Chapter 1

One complex variable, for adults

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1.1 Introduction

The object of these lectures is to give an introduction to the theory of holomorphic functions of several variables. Unlike the theory for functions of one complex variable, there are no 'physical problems' underlying this subject. The earliest work in the subject is due primarily to Poincaré. He was interested in filling gaps in the proofs of several important results in the theory of Riemann surfaces. These questions involved the study of mermorphic functions on the Jacobian variety which could in turn be rephrased as questions about meromorphic functions on $\mathbb{C}^n$. Among other things Poincaré proved a generalization of the following theorem of Weierstraß :

Theorem 1.1.1. *Suppose that $f(z)$ is a meromorphic function in the whole complex plane $\mathbb{C}$ then there exist two entire functions $p(z), q(z)$ such that*

$$f(z) = \frac{p(z)}{q(z)}.$$

He also considered the problem of generalizing the Riemann mapping theorem:

Theorem 1.1.2. *Suppose that $D \subset \mathbb{C}$ is simply connected and ∂D consists of more than two points, then there exists a holomorphic function defined in D which carries D one to one, onto the unit disk.*

While the Weierstraß theorem generalized, an entirely different proof was required. The Riemann mapping theorem is simply false in more than one variable. The problems of uniformization and biholomorphic equivalence remain active areas of research.

In addition to the two problems considered by Poincaré , there were two other, one variable theorems that formed the main impetus in the development of several complex variables. The first was the Mittag Leffler theorem

Theorem 1.1.3. *Let $\{z_k\}$ be a set of points in $\mathbb{C}$ with no finite point of accumulation and $\{p_k(z)\}$ a set of polynomials without constant term. Then there exists a meromorphic function $f(z)$ with singularities at $\{z_k\}$ and principal part at z_k given by $p_k(\frac{1}{z-z_k})$.*

The other was the theory of analytic continuation and its extension to the theory of Riemann surfaces. The theory of compact Riemann surfaces, should more properly be considered as a motivation for algebraic geometry whereas in several complex variables one is generally more concerned with the non–compact case. There are of course, many points of contact between these two subjects though they remain quite distinct.

The first few lectures are devoted to a quick review of the one variable theory. The proofs we give for the main theorems are not the usual, one variable arguments but rather introduce techniques which generalize to the many variables case. As will soon become apparent the main thrust is directed towards the analysis of the $\overline{\partial}$–equation. In one complex dimension it reads:

$$\partial_{\bar{z}} u = \frac{1}{2}(\partial_x + i\partial_y)u = f.$$

In one form or the other we will be studying this equation for most of the semester.

The next order of business is to understand the local theory of holomorphic functions of several variables. As in the one variable case, there are several different characterizations of holomorphy, though their equivalence is a bit subtler. Much of the local theory follows from a simple generalization of the Cauchy integral formula. However there are some notable differences which start to appear at this stage. They mostly have to do with the problem of extending holomorphic functions. Next we introduce several different different notions of geometric and analytic convexity. Of course we will consider the local theory of the $\overline{\partial}$–equation which is already a good deal richer than in the one variable case.

The Bergmann kernel function and its elementary properties will be introduced. We then consider the analytic geometry of the unit ball and the polydisk in order to show that these are biholomorphically inequivalent. The next several topics are purely several variables and have no one dimensional analogues. These are concerned with the boundary behavior of holomorphic functions, CR–structures and the Lewy extension theorem. These results give an indication of the importance of geometric notions like pseudoconvexity.

Finally we present Hörmander's L^2–method for solving the $\overline{\partial}$–equation in pseudoconvex domains. These results are then applied to solve the "Cousin problems" in this context. The solutions of these problems bring us very close to generalizations of the Weierstraß and Mittag–Leffler theorems. However to complete the picture we need some basic understanding of the geometry of analytic varieties. The fundamental tools for this study are the Weierstraß preparation and division theorems and some elementary commutative algebra. Hopefully we will finish the course by proving that, under suitable hypotheses, a mermorphic function defined in a pseudoconvex domain in $\mathbb{C}^n$ is the quotient of two holomorphic functions.

1.2 The Cauchy Integral Formula

The fundamental tool for studying holomorphic functions in one variable is the Cauchy integral formula. This formula is actually a special case of the Stokes formula. Recall that if $\omega = adx + bdy$ is a smooth one form defined in a neighborhood of a bounded domain, $\Omega \subset \mathbb{R}^2$ with a smooth boundary then

$$d\omega = (b_x - a_y)dx \wedge dy.$$

Stokes' formula states that

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$

In order to evaluate the line integral we need specify an orientation for the boundary of Ω. The ordered basis $\{\partial_x, \partial_y\}$ defines a global orientation for $\mathbb{R}^2$. Along $\partial\Omega$ we can define an outward pointing unit normal vector field ν. The orientation for $\partial\Omega$ is fixed by choosing the unit tangent vector field τ so that the ordered basis, $\{\nu, \tau\}$ give the aforementioned orientation to $\mathbb{R}^2$ along $\partial\Omega$.

If $\partial\Omega$ has a single component, then we can parametrize it by a map $(x(t), y(t)) : [0, 1] \to \partial\Omega$. The map has the correct orientation provided

$$(x'(t), y'(t)) = \lambda(t)\tau(t),$$

where $\lambda(t) > 0$. The line integral above is simply:

$$\int_0^1 [a(x(t), y(t))x'(t) + b(x(t), y(t))y'(t)]dt.$$

More useful in the study of functions of a complex variable is a version of Stoke's theorem making use of complex notation. Essentially we introduce complex valued coordinates for points in the plane:

$$z = x + iy, \; \bar{z} = x - iy.$$

Since $x = \frac{1}{2}(z + \bar{z})$ and $iy = \frac{1}{2}(z - \bar{z})$ any function in $\mathbb{R}^2$ can be thought of as a function of these variables,

$$F(z, \bar{z}) = f(\frac{1}{2}(z + \bar{z}), -i\frac{1}{2}(z - \bar{z})).$$

We have coordinate vector fields and differentials given by:

$$\begin{gathered}\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \; \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \\ dz = (dx + idy), \; d\bar{z} = (dx - idy).\end{gathered} \tag{1.2.1}$$

The Taylor series of a smooth function in complex notation is

$$f(z, \bar{z}) \sim \sum_{0 \leq i,j} \partial_z^i \partial_{\bar{z}}^j f(a, \bar{a}) \frac{(z-a)^i (\bar{z} - \bar{a})^j}{i! j!}.$$

Using these coordinates we can express the differential of a function

$$df = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}. \tag{1.2.2}$$

Any one form ω can be expressed relative to this basis:

$$\omega = a dz + b d\bar{z}.$$

We call $a\,dz$ the $(1,0)$–part and $b\,d\bar{z}$ the $(0,1)$–part. This splits $\Lambda^1\mathbb{R}^2$ into two subbundles denoted by $\Lambda^{1,0}\mathbb{C}$ and $\Lambda^{0,1}\mathbb{C}$. Using this splitting we define two differential operators

$$\begin{aligned}\partial f &= \{ \text{ the } (1,0) \text{ part of } df \} = \partial_z f dz, \\ \overline{\partial} f &= \{ \text{ the } (0,1) \text{ part of } df \} = \partial_{\bar{z}} f d\bar{z}.\end{aligned}$$

Evidently $d = \partial + \overline{\partial}$, we can extend these definitions to one forms as well:

$$\partial\omega = \partial_z b \, dz \wedge d\bar{z}, \quad \overline{\partial}\omega = \partial_{\bar{z}} a \, d\bar{z} \wedge dz. \tag{1.2.3}$$

Note that

$$\partial\partial = 0, \; \overline{\partial\partial} = 0$$

and since $d = \partial + \overline{\partial}$ these imply that

$$\partial\overline{\partial} + \overline{\partial}\partial = 0.$$

If ω is a $(1,0)$–form, then $d\omega = \overline{\partial}\omega$. From this observation we derive the form of Stokes' theorem used in one complex variable.

Proposition 1.2.4. *Let $D \subset \mathbb{C}$ be a smooth bounded domain and let ω be a $(1,0)$–form C^1 in $\overline{D}$ then*

$$\int_D \overline{\partial}\omega = \int_{\partial D} \omega. \tag{1.2.5}$$

The boundary is given the induced orientation.

Mostly we will use the following special case of (1.2.5):

Cauchy's Integral Formula 1.2.6. *Let D be a smoothly bounded domain in $\mathbb{C}$ and let $f \in C^1(\overline{D})$ then for $z \in D$ we have*

$$f(z) = \frac{1}{2\pi \imath}\left[\iint\limits_{D} \frac{dw \wedge \overline{\partial} f}{w - z} + \int\limits_{\partial D} \frac{f dw}{w - z}\right]. \tag{1.2.7}$$

Proof. This formula follows from (1.2.5) by taking $\omega = \frac{f\,dw}{w-z}$ in the domain $D \setminus B(z, \epsilon)$. In such a domain $\partial_{\bar{w}} \frac{1}{w-z} = 0$ and furthermore $\frac{1}{w-z}$ is locally integrable, allowing $\epsilon \to 0$ gives (1.2.7).

At this point we are ready to define holomorphic functions. The definition we give is one that generalizes most easily.

Definition 1.2.8. For a domain $D \subset \mathbb{C}$ a function $f \in C^1(D)$ is said to be holomorphic (or analytic) if

$$\overline{\partial} f = 0. \tag{1.2.9}$$

The set of holomorphic functions in D will be denoted by $H(D)$.

To study holomorphic functions it is essential that we have a method for solving the inhomogeneous $\overline{\partial}$–equation:

$$\overline{\partial} u = f$$

First suppose that $f \in \mathcal{C}_c^\infty(\mathbb{C})$ and set

$$u(z) = \frac{1}{2\pi \imath}\left[\iint\limits_{D} \frac{f(w) dw \wedge d\bar{w}}{w - z}\right].$$

By changing variables we see that

$$u(z) = \frac{1}{2\pi \imath}\left[\iint\limits_{D} \frac{f(w+z) dw \wedge d\bar{w}}{w}\right];$$

This can evidently be differentiated under the integral sign to give

$$\overline{\partial} u = \frac{1}{2\pi \imath}\left[\iint\limits_{D} \frac{\partial_{\bar{w}} f(w+z) dw \wedge d\bar{w}}{w - z}\right].$$

Applying (1.2.7), recalling that f is compactly supported we see that

$$\overline{\partial} u = f.$$

From the computation it is clear that the argument only requires f to be once differentiable and compactly supported.

A more general result is summarized in the following proposition

Proposition 1.2.10. *Suppose that $d\mu$ is a finite measure supported in a compact set $K \subset \mathbb{C}$ then*

$$u(z) = \frac{1}{2\pi \imath}\iint \frac{d\mu(w)}{w - z}$$

is holomorphic outside K. If $U \subset K$ is a open set in which $d\mu = \phi dz \wedge d\bar{z}$ for $\phi \in C^k(U)$, then $u \in C^k(U)$ and

$$\overline{\partial} u = \phi d\bar{z}. \tag{1.2.11}$$

Proof. The first assertion follows easily by differentiating under the integral. In fact, off of the support of $d\mu$ we can differentiate arbitrarily often, so $u(z) \in \mathcal{C}^\infty(K^c)$. To prove the other assertion, fix a point $p \in U$ and a positive number r such that $B(p,r) \subset\subset U$. Choose a smooth function ψ compactly supported in $B(p,r)$ and identically 1 in $B(p, \frac{1}{2}r)$. Clearly

$$u(z) = \frac{1}{2\pi\imath} \iint \frac{\psi\phi dw \wedge d\bar{w}}{w-z} + v(z), \tag{1.2.12}$$

where by the above argument $v(z)$ is in $H(B(p, \frac{1}{2}r)) \cap \mathcal{C}^\infty(B(p, \frac{1}{2}r))$. Denote the first term by $u'(z)$. It suffices to show that $u'(z)$ is in $C^k(B(p, \frac{1}{2}r))$ and satisfies

$$\partial_{\bar{z}} u' = \psi\phi. \tag{1.2.13}$$

To obtain the smoothness we change variables in the integral by setting variables $w' = w - z$, then we can differentiate k times under the integral and obtain a continuous integrand. It follows from (1.2.7) that

$$\overline{\partial} \iint\limits_{B(p,r)} \frac{(\psi\phi d\bar{w} - \overline{\partial} u')dw}{w-z} = 0, z \in B(p,r). \tag{1.2.14}$$

Since each term is separately differentiable, (1.2.13) follows from the special case considered above.

As a corollary of (1.2.7) and (1.2.10) we have the standard form of the Cauchy integral formula for holomorphic functions:

Corollary 1.2.15. *If $f(z)$ is a holomorphic function in an a neighborhood of a compact set K with smooth boundary, then*

$$f(z) = \frac{1}{2\pi\imath} \int\limits_{\partial K} \frac{f(w)\, dw}{w-z}. \tag{1.2.16}$$

From this corollary most of the important local properties of holomorphic functions follow easily.

1.3 Elementary facts about holomorphic functions of one variable

Most of these facts follow by standard arguments from the Cauchy integral formula so detailed proofs are usually omitted.

Corollary 1.3.1. *If $u \in H(\Omega)$ then $u \in \mathcal{C}^\infty(\Omega)$ as well and therefore $\partial_z u \in H(\Omega)$.*

Proof. Immediate from (1.2.10) with $d\mu$ a measure concentrated on a curve as in (1.2.16).

An interesting fact about holomorphic functions is summarized in the following theorem

Theorem 1.3.2. *For every compact subset $K \subset\subset \Omega$, every open neighborhood $K \subset\subset U \subset\subset \Omega$ and $j \in \mathbb{N}$ there is a constant C_j such that*

$$\sup_{z \in K} |\partial_z^j u(z)| \leq C_j \|u\|_{L^1(U)}, \forall\, u \in H(\Omega). \tag{1.3.3}$$

Proof. We choose a function $\psi \in \mathcal{C}_c^\infty(U)$ such that $\psi = 1$ on a neighborhood of K. Then it follows from (1.2.7) that

$$\psi(z)u(z) = \frac{1}{2\pi\imath} \int\limits_{U \setminus K} \frac{u\overline{\partial}\psi \wedge dz}{z-w}. \tag{1.3.4}$$

Since $\overline{\partial}\psi = 0$ in a neighborhood of K, we can differentiate under the integral in (1.3.4) so long as $z \in K$. The assertion of the theorem follows easily from this.

Corollary 1.3.5. *If u_n is a sequence of functions in $H(\Omega)$ that converge locally uniformly to a function $u(z)$ then $u \in H(\Omega)$.*

Proof. The previous theorem allows us to estimate the derivatives $\partial_z(u_n - u_m)$ on compact subsets of Ω, since $\overline{\partial}_z(u_n - u_m) = 0$ it follows that all first derivatives of the sequence u_n also converge locally uniformly in Ω. Therefore the limit u is in $C^1(\Omega)$ and

$$\overline{\partial}_z u = \lim_{n\to\infty} \overline{\partial}_z u_n = 0.$$

Corollary 1.3.6. *If $u_n \in H(\Omega)$ and the sequence $|u_n|$ is uniformly bounded on compact sets then there is a convergent subsequence.*

Proof. By (1.3.3) the first derivatives of u_n are uniformly bounded on compact subsets of Ω. Thus we can apply the Arzela–Ascoli theorem to extract a uniformly convergent subsequence on such a compact subset. By exhausting Ω by a nested family of compact subsets and applying a diagonal argument we obtain a locally uniformly convergent subsequence on Ω.

Corollary 1.3.7. *The sum of a power series*

$$u(z) = \sum_{n=0}^{\infty} a_n z^n$$

is holomorphic in the interior of the circle of convergence.

This is an alternative definition of a holomorphic function: one that is represented in a neighborhood of every point by a convergent power series in z. In fact this is equivalent to the previous definition. The easy direction follows from (1.3.7) the harder direction is

Theorem 1.3.8. *If $u(z)$ is holomorphic in $B(0,r)$ then*

$$u(z) = \sum_{n=0}^{\infty} \frac{u^{[n]}(0)z^n}{n!}. \tag{1.3.9}$$

with uniform convergence on every compact subset of $B(0,r)$.

Proof. We give a slightly different proof from the usual. The Cauchy integral formula can be differentiated to give

$$u^{[n]}(z) = \frac{n!}{2\pi\imath} \int\limits_{|w|=r_2} \frac{u(w)dw}{(w-z)^{n+1}}, \tag{1.3.10}$$

for an $r_2 \leq r$. If

$$M(\rho) = \sup_{|z|=\rho} |u(z)|$$

then (1.3.10) implies that

$$|u^{[n]}(z)| \leq \frac{r_2 M(r_2) n!}{(r_2 - |z|)^{n+1}}. \tag{1.3.11}$$

These are called the Cauchy Estimates. From (1.3.11) it is immediate that (1.3.9) converges uniformly on any $B(0,r_1)$ with $r_1 < r_2 < r$. To see that it converges to $u(z)$ we use the integral form of the remainder term in Taylor's theorem

$$u(\rho e^{i\theta}) - \sum_{j=0}^{n} \frac{u^{[j]}(0)(\rho e^{i\theta})^j}{j!} = R_n(\rho e^{i\theta}) = \frac{1}{n!} \int\limits_0^{\rho} e^{i(n+1)\theta} u^{[n+1]}(se^{i\theta})(\rho - s)^n ds. \tag{1.3.12}$$

Using the estimates (1.3.11) in (1.3.12), for $|z| < r_2$, we derive

$$|R_n(z)| \leq \frac{C(n+1)}{(r_2 - |z|)^2} \left(\frac{|z|}{r_2}\right)^{n+1}. \tag{1.3.13}$$

From (1.3.13) we deduce that

$$\lim_{n\to\infty} |R_n(z)| = 0, \text{ if } |z| < r_2.$$

This establishes the equivalence of the two definitions of holomorphic. From the power series representation we derive the uniqueness of analytic continuation.

Corollary 1.3.14. *If $u(z)$ is holomorphic in a connected set Ω and there exists a $z_0 \in \Omega$ at which*

$$u^{[n]}(z_0) = 0 \,\forall\, n \in \mathbb{N}_0 \tag{1.3.15}$$

then u is identically zero in Ω.

Proof. The set of points which satisfy (1.3.15) is evidently a closed subset of Ω. From (1.3.8) it is also open, by hypothesis it is non–empty and therefore it must be all of Ω.

Using Cauchy's formula we can prove an very important result of Riemann's on the continuation of an analytic function defined in a punctured neighborhood:

The Riemann Extension Theorem 1.3.16. *Suppose that $f(z)$ is holomorphic in $B(a,r) \setminus \{a\}$ and*

$$|f(z)| = o(|z-a|^{-1})$$

as z approaches a, then f has an extension as an analytic function on $B(a,r)$.

Remark. A more precise formulation would be: there exists an analytic function, F defined on $B(a,r)$ such that

$$F \restriction_{B(a,r)\setminus\{a\}} = f.$$

The extension is obviously unique. This result is essentially optimal as $f(z) = \frac{1}{z}$ just barely fails to satisfy the necessary estimate and has no extension across 0.

Proof. The proof is an elementary application of the Cauchy formula: for each $0 < \epsilon < \rho$ we can express $f(w)$ for $\rho < |w| < r$ as the integral:

$$f(w) = \frac{1}{2\pi\imath}\left[\int\limits_{|z-a|=r} \frac{f(z)dz}{z-w} - \int\limits_{|z-a|=\epsilon} \frac{f(z)dz}{z-w}\right]. \tag{1.3.17}$$

The second integral in (1.3.17) is estimated by

$$\frac{\epsilon M_\epsilon}{\rho - \epsilon}.$$

Here $M_\epsilon = \max\limits_{|z|=\epsilon} |f(z)|$. The estimate on $|f(z)|$ as $z \to a$ implies that this term tends to zero as $\epsilon \to 0$. Hence we have a representation for $f(w)$ given by

$$f(w) = \frac{1}{2\pi\imath} \int\limits_{|z-a|=r} \frac{f(z)dz}{z-w}.$$

But the integral on the right hand side evidently defines a function analytic in $B(a,r)$. This completes the proof of the theorem.

Finally we have the maximum principle

The Maximum Principle 1.3.18. *Let Ω be a bounded domain and let $u \in C(\overline{\Omega}) \cap H(\Omega)$ then the maximum of $|u|$ is attained on the $\partial\Omega$.*

Proof. If $B(z_0, \rho) \subset \Omega$ we can use the Cauchy integral formula to express

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta.$$

Taking absolute values we obtain that

$$|u(z_0)| \leq \max_{|z-z_0|=\rho} |u(z)|.$$

The inequality is strict unless $u(z) = u(z_0) \, \forall \, z \in B(z_0, \rho)$. Therefore $|u(z)|$ does not assume its maximum at an interior point unless $u(z)$ is constant in that connected component of Ω.

From the power series representation of a holomorphic functions we deduce the following "normal form" theorem

Corollary 1.3.19. *For a function, u holomorphic in $B(0, r)$ there is a unique $n \geq 0$ and a holomorphic function $v(z)$ with $v(0) \neq 0$ such that*

$$u(z) = z^n v(z)$$

in $B(0, r)$.

In fact we can introduce a new complex parameter $w = z(v(z))^{\frac{1}{n}}$. This defines a nonsingular change of variable near to $w = z = 0$. In terms of this new coordinate

$$u(w) = w^n. \tag{1.3.20}$$

From (1.3.20) we see that, up to a local change of variable, the mapping of $\mathbb{R}^2$ to itself defined by a holomorphic function is locally determined the integer n appearing in (1.3.19)

There is one further characterization of holomorphic functions. A differentiable function is holomorphic if the induced mapping of $\mathbb{R}^2$ is conformal relative to the flat metric. Computing the Jacobian shows that this is equivalent to the Cauchy–Riemann equations. However this geometric characterization does not generalize to higher dimensions. So we will not pursue it here.

We conclude this section with one of the most celebrated theorems of one complex variable and perhaps all of mathematics, the Riemann mapping theorem.

Definition 1.3.21. Two domains Ω_1, Ω_2 are *biholomorphically equivalent* if there is a mapping

$$\Phi : \Omega_1 \longrightarrow \Omega_2$$

such that both Φ and Φ^{-1} are holomorphic.

As a simple application of (1.3.19) we see that a holomorphic mapping is locally one to one if and only if its derivative never vanishes. Thus it follows easily from the inverse function theorem that if Φ is holomorphic and one to one then Φ^{-1} is automatically holomorphic as well.

The Riemann Mapping Theorem 1.3.22. *Suppose that Ω is a simply connected open subset of $\mathbb{C}$ with $\partial\Omega \neq \emptyset$ then Ω is biholomorphically equivalent to $B(0, 1)$.*

The proof of this theorem would take us too far afield, suffice to say that it does not generalize to higher dimensions. This theorem implies that any two simply connected domains (not all of $\mathbb{C}$) are biholomorphically equivalent. Notice that it makes no assumptions whatsoever about the regularity or even the topology of $\partial\Omega$. We will see that this statement fails completely in higher dimensions and that there is an infinite dimensional "space" of biholomorphically inequivalent domains diffeomorphic to the unit ball.

1.4 The Runge approximation theorem, the holomorphic convex hull

In this section we consider a theorem which describes the relationship of functions of $H(\Omega_1)$ and $H(\Omega_2)$ when $\Omega_1 \subset\subset \Omega_2$. As we shall see, this is a fundamental step in understanding the solvability properties of the equation

$$\overline{\partial} u = f$$

on planar regions. This should not be too surprising as $H(\Omega) = \ker \overline{\partial} \restriction_{\Omega}$. This in turn will allow us to study the types of meromorphic functions which can be defined on very general planar regions. The methods we employ generalize to both the Riemann surface case and to several variables.

The basic observation is that given $f(z)$, holomorphic in $B(0, r)$, for any $r_1 < r$ and any $M > 0$, we can find a polynomial $p_M(z)$ such that

$$\max_{z \in B(r_1, 0)} |f(z) - p_M(z)| \leq \frac{1}{M}.$$

In this simple case we merely take some sufficiently large partial sum of the Taylor series of $f(z)$ about 0. On the other hand for $f(z)$ holomorphic in the annular region, $r < |z| < R$, we do not expect to be able to approximate on compact subannuli by polynomials but rather by meromorphic functions of the form

$$\sum_{n=-M}^{N} a_n z^n.$$

These are functions holomorphic in $\mathbb{C} \setminus \{0\}$. The situation is summed up by the following theorem

Runge Approximation Theorem 1.4.1. *Let Ω be an open set in $\mathbb{C}$ and K a compact subset of Ω. The following conditions on Ω and K are equivalent:*

1.4.2 *Every function which is analytic in a neighborhood of K can be approximated uniformly by functions in $H(\Omega)$.*

1.4.3 *The open set $\Omega \setminus K = \Omega \cap K^c$ has no component which is relatively compact in Ω*

1.4.4 *For every $z \in \Omega \setminus K$ there is a function $f \in H(\Omega)$ such that*

$$|f(z)| > \sup_K |f|. \tag{1.4.5}$$

Proof. The easiest implication is that (1.4.4)$\Longrightarrow$(1.4.3). For suppose that $\Omega \setminus K$ has relatively compact component L. This implies that $\partial L \subset \partial K$. The maximum principle implies that for $u \in H(\Omega) \subset H(\overline{L})$

$$\sup_L |u| \leq \sup_{\partial L} |u| \leq \sup_K |u|.$$

However this contradicts (1.4.4).

Next we will verify that (1.4.2)$\Longleftrightarrow$(1.4.3). First suppose that (1.4.2) holds but that $\Omega \setminus K$ has a relatively compact component L. Let $w \in L$; the function $h(z) = (z - w)^{-1}$ is holomorphic in a neighborhood of K. Thus we can find sequence $f_n \in H(\Omega)$ for which

$$\sup_K |f_n - h| \leq \frac{1}{n}. \tag{1.4.6}$$

Evidently it follows from (1.4.6) that

$$\sup_K |(z - w)f_n - (z - w)h| \leq \frac{C}{n}, \tag{1.4.7}$$

for some fixed constant C. As before $\partial L \subset \partial K$ and therefore by the maximum principle, (1.4.7) must hold on $K \cup \overline{L}$. From this we conclude that $(z - w)f_n$ converges to $(z - w)h$ uniformly for $z \in L$. However

$$(z - w)f_n(z) \restriction_{z=w} = 0, \; \forall\, n, \text{ but } (z - w)h(z) \restriction_{z=w} = 1.$$

This contradiction proves the assertion.

To obtain the other direction we use the Hahn–Banach Separation Theorem,(1.A.30). We recall what this theorem says in the case at hand: Let that S_K denote the closure, in the sup–norm topology on K of the linear subspace of $C^0(K)$ defined by the restrictions of functions holomorphic in a neighborhood of K. We also let S_Ω denote the closure in $C^0(K)$ of the linear subspace defined by restrictions to K of functions in $H(\Omega)$. Clearly $S_\Omega \subset S_K$, if they are not equal then there is a function $f \in S_K \setminus S_\Omega$. The Hahn Banach theorem would ensure the existence of a continuous linear function $\ell : C^0(K) \to \mathbb{C}$ such that

$$\ell \restriction_{S_\Omega} = 0 \text{ but } \ell(f) = 1.$$

By the Riesz representation theorem the linear functional ℓ would be represented by a finite measure supported on K,

$$\ell(g) = \int_K g d\mu.$$

To show that $S_\Omega = S_K$ it therefore suffices to show that any finite measure supported on K, $d\mu$ such that

$$\int u(z) d\mu(z) = 0, \ \forall\, u \in H(\Omega)$$

also annihilates any function which is holomorphic in a neighborhood of K.

Let $d\mu$ be such a measure then the function

$$\phi(z) = \int \frac{d\mu(w)}{z - w}, \tag{1.4.8}$$

vanishes for $z \in \Omega^c$. This function is holomorphic for $z \in K^c$. By our assumption that $\Omega \setminus K$ has no relatively compact components it follows that every component of K^c meets a component of Ω^c. From this we conclude that

$$\phi(z) = 0 \text{ for } z \in K^c. \tag{1.4.9}$$

If ψ is a function, holomorphic in a neighborhood of K, then we can choose a $\mathcal{C}^\infty$ function χ supported in the domain of holomorphy of ψ and identically equal to 1 on K. From (1.2.7) we have the representation of ψ for $z \in K$:

$$\psi(z) = \frac{1}{2\pi\imath}\left[\iint_{K^c} \frac{\psi(w) dw \wedge \overline{\partial}\chi}{w - z}.\right]. \tag{1.4.10}$$

Combining this representation with (1.4.9) and changing the order of the integrations we obtain that

$$\int \psi(w) d\mu(w) = 0.$$

This was what we needed to establish.

To complete the proof we need to show that (1.4.2)$\Longrightarrow$(1.4.4). Choose a point $w \in K^c$. Since we have shown the equivalence of the first two assertions, it follows that no component of $\Omega \setminus K$ is relatively compact in Ω. From this it is clear that we can find an $\epsilon > 0$ so that the domain $K' = K \cup B(w, \epsilon)$ also has this property. Again by the equivalence of the first two assertions we conclude that $H(\Omega)$ is uniformly dense in the functions holomorphic in a neighborhood of K'. Therefore the function defined by

$$f(z) = \begin{cases} 0 & z \in K \\ 1 & z \in B(w, \epsilon) \end{cases}$$

is uniformly approximated by functions in $H(\Omega)$. In particular we can find a function $u \in H(\Omega)$ such that $|u(w)| > \frac{1}{2}$ whereas

$$\sup_K |u| < \frac{1}{2}.$$

This completes the proof

In proving the equivalence of (1.4.3) and (1.4.4) we verify that a condition of holomorphic convexity is equivalent to a ‘topological convexity’.

Definition 1.4.11. If $K \subset\subset \Omega$ is a compact subset of an open set then we define the holomorphic convex hull of K (relative to Ω) to be

$$\widehat{K}_\Omega = \{z \in \Omega : |f(z)| \le \sup_K |f|, \forall f \in H(\Omega)\}.$$

By considering special functions in $H(\Omega)$ we can study some of the gross properties of the operation $K \longrightarrow \widehat{K}_\Omega$. When no confusion can arise we omit explicit reference to Ω. Clearly

$$\widehat{\widehat{K}} = \widehat{K}. \tag{1.4.12}$$

Let $w \in \Omega^c$ then the function $(z-w)^{-1} \in H(\Omega)$, thus we easily obtain that

$$\operatorname{dist}(K, \Omega^c) = \operatorname{dist}(\widehat{K}, \Omega^c). \tag{1.4.13}$$

Further since $e^{az} \in H(\Omega)$ we can show that

$$\widehat{K} \subset \text{ usual convex hull of } K. \tag{1.4.14}$$

Exercise 1.

(a) Prove that if $K_1 \subset K_2$ then $\widehat{K_1} \subset \widehat{K_2}$. If $K \subset\subset \Omega_1 \subset \Omega_2$ then $\widehat{K}_{\Omega_2} \subset \widehat{K}_{\Omega_1}$.
(b) Prove (1.4.12), (1.4.13), (1.4.14).
(c) Let L denote the union of the relatively compact components of $\Omega \setminus K$, prove that $\widehat{K}_\Omega = K \cup L$.
(d) Can you explain why *holomorphic convex hull* is a reasonable name for $\widehat{K}$.

From (1.4.13) and (1.4.14) one can easily deduce that the holomorphic convex hull of K relative to Ω, is also a compact subset of Ω. From elementary point set topology we know that an open subset can be exhausted by a nested sequence of compact subsets. Let L_j be such a collection

$$L_j \subset L_{j+1}, \, \Omega = \cup L_j. \tag{1.4.15}$$

Exercise 2. Prove the existence of compact sets $\{L_j\}$ which satisfy (1.4.15).

If we set $K_j = \widehat{L}_j$ then

$$K_j \subset\subset K_{j+1}, \quad \widehat{K}_j = K_j, \quad \Omega = \cup K_j.$$

This proves

Proposition 1.4.16. *Any open subset of $\mathbb{C}$ can be exhausted by a nested sequence of compact subsets, K_i which satisfy $\widehat{K}_i = K_i$.*

One says that an open subset $\Omega \subset \mathbb{C}$ is holomorphically convex if for every compact subset $K \subset\subset \Omega$, $\widehat{K}_\Omega \subset\subset \Omega$. As we've just seen every open subset of $\mathbb{C}$ is holomorphically convex. This simple geometric proposition plays a crucial role in the analysis of holomorphic functions in open subsets of $\mathbb{C}$. This is a point where the difference between one and several variables is the most marked. As we shall see, not every open subset of $\mathbb{C}^n$, $n \ge 2$, is holomorphically convex. In fact the characterization of such domains by a relatively simple geometric property formed a central problem around which several complex variables grew. This is called the Levi problem.

1.5 Solving the $\overline{\partial}$–equation

The most powerful analytic techniques for the study of holomorphic functions of several variables revolve around solving the $\overline{\partial}$–equation

$$\overline{\partial} u = f d\bar{z}. \tag{1.5.1}$$

In this section we prove the following basic existence theorem

Theorem 1.5.2. *Suppose that Ω is an open subset of $\mathbb{C}$ and $f \in \mathcal{C}^\infty(\Omega)$ then there exists $u \in \mathcal{C}^\infty(\Omega)$ such that*

$$\overline{\partial} u = f d\bar{z}. \tag{1.5.3}$$

Remark. In §1.2 we obtained a solution to equation (1.5.3) for compactly supported f. This theorem is a global statement to the effect that we can solve this equation throughout a specified domain. Note that no hypothesis is made on either the regularity of $\partial\Omega$, nor on the behavior of f near $\partial\Omega$. Moreover our technique of solution gives no information about the behavior of u as we approach the boundary. This suffices for many applications, however there are techniques which, under suitable hypotheses of Ω and f give a solution u which is globally estimated in terms of f. Similar results with $f \in \mathcal{C}^k$ can easily be deduced from this theorem. Finally note that the solution is never unique as we can always add functions in $H(\Omega)$ to obtain other solutions.

Proof. Let $K_j, j = 1, \dots$ define a exhaustion of Ω by holomorphically convex, compact subsets as in (1.4.16). Let $\psi_j \in \mathcal{C}_c^\infty(K_{j+1})$ with $\psi_j = 1$ in a neighborhood of K_j. Set

$$u_1 = \frac{1}{2\pi\imath} \iint \frac{\psi_1 f dw \wedge d\bar{w}}{w - z}. \tag{1.5.5}$$

From (1.2.10) it follows that $u_1 \in \mathcal{C}^\infty(\mathbb{C})$ and $\overline{\partial} u_1 = \psi_1 f d\bar{z}$. We will find solutions to our problem on larger and larger sets using the Runge theorem to control the convergence of our sequence.

As the second step we set

$$\widetilde{u}_2 = \frac{1}{2\pi\imath} \iint \frac{\psi_2 f dw \wedge d\bar{w}}{w - z}.$$

Then as before $\widetilde{u}_2 \in \mathcal{C}^\infty(\mathbb{C})$ and $\overline{\partial}\widetilde{u}_2 = \psi_2 f d\bar{z}$. However we have no a priori control on $u_1 - \widetilde{u}_2$. By our choice of of ψ_j it follows that

$$\overline{\partial}(\widetilde{u}_2 - u_1) = 0$$

on a neighborhood of K_1. Since K_1 is assumed to be holomorphically convex, (1.4.1) implies that we can find a $p_2 \in H(\Omega)$ such that

$$\sup_{K_1} |\widetilde{u}_2 - u_1 - p_2| \le \frac{1}{2}. \tag{1.5.6}$$

Set $u_2 = \widetilde{u}_2 - p_2$; it satisfies

$$u_2 \in \mathcal{C}^\infty(\Omega), \overline{\partial} u_2 = \psi_2 f d\bar{z}, \sup_{K_1} |u_2 - u_1| \le \frac{1}{2}. \tag{1.5.7}$$

As an inductive step suppose that we can find $u_j \in \mathcal{C}^\infty(\Omega)$ such that

$$\overline{\partial} u_j = \psi_j f d\bar{z}, \sup_{K_{j-1}} |u_j - u_{j-1}| \le \frac{1}{2^{j-1}}. \tag{1.5.8}$$

To verify the inductive hypothesis we argue exactly as above, set

$$\widetilde{u}_{j+1} = \frac{1}{2\pi\imath} \iint \frac{\psi_{j+1} f dw \wedge d\bar{w}}{w - z}.$$

From (1.2.10) we deduce that $\widetilde{u}_{j+1} \in \mathcal{C}^\infty(\mathbb{C})$ and that

$$\overline{\partial}\widetilde{u}_{j+1} = \psi_{j+1} f d\bar{z}. \tag{1.5.9}$$

From (1.5.9) and the inductive hypothesis we obtain that $\widetilde{u}_{j+1} - u_j$ is holomorphic in a neighborhood of K_j and therefore by (1.4.1) we can find $p_{j+1} \in H(\Omega)$ so that

$$\sup_{K_j} |\widetilde{u}_{j+1} - u_j - p_{j+1}| \leq \frac{1}{2^j}. \tag{1.5.10}$$

Letting $u_{j+1} = \widetilde{u}_{j+1} - p_{j+1}$ and applying (1.5.9) and (1.5.10) completes the proof of the inductive hypothesis.

From (1.5.8) it follows that $\{u_k \restriction_{K_j}\}$ is a Cauchy sequence in the uniform topology for any j. Thus the sequence has a continuous limit, $u \in C(\Omega)$. If we consider the subsequence $u_k - u_j$ then we have a uniformly convergent sequence of *holomorphic* functions on K_j. Denote the limit of this sequence by h_j. By (1.3.5) we conclude that $h_j \in H(K_j)$. As

$$u \restriction_{K_j} = h_j + u_j,$$

it is immediate that $u \in \mathcal{C}^\infty(K_j)$. Since j is arbitrary,

$$u \in \mathcal{C}^\infty(\Omega), \overline{\partial} u = f d\bar{z}.$$

Exercise 3.

(1) Show that the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$ satisfies

$$\Delta = 4\partial_z \partial_{\bar{z}} = 4\partial_{\bar{z}} = \partial_z.$$

(2) Show that $\partial_z f = \overline{\partial_{\bar{z}} \bar{f}}$.

(3) Using these two facts and Theorem 1.5.2 show that for any open set $\Omega \subset \mathbb{C}$ and any $f \in \mathcal{C}^\infty(\Omega)$ there is a solution to the equation

$$\Delta u = f.$$

(4) Suppose that $u \in \mathcal{C}^2(\Omega)$ satisfies $\Delta u = 0$. Use the two facts above and Theorem 1.5.2 to prove that $u \in \mathcal{C}^\infty(\Omega)$.

Exercise 4. Can you formulate and prove a "Runge Theorem" for harmonic functions? Hint: Use the representation formula for harmonic functions that follows from Green's formula:

$$\iint\limits_{\Omega} [u\Delta v - v\Delta u] dA = \int\limits_{\partial\Omega} [u\partial_\nu v - v\partial_\nu u] ds.$$

Recall that $\Delta \log |z| = 2\pi\delta_0(z)$.

1.6 The Mittag–Leffler and Weierstraß Theorems

In this subsection we use the solvability of the $\overline{\partial}$–equation on an open set in $\mathbb{C}$ to prove the two main results from the function theory on such domains. These results describe the sorts of meromorphic functions which can be defined on such domains. First we give a definition of a meromorphic function which generalizes to higher dimensions.

Definition 1.6.1. A function f is meromorphic in a domain Ω provided there is a covering of Ω by open sets U_i such that in each U_i we have functions $g_i, h_i \in H(U_i), h_i \not\equiv 0$ satisfying

$$f \restriction_{U_i} = \frac{g_i}{h_i}. \tag{1.6.2}$$

A function is meromorphic if it is locally the quotient of holomorphic functions. The set of meromorphic functions on Ω is denoted by $\mathcal{M}(\Omega)$. As a simple corollary of the definition we have the following alternate local representation

Corollary 1.6.3. *Let $f \in \mathcal{M}(\Omega)$ and $w \in \Omega$ then f has a unique representation in a neighborhood of w as either*

$$f(z) = \begin{cases} \sum_{j=1}^{m} \frac{a_j}{(z-w)^j} & +G(z), \text{ for an } m > 0 \text{ or,} \\ & (z-w)^n G(z), \text{ for an } n \geq 0. \end{cases} \tag{1.6.4}$$

The function $G(z)$ is holomorphic and nonvanishing at $z = w$.

Proof. This follows from the definition and (1.3.19).

If the first case in (1.6.4) pertains then f has a pole at w and the polynomial in $(z-w)^{-1}$ is called the principle part at w. The Mittag–Leffler theorem deals with the problem of specifying the principle parts of a meromorphic function. In several variables one studies this problem in a slightly round about manner which we also adopt in the one variable case for obvious pedagogic reasons. First we state the result that generalizes directly to several variables; before proving it we show how it solves the Mittag–Leffler problem.

Theorem 1.6.5. *Suppose that Ω is an open subset of $\mathbb{C}$ and U_i is a locally finite open cover. Suppose that on the overlaps $U_i \cap U_j$ we specify functions $f_{ij} \in H(U_i \cap U_j)$ that satisfy the following properties:*

(1.6.6) $f_{ij} = -f_{ji}$

(1.6.7) $f_{ij} + f_{jk} = f_{ik}$ *whenever $U_i \cap U_j \cap U_k \neq \emptyset$. (cocycle condition)*

Then there are functions $f_i \in H(U_i)\, \forall\, i$, such that

$$f_{ij} = f_i - f_j. \tag{1.6.8}$$

Before proving this we show how it implies the classical Mittag–Leffler theorem

Mittag–Leffler Theorem 1.6.9. *Let $\{w_i\}$ be a discrete subset of the domain Ω and let $\{p_i\}$ be a collection of polynomials without constant term. Then there is a function $f \in \mathcal{M}(\Omega)$ such that the poles of $f(z)$ are exactly the $\{w_i\}$ and the principle part at w_i is given by $p_i((z-w_i)^{-1})$.*

Proof. Since w_i is a discrete subset of Ω we can choose open disks U_i such that $w_i \in U_i$ and $U_i \cap U_j = \emptyset$ if $i \neq j$. Let U_0 denote an open set so that $w_i \notin U_0, i = 1, \ldots$ and $\{U_i, i = 0, 1, \ldots\}$ is an open cover of Ω. The only nontrivial intersections are of the form $U_i \cap U_0$. We define $f_{i0} = p_i((z-w_i)^{-1}), f_{0i} = -p_i((z-w_i)^{-1})$ and all other $f_{ij} = 0$. These clearly satisfy (1.6.6) and (1.6.7) so the previous theorem implies that we can find $f_i \in H(U_i), i = 0, 1, \ldots,$ such that

$$p_i((z-w_i)^{-1}) - f_i = -f_0 \text{ in } U_i \cap U_0. \tag{1.6.10}$$

If we set

$$f \restriction_{U_0} = -f_0,\ f \restriction_{U_i} = p_i((z-w_i)^{-1}) - f_i,$$

then it follows from (1.6.10) that f is well defined and satisfies the hypotheses of (1.6.9).

Now we turn to the proof of (1.6.5).

Proof. The proof of this theorem follows a general pattern: first we solve the problem in $\mathcal{C}^\infty$ then we use the solvability of the $\overline{\partial}$–equation to correct the $\mathcal{C}^\infty$ solution and obtain a holomorphic solution. To solve the problem in $\mathcal{C}^\infty$ we choose a partition of unity subordinate to the cover U_i. This is a family of non–negative smooth functions $\{\psi_i\}$ with $\operatorname{supp} \psi_i \subset U_i$ and

$$\sum \psi_i \equiv 1. \tag{1.6.11}$$

Clearly ψ_j vanishes to infinite order along $\partial U_j \cap U_i, \forall\, i \neq j$. Thus the extension of $\psi_j f_{ij}$ to all of U_i by zero is in $\mathcal{C}^\infty(U_i)$. Condition (1.6.6) implies that $f_{ii} = 0$, $\forall i$. Then we define

$$h_i = \sum_j f_{ij} \psi_j.$$

Since the cover is locally finite the sum defines a function in $\mathcal{C}^\infty(U_i)$. It follows from (1.6.7) that these functions satisfy (1.6.8):

$$\begin{aligned} h_i - h_j \restriction_{U_i \cap U_j} =& \sum_k (f_{ik} - f_{jk})\psi_k \\ =& f_{ij} \sum_k \psi_k \\ =& f_{ij}. \end{aligned}$$

The last equality follows from (1.6.11).

To use the solution of $\overline{\partial}$ we need to consider the error introduced in the previous step. Let $\alpha_i = \overline{\partial} h_i$ on U_i. Since the difference $h_i - h_j$ is holomorphic on $U_i \cap U_j$ it follows that

(1.6.12) $$\alpha_i \restriction_{U_i \cap U_j} = \alpha_j \restriction_{U_i \cap U_j} .$$

Thus we obtain a globally defined $(0,1)$–form by setting

$$\alpha \restriction_{U_i} = \alpha_i.$$

We can now apply (1.5.2) to obtain a function $u \in \mathcal{C}^\infty(\Omega)$ such that

(1.6.13) $$\overline{\partial} u = \alpha.$$

If we set $g_i = h_i - u$ then (1.6.12) and (1.6.13) imply that

$$g_i \in H(U_i)$$

and of course

$$g_i - g_j \restriction_{U_i \cap U_j} = h_i - h_j \restriction_{U_i \cap U_j} = f_{ij}.$$

This completes the proof of the theorem.

The techniques introduced in the proof of (1.6.5) form a part of the study of *Čech cohomology.* We will discuss this latter in the course.

Using the same general result we can prove the Weierstraß theorem. This theorem shows that we can entirely specify the poles and zeros of a meromorphic function. Let us suppose that $\{U_i\}$ is a locally finite open cover of Ω such that the intersections $U_i \cap U_j$ are simply connected. Suppose that $f_i \in \mathcal{M}(U_i)$ and that

$$f_{ij} = \frac{f_i}{f_j} \in H(U_i \cap U_j)$$

is non–vanishing. Then we seek a meromorphic function $f \in \mathcal{M}(\Omega)$ whose poles and zeros on U_i agree with those of f_i. More precisely

$$\frac{f}{f_i} \in H(U_i)$$

and is non–vanishing. The existence of such a function is easily reduced to the result given (1.6.5). Since $U_i \cap U_j$ is simply connected and f_{ij} is non–vanishing on $U_i \cap U_j$ we can define

$$g_{ij} = \log f_{ij} \in H(U_i \cap U_j).$$

Since $f_{ij} f_{jk} = f_{ik}$ it is clear that g_{ij} also satisfy a cocycle condition:

$$n_{ijk} = g_{ij} + g_{jk} - g_{ik} \in 2\pi i \mathbb{Z}.$$

To apply the previous result it is necessary that $n_{ijk} = 0$.

At this point we need to use a little algebraic topology to obtain new $\{g'_{ij}\}$ which satisfy (1.6.6) and (1.6.7). Using Čech cohomology one can show that $n_{ijk}/(2\pi i)$ defines a class in $H^2(\Omega;\mathbb{Z})$. It is a standard fact that this group is zero for Ω an open subset of $\mathbb{C}$. For the case of a domain with a smooth boundary this is easily seen: This group is homotopy invariant and such a domain can be retracted onto a one dimensional simplicial complex. Since we are using Čech cohomology the triviality of $H^2(\Omega;\mathbb{Z})$ means that we can find integers n_{ij} such that

$$n_{ijk} = 2\pi i(n_{ij} + n_{jk} - n_{ik}).$$

If we set $g'_{ij} = g_{ij} - 2\pi i n_{ij}$ then

$$e^{g'_{ij}} = f_{ij} \text{ in } U_i \cap U_j \text{ and } g'_{ij} + g'_{jk} - g'_{ik} = 0.$$

Therefore we can apply (1.6.5) to obtain $g_i \in H(U_i)$ so that $g'_{ij} = g_i - g_j$. If we set

$$h_i = \exp(-g_i)f_i,$$

then

$$h_i \restriction_{U_i \cap U_j} = h_j \restriction_{U_i \cap U_j}.$$

This completes the proof of

The Weierstraß Theorem 1.6.14. *Let $\{w_i\}$ be a discrete subset of Ω and $\{n_i\}$ a set of integers then there exists a function $f \in \mathcal{M}(\Omega)$ such that the poles and zeros of f are contained in $\{w_i\}$ and $f(z)(z-w_i)^{-n_i}$ is holomorphic and non–vanishing in a neighborhood of w_i.*

Proof. We choose a locally finite cover of Ω by disks, $\{U_i\}$ each of which contains at most one w_i in its interior. We then set $f_i = (z-w_j)^{n_j}$ if $w_j \in U_i$. Otherwise we set $f_i = 1$. The argument given above produces the desired function.

Remarks. If you are uncomfortable with the use of Čech cohomology then you should consult an alternate proof given in Hörmander. I included it to show, in a simple context, how topological obstructions might arise in the solution of analytic problems. Indeed if Ω is a compact Riemann surface then $H^2(\Omega;\mathbb{Z}) \simeq \mathbb{Z}$ and the element of this group defined by $\mathfrak{n} = \{n_{ijk}/2\pi\imath\}$ is not always 0. The functions $\{f_{ij}\}$ define the transitions funtions for a complex line bundle and $\mathfrak{n}$ is precisely the Euler class of the line bundle. This is, of course a topological invariant of the line bundle. A meromorphic function satisfying the conclusions of the theorem defines a nonvanishing section of the associated line bundle. Such a section can exist only if the line bundle is topologically trivial and thus the Euler class is 0.

Using a bit more cohomology theory one can show that the class defined by $\mathfrak{n}$ is equal to $N-P$ where N is the sum of the orders of the zeros of the $\{f_i\}$ and P the sum of the orders of the poles. It is a relatively simple matter to prove that for a meromorphic function on a compact Riemann surface $N-P=0$. Thus we have an analytic interpretation for this topological obstruction to finding a mermorphic function with prescribed zeros and poles. For more on these topics see either [Fo] or [Ch].

This result has many important corollaries

Corollary 1.6.15. *Suppose that $f \in \mathcal{M}(\Omega)$ then there exists a pair of functions $p, q \in H(\Omega)$ such that*

$$f = \frac{p}{q}.$$

Thus the local definition of meromorphic functions given above, (1.6.1) actually leads to the same class of functions as produced by the naive notion of a meromorphic functions.

As another application we construct a function $f \in H(\Omega)$ which cannot be extended, even as a meromorphic function, across any boundary point of Ω. By this we mean that if U is any open subset such that $U \not\subset \Omega$ and $U \cap \Omega$ is non–empty then there does not exist $h \in \mathcal{M}(U)$ such that

$$h \restriction_{\Omega \cap U} = f \restriction_{\Omega \cap U}.$$

Corollary 1.6.16. *For any open set $\Omega \subset \mathbb{C}$ there exists a function $f \in H(\Omega)$ which cannot be extended, even as a meromorphic function across any boundary point of Ω.*

Proof. Choose a discrete subset $D = \{w_i\}$ of Ω such that the closure of D contains $\partial\Omega$. It follows from (1.6.14) that there exists an $f \in H(\Omega)$ such that $f(w_i) = 0$ but $f \not\equiv 0$. Suppose that U is an open set with $U \cap \Omega \neq \emptyset$ then U must contain an accumulation point of D in its interior. Thus any function in $\mathcal{M}(U)$ which agrees with f on $U \cap \Omega$ must necessarily vanish identically.

The property described in (1.6.16) is very important in higher dimensions though the generalization of Corollary is very complicated.

Definition 1.6.17. A domain in $\mathbb{C}$ on which there is a holomorphic function defined which cannot be continued across any boundary point is called a domain of holomorphy.

We can rephrase the previous corollary as

Corollary 1.6.16'. *Any open subset of $\mathbb{C}$ is a domain of holomorphy.*

The interest in this property stems in large part from the fact that a Riemann surface can be constructed by analytically continuing the germ of a single meromorphic function. A maximal surface is then the largest domain of definition of the meromorphic function defined by this germ. The simplest case is then to understand which planar domain are maximal surfaces for some meromorphic function germ. The corollary says that any such domain is. In several variables we will quickly learn that this is not the case but that there are domains $D \subset \mathbb{C}^n$ with the property that any function $f \in H(D)$ actually extends to be holomorphic on a larger open set D'. A considerable part of the theory of several complex is devoted to characterizing domains of holomorphy.

Large parts of §1.2–§1.6 are taken from [Hö] where addtional results can be found.

1.A. Review of Functional Analysis

In much of complex variables one deals with spaces of functions that are defined on open sets, as a consequence they do not have the structure of a Banach space but only that of a Frechet space. As these are a little less familiar we begin by reviewing the basic facts about such spaces.

Let X be a vector space, we want to define a topology on X such that

(1.A.1) every point of X is a closed set
(1.A.2) the vector space operations are continuous.

A vector space with such a topology is called a topological vector space. Such spaces have some properties but not many. Our topologies will usually be defined by seminorms. A function $p : X \longrightarrow \mathbb{R}$ is a seminorm provided

(1.A.3) $p(x + y) \leq p(x) + p(y),\ x, y \in X$
(1.A.4) $p(\alpha x) = |\alpha| p(x),\ x \in X, \alpha \in K$

Here K is the scalar field over which X is a vector space. If, in addition to these p also satisfies

$$p(x) = 0 \Longleftrightarrow x = 0 \tag{1.A.5}$$

then p is a norm. A topological vector space with the topology defined by a norm is called a normed linear space, if it is complete, then it is called a Banach Space. A family of seminorms $\mathcal{P}$ is called separating if for every $x \in X$ there is at least one $p \in \mathcal{P}$ such that $p(x) \neq 0$. We have the following elementary properties for seminorms

Proposition 1.A.5. *Suppose that p is a seminorm on a vector space X. Then*

(1.A.6)
$$p(0) = 0,$$
$$|p(x) - p(y)| \leq p(x - y),$$
$$p(x) \geq 0,$$
The set $\{x : p(x) = 0\}$ is a subspace of X,
The set $B_p = \{x : p(x) < 1\}$ is a convex set.

We leave the proofs of these facts as exercises. We can use a separating family of seminorms to define a topology on X as follows:

$$\text{For each } n \in \mathbb{N}, p \in \mathcal{P} \text{ define } V(p,n) = \{x \in X : \, p(x) < \frac{1}{n}\}. \tag{1.A.7}$$

If we use finite intersections of the collection of sets defined in (1.A.7) to define a local base for a topology on X then, with this topology, X is a locally convex topological vector space. This means that the open sets are simply unions of translates of the sets defined in (1.A.7). The translate of a set S by an element $a \in X$ is simply the set

$$S + a = \{x + a : \, x \in S\}.$$

X is locally convex because (1.A.3) and (1.A.4) imply that each of the sets $V(p,n)$ is a convex set. One should also observe that each seminorm in $\mathcal{P}$ defines a continuous function on X with respect to this topology.

There is only one further property which we require, which is that the separating collection of seminorms be countable, i.e.

$$\mathcal{P} = \{p_i : \, i \in \mathbb{N}\}.$$

In this situation we can actually define a metric on X which induces the same topology as the local base defined in (1.A.7):

$$d(x,y) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(x-y)}{1 + p_i(x-y)}. \tag{1.A.8}$$

This metric clearly satisfies

$$d(x+z, y+z) = d(x,y), \forall\, x, y, z \in X.$$

In general a metric which satisfies this condition is called an invariant metric.

Definition 1.A.9**.** A locally convex, complete topological vector space with the topology defined by an invariant metric is called a *Frechet space.*

At this point we should consider some simple examples:

Example 1.A.10**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set. By considering the distance to Ω^c we can construct a continuous function ψ defined on $\mathbb{R}^n$ such that $\psi(x) \geq 0$ and

$$\Omega^c = \{x : \, \psi(x) = 0\}. \tag{1.A.11}$$

Since $\psi(x)$ is continuous the sets

$$K_a = \{x : \, \psi \geq a^{-1}\}, \, a > 0$$

are closed and bounded and therefore relatively compact subsets of Ω and

$$\Omega = \cup_{a>0} K_a.$$

Let $C(\Omega)$ denote the continuous functions on Ω, for each $n \in \mathbb{N}$ define the function $p_n : C(\Omega) \longrightarrow \mathbb{R}^+$ by

$$p_n(f) = \sup_{x \in K_n} |f(x)|.$$

It is elementary to verify that each p_n defines a seminorm. Let $d_\Omega(f,g)$ denote the metric defined by these seminorms as in (1.A.8). The family is separating since, if $f \in C(\Omega)$ is not identically zero then, there is an $\epsilon > 0$ and a ball $B(p,r) \subset\subset \Omega$ on which $f \geq \epsilon$. From (1.A.11) we conclude that $B(p,r) \subset\subset K_n$, for some n. Thus this family of seminorms makes $C(\Omega)$ into a locally convex metric space. All we need to check is that it is complete.

Let $\{f_m\}$ be a Cauchy sequence. From the definition of the metric it is clear that for each i

$$\lim_{m,n \to \infty} p_i(f_m - f_n) = 0. \tag{1.A.12}$$

From the elementary properties of continuous functions on compact sets it is clear that $\{f_m \restriction_{K_j}\}$ converges uniformly for each j. Let f denote the common limit function. It is clear that f is continuous and

$$\lim_{n\to\infty} d_\Omega(f, f_n) = 0.$$

Thus $C(\Omega)$ with the topology defined by the seminorms is a Frechet space. It is also easy to show that if L_i is another nested family of compact subsets of Ω such that

$$\Omega = \cup_{i=1}^{\infty} L_i,$$

then the topology defined by uniform convergence on the L_i agrees with that defined above.

For a Frechet space, X with topology defined by a family of seminorms $\{p_i\}$ there is a very simple criterion for a linear functional

$$l : X \longrightarrow K$$

to be continuous:

Proposition 1.A.12. *A linear functional l is continuous if and only if there exist a set if indices $i_1, \dots, i_k$ and a constant M such that*

$$l(x) \le M \max_{j\in 1,\dots,k} p_{i_j}(x). \tag{1.A.13}$$

As a simple consequence we see that every continuous linear functional on $C(\Omega)$ is compactly supported. This means that we can find a j such that if $f \restriction_{K_j} = g \restriction_{K_j}$ then $l(f) = l(g)$. From this we conclude that l defines a linear functional on $C(K_j)$. The Riesz–Fischer theorem implies that there is a signed measure $d\mu$ with support on K_j such that for $f \in C(K_j)$

$$l(f) = \int f d\mu. \tag{1.A.14}$$

Clearly (1.A.14) holds for all $f \in C(\Omega)$.

The fundamental properties of linear transformations on Frechet spaces follow from a general result on complete metric spaces. Since we will have several occasions to use this result directly I include it here, along with a proof.

Definition 1.A.15. Let S be a topological space, a set $E \subset S$ is said to be nowhere dense if $\overline{E}$ has empty interior. A subset of S is of *the first category* if it is a countable union of nowhere dense sets. A subset of S not of the first category is of *the second category.*

This terminology is not very descriptive but it is ubiquitous. The following is usually referred to as the Baire Category Theorem

Theorem 1.A.16. *If S is a complete metric space then the intersection of every countable collection of dense open subsets of S is dense.*

Proof. Let $V_1, V_2, \dots,$ be a collection of dense open subsets. Let $p \in S$ and $r > 0$ be chosen arbitrarily. Then since V_1 is dense and open we can find a $p_1, 0 < r_1 < 1$ such that

$$\overline{B}(p_1, r_1) \subset V_1 \cap B(p, r).$$

Suppose that a $p_i, r_i, i = 1, \dots, n-1$ have been chosen so that

$$\overline{B}(p_i, r_i) \subset V_i \cap B(p_{i-1}, r_{i-1}) \text{ and } 0 < r_i < \frac{1}{i}. \tag{1.A.17}$$

Since V_n is a dense open set we can choose an p_n, r_n so that (1.A.17) holds for $i = n$ as well. Set

$$K = \cap \overline{B}(p_i, r_i).$$

The centers $\{p_i\}$ form a Cauchy sequence, since S is assumed to be complete it must converge to a limit. Clearly this limit belongs to K. Thus K is non–empty, moreover $K \subset V_n$ for every n. Since $K \subset B(p, r)$ it follows that $B(p, r)$ intersects $\cap V_n$. Since p, r are arbitrary this proves the theorem.

As an application of this theorem consider the following:

Example 1.A.18. Let $A(x)$ be a map from a closed interval I into finite rank operators on a Hilbert space. We suppose that $A(x)$ is continuous from the usual topology on I into the strong operator topology. As an exercise prove that the sets

$$I_n = \{x : \operatorname{rk} A(x) \leq n\}$$

are closed. Since $A(x)$ is finite rank operator for every x it follows that

$$I = \cup_{n=0}^{\infty} I_n.$$

From the Baire category theorem it follows that at least one of the sets has non–empty interior for otherwise the whole interval, would be of the first category. Thus we see that there exists a nonempty subinterval $J \subset I$ and an integer N such that

$$\operatorname{rk} A(x) \leq N, x \in J.$$

From the Baire category one deduces the three basic tools for the analysis of operators on Frechet space. In fact some of these results are true in slightly more general settings. Let Λ denote a linear mappings

$$\Lambda : X \longrightarrow Y,$$

where X, Y are topological vector spaces. A linear mapping, Λ, is continuous if for every neighborhood W of $0 \in Y$ there is a neighborhood V of $0 \in X$ such that

$$\Lambda(V) \subset W. \tag{1.A.20}$$

A family of linear mapping, Γ is called equicontinuous if for each W a fixed V works in (1.A.20) for all $\Lambda \in \Gamma$.

Uniform Boundedness Theorem 1.A.21. *If Γ is a collection of continuous linear mappings from a Frechet space X to a topological vector space Y such that for each $x \in X$.the set*

$$\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$$

is bounded in Y, then Γ is an equicontinuous family.

As an application of this theorem we have:

Example 1.A.22. Let $\mathcal{C}_K^\infty$ be $\mathcal{C}^\infty$ functions with support in the compact set K. Set

$$p_N(\phi) = \sup_K \max_{|\alpha| \leq N} |D^\alpha \phi|.$$

These seminorms define a complete Frechet structure on $\mathcal{C}_K^\infty$. Suppose that Λ_i is a sequence of continuous functionals such that $\Lambda_i(\phi)$ converges for every $\phi \in \mathcal{C}_K^\infty$. This implies that the sequence is bounded for each ϕ. Therefore (1.A.21) implies that there exists some N and a constant, M, so that $\Lambda_i(\phi) \leq M p_N(\phi) \quad \forall\, i.$

A mapping $f : X \longrightarrow Y$ is open if for every open set $U \subset X$, $f(U)$ is an open subset of Y. Our second fundamental theorem is

Open Mapping Theorem 1.A.23. *If Λ is a continuous linear mapping of a Frechet space onto a Frechet space then it is an open mapping.*

As a simple application of this theorem we have.

Corollary 1.A.24. *If Λ is a one to one linear mapping from a Frechet space X onto a Frechet space Y then Λ^{-1} is continuous.*

Proof. If $U \subset X$ is an open set then $V = \Lambda(U)$ is an open subset of Y by the open mapping theorem. Since

$$\Lambda(U) = (\Lambda^{-1})^{-1}(U),$$

this implies that Λ is continuous.

Note we could also have deduced this from the uniform boundedness theorem, as Λ^{-1} is a globally defined linear transformation from a Frechet space to a Frechet space.

The last main consequence is a very useful criterion for establishing the continuity of a linear transformation. If $f : X \longrightarrow Y$ is a map between topological spaces then the graph of f is the subset of $X \times Y$ given by

$$G_f = \{(x, y) \in X \times Y : y = f(x)\}.$$

If X is a topological space and Y satisfies the Hausdorff separation axiom then G_f is closed in the product topology on $X \times Y$ if f is continuous. The converse is true for linear maps between Frechet spaces.

Closed Graph Theorem 1.A.25. *If Λ is a linear map between Frechet spaces X and Y such that G_Λ is a closed subset of $X \times Y$ then Λ is continuous.*

As an application we have

Example 1.A.26. This result has many applications in the study of unbounded operators acting between Banach spaces. We give one such application. An operator T defined on a dense subspace S of a Banach space X with range in a Banach space Y is called closeable if for every sequence $\{x_n\} \subset S$ converging to zero for which $\{Tx_n\}$ is convergent

$$\lim_{n\to\infty} Tx_n = 0.$$

We can then define the closure of T as T extended to the subspace, $\widetilde{S}$ of X defined by the conditions

$$\{x_n\} \subset S, \lim_{n\to\infty} x_n = x^*, \lim Tx_n = y^* \text{ exists.}$$

The extended operator is then defined by $\widetilde{T}x^* = y^*$.

Suppose that for some $M > 0$, $T : S \longrightarrow Y$ satisfies

$$M\|x\|_X < \|Tx\|_Y. \tag{1.A.27}$$

From (1.A.27) it follows easily that T is closeable and furthermore range$(\widetilde{T})$ is a closed subspace of Y. It is also immediate from (1.A.27) that $\widetilde{T}$ is one to one. Thus we can define $\widetilde{T}^{-1} : \text{range}(\widetilde{T}) \longrightarrow X$. It follows from the closed graph theorem that this operator is continuous. We need to verify that if $y_n \in \text{range}(\widetilde{T})$ is convergent and so is $\widetilde{T}^{-1}y_n$ then

$$\lim_{n\to\infty} \widetilde{T}^{-1}y_n = \widetilde{T}^{-1} \lim_{n\to\infty} y_n. \tag{1.A.28}$$

This is an easy deduction from (1.A.27). Thus we've show that a densely defined operator satisfying (1.A.27) with a dense range has a continuous inverse. For example, if we consider the operator

$$Tf(x) = -f''(x), \text{ for } f \in \mathcal{C}_c^\infty([0,1]),$$

then applying the Cauchy-Schwarz inequality and integrating by parts we obtain

$$\|Tf\|_{L^2}\|f\|_{L^2} \geq |< Tf, f >| = \int_0^1 |f'(x)|^2 dx.$$

Since f is compactly supported and smooth it follows from the fundamental theorem of calculus and the Cauchy–Schwarz inequality that

$$\|f\|_{L^2}^2 \leq \frac{1}{2}\|f'\|_{L^2}^2.$$

Combining these two inequalities we obtain

$$\|Tf\|_{L^2} \geq 2\|f\|_{L^2}.$$

Thus the closure of T is boundedly invertible on its range. I leave it as an exercise to show that range T is dense in $L^2([0,1])$.

The Baire category theorem and its consequences rely crucially on the completeness of the spaces involved; the other fundamental property of Frechet spaces is local convexity. This means that there is a local base for the topology consisting of convex sets. This does not imply that the metric balls are convex. If the topology is defined by a collection of seminorms then this property follows from (1.A.3) and (1.A.4). The consequences of convexity are usually called the Hahn–Banach theorems. These fall into two groups: the first says that linear functionals defined on subspaces that are bounded by a seminorm can be extended to the whole space satisfying the same bound. The other group of theorems say, in essence, that non–intersecting closed convex subsets can be separated by a linear functional.

Extension Hahn–Banach Theorem 1.A.29**.** *Suppose that M is a subspace of a vector space X, p is a seminorm on X, and f is a linear functional defined on M such that*

$$|f(x)| \leq p(x) \quad \forall x \in M.$$

Then f extends to a linear functional F on X which satisfies

$$|F(x)| \leq p(x) \quad \forall x \in X.$$

Notice that no hypothesis is made concerning the topology of X. If follows form the uniform boundedness principle that F is automatically continuous if X is a Frechet space, whether or not $p(x)$ is a continuous seminorm. A useful special case of the separation theorem is

Separation Hahn–Banach Theorem 1.A.30**.** *If M is a subspace of a locally convex space X and $x_0 \in X$ is not in the closure of M then there is a continuous linear functional Λ such that*

$$\Lambda(x_0) = 1 \text{ and } \Lambda\restriction_M = 0.$$

We complete this section with an application of (1.A.29).

Example 1.A.31**.** Let Ω be an open connected subset of $\mathbb{C}$ and let $C(\Omega)$ be as defined in (1.A.10). If we let $H(\Omega)$ denote functions holomorphic on Ω, with a topology induced by the same seminorms as used to define the topology on $C(\Omega)$, then $H(\Omega)$ is easily seen to be a closed subspace of $C(\Omega)$. This is because a locally uniformly convergent sequence of holomorphic functions has a holomorphic limit. Suppose that l is a continuous linear functional defined on $H(\Omega)$. From (1.A.12) it follows that there exists a continuous seminorm $p(\psi)$ as in (1.A.13) such that

$$|l(\psi)| \leq Mp(\psi), \forall \psi \in H(\Omega). \tag{1.A.32}$$

Since we've used the same seminorms to define the topology on $H(\Omega)$, p is also a continuous seminorm on $C(\Omega)$. Thus we can apply (1.A.29) to find an extension of l to a continuous functional Λ defined on all of $C(\Omega)$. From (1.A.14) it therefore follows that there is a finite, complex measure $d\mu$ supported on a compact subset $K \subset\subset \Omega$ such that

$$\Lambda(f) = \int_K f d\mu. \tag{1.A.33}$$

Evidently l on $H(\Omega)$ must also be given by such a formula.

Most of the material in this appendix was taken from [Ru] by Walter Rudin, 1973. Proofs and more general results can be found there.

References

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