties analogous to those described in 2). To see this, it is convenient to work in the Klein model of $\mathbb{H}^2$ (see [Sp]). This model is a metric on the unit disk, D_1 which is not conformal to the Euclidean metric but the geodesics in this metric are Euclidean straight lines. Thus, if ℓ_1 and ℓ_2 are two lines in $\mathbb{R}^2$ which pass through D_1 then ℓ_1 intersects ℓ_2 as hyperbolic lines if $\ell_1 \cap \ell_2$ is in D_1. If $\ell_1 \cap \ell_2$ is in $D_1^c \setminus \partial D_1$ then ℓ_1 and ℓ_2 have a common perpindicular line, $\ell^{\perp}$. If ℓ_1 is a line through (0,0) and $z = \ell_1 \cap \ell_2$ then $1/z$ is the point of intersection: $\ell_1 \cap \ell^{\perp}$.

Another important feature of this model is that a set $\Omega \subset\subset D_1$ is convex with respect to hyperbolic geometry if and only if it is convex with respect to Euclidean geometry.

Let $\gamma = AP_1, \ldots P_n B$ be a chord convex polygonal arc in the Klein model. Let C denote the Euclidean line through A and B and C_i the Euclidean line through P_i and P_{i+1}.

Let $X_i = C_i \cap C$. These are cyclically ordered as described above and so divided into two sets

$$\mathcal{A} = \{X_1, \ldots, X_m\}$$

$$\mathcal{B} = \{X_{m+1}, \ldots, X_{n-1}\}$$

The points in $\mathcal{A}$ lie below A on AB while the points in $\mathcal{B}$ lie above B. The ordering is so that X_i lies further from A than X_{i-1} if $X_i \in \mathcal{A}$ while X_i lies further from B than X_{i+1} if $X_i \in \mathcal{B}$. See figure 3.

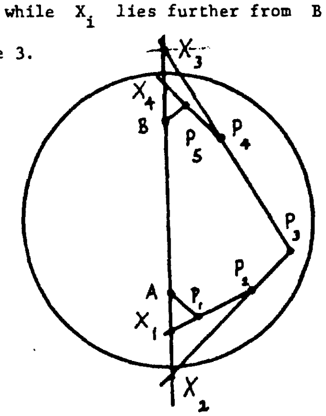


Figure 3

If we think of γ as a hyperbolic curve then the points $\{X_1 \ldots X_{n-1}\}$ fall into three classes:

$$\mathcal{S} = \{S_1, \ldots, S_p\}$$

$$\mathcal{Q} = \{Q_{p+1}, \ldots, Q_q\}$$

$$\mathcal{T} = \{T_{q+1}, \ldots, T_{n-1}\}$$

The set $\mathcal{S} = \mathcal{A} \cap D_1$ and $\mathcal{T} = \mathcal{B} \cap D_1$. The points in $\mathcal{Q}$ are the intersection points between $C \cap D_1$ and the common perpendicular line to $C \cap D_1$ and $C_i \cap D_1$. The points in $\mathcal{S}$ and $\mathcal{T}$ are ordered in the same fashion as their counterparts in $\mathcal{A}$ and $\mathcal{B}$. From the remarks above, it is clear that Q_i lies above Q_{i-1} on the line $C \cap D_1$ directed from A to B. Figure 4 illustrates this in the Poincaré model:

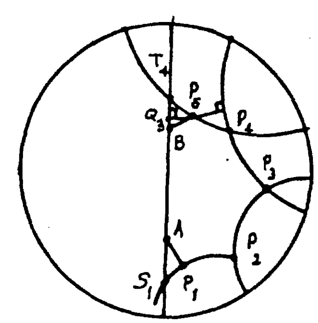


Figure 4

Now and henceforth let C denote the line in $\mathbb{H}^2$ through A and B and C_i the line in $\mathbb{H}^2$ through P_i and P_{i+1}. We think of $\mathbb{H}^2$ as sitting in $\mathbb{H}^3$. As $AP_1 \ldots P_n B$ is a planar curve, an edge, C_i which does not intersect C, actually has a common perpendicular plane with C. We will denote this plane by π_i. Note that $Q_i = C \cap \pi_i$.

In section 1, we present the proof of Theorem A for the special case $Q = \emptyset$. This is Schur's argument verbatim. In section 2, we present the additional analysis which is needed when $Q \neq \emptyset$. This case splits into a large number of subcases. In section 3, we prove a comparision theorem for the chordal lengths of two planar chord convex curves. In the last section, we prove the C^2-version of the theorem.

<u>Notation</u>

In the remainder of the paper, all geometric objects will be hyperbolic, that is lines, planes, distances, angles, arclengths, curvatures, etc.

AB - The hyperbolic line through A and B.

$\overline{AB}$ - The directed line segment from A to B.

$\overrightarrow{AB}$ - The directed half line from A towards B.

$|AB|$ - The length of the segment from A to B.

$\angle ABC$ - The angle from $\overline{AB}$ to $\overline{BC}$ measured in the counterclockwise direction at B.

We will think of the curvature of a space curve as having a sign; this makes sense if the curve lies in an oriented surface. Otherwise, the curvature of all space curves can be taken to be positive.

<u>Acknowledgements</u>

I would like to thank R. Coifman for an illuminating conversation and Angelica Faltings for helping me with Schur's German.

§1 Schur's Proof.

In this section, we consider a chord convex polygon $\gamma = AP_1 \ldots P_n B$ such that $C_i \cap C \neq \emptyset$ for every i. The argument in this case is identical to Schur's argument and serves to illustrate the method whereby the general case is treated in the next section. Let $\gamma' = A'P'_1 \ldots P'_n B$ be an admissible deformation of γ. We obtain γ' from γ by the following sequence of rotations:

1) AP_1 is rotated about $P_1 P_2$ to yield:
   $A^1 P_1 \cdots P_n B$

2) $A^1P_1P_2$ is rotated about P_2P_3 to yield

$A^2P_1^2P_2\cdots P_nB$

$\vdots$

n-1) $A^{n-2}P_1^{n-2}P_2^{n-2}\cdots P_{n-2}$ is rotated about $P_{n-2}P_{n-1}$ to yield $\gamma'=A^{n-1}P_1^{n-1}\cdots P_{n-2}^{n-1}P_{n-1}P_nB$.

By a recursive argument, we will show that

$$|A^kB| \geq |AB| \quad k=1,\ldots,n-1.$$

As $Q=\emptyset$ either $S=\emptyset$ or $S\neq\emptyset$. We will begin with the former case, the latter will be treated as part of another case.

As A^1 is the result of rotating A about an axis through P_1 and P_2 it follows that

$$|AS_1| = |A^1S_1|$$

This is because S_1 lies on the axis of rotation. By the triangle inequality, it follows:

$$\begin{aligned}|A^1B| &\geq |BS_1| - |A^1S_1|\\ &= |BS_1| - |AS_1|\\ &= |AB|.\end{aligned}$$

After one deformation, the chord is not decreased. If $S=\{S_1,\ldots,S_m\}$ then we will show inductively that

$$|A^mB| \geq |AB| \qquad (1.2)$$

by showing:

$$|A^mS_m| \leq |AS_m|, \qquad (1.3)$$

and applying the triangle inequality.

Assume we've shown that

$$|A^kS_k| \leq |AS_k|. \qquad (1.4)$$

As before S_{k+1} lies on the axis of rotation for the (k+1)-st deformation and therefore:

$$|A^{k+1}S_{k+1}| = |A^kS_{k+1}|. \qquad (1.5)$$

From the triangle inequality, (1.4) and (1.5), it follows that:

$$\begin{aligned}|A^{k+1}S_{k+1}| = |A^kS_{k+1}| &\leq |A^kS_k| + |S_kS_{k+1}|\\ &\leq |AS_k| + |S_kS_{k+1}|\\ &= |AS_{k+1}|. \qquad (1.6)\end{aligned}$$

Thus we've verified (1.4) for $k+1$; now we will use this and the triangle inequality to show:

$$|A^{k+1}B| \geq |AB|$$

as follows:

$$\begin{aligned}|BA^{k+1}| &\geq |BS_{k+1}| - |A^{k+1}S_{k+1}|\\ &\geq |BS_{k+1}| - |AS_{k+1}|\\ &= |BA|. \qquad (1.7)\end{aligned}$$

As we've verified (1.4) for $k=1$ the induction (1.4)-(1.6) implies (1.3). Using the estimate in (1.7) with $k+1=m$, we obtain (1.2). The sequence of steps (1.4)-(1.6) will be called the *S*-induction.

If $m=n-1$, then (1.2) is the conclusion of the theorem otherwise there are points in T and we need to argue further. Let A^{m+1} be the next point as before:

$$|A^{m+1}T_{m+1}| = |A^{m}T_{m+1}|. \tag{1.8}$$

(If $S=\emptyset$ then $|AT_1|=|A^1T_1|$.) We will show that

$$|A^{m+1}T_{m+1}| \geq |AT_{m+1}|. \tag{1.9}$$

We apply the triangle inequality, (1.3) and (1.8) to obtain (1.9):

$$\begin{aligned} |A^{m+1}T_{m+1}| = |A^{m}T_{m+1}| &\geq |T_{m+1}S_m|-|A^{m}S_m| \\ &\geq |T_{m+1}S_m|-|AS_m| \\ &= |AT_{m+1}|. \end{aligned} \tag{1.10}$$

From (1.9) it follows that

$$|BA^{m+1}| \geq |BA|$$

for:

$$\begin{aligned} |BA^{m+1}| &\geq |A^{m+1}T_{m+1}| - |BT_{m+1}| \\ &\geq |AT_{m+1}| - |BT_{m+1}| \\ &= |BA|. \end{aligned} \tag{1.11}$$

To treat the remaining cases, we use an inductive argument to show:

$$|A^{n-1}T_{n-1}| \geq |AT_{n-1}|. \tag{1.12}$$

Assume that:

$$|A^{k}T_k| \geq |AT_k|. \tag{1.13}$$

Then as before:

$$|A^{k+1}T_{k+1}| = |A^{k}T_{k+1}|. \tag{1.14}$$

From the triangle inequality, (1.13) and (1.14) we obtain:

$$\begin{aligned} |A^{k+1}T_{k+1}| = |A^{k}T_{k+1}| &\geq |A^{k}T_k| - |T_{k+1}T_k| \\ &\geq |AT_k| - |T_{k+1}T_k| \\ &= |AT_{k+1}|. \end{aligned} \tag{1.15}$$

We've verified (1.13) for $k=m+1$, thus (1.13)-(1.15) applied recursively implies (1.12), arguing as in (1.11) we obtain:

$$|A^{n-1}B| \geq |AB|;$$

this is the conclusion of the theorem. The argument (1.13)-(1.15) will be called the *T*-induction.

The structure of the proof when $Q\neq\emptyset$ is similar: for each type of incidence we derive a recursion involving A^k and the point of intersection which implies $|A^kB| \geq |AB|$. When the nature of the incidence changes a special argument like (1.10) is required to transfer the information from one induction to the next. Working from the top down, we will show that a given subcase reduces to a subcase which was already treated.

We will repeatedly use the fact that the shortest distance from a point P in $\mathbb{H}^3$ to a plane π is realized by the unique line through P orthogonal to π and the fact that if π and π' are disjoint planes then

$$\min_{\substack{A\in\pi \\ B\in\pi'}} |AB|$$

is realized by the unique line perpindicular to both π and π'. These facts follow easily from the law of cosines for hyperbolic space (see [Be]).

§2 The General Case

Now we suppose $Q\neq\emptyset$; the proof breaks into two main subcases:

I) $S=\emptyset$

II) $S\neq\emptyset$

First we will treat I). The subcases will be labelled according to a self explanatory system akin to the Dewey decimal system.

Case I is further subdivided into three subcases:

I,a) $Q_1\in\overrightarrow{B\infty}$,

I,b) $Q_1\in\overline{AB}$,

I,c) $Q_1\in\overrightarrow{-\infty A}$.

(Q_k=A or B is non-generic and can therefore be ignored.)

We will treat the subcases in this order. In each subcase there is either a single point or several, as the transition to the next case is the same in either case, we will tacitly assume that there are several points in each subcase and derive the induction.

In the illustrations accompanying the arguments, no attempt is made at 3-dimensional perspective. A dot indicates a point of intersection; a line indicates a line or a plane.

<u>I,a</u>

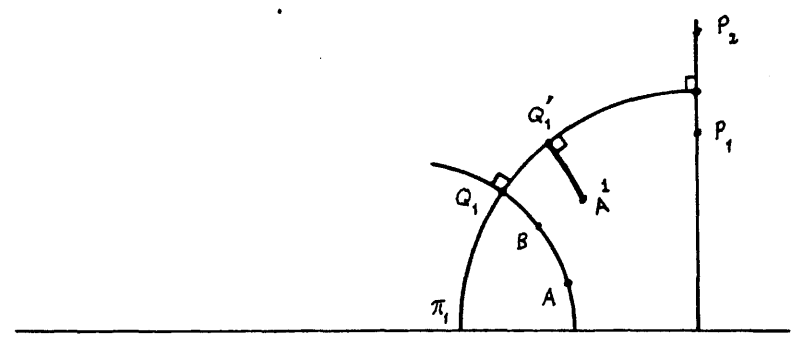


Figure 5

Let A^1 denote the image of A under the rotation about P_1P_2. Let Q_1' denote the point on π_1 closest to A^1. As π_1 is orthogonal to the axis of rotation, it is clear that

$$|A^1Q_1'| = |AQ_1|. \qquad (2.1)$$

From the triangle inequality and (2.1) it follows:

$$\begin{aligned} |BA^1| &\geq |A^1Q_1| - |BQ_1| \\ &\geq |A^1Q_1'| - |BQ_1| \\ &= |AQ_1| - |BQ_1| \\ &= |AB|. \end{aligned} \qquad (2.2)$$

In the second line, we've used the fact that:

$$|A^1Q_1'| \leq |A^1Q_1|.$$

We assume there are further planes $\{\pi_2,\dots,\pi_\ell\}$ with $\{Q_2,\dots,Q_\ell\}$ in $\overrightarrow{B\infty}$.

Let A^{k+1} be the point resulting from the (k+1)-st deformation. Let Q_{k+1}' be the point π_{k+1} nearest to A^{k+1} and Q_{k+1}'' the point on π_{k+1} nearest to A^k. We make the following inductive hypothesis:

$$|A^kQ_k'| \geq |AQ_k| \qquad (2.3)$$

A^k lies below π_k (that is on the same side as B) (2.4)

As above we know:

$$|A^kQ_{k+1}''| = |A^{k+1}Q_{k+1}'|, \qquad (2.5)$$

moreover A^{k+1} lies on the same side of π_{k+1} as A^k. The planes are nested and π_k lies below π_{k+1} hence it follows from (2.4) that A^{k+1} lies below π_{k+1}. As the planes are nested A^kQ_{k+1}'' intersects π_k; let P denote the point of intersection. We will verify (2.3) for k+1 using the triangle inequality, (2.3) and (2.5):

$$\begin{aligned} |A^{k+1}Q_{k+1}'| = |A^k\ Q_{k+1}''| &= |Q_{k+1}''P| + |PA^k| \\ &\geq |Q_{k+1}Q_k| + |Q_k'A^k| \\ &\geq |Q_{k+1}Q_k| + |Q_kA| \\ &= |Q_{k+1}A| \end{aligned} \qquad (2.6)$$

From (2.6), we estimate $|BA^{k+1}|$:

$$\begin{aligned} |BA^{k+1}| &\geq |Q_{k+1}A^{k+1}| - |BQ_{k+1}| \\ &\geq |Q_{k+1}'A^{k+1}| - |BQ_{k+1}| \\ &\geq |AQ_{k+1}| - |BQ_{k+1}| \\ &= |BA|. \end{aligned} \qquad (2.7)$$

As we've verified (2.3) and (2.4) for k=1 the inductive argument (2.3)-(2.6) implies:

$$|A^\ell Q'_\ell| \geq |AQ_\ell|. \qquad (2.8)$$

From (2.7), it follows that

$$|BA^\ell| \geq |BA|. \qquad (2.9)$$

If $T=\emptyset$ then $\ell=n-1$ and (2.9) is the conclusion of the theorem. Otherwise, we will show that (2.8) implies:

$$|A^{\ell+1}T_{\ell+1}| \geq |AT_{\ell+1}|. \tag{2.10}$$

This is the first step of the T-induction; the proof is then completed as in §1. The point $T_{\ell+1}$ lies either above or below the plane π_ℓ, we will label these subcases as I,a,+ and I,a,- respectively.

I,a,+

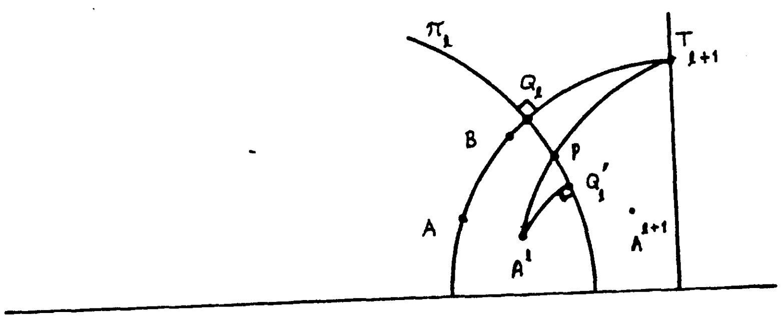


Figure 6

It follows from the above argument that A^ℓ lies below π_ℓ and as usual:

$$|A^{\ell}T_{\ell+1}| = |A^{\ell+1}T_{\ell+1}|. \tag{2.11}$$

Let P be the point of intersection between $A^\ell T_{\ell+1}$ and π_ℓ. We apply (2.8), (2.11) and the triangle inequality to prove (2.10):

$$\begin{aligned} |A^{\ell+1}T_{\ell+1}| = |A^{\ell}T_{\ell+1}| &= |A^{\ell}P| + |PT_{\ell+1}| \\ &\geq |A^{\ell}Q'_{\ell}| + |Q_{\ell}T_{\ell+1}| \\ &\geq |AQ_{\ell}| + |Q_{\ell}T_{\ell+1}| \\ &= |AT_{\ell+1}|. \end{aligned}$$

By applying the T-induction, we complete subcase I,a,+.

I,a,-

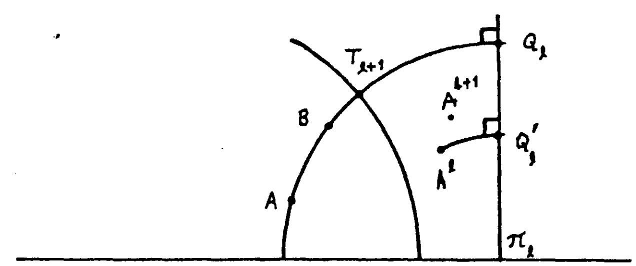


Figure 7

As in the previous case, A^ℓ lies below π_ℓ and

$$|A^\ell T_{\ell+1}|=|A^{\ell+1}T_{\ell+1}|. \qquad (2.12)$$

To prove (2.10), we apply (2.8), (2.12) and the triangle inequality:

$$\begin{aligned}|A^{\ell+1}T_{\ell+1}|=|A^\ell T_{\ell+1}| &\geq |A^\ell Q_\ell|-|T_{\ell+1}Q_\ell| \\ &\geq |A^\ell Q'_\ell|-|T_{\ell+1}Q_\ell| \\ &\geq |AQ_\ell|-|T_{\ell+1}Q_\ell| \\ &=|AT_{\ell+1}|.\end{aligned}$$

Again the T-induction completes the proof in this subcase.

This completes I,a. In the next case, we assume Q_1 is between A and B:

I,b

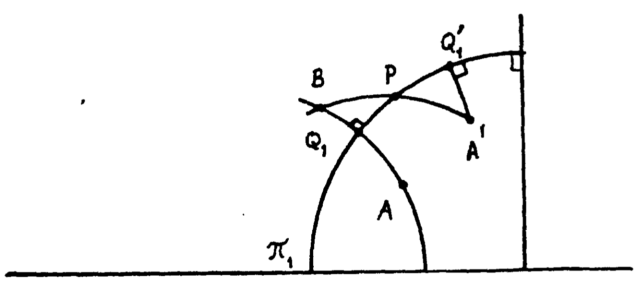


Figure 8

As before Q_1' denotes the point on π_1 nearest to A^1 and

$$|A^1Q_1'|=|AQ_1|. \qquad (2.13)$$

As A lies below π_1 so does A^1. Let P denote the intersection of BA^1 and π_1. It follows from (2.13) and the triangle inequality that:

$$\begin{aligned}|BA^1|&=|BP|+|PA^1| \\ &\geq |BQ_1|+|Q_1'A^1| \\ &=|BQ_1|+|Q_1A| \\ &=|BA| \qquad (2.13)\end{aligned}$$

Now we will derive a recursion to show that

$$|A^kQ_k'|\geq|AQ_k|. \qquad (2.14)$$

so long as Q_k lies between A and B. From this it follows as in (2.13) that $|BA^k|\geq|BA|$. The two inductive hypotheses are:

$$A^k \text{ lies below } \pi_k, \qquad (2.15)$$

$$|A^kQ_k'|\geq|AQ_k|. \qquad (2.16)$$

As in the previous section, π_{k+1} lies above π_k hence it follows from (2.15) that A^k and therefore A^{k+1} lies below π_{k+1}. As usual, we let Q_{k+1}' denote the point on π_{k+1} nearest to A^{k+1} and Q_{k+1}'' denote the point on π_{k+1} nearest to A^k and thus we have:

$$|A^kQ_{k+1}''|=|A^{k+1}Q_{k+1}'|. \qquad (2.17)$$

Let R denote the intersection between $A^kQ''_{k+1}$ and π_{k+1}; it follows from the triangle inequality, (2.16) and (2.17) that:

$$\begin{aligned}|A^{k+1}Q'_{k+1}| &= |A^kQ''_{k+1}| = |A^kR| + |RQ''_{k+1}| \\ &\geq |A^kQ'_k| + |Q_kQ_{k+1}| \\ &\geq |AQ_k| + |Q_kQ_{k+1}| \\ &= |AQ_{k+1}|. \qquad (2.18)\end{aligned}$$

From (2.18), we estimate $|BA^k|$ as in (2.13) to obtain:

$$|BA^{k+1}| \geq |BA|. \qquad (2.19)$$

Suppose that $\mathcal{Q}\cap\overline{AB}=\{Q_1,\ldots,Q_\ell\}$. From the inductive argument presented above, it follows that

$$|A^\ell Q'_\ell| \geq |AQ_\ell|. \qquad (2.20)$$

If $\mathcal{Q}=\overline{AB}\cap\mathcal{Q}$ and $\mathcal{T}=\emptyset$ then $\ell=n-1$ and the conclusion of the theorem follows from (2.19). The other two possibilites are

I,b,1) $\mathcal{Q}=\mathcal{Q}\cap\overline{AB}$ and $\mathcal{T}\neq\emptyset$,

I,b,2) $\mathcal{Q}\neq\overline{AB}\cap\mathcal{Q}$.

We treat them in this order:

<u>I,b,1</u>:

As $T\in\overrightarrow{B\infty}$ it is clear that $T_{\ell+1}$ lies above π_ℓ. Let P be the intersection of $A^\ell T_{\ell+1}$ and π_ℓ. Using (2.20) we will show that:

$$|A^{\ell+1}T_{\ell+1}| \geq |AT_{\ell+1}|. \qquad (2.21)$$

As usual

$$|A^{\ell+1}T_{\ell+1}| = |A^\ell T_{\ell+1}|. \qquad (2.22)$$

From (2.20), (2.22) and the triangle inequality we derive (2.21):

$$\begin{aligned}|A^{\ell+1}T_{\ell+1}| &= |A^\ell T_{\ell+1}| = |A^\ell P| + |PT_{\ell+1}| \\ &\geq |A^\ell Q'_\ell| + |Q_\ell T_{\ell+1}| \\ &\geq |AQ_\ell| + |Q_\ell T_{\ell+1}| \\ &= |AT_{\ell+1}|.\end{aligned}$$

In this case the $\mathcal{T}$-induction completes the proof.

<u>I,b,2</u>:

As $\mathcal{Q}\neq\mathcal{Q}\cap\overline{AB}$ there is a point $Q_{\ell+1}$ on $\overrightarrow{B\infty}$. We will show that $A^{\ell+1}$ lies below $\pi_{\ell+1}$ and

$$|A^{\ell+1}Q'_{\ell+1}| \geq |AQ_{\ell+1}|. \qquad (2.23)$$

This will allow us to use the induction presented in I,a to complete the proof.

The plane $\pi_{\ell+1}$ lies above π_ℓ and thus A^ℓ lies below $\pi_{\ell+1}$ hence $A^{\ell+1}$ does as well. As usual:

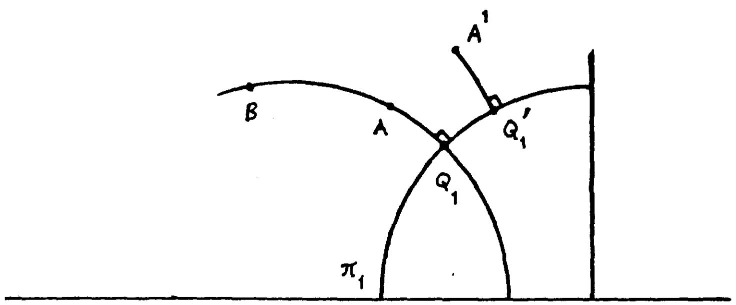
$$|A^{\ell+1}Q'_{\ell+1}| = |A^\ell Q''_{\ell+1}|. \qquad (2.24)$$

Let P be the point of intersection between $\overline{A^\ell Q''_{\ell+1}}$ and π_ℓ. From (2.20), (2.24) and the triangle inequality, we prove (2.23):

$$\begin{aligned}|A^{\ell+1}Q'_{\ell+1}| &= |A^\ell Q''_{\ell+1}| = |A^\ell P| + |PQ''_{\ell+1}| \\ &\geq |A^\ell Q'_\ell| + |Q_\ell Q_{\ell+1}| \\ &\geq |AQ_\ell| + |Q_\ell Q_{\ell+1}| \\ &= |AQ_{\ell+1}|.\end{aligned}$$

The proof is again completed as in I,a; this completes subcase I,b. Now we assume that $Q_1 \in \overrightarrow{-\infty A}$:

I,c



The deformed point A^1 lies above π_1 as A does moreover:

$$|AQ_1| = |A_1 Q'_1|. \tag{2.25}$$

Using the triangle inequality and (2.25), we estimate $|BA^1|$:

$$\begin{aligned} |BA^1| &\geq |BQ'_1| - |Q'_1 A^1| \\ &\geq |BQ_1| - |Q_1 A^1| \\ &= |BA|. \end{aligned} \tag{2.26}$$

We assume that $Q \cap \overrightarrow{-\infty A} = \{Q_1, \ldots, Q_\ell\}$. This subcase splits into two subcases:

I,c,+) All A^k lie above π_{k+1} $k=1,\ldots \ell-1$.

I,c,-) For some $k \in \{2,\ldots,\ell\}$, A^{k-1} lies below π_k.

The second case is essentially trivial; we will treat the other case first:

I,c,: We will construct an inductive argument to show:

$$|AQ_k| \geq |A^k Q'_k|. \tag{2.27}$$

Assume (2.27) for k and assume that A^k lies above π_{k+1} and that

$$|A^{k+1} Q'_{k+1}| = |A^k Q''_{k+1}|. \tag{2.28}$$

Let R denote the intersection of $A^k Q'_k$ and π_{k+1}. From (2.27) and the triangle inequality it follows:

$$\begin{aligned} |AQ_k| &\geq |A^k Q'_k| = |A^k R| + |RQ'_k| \\ &\geq |A^k Q''_{k+1}| + |Q_{k+1} Q_k|. \end{aligned} \tag{2.29}$$

This estimate implies

$$\begin{aligned} |A_k Q''_{k+1}| &\leq |AQ_k| - |Q_{k+1} Q_k| \\ &= |AQ_{k+1}| \end{aligned} \tag{2.30}$$

The estimate (2.27) follows from (2.28) and (2.30). We apply (2.27)-(2.30) recursively to obtain:

$$|A^\ell Q'_\ell| \leq |AQ_\ell|, \tag{2.31}$$

and

$$A^\ell \text{ lies above } \pi_\ell. \tag{2.32}$$

From the triangle inequality and (2.31) we obtain:

$$|BA^{\ell}| \geq |BQ'_{\ell}| - |Q'_{\ell}A^{\ell}|$$
$$\geq |BQ_{\ell}| - |Q_{\ell}A|$$
$$= |BA|. \qquad (2.33)$$

If $\mathcal{Q}=\mathcal{Q}\cap\overrightarrow{-\infty A}$ and $T=\emptyset$ then $\ell=n-1$ and (2.33) is the conclusion of the theorem. Otherwise, we need to consider three cases:

I,c,+ 1) $\mathcal{Q}\cap\overline{AB}\neq\emptyset$,

I,c,+,2) $\mathcal{Q}\cap\overline{AB}=\emptyset$ but $\mathcal{Q}\cap\overrightarrow{B\infty}\neq\emptyset$,

I,c,+,3) $\mathcal{Q}=\mathcal{Q}\cap\overrightarrow{-\infty A}$ but $T\neq\emptyset$.

We will treat them in this order:

<u>I,c,+,1</u>:

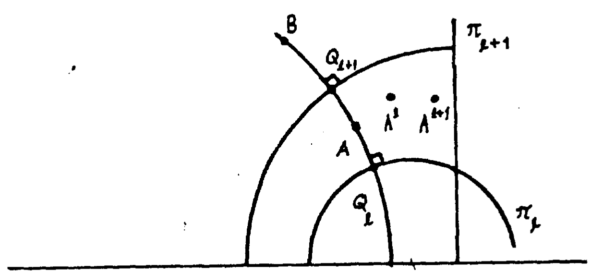


Figure 9

To complete the proof, we will verify the inductive hypotheses of case I,b:

a) $A^{\ell+1}$ lies below $\pi_{\ell+1}$

b) $|A^{\ell+1}\,Q'_{\ell+1}| \geq |AQ_{\ell+1}|$.

The first assertion is clear for A and Q_{ℓ} lie below $\pi_{\ell+1}$ and (2.31) therefore implies that A^{ℓ} does as well. From this, a) is immediate and

$$|A^{\ell+1}\,Q'_{\ell+1}| = |A^{\ell}Q''_{\ell+1}|. \qquad (2.34)$$

To verify b), we use (2.31), (2.34) and the triangle inequality:

$$|A^{\ell+1}\,Q'_{\ell+1}| = |A^{\ell}\,Q''_{\ell+1}| \geq |Q''_{\ell+1}\,Q'_{\ell}| - |AQ'_{\ell}|$$
$$\geq |Q_{\ell+1}\,Q_{\ell}| - |AQ_{\ell}|$$
$$= |AQ_{\ell+1}|.$$

The proof is completed as in I,b.

<u>I,c,+,2</u>:

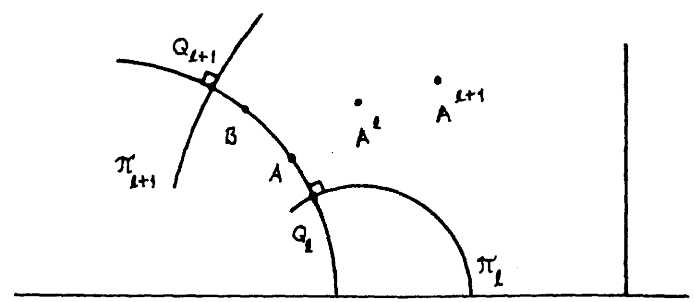


Figure 10

In this case, we will verify the inductive hypotheses for case I,a:

a) $A^{\ell+1}$ lies below $\pi_{\ell+1}$,

b) $|A_-^{\ell+1} Q'_{\ell+1}| \geq |AQ_{\ell+1}|$.

The first assertion follows exactly as in the previous subcase. To verify b), we apply (2.31), (2.34) and the triangle inequality:

$$|A^{\ell+1} Q'_{\ell+1}| = |A^{\ell} Q''_{\ell+1}| \geq |Q''_{\ell+1}Q'_{\ell}| - |AQ'_{\ell}|$$
$$\geq |Q_{\ell+1}Q_{\ell}| - |AQ_{\ell}|$$
$$= |AQ_{\ell+1}|.$$

The proof is completed as in case I,a.

The final subcase of I,c,+ is:

<u>I,c,+,3</u>:

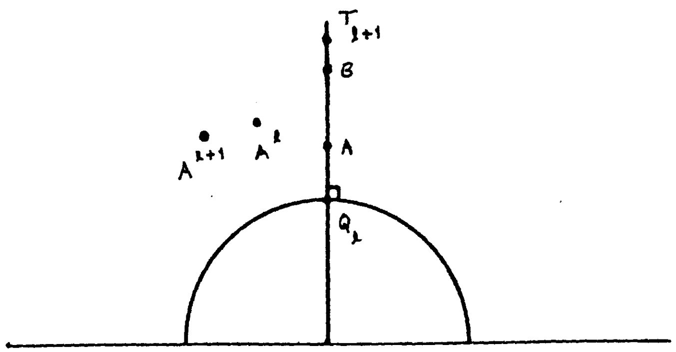


Figure 11

We will verify the inductive hypothesis for the T-induction:

$$|A^{\ell+1}T_{\ell+1}| \geq |AT_{\ell+1}|. \tag{2.35}$$

As usual $|A_-^{\ell+1}T_{\ell+1}| = |A^{\ell}T_{\ell+1}|$. We use this, (2.31) and the triangle inequality to prove (2.35):

$$|A^{\ell}T_{\ell+1}| \geq |T_{\ell+1}Q'_{\ell}| - |A^{\ell}Q'_{\ell}|$$
$$\geq |T_{\ell+1}Q_{\ell}| - |AQ_{\ell}|$$
$$= |AT_{\ell+1}|.$$

The proof is completed as in §1.

This completes the proof in case I,c,+.

<u>I,c,-</u> :

Because the planes $\{\pi_k\}$ are nested, it follows that if A^k lies below π_k then A^{k+1} lies below π_{k+1}. If $\mathcal{Q}\cap\overrightarrow{-\infty A} = \{Q_1,\ldots,Q_\ell\}$ and A^k lies below π_k for some $k\leq\ell$ then evidently A^{ℓ} lies below π_{ℓ}.

It is clear that $|BA^{\ell}| \geq |BA|$ if $T=\emptyset$ we are done.

Otherwise, the remainder of the proof breaks into the same subcases as in case I,c,+:

I,c,-,1) $\mathcal{Q}\cap\overline{AB} \neq \emptyset$,

I,c,-,2) $\mathcal{Q}\cap\overline{AB} = \emptyset$ but $\mathcal{Q}\cap B\infty \neq \emptyset$,

I,c,-,3) $\mathcal{Q}\cap\overrightarrow{A\infty} = \emptyset$ but $T \neq \emptyset$.

Case I,c,+,3 is trivial so we provide only a figure:

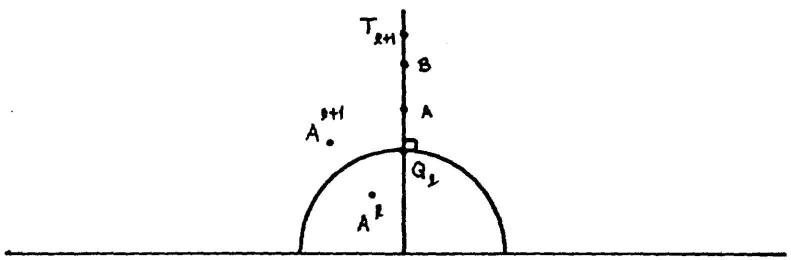


Figure 12

$$|A^{\ell+1}T_{\ell+1}| = |A^{\ell}T_{\ell+1}| > |AT_{\ell+1}|.$$

The T-induction completes the proof in this case. The other two cases are identical so we will omit the proof of I,c,-,2.

<u>I,c,-,1</u>

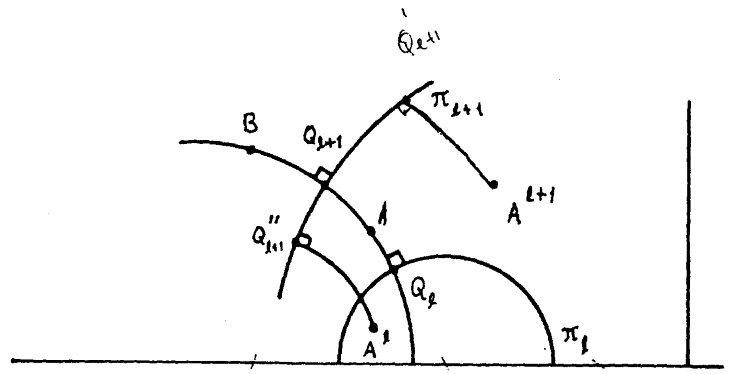


Figure 13

A^{ℓ} lies below $\pi_{\ell+1}$ and therefore so does $A^{\ell+1}$. As usual

$$|A^{\ell}Q''_{\ell+1}| = |A^{\ell+1}Q'_{\ell+1}|.$$

We need to verify

$$|A^{\ell}Q''_{\ell+1}| \geq |AQ_{\ell+1}|.$$

Evidently we have:

$$|A^{\ell}Q''_{\ell+1}| \geq |Q_{\ell+1}Q_{\ell}|$$
$$\geq |AQ_{\ell+1}|.$$

The proof is completed as in I,b.

This completes the proof in case I.

Now we must consider the cases where $S\neq\emptyset$ and $Q\neq\emptyset$. Suppose that:

$S=\{S_1,\ldots,S_m\}$.

Applying the S-induction we conclude that

$$|A^m S_m| \leq |AS_m|. \tag{2.36}$$

There are 3 subcases:

II,a) $Q_{m+1}\in \overrightarrow{-\infty A}$,

II,b) $Q_{m+1}\in \overline{AB}$,

II,c) $Q_{m+1}\in \overrightarrow{B\infty}$.

II,a) This case splits into two cases:

II,a,+) A^m lies above π_{m+1},

II,a,−) A^m lies below π_{m+1}.

Case II,a,− is trivial and is treated exactly as in I,c,−; we omit the details. Case II,a,+ further splits into two cases:

II,a,++) S_m lies above π_{m+1},

II,a,+−) S_m lies below π_{m+1}.

II,a,++

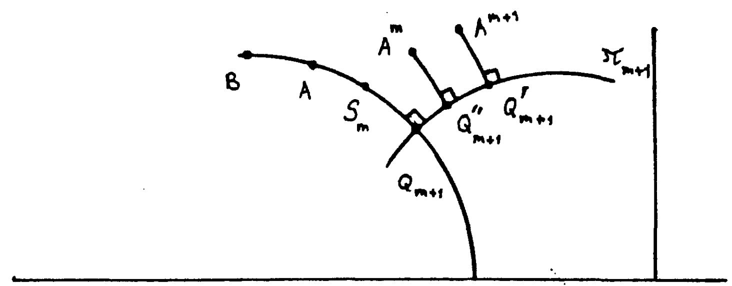


Figure 14

To complete the proof we verify the inductive hypothesis for case I,c:

a) A^{m+1} lies above π_{m+1},

b) $|A^{m+1}Q'_{m+1}| \leq |AQ_{m+1}|$.

The first hypothesis is evident as A^m is assumed to lie above π_{m+1}. As usual $|A^mQ''_{m+1}| = |A^{m+1}Q'_{m+1}|$. Using the triangle inequality and (2.36) we verify b):

$$\begin{aligned} |A^{m+1}Q'_{m+1}| = |A^mQ''_{m+1}| &\le |A^mQ_{m+1}| \\ &\le |A^mS_m| + |S_mQ_{m+1}| \\ &\le |AS_m| + |S_mQ_{m+1}| \\ &= |AQ_{m+1}|. \end{aligned}$$

The proof is completed as in case I,c.

II,a,+

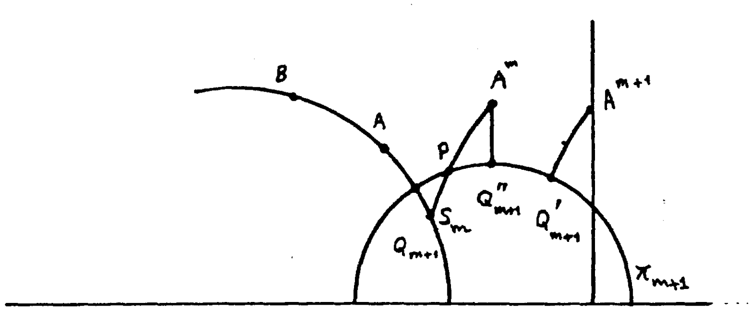


Figure 15

We will again verify a) and b) above. Let P denote the intersection of $\overline{A^mS_m}$ and π_{m+1}. From (2.36) and the triangle inequality we obtain:

$$\begin{aligned} |AS_m| \ge |A^mS_m| &= |A^mP| + |PS_m| \\ &\ge |A^mQ''_{m+1}| + |Q_{m+1}S_m|. \end{aligned} \qquad (2.37)$$

From (2.37) we obtain:

$$\begin{aligned} |A^{m+1}Q'_{m+1}| = |A^mQ''_{m+1}| &\le |AS_m| - |S_mQ_{m+1}| \\ &= |AQ_{m+1}|. \end{aligned}$$

This completes case II,a.

II,b

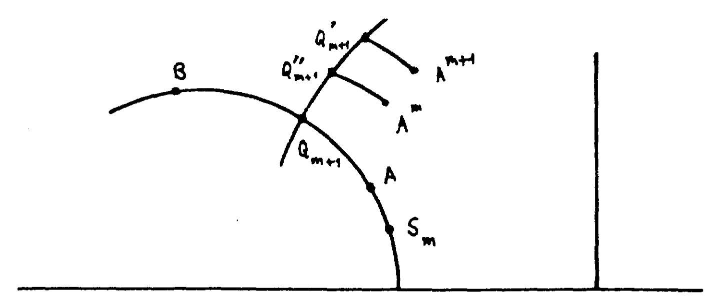


Figure 16

To complete this case, we verify the inductive hypothesis for case I,b:

a) A^m lies below π_{m+1}

b) $|A^{m+1}Q'_{m+1}| \geq |AQ_{m+1}|$.

That A^m lies below π_{m+1} follows from (2.36) and the fact that S_m lies below A whereas Q_{m+1} lies above. We prove b) by applying (2.36) and the triangle inequality:

$$\begin{aligned} |A^{m+1}Q'_{m+1}| = |A^m Q''_{m+1}| &\geq |Q''_{m+1}S_m| - |S_m A^m| \\ &\geq |Q_{m+1}S_m| - |S_m A| \\ &= |AQ_{m+1}|. \end{aligned}$$

This completes case II,b.

<u>II,c</u>

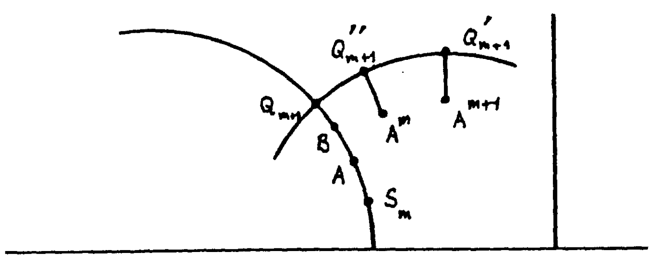


Figure 17

To complete this case we verify the inductive hypotheses used in I,a:

a) A^m lies below π_{m+1},

b) $|Q'_{m+1}A^{m+1}| \geq |Q_{m+1}A|$.

As before a) follows from (2.36). We prove b) using (2.36) and the triangle inequality:

$$\begin{aligned} |Q'_{m+1}A^{m+1}| = |Q''_{m+1}A^m| &\geq |Q''_{m+1}S_m| - |S_m A^m| \\ &\geq |Q_{m+1}S_m| - |S_m A| \\ &= |Q_{m+1}A|. \end{aligned}$$

The proof of Theorem A is now complete for any generic chord convex polygonal arc. As the intersections of the planes, $\{\pi_K\}$ with C depend continuously on the vertices, it follows that we can uniformly approximate a non-generic polygon,

$\gamma = AP_1 \cdots P_n B$ by a sequence of generic polygons $\{\gamma_m = A_m P_{1m} \ldots P_{nm} B_m : m \in \mathbb{N}\}$. If γ' is a polygonal space curve obtained from γ by allowable deformations, then $\{\gamma'_m\}$, the sequence of space curves obtained from γ_m by the same sequence of deformations, converge uniformly to γ'. The theorem applies to each approximant to conclude:

$$|A'_m B'_m| \geq |A_m B_m|. \qquad (2.38)$$

As m tends to infinity (2.38) tends to

$$|A'B'| \geq |AB|.$$

Thus the proof of Theorem A is complete.

§3 Comparisons of planar arcs:

To make an effective tool out of Schur's theorem it is better to compare a space curve to a planar curve with different angles, for example a curve with constant exterior angles. To that end, we use a classical result of Cauchy, proved for general space forms by Stoker:

Proposition 3.1: Let $\gamma_1 = AP_1 \ldots P_n B$ be a chord convex arc with exterior angles $\{\theta_i\}$, let $\gamma_2 = A'P_1' \ldots P_n' B'$ be a second chord convex arc with the same side lengths and exterior angles $\{\phi_i\}$ such that

$$0 \leq \phi_i \leq \theta_i < \pi \qquad i = 1, \ldots, n$$

then

$$|A'B'| \geq |AB| .$$

For the proof see pages 135-138 of [St].

Combining Proposition 3.1 and Theorem A we obtain:

Theorem 3.2 Let γ_2 be a polygon in $\mathbb{H}^3$ with exterior angles $\{\theta_i\}$ and let γ_1 be a chord convex polygon in $\mathbb{H}^2$ with the same side lengths as γ_2 and exterior angles $\{\phi_i\}$ which satisfy:

$$|\theta_i| \leq \phi_i \qquad i = 1, \ldots, n .$$

Suppose that $\tilde{\gamma}_2$, the planar arc with the same side lengths and exterior angles as γ_2 is also chord convex.

If d_1 is the chord length of γ_1 and d_2 the chord length of γ_2 then

$$d_2 \geq d_1 .$$

§4 Smooth Curves

The theorems for polygonal arcs are easily applied to prove corresponding theorems for smooth or piecewise smooth curves in $\mathbb{H}^3$. The connection is via the geometric definition of geodesic curvature of a smooth arc; it is as follows:

Let γ be a smooth space curve in $\mathbb{H}^3$ and p a point on γ. Choose two sequences $\{q_n^+\}$ and $\{q_n^-\}$ on γ converging to p such that $\{q_n^+\}$ approaches p from one side and $\{q_n^-\}$ approaches from the other. Each triple (q_n^-, p, q_n^+) determines a plane, π_n in $\mathbb{H}^3$. In that plane there is a unique curve of constant curvature, C_n through (q_n^-, p, q_n^+). Let κ_n be the curvature of C_n. If $\theta_n = \angle q_n^- p q_n^+$ and

$$\sigma_n^{\pm} = |pq_n^{\pm}|$$

then κ_n is given by the formula:

$$\kappa_n = \frac{2\sin\theta_n}{(th^2(\sigma_n^+/2)-2\cos\theta_n th(\sigma_n^+/2)th(\sigma_n^-/2)+th^2(\sigma_n^-/2))^{\frac{1}{2}}} \qquad (4.1)$$

$$= k\ (\theta_n, \sigma_n^+, \sigma_n^-).$$

The curvature of γ at p is the limiting value of κ_n as $n\to\infty$:

$$\kappa(p) = \lim_{n\to\infty} k\ (\theta_n, \sigma_n^+, \sigma_n^-).$$

The quantity $k(\theta, \sigma_1, \sigma_2)$ is positive if $\theta\in[0,\pi]$ and negative if $\theta\in[-\pi,0]$; in our applications θ will always be in $[0,\pi]$.

First we prove a comparison result for planar curves.

<u>Proposition 4.1</u> Let $\gamma_1(t)$ and $\gamma_2(t)$ be arclength parametrized planar curves of the same length, L. Let the curvatures be $\kappa_1(t)$ and $\kappa_2(t)$ respectively. Suppose both are chord convex and that:

$$0\le\kappa_2(t)\le\kappa_1(t) \qquad t\in[0,L],$$

then

$$|\gamma_2(0)\gamma_2(L)|\ge|\gamma_1(0)\gamma_1(L)|. \qquad (4.2)$$

<u>Proof</u>: We approximate the two curves by sequences of polygons: for each $n\in\mathbb{N}$, let λ_n^i be the polygonal arc obtained by approximating γ_i by chords of length L/n, (see fig. 19).

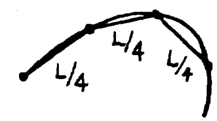


Figure 19

Let $\{P^i_{nm_i} : m=1,\ldots,M_i(n)\}$ be the vertices of λ_n^i. Define:

$$\sigma_n = L/n,$$
$$\theta^i_{nm} = P^i_{nm-1}\, P^i_{nm}\, P^i_{nm+1},$$

and

$$M(n) = \min\{M_1(n), M_2(n)\}.$$

As both curvatures are non-negative and continuous, it follows that both curves are locally convex and therefore

$$0 < \theta^i_{mn} \le \pi.$$

As both curves are smooth these angles tend uniformly to π as n tends to ∞.

Assume for the moment that

$$\kappa_2(t) < \kappa_1(t) \qquad t\in[0,L] \qquad (4.3)$$

Then it follows from formula (4.1) that for sufficiently large n

$$\theta^1_{nm} < \theta^2_{nm},\ m=1,\ldots,M(n).$$

Since λ_n^i converges to γ_i it follows that λ_n^i will be chord convex for large enough n. We can apply Proposition 3.1 to conclude:

$$|P^2_{n0}P^2_{nM(n)}| \geq |P^1_{n0}P^1_{nM(n)}|. \tag{4.4}$$

Because $P^i_{n0} = \gamma_i(0)$ and

$$\lim_{n\to\infty} P^i_{nM(n)} = \gamma_i(L)$$

it follows from (4.4) that:

$$|\gamma_2(0)\gamma_2(L)| \geq |\gamma_1(0)\gamma_1(L)|. \tag{4.5}$$

To obtain the result without assuming (4.3), we observe that if $\gamma_1(t)$ is chord convex and $\gamma_1(0)\neq\gamma_1(L)$ then the plane curve $\gamma^\varepsilon_1(t)$ with curvature $\kappa_1(t)+\varepsilon$ is also chord convex for sufficiently small, positive ε. Clearly

$$\lim_{\varepsilon\to 0} |\gamma^\varepsilon_1(0)\gamma^\varepsilon_1(L)| = |\gamma_1(0)\gamma_1(L)|. \tag{4.6}$$

We can apply the above reasoning to γ^ε_1 to conclude:

$$|\gamma_2(0)\gamma_2(L)| \geq |\gamma^\varepsilon_1(0)\gamma^\varepsilon_1(L)| \tag{4.7}$$

The conclusion follows easily from (4.6) and (4.7).

Now we prove the analogue of Theorem 3.2 for smooth curves:

Theorem B Let $\gamma_1(t)$ and $\gamma_2(t)$ be arclength parametrized curves of the same length, L. Let $\kappa_1(t)$ and $\kappa_2(t)$ denote the curvatures of γ_1 and γ_2 respectively. Suppose that $\gamma_1(t)$ and the planar curve with curvature $|\kappa_2|$ are a chord convex planar curves and that:

$$|\kappa_2(t)| \leq \kappa_1(t), \qquad t\in[0,L].$$

Then

$$|\gamma_1(0)\gamma_1(L)| \leq |\gamma_2(0)\gamma_2(L)|. \tag{4.8}$$

Proof: We will first compare $\gamma_2(t)$ to the planar curve $\tilde{\gamma}_2(t)$ with curvature $|\kappa_2(t)|$. As $\tilde{\gamma}_2(t)$ is chord convex it follows from Proposition 4.1 that

$$|\gamma_1(0)\gamma_1(L)| \leq |\tilde{\gamma}_2(0)\tilde{\gamma}_2(L)|. \tag{4.9}$$

To complete the proof we only need to show:

$$|\tilde{\gamma}_2(0)\tilde{\gamma}_2(L)| \leq |\gamma_2(0)\gamma_2(L)|. \tag{4.10}$$

As before we approximate γ_2 by polygonal arcs. It is also convenient to introduce the planar curves $\gamma^\varepsilon_2(t)$. These are the planar curves with curvature $|\kappa_2(t)|+\varepsilon$. If $\tilde{\gamma}_2(0)\neq\tilde{\gamma}_2(L)$ then γ^ε_2 will be chord convex for sufficiently small, positive ε. If $\tilde{\gamma}_2(0)=\tilde{\gamma}_2(L)$ then (4.10) is immediate.

Let λ_n be the polygonal arc obtained by approximating γ_2 by chords of length L/n and let λ^ε_n be the planar polygonal arc obtained by approximating γ^ε_2 by chords of length L/n. Let $\{P_{nm}: m=0,\ldots,M(n)\}$ denote the vertices of λ_n and $\{P^\varepsilon_{nm}: m=0,\ldots,M^\varepsilon(n)\}$ denote the vertices of λ^ε_n. We denote the interior angles by:

$$\theta_{nm} = \angle P_{nm-1}\, P_{nm}\, P_{nm+1}$$

and

$$\phi_{nm} = \angle P^\varepsilon_{nm-1} P^\varepsilon_{nm}\, P^\varepsilon_{nm+1}.$$

As γ_2 and γ^ε_2 are smooth curves it follows that λ_n and λ^ε_n tend iformly to γ_2 and γ^ε_2 respectively as n tends to infinity, moreover $\{\theta_{nm}\}$ and $\{\phi^\varepsilon_{nm}\}$ tend to uniformly to π. For fixed

$\epsilon>0$ and sufficiently large n it follows as in the previous proof that:

$$|\theta_{nm}| \leq \phi^{\epsilon}_{nm} \; m=0,\dots,\bar{M}(n),$$

here

$$\bar{M}(n) = \min\{M(n), M^{\epsilon}(n)\}.$$

As λ^{ϵ}_n is chord convex it follows from Theorem 3.2 that

$$|P_{n0}\, P_{n\bar{M}(n)}| \geq |P^{\epsilon}_{n0}\, P^{\epsilon}_{n\bar{M}(n)}| \qquad (4.11)$$

for sufficiently large n. Allowing n to tend to infinity in (4.11) we obtain:

$$|\gamma_2(0)\gamma_2(L)| \geq |\gamma^{\epsilon}_2(0)\gamma^{\epsilon}_2(L)| \qquad (4.12)$$

Allowing ϵ to tend to zero in (4.12) we obtain (4.10) and therefore the proof of Theorem B is complete.

Remarks:

1) In Euclidean space the hypothesis that $\gamma_1(t)$ and $\tilde{\gamma}_2(t)$ be chord convex can be tricky to check; in hyperbolic space a planar curve $\gamma_3(t)$ whose curvature, $\kappa_3(t)$ satisfies

$$0 \leq \kappa_3(t) \leq \alpha \leq 1$$

is chord convex regardless of its length, see [Ep 1]. Thus Theorem B gives a non-trivial estimate for the length of the chord of a space curve $\gamma_2(t)$ whose curvature $\kappa_2(t)$ satisfies:

$$|\kappa_2(t)| \leq \alpha < 1.$$

This estimate is:

$$\cosh(\ell_2(t)) \geq 1 + (e^{kt}-1)^2/(2k^2 e^{kt}) \qquad (4.13)$$

i.e. $k=\sqrt{1-\alpha^2}$ $0 \leq \alpha < 1$ and

$$\ell_2(t) = |\alpha_2(0)\alpha_2(t)|.$$

Letting α tend to one in (4.13) we obtain:

$$\cosh(\ell_2(t)) \geq 1 + \frac{t^2}{2}\, ; \quad \alpha=1.$$

2) In a higher rank, non-compact symmetric space one can always find totally geodesic, flat embeddings of $\mathbb{R}^2$. In this context, Schur's argument applies without any modification to estimate the chordal length of any polygonal space curve obtainable from a planar chord convex curve via admissible deformations. As the underlying space is not isotropic it is necessary to characterize these space curves and their C^2 limits.

3) In Theorem B it is not necessary for γ_1 and γ_2 to be C^2, they could have exterior angles $\{\theta^1_i\}$ and $\{\theta^2_i\}$ respectively occurring at arclengths $\{L_i\}$. These angles must satisfy:

$$|\theta^2_i| \leq \bar{\theta}^1_i.$$

4) Using the representation of $\mathbb{H}^3$ as a hyperboloid in 4-dimensional Minkowski space it should be possible to prove a complete analogue of Schur's Theorem, directly, in a way analogous to the proof of Schur's theorem presented in [Ch].

5) The restriction that γ_2 lie in $\mathbb{H}^3$ is not necessary, γ_2 may lie in $\mathbb{H}^n$ for any $n \geq 2$.

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