Complex numbers. Motivating problem: you can write down equations which don't have solutions, like $x^2+1=0$. Introduce a (formal) solution i, where $i^2=-1$. Define the set $\mathbb{C}=\{\alpha+i\beta : \alpha,\beta\in\mathbb{R}\}$. Can put a ring structure on this.

$$\begin{aligned}(\alpha_1+i\beta_1)+(\alpha_2+i\beta_2) &\stackrel{\text{def}}{=} (\alpha_1+\alpha_2)+i(\beta_1+\beta_2)\\ (\alpha_1+i\beta_1)\cdot(\alpha_2+i\beta_2) &\stackrel{\text{def}}{=} (\alpha_1\alpha_2-\beta_1\beta_)+i(\alpha_1\beta_2+\alpha_2\beta_1)\end{aligned}$$

One verifies that it's associative, distributive, and has the expected units. This defines an integral domain.

[Geometric proof:

$$\begin{pmatrix}\alpha_1 & -\beta_1\\ \beta_1 & \alpha_1\end{pmatrix}\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}=0.$$

Check out the kernel.]

We'd like to show that $\mathbb{C}$ is actually a field. To show this, we need to show that every non-zero element has a multiplicative inverse.

$$(\alpha+i\beta)^{-1}=\frac{\alpha}{\alpha^2+\beta^2}-i\frac{\beta}{\alpha^2+\beta^2}.$$

Moreover, we can identify $\mathbb{R}$ as a subfield of $\mathbb{C}$; look at $\{\alpha+i0\}\cong\mathbb{R}$ as a field. $\mathbb{C}=\mathbb{R}[x]/x^2+1$.

In general, write $z=(x+iy)$. Can we, in general, solve $z^2-(\alpha+i\beta)=0$? Well, $^2=(x+iy)^2=x^2-y^2+2ixy=\alpha+i\beta$. So we have the two equations

$$\begin{aligned}x^2=y^2 &= \alpha\\ 2xy &= \beta\end{aligned}$$

We know $2x^2=\alpha+\sqrt{\alpha^+\beta^2}$, and $2y^2=\sqrt{\alpha^2+\beta^2}-\alpha$. So

$$\begin{aligned}x &= \pm\sqrt{(\alpha+\sqrt{\alpha^2+\beta^2})/2}\\ y &= \pm\sqrt{(\sqrt{\alpha^2+\beta^2}-\alpha)/2}\end{aligned}$$

How do we figure out which signs will work? The problem is, we lost information when we squared. *Blah*

We've shown that every number $z \in \mathbb{C}$ has a square root.[1]

Indeed, the fundamental theorem of algebra says that every polynomial $p(z)$ of degree n has a solution $p(z_0) = 0$, where the coefficients and z_0 are in $\mathbb{C}$.

There's a sense in which $\mathbb{C} \cong \mathbb{R}^2$. This is since $\mathbb{C}$ has an underlying vector space structure, $(x + iy) \mapsto (x, y)$. Gotta show that $a(x + iy) \leftrightarrow a(x, y)$, and addition is preserved as well. So we identify a complex number with a vector, and complex addition is vector addition in $\mathbb{R}^2$. Can also use polar coordinates; $\rho = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(y/x)$. Then $(x, y) \leftrightarrow (\rho \cos\theta, \rho \sin\theta)$. Then

$$(\rho \cos\theta + i\rho \sin\theta) \cdot (r \cos\phi + ir \sin\phi) = (\rho r \cos(\theta + \phi), \rho r \sin(\theta + \phi)).$$

So you multiply the lengths and add the angles. Define $\arg z = \tan^{-1}(y/x)$. We see that $\arg z_1 z_2 = \arg z_1 + \arg_2$. Define $|z| = \sqrt{x^2 + y^2}$.

Complex conjugation is a map $\mathbb{C} \to \mathbb{C}$ $(x + iy) \mapsto (x - iy)$ $z \mapsto \overline{z}$. We then have

$$x = \frac{z + \overline{z}}{2} = \Re z$$

and

$$y = \frac{z - \overline{z}}{2i} = \Im z.$$

Then we think of $f(x, y)$ as $g(z, \overline{z}) = f((z + \overline{z})/2, (z - \overline{z})/2i)$. Note that we have $|z|^2 = z\overline{z}$, and $\overline{zw} = \overline{z} \cdot \overline{w}$; extend by induction to finite products. $\overline{z} = z \iff z \in \mathbb{R} \hookrightarrow \mathbb{C}$.

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z} + \overline{w}) \\ &= |z|^2 + w\overline{z} + \overline{z}w + |w|^2 \\ &= |z|^2 + 2\Re w\overline{z} + |w|^2 \end{aligned}$$

With suitable persuasion, this gives us

[1] Actually, it has two of them, since $w^2 = z \Rightarrow (-w)^2 = z$.

$$|\Re w\overline{z}| \leq \frac{|z|^2 + |w|^2}{2}$$

or

$$\Re wz \leq \frac{|z|^2 + |w|^2}{2}.$$

More useful stuff: $-|w| \leq \Re w, \Im w \leq |w|$. Also, $|z+w|^2 \leq (|z| + |w|)^2$. Triangle inequality. Equality holds when $\arg z = \arg w$. *Blah.* $\Re\overline{z}w = |zw|$.

So anyways, we've got the Cauchy-Schwarz inequality:

$$\Re\overline{z}w \leq |zw|.$$

There's a generalization:

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2.$$

You prove it by looking at $0 \leq \sum \left|a_i - \lambda\overline{b_i}\right|^2 = \sum |a_i|^2 - 2\Re\overline{\lambda}a_i b_i + |\lambda|^2 |b_i|^2$. Set $\lambda = \frac{\sum a_i b_i}{\sum |b_i|^2}$.

$$\sum |a_i|^2 - 2\sum \Re\frac{\sum \overline{a_i b_j}}{\sum |b_i|^2} a_i b_i + \frac{|\sum a_i b_i|^2}{(\sum |b_i|^2)} \sum |b_i|^2.$$

Whatever. And $0 \leq \sum |a_i|^2 - \frac{|\sum a_i b_i|^2}{\sum |b_i|^2}$.

If we write $z = \rho(\cos\theta + i\sin\theta)$, then $z^2 = \rho^2(\cos 2\theta + i\sin 2\theta)$, and $z^n = \rho^n(\cos n\theta + i\sin n\theta)$. This certainly works for $n > 0$. We know that $z^{-1} = \rho^{-1}(cos\theta - i\sin\theta)$, so the formula works for any $n \in \mathbb{Z}$. Can use this on a root of unity to get

$$\sum (nCj)\cos^2\theta(i\sin\theta)^{n-k} = \cos n\theta + i\sin n\theta.$$

Consider the equation $z^n = w$. Write $z = \rho\cos\theta + i\rho\sin\theta$, and $w = r\cos\phi + ir\sin\phi$. Then we have $\rho^n\cos n\theta + i\rho^n\sin n\theta = r\cos\phi + ir\sin\phi$. So $\rho = \sqrt{r}n$, and $\theta = \phi/n + 2\pi j/n$ for $j =$ *Blah.*

I've lost a lot of stuff due to power failure.

We open with something on stereographic projection. Let N be the north pole, and S the south pole. Then we get maps $\Sigma_N : S^2 - N \to \mathbb{C}$ and $\Sigma_S : S^2 - P \to \mathbb{C}$. It turns out that $\Sigma_N \circ \Sigma_S^{-1}(z)$ is an analytic function.[2]

We defined $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. A complex function f is analytic if $\partial_{\overline{z}} f = 0$. We could say that

$$\partial_{\overline{z}} f = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} + i\frac{f(x,y+h)-f(x,y)}{h}.$$

We say that $f(z)$ is analytic at z_0 if $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists. Here, h is a complex number.

Example. $f(z) = z$. $\lim((z+h)-z)/h = 1$. $f(z) = \overline{z}$. $\lim((\overline{z}+\overline{h})-\overline{z})/h - \lim_{h\to 0} \overline{h}/h$.

Does $\lim \frac{h}{\overline{h}}$ exist?

$$\begin{aligned} \lim_{h\to 0} \frac{h}{\overline{h}} &= \lim_{h\to 0} \frac{h2}{|h|^2} \\ &= \lim_{h\to 0} \cos 2\theta + i \sin 2\theta \end{aligned}$$

[3]

which doesn't exist.

But $\partial_{\overline{z}} \overline{z} = 1 \neq 0$.

Look at $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$, which is

$$\lim \frac{u(x+h,y)+iv(x+h,y)-(u(x,y)+iv(x,y))}{h} = \partial_x u + i\partial_x v.$$

But the right-hand side is also $\lim_{h\to 0} \frac{f(z+ih)-f(z)}{ih} = \lim_{h\to 0} \frac{u(x,y+h)+iv(x,y+h)-(u(x,y)+iv(x,y))}{ih} = \frac{1}{i}(u_y + iv_y)$.

This tells us that $\partial_x u + i\partial_x v = -i(\partial_y u + i\partial_y v)$, so $\partial_x(u+iv) = -i\partial_y(u+iv)$, or $(\partial_x + i\partial_y)(u+iv) = 0$. This equation is thus equivalent to $\partial_{\overline{z}}(u+iv) = 0$.

$\Leftarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists. If you assume the partials are continuous, you get the implication both ways.

If $\partial_{\overline{z}} f = 0$, then $\partial_z f = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$.

[2] What function is it? Should be able to do it without computation.

[3] $h^2 = \rho^2(\cos 2\theta + i \sin 2\theta)$

This gives us a product rule, quotient rule, and sum rule. If f and g are analytic, then so are fg, f/g (where $g \neq 0$), and $f \pm g$. As before, we have, e.g., $\partial_z(fg) = (\partial_z f)g + (\partial_z g)f$. This shows that the set of holomorphic functions is closed under certain algebraic operations.

Can think of this algebraically; $\partial_{\bar{z}} fg = g\partial_{\bar{z}} f + f\partial_{\bar{z}} g$; if something is in the kernel, then so is its product. $0 = \partial_{\bar{z}} g\frac{1}{g} = g\partial_{\bar{z}}\frac{1}{g} + \frac{1}{g}\partial_{\bar{z}} g$. Thus, $g\partial_{\bar{z}}\frac{1}{g} = 0$, and $\partial_{\bar{z}}\frac{1}{g} = 0$ if $g(z) \neq 0$.

So $f(z) = z$ is analytic; we proved this. $\Rightarrow$ every polynomial in z is also analytic. So is $1/p(z)$ for any polynomial $p(z)$, so long as we avoid the zeros of p. We have a nice class of analytic functions, namely, rational functions $p(z)/q(z)$.

The Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ have some interesting consequences. F'rinstance, $u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$. Similarly, $v_{xx} + v_{yy} = 0$. This gives us a second-order differential operator, the Laplace operator $\Delta = (\partial_x^2 + \partial_y^2)$. So if $f(x, y) = u + iv$ is analytic, then $\Delta u = \Delta v = 0$. If $\Delta u = 0$, we say that u is harmonic. The converse is false.

Suppose that u is harmonic in an open subset $\Omega \subset \mathbb{C}$. Green's theorem is useful here. If $v(x, y)$ is C^1, then we can reconstruct $v(x, y)$ from $v_x, v_y)$ by integration; $v(x, y) = v(x_0, y_0) + \int_{x_0,y_0}^{x,y} v_x dx + v_y dy$. Some stuff about what this means; pick a path $(x(t), y(t))$. Then $dx = \frac{dx}{dt}dt$, and similarly for y. Then the integral is computed as $\int_0^1 v_x(x(t), y(t))\frac{dx}{dt}dt + v_y(x(t), y(t))\frac{dy}{dt}dt$.

Suppose u is given, $\Delta u = 0$. We'll try to define a function v with $\nabla v = (-u_y, -u_x)$. So use

$$v(x, y) = v(x_0, y_0) + \int_{x_0,y_0}^{x,y} (-u_y dx + u_x dy).$$

Suppose there are two paths from (x_0, y_0) to (x, y). If we look for the difference between integrating these two paths, we get a path integral γ around a domain D. Green's theorem says

$$\int_\gamma (-u_y dx + u_x dy) = \int_D (\partial_y(u_y) + \partial_x(u_x))dxdy = \int_D \Delta u dxdy = 0.$$

This all assumes that u is twice diferentiable on all of D. We've shown that if u is defined in a region D without any holes, and $\Delta u = 0$ in D, then there is a differentiable function v defined as above. Thus, $v(x, y) = \int_{x_0,y_0}^{x,y} -u_y dx + u_x dy$. There's considerable freedom in the choice of path; choose one which is horizontal near (x, y). Then

$$v(x + h, y) - v(x, y) = \int_x^{x+h} -u_y(s, y)ds.$$

Divide by h, and get $\partial_x v = -u_y$. Similarly, one computes that $\partial_y v = u_x$. Therefore, $u + iv$ is an analytic function in the domain D. Here v is called the harmonic conjugate of u. It, too, is a harmonic function.

Recall that a half-plane is something given by $\Im \frac{z-a}{b} > 0$. Suppose that $P(z)$ is a polynomial of degree n. For now, assume that $P(z) = A \prod_{i=1}^{n}(z - \alpha_i)$. Lucas' Theorem: where are the roots of $P'(z)$?

Do an example. If we have a function with only real roots, then the zeros of the derivative fall between the real solutions. Lots-o-information there.

Theorem [Lucas] If all the roots of $P(z)$ lie in a half-plane, then so do the roots of $P'(z)$.

Proof Look at $P'(z)/P(z) = \sum_1^n \frac{1}{z-\alpha_i}$. Let's assume that $\Im \frac{\alpha_i - a}{b} > 0$, but $\Im \frac{z-a}{b} < 0$. Consider $\Im \frac{z-\alpha_i}{b} = \Im \frac{z-a}{b} - \Im \frac{\alpha_i - a}{b} < 0$. Now, if $\Im \frac{z-\alpha_i}{b} < 0$, then $\Im \frac{b}{z-\alpha_i} > 0$. We see that $\Im \frac{bP'(z)}{P(z)} = \sum \Im \frac{b}{z-\alpha_i} > 0 \Rightarrow \Im \frac{bP'(z)}{P(z)} \neq 0$, so $P'(z) \neq 0$. $\diamond$

Corollary If C is the convex hull of the roots, then all roots of $P'(z)$ lie in C, as well.

Corollary The roots of $P^{(k)}$ are also contained in C for all $k < n$.

We now move on to rational functions. Let

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0}{b_m z^m + \cdots b_0}.$$

Let's assume that P and Q are relatively prime. The roots of P are the roots of R. The roots of Q are called the poles of R. We can write $P(z) = A \prod^k (z - \alpha_i)^{n_i}$ where the α_i are distinct. We say that α_i is a root of P of order n_i. Similarly, write $Q(z) = \prod^l (z - \beta_i)^{m_i}$. Then $R(z)$ has a pole of order m_i at β_i. We say that the order of β as a pole of $R(z)$ is the least value of k so that $(z - \beta)^k R(z)$ is bounded in a neighborhood of β.

Suppose $m = n$. Then $\lim_{z\to\infty} R(z) = \frac{a_n}{b_n}$. So $R(z)$ may be extended to the point at infinity; $R(\infty) = a_n/b_n$.

Suppose $n < m$. Then

$$\lim_{z\to\infty} R(z) = \lim \frac{1}{z^{m-n}} \frac{a_n + a_{n-1} z^{-1} + \cdots + a_0 z^{-n}}{b_m + b_{m-1} z^{-1} + \cdots b_0 z^{-m}} = 0,$$

and $R(\infty) = 0$; ∞ is a zero of order $m-n$. R has exactly m zeros and m poles in $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$, the extended complex plane.

If $n > m$, then $\lim_{z\to\infty} R(z) = \infty$, and it's a pole of order $n-m$. Once again, n zeros and n poles.

Theorem If $R(z)$ is a rational function, and $d = \max\{\deg P, \deg Q\}$, then for any point $b \in \hat{\mathbb{C}}$ there are d solutions, counted with multiplicity, to $R(z) = b$.

Proof We've proved this for $b = 0, \infty$. $R(z) - b$ has the same poles as $R(z)$; therefore, it has the same number of zeros. $\diamond$

From the geometric point of view, a rational function gives a map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. If you count with multiplicities, then you get a d-to-one map. We call d the degree of R.

For a moment, let's work with $R(z) = P(z)/Q(z)$, $\deg P > \deg Q$. Using a Euclidean kind of argument, can write $R = G_\infty + H$ where $G_\infty(z)$ is a polynomial, and $H(z)$ is a rational function which is not singular at ∞. Suppose that β_i is a finite root of Q. Look at $R(\frac{1}{\zeta} + \beta_i)$. This rational function (of ζ_i) has a pole at ∞. Write $R(\frac{1}{\zeta} + \beta_i) = G_i(\zeta) + H_i(\zeta)$. Let $z = \frac{1}{\zeta} + \beta_i$, so $\frac{1}{z-\beta_i} = \zeta$. Then $R(z) = G_i(\frac{1}{z-\beta_i}) + H_i(\frac{1}{z-\beta_i}$. Subtract off the poles from R; $R(z) - (G_\infty(z) + \sum_{i=1}^{l} G_i(\frac{1}{z-\beta_i}))$. This thing is bounded for all $z \in \hat{\mathbb{C}}$. So it's actually a constant. The conclusion is that

$$R(z) = G_\infty(z) + \sum_{i=1}^{l} G_i(\frac{1}{z-\beta_i}).$$

This is called the partial fractions decomposition. (Note that G_∞ has eaten the constant.) A rational function is determined by its behavior at the singularities.

Hopefully, this will work for other sorts of functions.[4] We'll look at power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Hadamard Criterion: Set $\rho = \limsup |a_n|^{1/n}$. The series converges absolutely and uniformly in any disk $\{z||z| < r\}$ for $r < \frac{1}{\rho}$.

Proof Comparison with the geometric series. Given $\epsilon > 0$, $\exists N$ so that $|a_n|^{1/n} < \rho + \epsilon$. If we choose an $r < 1/\rho$, then there's an ϵ so that $|(\rho+\epsilon)r| < 1$. So $\left|\sum_{n=N+1}^{M} a_n z^n\right| \leq$

[4] For the sequel should know about convergence, absolute convergence, and uniform convergence.

$\sum_{n=N+1}^{M} |a_n z^n| \leq \sum_{N+1}^{M} |(\rho+z)r|^n$. Thus, $\left|\sum_{N+1}^{\infty} a_n z^n\right| \leq \sum_{N+1}^{\infty} |(\rho+z)r|^n$, a geometric series.

In a way this is like a Cauchy test. Given $\eta > 0$, there's an M so that if $n, m > M$, then $|\sum_n^m a_j z^j| < \eta$, provided that $|z| \leq r$.

We're playing with power series $f(z) = \sum_0 a_n z^n$. There's a theorem of Hadamard which says that if $\rho = \overline{\lim}_{n\to\infty} |a_n|^{1/n}$, then the series converges uniformly in $\{|z| < R < \frac{1}{\rho}\}$. This is cooler than the ratio test, in that it is more widely applicable.

Formally, we can differentiate $f(z)$ and take the formal derivative:

$$f_1(z) = \sum_1 n a_n z^n.$$

If $\rho < \infty$, then $\overline{\lim}_{n\to\infty} |na_n|^{1/n} = \rho$, since $|n|^{1/n} = 1 + \delta_n$, where $\delta_n \to 0$ as $n \to \infty$. The only issue is to show that

1. $f(z)$ is differentiable.
2. $f'(z) = f_1(z)$.

Let $S_n(z) = \sum_0^n a_m z^m$, the partial sum. Look at $\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)$. Write this as

$$\frac{\sum_0^\infty a_n z^n - \sum_0^\infty a_n z_0^n}{-_0} - \sum_0^\infty n a_n z^{n-1} = \sum_0^m \frac{a_n z^n - a_n z_0^n}{z - z_0} + \sum_{m+1}^\infty \frac{a_n z^n - a_n z_0^n}{z - z_0} - (\sum_0^m n a_n z_0^{n-1} + \sum_{m+1}^\infty n a_n z_0^n).$$

To estimate the first term, use $z_n - z_0^n = (-_0)(z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1})$. So

$$\left| \frac{z_n - z_0^n}{z - z_0} \right| \leq |z|^{n-1} + |z|^{n-2}|z_0| + \cdots + |z_0|^{n-1} \leq n r^{n-1}$$

where $\frac{1}{\rho} > r > \max\{|z|, |z_0|\}$. So the big thing is

$$= \sum_0^m \frac{a_n z^n - a_n z_0^n}{z - z_0} - \sum_0^m n a_n z_0^{n-1} + O(\sum^n n|a_n| r^{n-1}).$$

Now, estimate

$$\overline{\lim}_{z\to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \right| \leq \overline{\lim}_{z\to z_0} \left| \sum^m \frac{a_n z^n - a_n z_0^n}{z - z_0} - \sum_0^m n a_n z_0^{n-1} \right| + O(\sum_{m+1}^\infty n|a_n| r^{n-1}).$$

Since $r < 1/\rho$, given $\epsilon > 0$ we can choose an M so that O-term is $< \epsilon$. The finite term goes to zero as $z \to z_0$. Thus, $\overline{\lim}_{z\to z_0} \left| \frac{f(z)-f(z_0)}{z-z_0} - f_1(z) \right| \leq \epsilon$ for all ϵ, so f is differentiable and $f'(z_0) = f_1(z_0)$.

By applying this result to f_1, we conclude that $f_1(z)$ is differentiable, and $f''(z) = \sum n(n-1) a_n z^{n-2}$.

Theorem If $f(z) = \sum_0^\infty a_n z^n$ with $\rho < \infty$, then $f(z)$ is analytic and indeed infinitely differentiable, with $f^{(k)}(z) = \sum_0^\infty a_n \frac{d^k}{dz^k} z^n$.

Partial Summations We'll look at $\sum_0^\infty a_n b_n$, and the partial sums $\sum_{m+1}^{m+p} a_n b_n$. Now, in calculus we have $\int_a^b f'(x)g(x)dx = fg|_a^b - \int_a^b fg'dx$. Let $s_m = a_0 + a_1 + \cdots a_m$. Note that $s_m - s_{m-1} = a - m$. Then

$$\sum_{m+1}^{m+p} a_n b_n = \sum_{m+1}^{m+p} s_n(b_n - b_{n+1}) + s_{n+p} b_{m+p+1}.$$

Corollary Suppose that $|s_n| \leq M$ for all n, and $b_n \geq b_{n+1} \geq \cdots$, and $\lim_{n\to\infty} b_n = 0$. Then $\sum_0^\infty a_n b_n$ converges.

Proof

$$\begin{aligned}
\left|\sum_{m+1}^{m+p} a_n b_n\right| &\leq \sum_{m+1}^{m+p} |s_n|(b_n - b_{n+1}) + |s_{m+p}||b_{m+p+1}| + |s_m||b_{m+1}| \\
&le\; M\sum_{m+1}^{m+p}(b_n - b_{n+1}) + M(b_{m+p+1} + b + m + 1) \\
&\leq m(b_{m+1} - b_{m+p+1}) + M(b_{m+p+1} + b_{m+1}) \\
&= 2Mb_{m+1}.
\end{aligned}$$

$\diamond$

What happens at the boundary of the circle of convergence?

- Suppose that $\sum_0^\infty a_n z^n$ converges, where $|z| = 1/\rho$. Do we have $\lim_{z\to z_0} f(z) = \sum_0^\infty a_n z_0^n$?
- Is there necessarily a point on the boundary of the circle of convergence where the series diverges? Example: $\sum_1^\infty \frac{z^n}{n^2}$.

The answer to the first question is affirmative; this is Abel's theorem. We can assume that $\rho = 1$, $z_0 = 1$, $\sum_0^\infty a_n$ exists. Then for every $k > 0$,

$$\lim_{z\to 1 \text{and} |1-z|<k(1-|z|)} = \sum_0^\infty a_n.$$

Now, $|1-z|/1-|z|<k$ is a cone, symmetric around the real axis. So we're only letting z approach 1 in a nontangential way. In other words, the distance from z to 1 is regulated by the distance from z to the circle. The proof works like this.

We can assume that $\sum_0^\infty a_n = 0$. Let $s_n = \sum_0^n a_j$. We'll rewrite

$$\begin{aligned}\sum_0^m a_n z^n &= \sum_0^m s_n(z^n - z^{n-1}) + s_m z^m \\ &= (1-z)\sum_0^m s_n z^n + s_m z^m.\end{aligned}$$

Assume that $|z|<1$. Then $\sum_0^m a_n z^n = (1-z)\sum_0^\infty s_n z^n = (1-z)\sum_0^m s_N^n + (1-z)\sum -m+1^\infty s_n z^n$. The first term won't cause any trouble; it's a finite sum, and disappears as $z \to 1$. So we lean on $\left|(1-z)\sum_{m+1}^\infty s_n z^n\right|$. Since $s_n \to 0$, given $\epsilon > 0$ there's an m so that $|s_n| < \epsilon$ if $n > m$. So

$$\left|\sum_{m+1}^\infty s_n z^n\right| \le \sum_{m+1}^\infty |s_n||z|^n \le \epsilon\frac{|z|^m}{1-|z|}.$$

Thus, $\left|(1-z)\sum_{m+1}^\infty s_n z^n\right| \le \epsilon\left|\frac{1-z}{1-|z|}\right| \le \epsilon k$ by assumption. Thus, $\overline{\lim}|\sum a_n z^n| \le \epsilon k$ for any ϵ. That is, $\lim_{z\to 1}\sum a_n z^n = 0 = \sum_0^\infty a_n$. That's the Abel summation theorem.

The converse is false. There are series a_n so that $\lim_{z\to 1}\sum_0^n a_n z^n$ exists, but $\sum_0^\infty a_n$ does not exist. The former limit is sometimes called the Abel sum. Cool; we can assign a value to a divergent series. The classic example is $a_n = (-1)^n$. Then the Abel sum is 1/2.

Exponential and trigonometric functions Maybe the easiest way to define the exponential function is as the solution to $f'(z) = f(z)$. Then if $f(z) = \sum a_n z^n$, then $(n+1)a_{n+1} = n$, or $a_{n+1} = \frac{a_n}{n+1}$. So we find shortly that $a_n = \frac{0}{n!}$. So $f(z) = a_0\sum_0^\infty \frac{z^n}{n!}$ is a solution to the differential equation. We define $e^z = \sum_0^\infty \frac{z^n}{n!}$. One easily shows that $\limsup(1/n!)^{1/n} = 0$. The theorem at the beginning of the lecture justifies this definition, and we have $\frac{de^z}{dz} = e^z$. This does what we want, e.g.,

$$\begin{aligned}\frac{d}{dz}e^{c-z}e^z &= \frac{de^{c-z}}{dz}e^z + e^{c-z}\frac{de^z}{dz} \\ &= -e^{c-z}e^+ e^{c-z}e^z \\ &= 0.\end{aligned}$$

So $e^{c-z}e^z$ is a constant function. If we set $c = 0$, we get $e^{-z}e^z = e^0 =$. Thus, $e^{-z} = \frac{1}{e^z}$. Can also specialize to get $e^{a_b} = e^a e^b$.

Define $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$. We see that $e^{iz} = \cos z + i \sin z$.

$$\begin{aligned}
\sin z &= \frac{1}{2i}\sum_0^\infty (\frac{(iz)^n}{n!} - \frac{(-iz)^n}{n!} \\
&= \sum_0^\infty \frac{z^{2n+1}(-1)^n}{(n+1)!}
\end{aligned}$$

and

$$\cos z = \sum_0^\infty (-1)^n \frac{2n}{(2n)!}.$$

Can deduce the normal trig formulations of sin and cos by working in the complex plane. This definition is nice, since it makes it easy to prove that, say, $\cos(z+w) = \cos z \cos w - \sin z \sin w$. We would like to prove that there exists a number c so that $e^{z+c} = e^z$, or $e^c = 1$. If we could show that there is a number y_0 so that $\cos(y_0) = 0$, then $\sin(y_0) = \pm 1$, and so $e^{iy_0} = \pm i$, and $e^{4iy_0} = 1$. We'll look at thefunction $w(y) = \frac{\sin y}{\cos y}$, where y is real.

$$\frac{dw}{dy} = 1 + w^2 > w^2.$$

Assume for contradiction that $w(y) > 0$ for $Y > 0$. Then integrate both sides above, and get $\int_a^y dw/w^2 > \int_a^y dy$, so

$$\begin{aligned}
\frac{1}{w(a)} - \frac{1}{w(y)} &> (y-a) \\
\frac{1}{w(a) - (y-a)} &> \frac{1}{w(y)}
\end{aligned}$$

a contradiction. Thus, there's a y_0 so that $\cos y_0 = 0$. Can pick the smallest positive such y_0. Then $\sin y_0 = 1$. We have $e^{iy_0} = i$, and $4iy_0$ is the period of the exponential. One can see fairly easily that all periods must be a multiple of this period. We call this number $4y_0 = 2\pi$.

Finally, e^z has an inverse function called $\log z$. The logarithm is complicated because e^z is periodic; the exponential isn't one-to-one. Since $e^z \neq 0$ for any z, the logarithm is a function on the punctured plane. Now, $e^{x+iy} = e^x e^{iy}$, and $|e^{iy}|^2 = e^{iy}e^{-iy}$. So $\overline{e^z} = e^{\overline{z}}$.

So we should have $\log(x+iy) = \log|x| + i\arg(x+iy)$. This is only determined up to an integer multiple of 2π. We certainly don't want the function to be multiple-valued; so define $\log z$ as a single valued analytic functionon $\mathbb{C} - \mathbb{R}_-$. We have to specify $\log 1 = 2\pi i n$. The principal branch of the logarithm is $n = 0$.

We don't even have to delete a ray; just have to remove a curve which prevents us from making a circuit around the origin.

Can use this to find inverse trig functions. If $\sin z = \frac{e^{iz}-e^{-iz}}{2i} = w$, then $e^{iz} - 2ie^{iz}w - 1 = 0$, and

$$e^{iz} = \frac{2iw \pm \sqrt{2iw^2+4}}{2} = iw \pm \sqrt{1-w^2}.$$

So $\sin^{-1} w = \frac{1}{i}\log(iw \pm \sqrt{w^2-1})$.

Complex integration We have to give some meaning to the formal expression $\int_\gamma f(z)dz$, where γ is some curve. We usually insist that it's piecewise differentiable; $\gamma = \{x(t) + iy(t)\}_{t\in[\alpha,\beta]}$ where $x(t)$, $y(t)$ are continuously differentiable on $[\alpha,\beta] - \{t_1,\cdots,t_n\}$. We define the integral to be $\int_\alpha^\beta f(x(t)+iy(t))(x'(t)+iy'(t))dt$, or $\sum_{i=0}^n \int_{t_i}^{t_{i+1}} f(x+iy(t))(x'(t)+iy'(t))dt$, where $t_0 = \alpha$ and $t_{m+1} = \beta$. We have to show that this is well-defined, i.e., that the integral is independent of the parametrization.

Say we reparameterize via $t(\tau) : [a,b] \to [\alpha,\beta]$. The standard theorem would say that $\int_\alpha^\beta f(z(t(\tau))dz = \int f(z(t(\tau))(x'(t(\tau)) + iy'(t(\tau)))\frac{dt}{d\tau}d\tau = \int_a^b f(z(t(\tau)))(\frac{d}{dz}z(t(\tau))d\tau$. So this integral is well-defined indepenently of the choice of parameter.

If we have an arclength parameter, can say $\int f(s)ds = \int_a^b f(z)|dz| = \int f(z)|z'(t)|dt$, where the curve γ is given by $x(t)+iy(t)$. Then $ds = \sqrt{(x')^2+(y')^2}dt = |z'(t)|dt$. Of course, we have

$$\left|\int_\alpha^\beta f(z)dz\right| \leq \int |f(z)||dz|.$$

For any $e^{i\theta}$, we have $\Re e^{i\theta}\int_\alpha^\beta f(z)dz = \int_\alpha^\beta \Re(e^{i\theta}f(z)dz)$. But we know that $\left|\Re e^{i\theta}f(z)dz\right| \leq |f(z)||dz|$. Use this, and we're done; choose $e^{i\theta}$ to rotate the integral to the real axis. It follows that $\left|\int f(z)dz\right| \leq \int |f(z)||dz|$. Essentially, this is just the triangle inequality.

If we want, we can view f as inducing a map $\gamma \mapsto \int_\gamma f(z)dz$. Define a map on formal sums of arcs: $\sum^n \gamma_i \mapsto \int_{\sum^n \gamma_i} f(z)dz = \sum \int_{\gamma_i} f(z)dz$.

For example, $\int_{|z|=1} \frac{dz}{z}$. Well, $z = e^{i\theta}$, so $dz = ie^{i\theta}d\theta$, and

$$\int_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}d\theta}{e^{i\theta}} = 2\pi i.$$

If we let $\gamma_1 = \{z : |z| = 1\}$, then

$$\int_{\gamma_1+\gamma+\gamma_1} \frac{dz}{z} = 3\int_{\gamma_1} \frac{dz}{z};$$

we just integrate around the unit circle three times. Can also give orientation, and have $-\gamma$ is γ with the opposite orientation. So if $\int_\gamma f(z)dz = \int_\alpha^\beta f(z(t))dz(t)$, then $\int_{-\gamma} f(z)dz = \int_\beta^\alpha f(z(t))dz(t)$.

We have $\int_\gamma f(z)dz \stackrel{\text{def}}{=} \int_a^b f(z(t))z'(t)dt$, where $z(t) : [a,b] \to \gamma$. Can write this as

$$\int f(z(t))(\frac{dx}{dt} + i\frac{dy}{dt})dt = \int f\frac{dx}{dt} + if\frac{dy}{dt}dt.$$

Can also say that $\int pdx + qdy \stackrel{\text{def}}{=} \int_a^b p\frac{dx}{dt}dt + q\frac{dy}{dt}dt$ if $(x(t), y(t)) : [a,b] \to \gamma$. Reminder:

Green's theorem Assume we have an open set Ω, and a region D inside that open set. If $pdx + qdy$ is a differential on Ω with C^1 coefficients, then $\int_{\partial D} pdx + qdy = \int\int_D (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})dxdy$. The orientation is such that if you're standing on the boundary, facing with the orientation, then the domain is on your left. We give the boundary ∂D the induced orientation.

Cauchy Theorem Suppose that $f(z)$ is analytic in an open set Ω, *and* $f'(z)$ is continuous. Then if $\gamma = \partial D$ where D is a compact subset of Ω, then $\int_\gamma f(z)dz = 0$.

Proof

$$\begin{aligned}
\int_\gamma f(z)dx + if(z)dy &= \int_{\partial D} f(z)dx + if(z)dy \\
&= \int\int_D (i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y})dxdy \\
&= \int\int_D i(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})dxdy \\
&= \int\int_D i\partial_{\bar z} f dxdy \\
&= 0.
\end{aligned}$$

Now, we will *not* assume that $f'(z)$ is continuous. We have a version of Green's theorem without differentiability:

Proposition Suppose that p and q are continuous in an open set, and there is a function $U(x,y)$ defined on the disk so that $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$. Then $\int_\gamma pdx + dy = 0$ for any closed curve γ contained in the set. The converse holds, as well.

Proof Suppose there is such a U. Consider

$$
\begin{aligned}
\int_\gamma pdx + qdy &= \int_a^b p(x(t), y(t))\frac{dx}{dt}dt + q(x(t), y(t))\frac{dy}{dt}dt \\
&= \int_a^b (\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt})dt \\
&= \int_a^b \frac{d}{dt}U(x(t), y(t))dt.
\end{aligned}
$$

By the fundamental theorem of calculus, we have $\int_\gamma pdx + qdy = U(x(b), y(b)) - U(x(a), y(a)) = 0$.

For the converse, we simply define $U(x, y) = \int_{(x_0,y_0)}^{(x,y)} pdx + qdy$. Take the path to be all horizontal, and then a little vertical. We get $\frac{\partial U}{\partial x} = p$. Similarly, get $\frac{\partial U}{\partial y} = q$. $\diamond$

If we're looking at $(z-a)^n$ with $n > 0$, then $(z-a)^n = \frac{\partial}{\partial z}\frac{(z-a)^{n+1}}{n+1}$. Let $f(z) = u + iv$. Then $u_x = v_y$ and $u_y = -v_x$. Then $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$. Apply to $(u+iv)$, and get $\frac{1}{2}(u_x + v_y + i(v_x - u_y))$. Plug in from Cauchy-Riemann, and get $u_x + iv_x$. The complex derivative may be computed just from the x derivatives; or y derivatives, for that matter; we have $u_x + iv_x v_y - iu_y = -i(u_y + iv_y)$.

So

$$
\begin{aligned}
f(z)dx + idy &= f(z)dx + if(z)dy \\
f(z) &= \frac{\partial U}{\partial x} \\
if(z) &= \frac{\partial U}{\partial y} \\
-f(z) &= -i\frac{\partial U}{\partial y} \\
f(z) &= \frac{1}{2}(\frac{\partial}{\partial z} - i\frac{\partial}{\partial y})U \\
&= \partial_z U.
\end{aligned}
$$

Thus, can apply this to $(z-a)^n$ for any $n \neq -1$.

We get special cases of Cauchy's theorem: $\int_\gamma (z-a)^n dz = 0$ for any closed curved γ with $a \notin \gamma$ and $n \neq -1$. The problem with $n = -1$is that $\partial z \log(z-a) = (z-a)^{-1}$, and log isn't analytic on a closed circle around a. Last class, we showed that $\int_{|z-a|=r} \frac{dz}{z-a} = 2\pi i$.

We have a Cauchy theorem for rectangles, due to Goursat.

Proposition Suppose that $f(z)$ is analytic in an open set which contains the rectangle R. Then $\int_{\partial R} f(z)dz = 0$.

Remember, analytic just means that $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all z_0 in the open set containing R.

Proof We divide the rectangle into four equal parts; number them $\begin{pmatrix} R^1 & R^2 \\ R^3 & R^4 \end{pmatrix}$, and the thing has length L and height H. Then

$$\int_{\partial R} f(z)dz = \int_{\partial R^1} f(z)dz + \int_{\partial R^2} f(z)dz + \int_{\partial R^3} f(z)dz + \int \partial R^4 f(z)dz,$$

since the integrals on the edges not on the boundary cancel each other. If $\eta(\partial R) \neq 0$ then for some R^i, $|\eta(\partial R)| \geq \frac{1}{4}|\eta(\partial R^i)|$. Let's call this rectangle R_1. Subdivide R_1 as before; there's an R^2 (a quarter of R_1) so that $|\eta(R_2)| \geq \frac{1}{4}|\eta(R_1)|$. Inductively, we get a sequence of rectangles with $R \supset R_1 \supset R_2 \supset \cdots$ with R_n a quarter of R_{n-1}, and $|\eta(R_1)| \geq (\frac{1}{4})^n|\eta(R)|$.

So we've got a nested sequence of rectangles, and $\cap_1^\infty R_n = z^*$. We know that $\lim_{z\to z^*} \frac{f(z)-f(z^*)}{z-z^*} = f'(z*)$ exists. We have $|f(z) - f(z^*) - (-z^*)f'(z)| < \epsilon|z-z^*|$ if $|z-z^*|$ is small enough. Now we're interested in $\int \int_{\partial R_n} f(z)dz$. We know that $\int_{\partial R_n} dz = \int_{\partial R_n} azdz = 0$.

$$\begin{aligned}
\int_{\partial R_n} f(z)dz &= \int_{\partial R_n} (f(z) - f(z^*) - (z-^*)f'(z^*))dz \\
\left|\int_{\partial R_n} f(z)dz\right| &\leq \int_{\partial R_n} |f(z) - f(z^*) - (z-z^*)f'(z^*)||dz| \\
&\leq \epsilon \int_{\partial R_n} |z-z^*||d|
\end{aligned}$$

for n large enough. We know that $\int_{\partial R_n} |d| = \frac{2(L+H)}{2^n}$. But that's not really telling us very much; that's why we have to use the ϵ factor. Now, $|z-z^*| \leq \frac{\sqrt{l^2+H^2}}{2^n}$, simply from the geometry. So $\epsilon \int_{\partial R_n} |z - z^*||dz| \leq \epsilon \frac{2(LH)\sqrt{L^2+H^2}}{4^n}$. But it's also $\geq |\eta(R_n)|4^n$. So $|\eta(R)| \leq C\epsilon$. Since ϵ is arbitrary, we have $|\eta(R)| = 0$. $\diamond$

Now, suppose that $f(z)$ is not necessarily analytic at every point of R. Indeed, we'll allow a finite set of points $\zeta_1, \cdots, \zeta_n$ where all that is assumed is that $\lim_{z \to \zeta_i} |(z - \zeta_i)f(z)| = 0$. In other words, f is a little less singular then $1/z$. With these conditions, the theorem still holds.

Proof This is similar. We've got a rectangle. Cut it up to isolate these points. (Only one pseudosingularity per cell) $\int_{\partial R} f dz = \sum \int_{\partial R_i} f dz$. So it's enough to prove the theorem assuming only one bad point. We cut the rectangle into nine subrectangles so that the bad point, a, is in the center. Call that one R_0. Then $\int f(z)dz = \sum_0^8 \int_{\partial R_i} f(z)dz = \int_{\partial R_0} f(z)dz$. But we have the estimate $|f(z)| < \epsilon/|z-a|$. Thus, $\left|\int_{\partial R_0} f(z)dz\right| \leq \int_{\partial R_0} \frac{\epsilon|dz|}{|-a|}$. Choose R_0 to be a square with side length $2h$. Then $\frac{1}{|z-a|} \leq \frac{1}{h}$. Thus, $\int_{\partial R_0} \frac{\epsilon|dz|}{|z-a|} \leq \frac{\epsilon}{h} \int_{\partial R_0} |dz| \leq 8\epsilon$, and so the theorem is true. $\diamond$

We'll extend Cauchy's theorem to curves which lie in a disk, D. Given $f(z)$ analytic in D, define $F(z)$ so that $\partial_z F = f$. This means we can define $F(w) = \int_a^w f(z)dz$, where the path of integration is one vertical path and one horizontal.

For $f(z)$ analytic in a disk, $\int_\gamma f(z)dz = 0$ for all closed curves γ contained in the disk. We can extend the argument functions which have a finite number of bad points where $\lim_{z-\zeta} |z-\zeta||f(\zeta)| = 0$. This isn't much harder. We might have to use three horizontal or vertical pieces to get between two points, but that's no big deal. In this context we can define $F(w) = \int_{z_0}^w f(z)dz$, where the path consists of three horizontal or vertical segments. F is well-defined in the complement of the singular set, and satisfies $\partial_z F = f$. So Cauchy's theorem applies to any closed curve which avoids the singular set.

We need to understand $n(\gamma, a) \stackrel{\text{def}}{=} \int_\gamma \frac{dz}{z-a}$ where γ is a closed curve.

1. $n(\gamma, a) = 2\pi i m$ for some $m \in \mathbb{Z}$.
2. Let $\mathbb{C} - \gamma = U_1 \cup \cdots \cup U_k$, where U_i are connected components. For $a \in U_i$, $n(\gamma, a)$ is constant.

Proof

1. $\int_a^b \frac{'(t)dt}{z(t)-a}$. Define $h'(t) = \frac{z'(t)}{z(t)-a}$. Define $g(t) = e^{-h(t)}(z(t)-a) = g(t)$. Differentiate, and get $\frac{dg}{dt} = e^{-h(t)}(-h'(t)(z-a) + z'(t)) = e^{-h}(-z' + z') = 0$. $h(a) = 0$. We know that $e^{-h(t)}(z(t) - \alpha) = (z(a) - \alpha)$, or $e^{h(t)} = \frac{z(t)-\alpha}{z(a)-\alpha}$. Well, $e^{h(b)} = \frac{z(b)-\alpha}{z(a)-\alpha} = 1$, so $h(b) = 2\pi i m$ for some $m \in \mathbb{Z}$.

2. Suppose we have two points in a connected component. We can join them with a smooth curve. We're looking at $\int_\gamma \frac{dz}{z-\alpha_t}$. We have $n(\gamma,\alpha_t) - n(\gamma,\alpha_{t_1}) = \int \frac{dz(\alpha_t-\alpha_{t_1})}{(z-\alpha_t)(z-\alpha_{t_1})}$. Thus we see that $n(\gamma,\alpha_t)$ is continuous in t. On the other hand, $n(\gamma,\alpha_t) \in 2\pi i\mathbb{Z}$. Thus, $n(\gamma,\alpha_t)$ is constant.

$\frac{n(\gamma,a)}{2\pi i}$ is called the winding of γ relative to a; it counts the number of times the curve goes around a. If $\mathbb{C} - \gamma = U_0 \cup \cdots \cup U_k$, there is a unique component which includes ∞. We claim that $n(\gamma, a) = 0$ for all $a \in U_0$, that component. For

$$\begin{aligned}
\left|\int_\gamma \frac{d}{z-a}\right| &\leq \int_\gamma \frac{|d|}{|z-a|} \\
&\leq \int_\gamma \frac{dz}{|a|-|z|} \\
&\leq \frac{K}{|a|-r} \\
&\to 0
\end{aligned}$$

as $a \to \infty$. $\diamond$

Let $f(z)$ be analytic in D and let $z_0 \in D$. Consder the function $\frac{f(z)-f(z_0)}{z-z_0}$. Well,

$$\lim_{z\to z_0} \left|(z-z_0)\frac{f(z)-f(z_0)}{z-z_0}\right| = 0.$$

So we can apply Cauchy's theorem to conclude that

$$\int_\gamma \frac{f(z)-f(z_0)}{z-z_0} dz = 0.$$

so long as $z_0 \notin \gamma$. And actually, we have

$$\begin{aligned}
\int_\gamma \frac{f(z)-f(z_0)}{z-z_0} dz &= \int_\gamma \frac{f(z)}{z-z_0} d - \int_\gamma \frac{(z_0)}{z-z_0} dz \\
\int \frac{f(z)}{z-z_0} dz &= f(z_0) n(\gamma, z_0)
\end{aligned}$$

the Cauchy integral formula. Also, $\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-z_0} = f(z_0)$.

Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - z_0}$$

if $n(\gamma, z_0) = 1$.

From this, we can conclude that f is an infinitely differentiable function; and in fact, we'll have

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)dz}{(z - z_0)^{k+1}}.$$

First, f is differentiable.

$$\begin{aligned} \frac{f(z_0) - f(z_1)}{z_0 - z_1} &= \frac{1}{z_0 - z_1}\int f(z)\frac{dz}{2\pi i}\left(\frac{1}{z - z_0} - \frac{1}{z - z_1}\right) \\ &= \int_\gamma \frac{f(z)dz}{(z - z_0)(z - z_1)2\pi i} \end{aligned}$$

and then you yell at the thing at the end until you know that it's $\leq C|z_0 - z_1|$. So $f'(z) = \int_\gamma \frac{f(z)dz}{(z-z_0)^2}$. Proceed by induction to arbitrary derivatives.

A consequence of the Cauchy integral formula is that an analytic function is infinitely differentiable.

Suppose we have a circle centered at z_0, and $|f(z)| \leq M$ on the boundary of the disk, $\partial D(z_0, r)$.[5] We get

$$\left|f^{(k)}(z_0)\right| \leq \frac{k!}{2\pi i}\int_{|z-z_0|=r} \frac{|f(z)||dz|}{|z - z_0|^{k+1}} \leq \frac{k!M}{r^k}.$$

Cauchy estimate If $|f(z)| \leq M$ on $\partial D(z_0, r)$, then $\left|f^{(k)}(z_0)\right| \leq k!Mr^{-k}$.

Corollary Liouville If $f(z)$ is holomorphic in the complex plane, and $|f(z)| \leq M$, then $f(z) = c$ a constant.

[5] Disk centered at z_0 of radius r.

Proof $|f'(z_0)| \leq \frac{M}{r}$. We can let $r \to \infty$, and then $f'(z) \to 0$.

Corollary Fundamental theorem of algebra Let $P(z) = \sum^n a_j z^j$. Then $P(z)$ has a root.

Proof Suppose $P(z) \neq 0$ for any finite z. Well, $|P(z)| \geq |a_n||z|^n(1 - \frac{a_{n-1}}{|a_n|} - \cdots - \frac{a_0}{|z|^n}) \geq |a_n||z|^n(1 - \epsilon)$ for $|z| \gg R$. Thus, we conclude that $\frac{1}{P(z)}$ is bounded. Liouville's theorem implies that $P(z)$ is a constant.

Last time, we had the Cauchy integral formula $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$ if $n(z,\gamma) = 1$, and f is analytic in a disk $D(a,r)$, and $\gamma \subset D(a,r)$.

From this formula, we showed that

- $f(z)$ analytic $\Rightarrow$ f is infinitely differentiable.
- $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)^{n+1}}$.

It's easy to *derive* these formulas, but it's harder to prove that they're correct. For example,

$$\begin{aligned}
\frac{f(z)-f(z_1)}{z-z_1} &= \frac{1}{2\pi i}\int_\gamma f(\zeta)(\frac{1}{\zeta-z} - \frac{1}{\zeta-z_1})d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma f(\zeta)(\frac{z-z_1}{(\zeta-z)(\zeta-z_1)})d\zeta\frac{1}{z-z_1}
\end{aligned}$$

As before, we assume $n(\gamma,z) = n(\gamma,z_1) = 1$. We know what the limit should be; estimate $\left|\frac{f(z)-f(z_1)}{z-z_1} - f\frac{1}{2\pi i}\int\frac{f(\zeta)}{(\zeta-z_1)^2}d\zeta\right|$, using the equality derived above.

We know there's an $\epsilon > 0$ so that if $d(z,z_1) < \epsilon$ then for some $\delta > 0$, $d(z,\gamma), d(z_1,\gamma) > \delta$. So we're trying to estimate

$$\int_\gamma \left|f(\zeta)\frac{z-z_1}{(\zeta-z)^2(\zeta-z_1)^2}\right|d\zeta \leq \frac{l(\gamma)M|z-z_1|}{\delta^3} \to 0.$$

Back to the general formula, the one for $f^{(n)}(z)$. Let $M = \max_{|z-a|=r}|f(z)|$. Then we have that $\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi}\int_\gamma \frac{|f(\zeta)||d\zeta|}{|(\zeta-z)^{n+1}|}$. Last time, we specialized at $z = a$ to find $\left|f^{(n)}(a)\right| \leq n!Mr^{-n}$. Can also use this to get an estimate for any point inside the circle. Let z be any point in the disk. The shortest path from z to boundary of the disk is the radius through z. We can rotate things so that z sits on the real line; then the distance to the boundary is $r - |z-a|$. We say that on γ, $|\zeta - z| \geq r - |z-a|$. This yields the following estimate:

$$\left|f^{(n)}(z)\right| \leq \frac{n!M2\pi r}{(r-|z-a|)^{n+1}}.$$

Morera's Theorem Suppose $f(z)$ is continuous in an open connected set $\Omega \subset \mathbb{C}$, and $\int_\gamma f(z)dz = 0$ for every piecewise differentiable closed curve $\gamma \subset \Omega$. Then $f(z)$ is analytic.

Proof Let $z_0 \in \Omega$. Define $F(z) = \int_{z_0}^{z} f(w)dw$. This is well-defined by assumption. Now, $F'(z) = f(z)$. For

$$\begin{aligned}\left|\frac{F(z_1) - F(z)}{z_1 - z} - f(z_1)\right| &= \left|\frac{\int_z^{z_1} f(w)dw}{z_1 - z}\frac{\int_z^{z_1} f(z_1)dw}{z_1 - z}\right| \\ &= \left|\frac{\int_z^{z_1}(f(w) - f(z_1))dw}{z - z_1}\right| \\ &\leq \frac{\int_z^{z_1} |f(w) - f(z_1)||dw|}{|z - z_1|}\end{aligned}$$

[6]

There's a $\delta > 0$ so that $|z - z_1| < \delta \Rightarrow |f(z) - f(z_1)| < \epsilon$.

So

$$\begin{aligned} &\leq \epsilon\frac{|z - z_1|}{|z - z_1|} \\ &= \epsilon\end{aligned}$$

for $|z - z_1| < \delta$. So $\lim_{z\to z_1} \frac{F(z_1)-F(z)}{z_1-z} = f(z)$. So F is analytic in Ω, and therefore $f = F'$ is as well. $\diamond$

Now, our goal is to show that every analytic function is represented by its power series.

Recall, we've shown that if f is analytic in $|z - a| \leq r$ and $|f(z)| \leq M$ on $|z - a| = r$, then $\left|f^{(n)}(a)\right| \leq n!Mr^{-n}$. The Taylor series is $\sum \frac{f^{(n)}(a)}{n!}(z - a)^n$. Consider

$$\sqrt[N]{\left|\frac{f^{(n)}(a)}{n!}\right|} \leq \frac{\sqrt[n]{M}}{r} \to \frac{1}{r}$$

By Hadamard's criterion, $\sum \frac{f^{(n)}(a)}{n!}(z - a)^n$ converges in $D(a, r)$.

We have $f(z) = f(a) + \int_0^1 \frac{d}{dt}f(a + tz)dt$, the fundamental theorem of calculus. So $f(z) = f(a) + \int_0^1 f'(a + tz)zdt$. This is actually a complex integral along $z(t) = a + tz$. So $f(z) =$

[6]For the first equality, remember that we're free to integrate along any contour, since the integral on a closed path is zero.

$f(a) + \int_a^t f'(w)dw$. Since we're assuming that the function is analytic, we can integrate by parts; $f(z) = f(a) - f'(z)(z-w)|_a^z + \int f''(z-w)dw$, or $f(a)+f'(a)(z-a)+\int_a^z f''(w)(z-w)dw$. This is the first-order expansion; the integral is the remainder term. We do this n times, and get

$$f(z) = f(a) + f'(a)(z-w) + \frac{f''(a)(z-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \int_a^z \frac{f^{(n+1)}(w)(z-w)^n dw}{n!}.$$

Now, if everything were real, we could apply the mean value theorem to the integral, $R_n(z)$; but as it isn't, we can't.

Use $f^{(n+1)}(w) \leq \frac{M 2\pi r (n+1)!}{(r-|w-a|)^{n+1}}$. So

$$|R_n(z)| \leq \int_a^z \left|f^{(n+1)}(w)\right| \frac{|z-w|^n}{n!} |dw|.$$

Let $\zeta = z - w$, and get

$$\int_0^{z-a} |\eta|^n |d\zeta| = \frac{|z-a|^{n+1}}{n+1}.$$

Substitute in, and get

$$|R_n(z)| \leq \frac{2\pi r M (n+1)!}{(r-|z-a|)^{n+1}} \frac{|z-a|^{n+1}}{(n+1)!} \leq 2\pi r M (\frac{|z-a|}{r-|z-a|})^{n+1}.$$

Thus, $f(z) = \sum \frac{f^{(j)}(a)(z-a)^j}{j!}$ if $|z-a| < \frac{r}{2}$.

In fact, $f(z) = \sum \frac{f^{(j)}(a)(z-a)^j}{j!}$ on the whole disk $D(a, r)$. We'll prove the following

Identity Theorem If Ω is an open connected subset of $\mathbb{C}$, and f, g analytic in Ω, and $f = g$ on some open subset of Ω, then $f \equiv g$ in all of Ω.

Proof Let $E_1 = \{z \in \Omega | f^{(j)}(z) = g^{(j)}(z) \forall j \in \mathbb{N}\}$ Let $E_2 = \Omega - E_1$. Well, E_1 is obviously a closed set; it's $\cap E_1^j$, where $E_1^j = (f^{(j)} - g^{(j)})^{-1}(\{0\})$.

Suppose $z_0 \in E_1$. This implies that $\sum f^{(j)}(z_0)z - z_0)^j/j! = \sum \frac{g^{(j)}(z_0)(z-z_0)^j}{j!}$ in $D(z_0, r)$. We know tha these are equal to $f(z)$ and $g(z)$, respectively, for $z \in D(z_0, r/2)$. This implies that $D(z_0, r/2) \subset E_1$. Thus, E_1 is open; since $\Omega = E_1 \cup E_2$, and Ω is connected, and $E_1 \neq \emptyset$, we have $\Omega = E_1$. $\diamond$

Corollary $f(z) = \sum \frac{f^{(j)}(a)(z-a)^j}{j!}$ for all $z \in D(a,r)$.

1. f and $\sum \frac{f^{(j)}(a)(z-a)^j}{j!}$ are both analytic in $D(a,r)$.
2. They are equal in $D(a,r/2)$.
3. They are equal in the whole disk, by the identity theorem. $\diamond$

We have shown that $\{f | \exists f' \in \Omega\} = \{$ functions representable by convergent power series around any point in $\Omega\}$.

Corollary If f and g are analytic in Ω and $f^{(j)}(a) = g^{(j)}(a)$ for all j and some $a \in \Omega$, then $f \equiv g$ in Ω.

Corollary If $f^{(j)} = 0$ for all j at some point a, then $f \equiv 0$.

Corollary Suppose $\{a_n\} \in \Omega$ is a sequence with a limit point $a^* \in \Omega$, and $f(a_n) = g(a_n)$. Then $f \equiv g$.

Taylor series remix We need to know about removable singularities. Suppose that $f(z)$ is analytic in $\Omega - \{a\}$, but $\lim_{z \to a} |z-a||f(z)| = 0$. Then $f(z)$ has an extension to Ω which is analytic.

The standard example is $\frac{f(z)-f(a)}{z-a} = f_1(z)$. Of course, $f_1(a) = f'(a)$.

We can sketch out a proof of this. Let $D(a,\gamma) \subset \Omega$. Then $f(z) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(\zeta)d\zeta}{\zeta - z}$ for $z \neq a$. But the denominator doesn't care about a, so the right-hand side is analytic in the whole disk. Furthermore, it agrees with f away from $z = a$; so it agrees with f everywhere, and that's how you can extend it to a. $\diamond$

This tells us that there is no analytic function with $|f(z)| \sim \frac{1}{(z-a)^\alpha}$ if $0 < \alpha < 1$.

On to something else [?]. Can look at $\frac{f_1(z)-f_1(a)}{z-a} = f_2(z)$, and $f_2(a) = f_1'(a) = f''(a)$. But $f_1' = \frac{f'(z-a)-(f(z)-f(a))}{(z-a)^2} = \frac{f'(z) - \frac{f(z)-f(a)}{(z-a)}}{-a} = f''(a)$.

If $f(z)$ is analytic in a disk $D(a,r)$, then $f(0) = \sum \frac{f^{(k)}(a)}{k!}(z-a)^k$ in $D(a,r)$. We did this in two steps; first, that this is well-defined; second, using the identity theorem, we get equality.

Suppose that the power series $\sum a_n z^n$ converges in $D(0,r)$ but not $D(0,r+\epsilon)$ for all $\epsilon > 0$. Then the function it represents is not analytic in $D(0,r+\epsilon)$ for any $\epsilon > 0$.

This isn't true if you're just working over $\mathbb{R}$.

Example $\frac{1}{1-z} = \sum z^n$. The radius of convergence is 1, and it doesn't converge anywhere on $|z| = 1$. But the only singularity is at $z = 1$. So the converse to the above statement is false.

Definition An entire function is a function analytic in all of $\mathbb{C}$.

If $f(z) = \sum_0^\infty a_n z^n$, then f is entire $\iff$ $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 0$.

Corollary If $\{z_n\} \subset \Omega$ is a sequence with an accumulation point, and f is analytic in Ω, and $f(z_n) = 0$, then $f \equiv 0$.

Proof [1] We'll suppose that $z_n \to z^* \in \Omega$. Consider the following expression:

$$\frac{a_0}{z-x_0} + \frac{a_1}{z-x_1} + \cdots + \frac{a_m}{z-x_m}.$$

Clear denominators; get

$$\frac{a_0(z-x_1)(z-x_2)\cdots(z-x_m) + \cdots + a_1(z-x-0)\cdots(z-x_{m-1})}{\prod_0^m (z-x_i)}.$$

Multiply out, and look at coefficients of z. From the highest term, we get $a_0 + a_1 \cdots a_m = 0$. For the second highest term, $a_0\sigma_1(\hat{x}_0) + a_1\sigma_1(\hat{x}_1) + \cdots + a_m\sigma_1(\hat{x}_m) = 0$. Here, σ_i are the elementary symmetric function on m variables, i.e., $\prod_1^m (z-x_i) = \sum (-1)^j \sigma_j(x_1, \cdots, x_m) z^{m-j}$. Also, $\hat{x}_j$ means delete x_j. One proves inductively that all of the equations are in terms of symmetric functions; $a_0\sigma_k(\hat{x}_0) + \cdots + a_m\sigma_k(\hat{x}_m) = 0$, for $k \le m-1$.

At this point, we've got to find the perp space of m vectors in $m+1$ space. [If $m = 2$, use cross products.]

$$\begin{pmatrix} \hat{i}_0 & \hat{i}_1 & \cdots & \hat{i}_m \\ v_{10} & v_{11} & \cdots & v_{1m} \\ v_{m0} & v_{m1} & \cdots & v_{mm} \end{pmatrix}$$

Take the determinant of this thing by expanding along first row, and you wind up with a vector $\sum a_i(x)\hat{i}_j$. Now the question is, are the $a_i(x)$'s nonzero?

Inductive argument to show

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ \sigma_1(\hat{x}_0) & & \sigma_1(\hat{x}_m) \\ \vdots & & \vdots \\ \sigma_m(\hat{x}_0) & \cdots & \sigma_m(\hat{x}_m) \end{pmatrix} = c_m \prod_{i<j}(x_i - x_j)$$

where $c_m \neq 0$. If $m = 2$, we have $\det\begin{pmatrix} 1 & 1 \\ x_1 & x_0 \end{pmatrix} = x_0 - x_1$, done.

Otherwise, set $x_0 = 0$. You get $\det = x_1 \cdots x_m c_{m-1} \prod_{i<j}(x_i - x_j)$.

With these $a_i(x)$'s, we have (from above)

$$\sum \frac{a_i}{z - x_i} = \frac{c_m \prod(x_i - x_j)}{\prod(z - x_i)}.$$

Call the numerator $n(x)$. We then have $\sum^m a_i(x)\frac{f(x_i)}{n(x_i)} = \sum \frac{1}{2\pi i}\int \frac{a_i(x)f(z)dz}{n(x)(z-x_i)} = \frac{c_m}{2\pi i}\int \frac{f(z)d}{\prod(z-x_i)}$. Thus,

$$\lim_{n\to\infty} \sum_0^m \frac{a_i(z_{n+1}\cdots z_{n+m} f(z_{n+1}))}{n(z_n, \cdots, z_{n+m})} = \frac{c_m}{2\pi i}\int \frac{f(z)dz}{(z - z^*)^{m+1}} = \widetilde{c_m}^{[m]} f(z^*).$$

$\diamond$

Corollary If $f(z_n) = 0$ for all n then $f^{[m]}(z^*) = 0$ for all m.

This has something to do with something called divided differences.

Proof [2] Suppose $f(z_0) = 0$, assume not identically zero. Then for some n, $f^{[j]}(z_0) = 0$ for $j = 0, 1, \cdots, n-1$ but $f^{[n]}(z_0) \neq 0$. Then can write $f(z) = \sum a_j(z - z_0)^j = a_n(z - z_0)^n \sum_{j=n}^{\infty} \frac{a_j}{a_n}(z - z_0)^{j-n}$. So we can write $f(z) = (z - z_0)^n f_n(z)$ where $f_n(z) = 1 + O(z - z_0)$. From this, we conclude that the zeros of an analytic function must be separated. So we now have $|f_n(z)| > \frac{1}{2}$ for $|z - z_0| < \delta$ for some δ. Thus, the zeros of $f(z)$ are isolated from one another. $\diamond$

At a point z_0 where f is analytic, there is a well-defined order, that is, the integer n defined above. Suppose $f(z)$ is analytic in a set $0 < |z - z_0| < \delta$. There are a couple possibilities.

1. It could be that $\lim_{z\to z_0} |z-z_0|^{\alpha}|f(z)| = 0$ for some $\alpha \in \mathbb{R}$.
2. $\lim_{z\to z_0} |z-z_0|^{\beta}|f(z)| = \infty$.
3. The limit doesn't exist for any $\alpha \in \mathbb{R}$.

In case 1, there's an $n \in \mathbb{Z}$ so that $n > \alpha$. We know that for some integer n, the function $(z-z_0)^n f(z)$ has a removeable singularity at $z = z_0$. This means that $(z-z_0)^n f(z) = a_0 + a_1(z-z_0) + \cdots a_{n-1}(z-z_0)^{n-1} + (z-z_0)^n \phi_n(z)$. Alternatively, $f(z) = \frac{a_0}{(z-z_0)^n} + \cdots + \frac{a_{n-1}}{-z_0} + \phi_n(z)$. *What happens next?*

In case 2, if $\lim_{z\to z_0} |z-z_0|^{\beta}|f(z)| = \infty$, then we can find an integer $n < \beta$ and $\lim_{z\to z_0} |z-z_0|^{n}|f(z)| = \infty$. So the function $\frac{1}{(z-z_0)^n f(z)}$ has a removeable singularity at $z = z_0$. This means that there is an analytic function $g(z)$ analytic near $z = z_0$ so that $\frac{1}{(z-z_0)^n f(z)} = g(z) = (z-z_0)^m h(z)$ where h is analytic at $z = z_0$ and nonvanishing. Thus, $f(z) = \frac{1}{h(z)} \frac{1}{(z-z_0)^{n+m}}$. Can rewrite this as $\frac{\widetilde{h}(z)}{(z-z_0)^{n+m}}$. So in case 2, $f(z)$ is either analytic at $z = z_0$, or has a pole of some finite order.

In either case one or two, there is a unique integer n so that $(z-z_0)^n f(z)$ is analytic at $z = z_0$ and nonvanishing there. This tells us that poles occur as integral powers. We'll never have $|z-z_0|^{\alpha}|f(z)| < K$ for some $\alpha \notin \mathbb{Z}$.

In case three, $f(z)$ is said to have an *essential* singularity at $z = z_0$.

Theorem [Casorati-Weierstrass Theorem] Suppose $f(z)$ has an essential singularity at $z = z_0$. Then $f(z)$ comes arbitrarily close to any complex number in any neighborhood of $z = z_0$. For any $\epsilon > 0$ and $w \in \mathbb{C}$, there's a $z_n \to z_0$ so that $|f(z_n) - w| < \epsilon$.

Proof Suppose this is false; then there's a $A \in \mathbb{C}$ and and $r > 0$ and $\delta > 0$ so that $|f(z) - A| > \delta$ for $z \in D(z_0, r)$. Look at $\frac{\lim_{z\to z_0} |z-z_0|^{\alpha}|}{f(z)-A =^{7} \infty}$ for all $\alpha < 0$. Thus, for some n, $\frac{(z-z_0)^n}{f(z)-A}$ is analytic in a neighborhood of $z = z_0$, and $\frac{z-z_0)^n}{f(z)-A} = h(z)$. Can choose n so that $h(z_0) \neq 0$. Then $f(z) = A = \frac{(z-z_0)^n}{h(z)}$, and $f(z) = \frac{(z-z_0)^n}{h(z)} + A$. So for α large enough, $\lim_{z\to z_0} |z-z_0|^{\alpha}|f(z)| = 0$, a contradiction.

If $f(z)$ is analytic in $0 < |z-z_0| < r$, then either

1. $f(z)$ extends to z_0 analytically.
2. $f(z) = \frac{a_0}{(z-z_0)^m} + \cdots + \frac{a_{n-1}}{z-z_0} + \phi_n(t)$ where $\phi_n(z)$ is analytic.

3. $f(z)$ has an essential singularity.

Let γ be a closed curve in an open set Ω, γ compact, f analytic. Suppose that $f(z)$ has only finitely many zeros inside Ω, say, $\{z_1, \cdots, z_m\}$, with multiplicities. Then $f(z) = \prod_1^n (z - z_i)h(z)$, where $h(z)$ is analytic in Ω and nonvanishing; $h(z) = \frac{f(z)}{\prod(z-z_i)}$.

Let's check $h(z)$ near $z = z_1$. Suppose that z_1 occurs n times. Then $\prod_1^n (z-z_i) = (z-z_1)^n g(z)$ where $g(z_1) \neq 0$. This happens if $f(z) = (z - z_1)^n k(z)$ where $k(z_1) \neq 0$. Then

$$\begin{aligned} \frac{f}{\prod(z - z_i)} &= \frac{(z - z_1)^n k(z)}{(z - z_1)^n g(z)} \\ &= \frac{k(z)}{g(z)}. \end{aligned}$$

We take the logarithmic derivative $\frac{d}{dz} \log f = \frac{f'}{f} = \sum_1^m \frac{1}{z-z_i} + \frac{h'}{h}$. Let's integrate. Then

$$\begin{aligned} \int_\gamma \frac{f'}{f} \frac{dz}{2\pi i} &= \sum_1^m \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_i} + \int_\gamma \frac{h'}{h} \frac{dz}{2\pi i} \\ &= \sum_1^m n(\gamma, z_i) + 0 \end{aligned}$$

Here, $n(\gamma, z_i)$ is the winding number, and h'/h is nonvanishing. The point is, $\frac{1}{\pi i} \int_\gamma \frac{f'}{f} dz = \sum_1^m n(\gamma, z_i)$. $\diamond$

Now, consider Ω a disk, γ a circle. Then $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)dz}{f(z)}$ is the number of zeros inside γ. If we let $w = f(z)$, then $\Gamma = f(\gamma)$, and $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)dz}{f(z)} = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w}$.

Theorem Suppose that $f(z)$ is analytic in a disk $D(a, r)$, and f has a zero of order n at $z = a$; that is, $f(z) = (z - a)^n h(z), h(a) \neq 0$. Then there's a $\delta > 0$ and $\epsilon > 0$ so that $f(z)w$ has n solutions in $|z - a| < \epsilon$.

Proof The number of solutions to $f(z) = w$ in $|z - a| < \epsilon = \frac{1}{2\pi i} \int_{|-a|=\epsilon} \frac{f'(z)dz}{f(z)-w}$. This function of w is continuous, so long as $f(z) - w$ doesn't vanish on $|z - a| = \epsilon$. So this equals $\frac{1}{2\pi i} \int_{|z-a|=\epsilon} \frac{f'(z)dz}{f(z)}$ if $|w| < \delta$.

If f is analytic in a disk, and γ a closed curve in the disk such that $f \neq 0$ at any point on γ, then

$$\int_\gamma \frac{f'}{f}\frac{dz}{2\pi i} = \sum_{\{z:f(z)=0\}} n(\gamma, z)m_z.$$

where m_z is the order of the zero that f has at z. Use this to prove:

Proposition If $f(z) - w_0$ has a zero of order n at z_0, then there's a $\delta > 0$ and $\epsilon > 0$ so that for $|w - w_0| < \delta$, $f(z) = w$ has n solutions in $|z - z_0| < \epsilon$.

Suppose $n = 1$. Then $f(z)$ is locally one-to-one and onto. This implies that $g(w) = f^{-1}(w)$ is defined in $|w - w_0| < \delta$. Can write

$$g(w) = \int_{|z-z_0|<\epsilon} \frac{f'(z)z}{f(z)-w}\frac{dz}{2\pi i}.$$

Why is this? Well, for w close enough to w_0 there's a unique $z_w \in |z - z_0| < \epsilon$ with $f(z_w) = w$. So $f(z) - w = a(z - z_w)h_w(z)$ where $h_w(z) \neq 0$ in $|z - _0| < \epsilon$ is a holomorphic function.

Look at $\frac{f'(z)}{f(z)-w}$. Since w is a constant, this is $\frac{1}{z-w} + \frac{h_w'(z)}{h_w(z)}$. So integrate;

$$\int \frac{f'(z)zdz}{f(z) - w \cdot 2\pi i} = \int_{|z-z_0|=\epsilon} (\frac{z}{z-z_w} + \frac{h_w'(z)z}{h_w(z)})\frac{dz}{2\pi i} = z_w.$$

$\diamond$

Proposition If $f(z)$ satisfies $f'(z_0) \neq 0$ then there is a disk $|z - z_0| < \epsilon$ in which $f(z)$ is 1-1 and the inverse function defined on $f(D(z_0, \epsilon))$ is analytic. Indeed, f is locally 1-1 $\iff$ $f'(z_0) \neq 0$.

If $f'(z_0) = 0$, then there is some integer n so that $f^{[j]} = 0$ for $1 \leq j \leq n-1$, and $f^{[n]}(z_0) \neq 0$.

Let $f(z_0) = w_0$. We know that $f(z) - w_0 = (z - z_0)^n h(z)$ where $h(z_0) \neq 0$. If $h(z_0) \neq 0$ then we can define, unambiguously, a branch of the logarithm $\log h(z)$ for z in some neighborhood of z_0. Therefore, we can define $h(z)^{1/n} = e^{\frac{\log h(z)}{n}}$ for z in this neighborhood. Define $\zeta(z) = (z - z_0)h(z)^{1/n}$. This map is locally 1-1, as $\frac{d\zeta}{dz}(z_0) \neq 0$. Using this map, we can write

$$f(z) = w_0 + {}^{8}\zeta(z)^n.$$

[8] I'm not sure that this should be a +.

Uh-oh. Picture time. ζ divides the preimage disk into n sectors; and then raising to the n^{th} power makes each sector cover the unit disk. Thus, in local coordinate, a function which vanishes to order n looks like $z \mapsto z^n$.

Contrast this with the map $(x, y) \mapsto (x^3, y)$. It's locally 1-1, but *not* locally invertible, e.g., at zero; for the Jacobian is $\begin{pmatrix} 3x^2 & 0 \\ 0 & 1 \end{pmatrix}$. This will never happen for a holomorphic function.

If we think of $f(z)$ as defining a mapping from an open set $U \subset \mathbb{C}$ to an open set $V \subset \mathbb{C}$, then f is locally 1-1 $\iff$ f is holomorphically invertible.

Proposition [Open Mapping Theorem] A nonconstant analytic function defines an open mapping from a subset of $\mathbb{C}$ to a subset of $\mathbb{C}$.

Proof We need to show that if $w \in \operatorname{im}(f)$ then there is an open set U such that $w \in U$ and $U \subset \operatorname{im}(f)$. But the proof of this is obvious, from the computation above and the fact that $z \mapsto z^n$ is an open map. $\diamond$

Look at $f(x, y) = x^2 y$. Consider the image of a disk centered on y-axis. It's image will be sort of folded in half. It's not an open map, as there are boundary points in the image which don't have open neighborhoods.

Theorem [Maximum Principle] If $f(z)$ is analytic in an open set Ω, then $|f(z)|$ never assumes an interior maximum, unless $f(z)$ is constant.

Proof Suppose that $|f(z)|$ assumed a local maximum at $z_0 \in \Omega$. Then $f(\Omega)$ contains a disk centered at $f(z_0)$ of positive radius. Somehow, this gives a contradiction.

Corollary For any $K \subset \Omega$ with K compact, $\max_{z \in K} |f(z)|$ occurs on ∂K. If there's a z_0 in the interior of K with $|f(z_0)| = \max_{z \in K} |f(z)|$, then $f(z)$ is constant.

Can also prove this with the Cauchy formula. Represent $f(z_0)$ in terms of $f(z_0 + \rho e^{i\theta})$ for some $\rho > 0$. We know that $f(z_0) = \frac{1}{2\pi} i \int_{|z-z_0|=\rho} \frac{f(z)dz}{z-z_0} = \frac{1}{2\pi i} \int f(z) + \rho e^{i\theta}) d\theta$. Therefore, $|f(z_0)| \leq \frac{1}{2\pi i} \left| \int_0^{2\pi} f(z + \rho e^{i\theta}) d\theta \right| \leq \frac{1}{2\pi i} \int_0^{2\pi} \left| f(z + \rho e^{i\theta}) \right| d\theta \leq \max_{0 \leq \theta \leq 2\pi} \left| f(z + \rho e^{i\theta}) \right|$. Equality will hold only if the modulus of f is constant on the boundary of the circle. Can show that we can keep shrinking the radius and still get the same number, i.e., the function is constant.

More verbosely, assme that $|f(z_0|$ is a local maximum. Then $|f(z_0)| \leq \max_{0 \leq \theta \leq 2\pi} \left| f(z_0) + \rho e^{i\theta} \right|$ for $\rho \in [0, \epsilon]$. But the maximum is $\leq |f(z_0)|$. This implies that $\left| f(z_0 + \rho e^{i\theta}) \right| = |f(z_0)|$ for $0 \leq \rho \leq \epsilon$ and $0 \leq \theta \leq 2\pi$. Therefore, the modulus is constant, and f is constant.

Theorem [Schwarz Lemma] Suppose that $f(z)$ is defined on $|z| < 1$ and $|f(z)| \leq 1$ on $|z| < 1$, $f(0) = 0$. Then $|f(z)| \leq |z|$, $|f'(0)| \leq 1$, and if there's a z_0 with $|z_0| < 1$ where $|f(z_0)| = |z_0|$, then $f(z) = cz$ for some $|c| = 1$.

Proof Consider $g(z) = f(z)/z$. This function has a removeable singularity at $z = 0$. Its value at zero is $f'(0)$. For any $\epsilon > 0$, we know that $\max_{|z|\leq 1-\epsilon} |g(z)| \leq \max_{|z|=1-\epsilon} \left|\frac{f(z)}{zz}\right| \leq \frac{1}{1-\epsilon}$. Since ϵ is arbitrary, we have $\max |z| < 1 |g(z)| \leq 1$. Now, $|f(z_0)| = |z_0| \iff |g(z_0)| = 1$, i.e., $g(z) = e^{i\theta}$ for all z. So $f(z) = e^{i\theta} z$ for all $|z| < 1$.[9] $\diamond$

Definition We say that $f_n \to f$ locally uniformly in Ω if for all $K \subset \Omega$ compact, $f_n|_K \to f|_K$ uniformly.

Theorem [Weierstrass] Suppose $\{f_n(z)\}$ is a sequence of analytic functions in an open set $\Omega \subset \mathbb{C}$. If $f_n(z)$ converge locally uniformly to $f(z)$ on Ω, then $f(z)$ is also analytic.

Proof [1] We'll use Morera's theorem.

- $f(z)$ is certainly continuous; for continuity is a local property.
- Analyticity is also a local property. For each $z_0 \in \Omega$, we can choose $D(z_0, r_{z_0}) \overset{\text{comp}\subset}{\Omega}$, so that if $\gamma \overset{\text{comp}\subset}{D} (z_0, r_{z_0})$ then $\int_\gamma f(z)dz = \lim_{n\to\infty} f_n(z)dz = 0$. So far, this is all real variable theory. By Morera's theorem, $f(z)$ is analytic. $\diamond$

Indeed, for any k, $f_n^{[k]}(z) \to f^{[k]}(z)$ locally uniformly.

Proof [2] $f_n^{[k]}(z) = k! \frac{1}{2\pi i} \int_{|\zeta - z_0| = \epsilon} \frac{f_n(\zeta) d\zeta}{(\zeta - z)^{k+1}}$. For $|z - z_0| < \epsilon/2$, $\frac{f_n(\zeta)}{(\zeta - z)^{k+1}}$ on $|\zeta - z_0| = \epsilon$. So $\lim_n f_n^{[k]}(z)$ exists, and is equal to $k! \frac{1}{2\pi i} \int_{|\zeta - z_0| < \epsilon} \frac{f(\zeta) d\zeta}{(z - \zeta)^{n+1}} = f^{[k]}(z)$. $\diamond$

Theorem [Hurwitz] Suppose $f_n(z) \to f(z)$ locally uniformly on $\Omega \subset \mathbb{C}$, and suppose that $f_n(z) \neq 0$ for any $z \in \Omega$. Then either

1. $f(z) \equiv 0$ on Ω

[9] Of course, if $|f'(0)| = 1$, then $f(z) = e^{i\theta}$,anyways.

2. $f(z) \neq 0$ for any $z \in \Omega$.

Suppose that there exists $z_0 \in \Omega$ so that $f(z_0) = 0$ but $f(z) \not\equiv 0$. Then there exists an $\epsilon > 0$ and $\delta > 0$ so that $|f(z)| > \delta$ if $|z - z_0| = \epsilon$¡ where $|z - z_0| \leq \epsilon \overset{\text{comp}\subset}{\Omega}$. So $f_n \to f$ uniformly on $|z - z_0| = \epsilon$. This means there's an N so that $|f_n(z) - f(z)| < \delta/2$ for $n > N$ and $|z - z_0| = \epsilon$.

Thus, $|f_n(z)| \geq |f(z)| - \delta/2$ on $|z - z_0| = \epsilon$, which is $\geq \delta/2$. From this, we conclude that $\frac{f_n'}{f_n} \to \frac{f'}{f}$ uniformly on $|z - _0| = \epsilon$. Now, $n = \int_{|z-z_0|=\epsilon} \frac{f'dz}{f} \frac{1}{2\pi i} = \lim_{n\to\infty} \int_{|-_0|=\epsilon} \frac{f_n' dz}{f_n} \frac{1}{2\pi i} = 0$. (We assumed that there weren't any zeros in the disk.) This yields a contradiction; $f \equiv 0$. $\diamond$

For example, $f_n(z) = \frac{z}{n}$ converges uniformly to zero, but it isn't zero.

Now, in real variable theory, we have $\lim_{n\to\infty}(1 + \frac{z}{n})^n = e^z$. We're going to prove that the LHS converges uniformly to the RHS. Now, $\log(1 + z/n)^n = n \log(1 + z/n)$. For $|z| < n$ we can define a single valued branch of $\log(1 + z/n)$ which equals zero at $z = 0$. The power series expansion is

$$\log(1 + w) = w + \frac{w^2}{2} + \frac{w^3}{3} + \cdots$$

So $\log(1 + z/n) = z/n + O((\frac{z^2}{n}))$ for $|z| < n - \epsilon$. This says that $|n \log(1 + z/n) - z| = |z + O(z^2/n) - z| = |O(z^2/n)|$ for $|z| < R$¡ and $n >> [R]$. Thus, $\lim_{n\to\infty} n \log(1 + z/n) = z$ locally uniformly in $\mathbb{C}$.

exercise If $\lim_{n\to\infty} f_n(z) = f(z)$ locally uniformly, then the limit $\lim_{n\to\infty} e^{f_n(z)} = e^{f(z)}$ locally uniformly.

We already know some power series, e.g., $e^z = 1 + z + z^2/2! + z^3/3! + \cdots$, $\sin(z) = z - z^3/3! + z^5/5! + \cdots$, etc.

Recall that $g(z)$ is $O(z^n)$ means that $\overline{\lim}_{z\to 0}|g(z)|/|z|^n < B < \infty$. So for example, $e^z = 1 + z + z^2/2 + O(Z^3)$. Thus, $O(z^n) \cdot O(z^m)$ is $O(^{n+m})$, and $O(z^n) \pm O(z^m)$ is $O(z^{\min(m,n)})$.

Newton showed that for any μ,

$$(1+z)^\mu = 1 + \mu z + \binom{\mu}{2} z^2 + \binom{\mu}{3} z^3 + \cdots$$

where $\binom{\mu}{n} = \frac{\mu(\mu-1)\cdots(\mu-n+1)}{n!}$. Now, there are actually many different branches; we're picking one by saying that $(1+z)^\mu|_{z=0} = 1$.

We defined $\log(1+z) = 1 - z + z^2/z -^3 /3 + z^4/4 + \cdots$. So $\frac{d}{dz}\log(1+z) = \frac{1}{1+}$, and $(\frac{d}{dz})^k(\frac{1}{1+z}) = \frac{(-1)^k k!}{(1+z)^{k+1}}$. One sees that the radius of convergence is 1; for $\lim_{k\to\infty}(\frac{1}{k})^{1/k} = 1$.

Also,

$$\begin{aligned}
\arctan(z) &= \int_0^z \frac{dw}{1+w^2} \\
&= \int_0^z \sum_{j=0}^{n} (-1)^j w^{2j} dw \\
&= \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{2j+1}
\end{aligned}$$

so long as $|z| < 1$.

What property characteries the Taylor polynomial of order n for a function $f(z)$? Any polynomial $p(z)$ with the property that $f(z) - p(z)$ has a zero of order n at $z = 0$ agrees with the Taylor polynomial in its order n subpart. We would say that $f(z) = p(z) + O(z^{n+1})$.[10]

Suppose that $f(z) = P_n(z) + O(z^{n+1})$, and $g(z) = Q_n(z) + O(z^{n+1})$. Then $fg = P_n \cdot Q_n + O(z^{n+1})$; extract the nth order piece of $P_n \cdot Q_n$, and it's the nth order Taylor polynomial of fg.

What about f/g? Compute $P_n/Q_n = R_n + O(z^{n+1})$, using the Euclidean algorithm. Then $P_n = R_n Q_n + O(z^{n+1})$. So $f + O(z^{n+1}) = R_n g + O(z^{n+1})$. The conclusion is that $f/g = R_n + O(z^{n+1})/g$. All of this works only if $g(0) \neq 0$. Then $f/g = R_n + O(z^{n+1})$.

[10] "Tendentious."

Now, composition. Let $f(w) = a_0 + a_1 w + a_2 w^2 + \cdots$, and $g(z) = b_1 z + b_2 z^2 + \cdots$.[11] What is the n^{th} order Taylor polynomial of $f(g(z))$? Assume $f(w) = P_n(w) + O(w^{n+1})$, and $g(z) = Q_n(z) + O(z^{n+1})$. Compute

$$\begin{aligned}
f(g(z)) &= f(Q_n(z)) + O(z^{n+1}) \\
&= P_n(Q_n(Z) + O(z^{n+1})) + O(((Q)n(z) + O(z^{n+1})))^{n+1}) \\
&= P_n(Qn(z) + O(z^{n+1})) + O(z^{n+1}) \\
&= P_n(Q_n(z)) + P_n(Q_n(z) + O(z^{n+1})) - P_n(Q_n(z)) \\
&= P_n(Q_n(z)) + \int_0^{O(z^{n+1})} P_n'(Q_n(z) + t)dt \\
&= P_n(Q_n(zz)) + O(z^{n+1}).
\end{aligned}$$

Suppose that $f(0) = 0$ and $f'(0) \neq 0$, so there's a function $g(z)$ analytic near $z = 0$ so that $f(g(z)) = z$. As always, we assume that $g(z) = a_1 z + \cdots + O(z^{n+1}) = Q_n(z) + O(z^{n+1})$. What we want is to construct a polynomial $P_n(w)$ of degree n such that $P_n(Q_n(z)) = z + O(z^{n+1})$. Observe that $P_n(g(z) + O(z^{n+1})) = P_n(Q_n(Z)) = z + O(z^{n+1})$. Remix: $P_n(g(z)) = P_n(Q_n(z)) + O(z^{n+1})) = P_n(Q_n(z)) + O(z^{n+1}) = z + O(z^{n+1})$.

Well, it's pretty clear that $P_1(z) = \frac{z}{b_1}$; for then $P_1(Q_1(z)) = z + O(z^2)$. Now, we assume that $P_{n-1}(w)$ is found so that $P_{n-1}(Q_{n-1}(z)) = z + O(z^n)$. And so we need to find c_n so that $P_n(w) = P_{n-1}(w) + c_n w^n$. Look at

$$\begin{aligned}
P_n(Q_n(z)) &= P_{n-1}(Q_n(z)) + c_n(Q_n(z))^n \\
&= P_{n-1}(Q_n(z)) + c_n(b_1 z)^n + O(z^{n+1}) \\
&= P_{n-1}(Q_{n-1} + b_n^n) + c_n(b_1 z)^n + O(z^{n+1}) \\
&= P_{n-1}(Q_{n-1}(z)) + P_{n-1}'(Q_{n-1}(z)) b_n z^n + c_n(b_1 z)^n + O(z^{n+1}) \\
&= \widetilde{P_{n-1}}(z) + d_n z^n + c_n b_1^n z^n + O(z^{n+1}).
\end{aligned}$$

We can set $c_n = -\frac{d_n}{b_1^n}$, and this gives $P_n(z)$ as desired.

Conformality A curve which is locally parameterized by a C^1 function $z(t)$ so that $z'(t) \neq 0$ is called regular; it has a tangent vector $\frac{dz}{dt}$ If ζ is some other curve, we can look at the

[11] We've normalized so that $g(0) = 0$.

angle of the intersection by the angle between $\frac{dz}{dt}$ and $\frac{d\zeta}{dt}$. If F is a nice map, you might not expect the angle of $F(\zeta)$ and $F(z)$ to relate to the original angle.

If $F = (u(x,y), v(x,y))$, then the derivative is

$$\begin{aligned}
\frac{d}{dt}F(z(t)) &= \left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right) \\
&= \begin{pmatrix} \frac{\partial u}{\partial x}(x(t),y(t)) & \frac{\partial u}{\partial y}(x(t),y(t)) \\ \frac{\partial v}{\partial x}(x(t),y(t)) & \frac{\partial v}{\partial y}(x(t),y(t)) \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} \\
&= F_* \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}
\end{aligned}$$

Now, to compute the angle between w and v we take $\langle w, v\rangle = \cos(\theta)/\|w\|\,\|v\|$, more or less; up to reflection around multiple of π.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. When is it true that for all pairs v and w, we have

$$\frac{\langle Av, Aw\rangle \|Av\|}{\|Aw\|} = \frac{\langle v, w\rangle \|v\|}{\|w\|}?$$[12]

If A is orthogonal i.e., $A^tA = I$, then $\langle Av, w\rangle = \langle A^tAv, w\rangle = \langle v, w\rangle$. It's an exercise to show that $A = \lambda\widetilde{A}$ where $\widetilde{A} \in O(2)$, the set of orthogonal 2×2 matrices.

If F is analytic, that means that $u_x = v_y$ and $u_y = -v_x$. So $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}$. Multiply this by the transpose, and get $\begin{pmatrix} u_x^2 + u_y^2 & 0 \\ 0 & u_x^2 + u_y^2 \end{pmatrix}$.

From all of this, we conclude that if F is analytic, then the mapping F_* on tangent vectors preserves the angles between curves.

Remix. Let $w(t) = F(z(t))$. Then $w'(t) = F'(z(t))z'(t)$. So $\arg w'(t) = \arg F'(z(t)) + \arg z'(t)$. So the difference of the arguments is maintained. Note that $F'(z(t))$ doesn't depend on the tangent vector.

We also note that $|w'(t)| = |F'(z(t)||z'(t)|$; it scales equally in both directions. This is another way of showing that F acts as a rotation and a scaling (equal in both directions).

A mapping $F : U \to V$ where U and V are open subsets of $\mathbb{R}^2$ is called conformal if

[12] May not have formula correct.

1. F_* preserves angles between tangent vectors, or
2. $\|F_*(z)v\| = \lambda \|v\|$ for some λ independent of v.

Suppose $w(t) = F(z(t), \overline{z(t)})$. Then $w'(t) = F_z z'(t) + F_{\overline{z}} \overline{z}'(t)$. So $\arg \frac{w'(t)}{z'(t)}$ is independent of $z'(t)$. But this argument is $\arg(F_z + F_{\overline{z}} \frac{\overline{z}'(t)}{z'(t)})$. A little futzing shows that $F_{\overline{z}} = 0$ is necessary for $\arg \frac{w'}{z'}$ to be independent of z'.

We have $\frac{w'}{z'} = F_z + F_{\overline{z}} \frac{\overline{z}'}{z'}$, and the norm of each side is independent of z'. So either $F_z \equiv 0$ or $F_{\overline{z}} \equiv 0$; F is either analytic or conjugate analytic [$F = \overline{f(z)}$].

This definition of analyticity does not generalize to higher dimensions. We could define holomorphic functions to be maps $f : U \to V$ so that f is conformal.

If we have a curve $z(t) = x(t) + iy(t)$ for $t \in [a, b]$, then the length of this curve is $\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt = \int_a^b |z'(t)| dt$. If we let $w(t) = f(z(t))$, then the length of w is $\int_a^b |w'(t)| dt = \int_a^b |f'(z(t))||z'(t)| dt$.

If we have a domain E, we can compute its area $\int\int_E dxdy$. Suppose $f : E \to E'$, $f = u + iv$. Then the area of E' is $\int\int_E dxdy = \int\int_E \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} dxdy = \int\int_E \det \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix} dxdy = \int\int_E |f'(z)|^2 dxdy$. Note that the area is positive; the orientation is preserved.

Fact Suppose $U, V \subset \mathbb{R}^3$, and $f : U \to V$ is conformal. Then there's a unique mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ so that $F|_U = f$, and the collection of all such conformal mappings of $\mathbb{R}^3$ to $\mathbb{R}^3$ is finite dimensional.

Fractional Linear Transformations [Or, alternatively, Möbius transformations.] The question is, what are the conformal maps S defined on $\hat{\mathbb{C}}$ which are 1-1 and onto $\hat{\mathbb{C}}$?

For example, if $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, then f is such a map.

These maps are called the conformal automorphisms of $\hat{\mathbb{C}}$. If we have two such transformations, we can compose them to get another map $f_1 \circ f_2$ which is a conformal automorphism of $\hat{\mathbb{C}}$. Lurking in the background is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$. Suppose we consider $(z_1, z_2) \in \mathbb{C}^2$. Look at the map $\mathbb{C}^2 - \{z_2 = 0\} \to \mathbb{C}$ via $(z_1, z_2) \mapsto \frac{z_1}{z_2}$. Set $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. We have a commutative map

$$\begin{array}{ccc} \mathbb{C}^2 & \to & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \mathbb{C} & \to & \mathbb{C} \end{array}$$

Note that $\frac{az_1+bz_2}{cz_2+d_2} = \frac{a(z_1/z_2)+b}{c(z_1/z_2)+d}$. Suppose we have $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

$$\begin{array}{ccccc} \mathbb{C}^2 & \overset{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\to} & \mathbb{C}^2 & \overset{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}{\to} & \mathbb{C}^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & \overset{\frac{az+b}{cz+d}}{\to} & \mathbb{C} & \overset{\frac{\alpha z+\beta}{\gamma z+\delta}}{\to} & \mathbb{C} \end{array}$$

and everything in sight commutes. The composition on the top row is simply $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Define $f_A(z) = \frac{az+b}{cz+d}$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We've shown that $f_{A \cdot B}(z) = f_A(f_B(z))$. This shows that the correspondence $A \mapsto f_A$ is a representation.

For starters, we'll talk about $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ [a translation], $\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$ a homothety, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ an inversion. They give, respectively, $z \mapsto z + \alpha$, $z \mapsto k^2 z$, and $z \mapsto \frac{1}{z}$.

Now, $\frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+d/c)} + \frac{a}{c}$. This is like doing

$$z \mapsto z + \frac{d}{c} \mapsto \frac{1}{z+\frac{d}{c}} \mapsto \frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}} \mapsto \frac{bc-ad}{c^2(z+d/c)} + \frac{a}{c}.$$

Thus, all of these maps are obtained through translation, homothety and inversion. The inversion map is *not* $\frac{1}{\bar{z}}$.

We actually have $f_A : SL_2(\mathbb{C}) \to \mathbf{M\ddot{o}b}(\mathbb{C})$, and $f_A = \mathrm{id} \iff A = \pm I$. By the way, $f_{A^{-1}} = f_A^{-1}$.

Given a set $Z = \{z_1, \cdots, z_n\}$ and $W = \{w_1, \cdots, w_n\}$, is there a fractional linear transformation $S : Z \to W$?

Suppose we have $\{z_2, z_3, z_4\}$ which we want to send to $\{1, 0, \infty\}$. Set $S(z) = \frac{z-z_3}{z-z_4}\frac{z_2-z_4}{z_2-z_3}$. This will indeed work. Obviously, we can use this to map any three points to any other three points.

Suppose $S_1, S_2 : \{z_2, z_3, z_4\} \to \{1, 0, \infty\}$. Then $T : S_1 \circ S_2^{-1} : \{1, 0, \infty\} \to \{1, 0, \infty\}$; there are three fixed points. We have $\frac{az+b}{cz+d}$. But 0 is fixed, so we have $\frac{az}{cz+d}$. Furthermore, $T(\infty) = \infty$; we have $\frac{az}{d}$. Finall, $T(1) = 1$, so $a = d$. There is therefore no nontrivial of $\mathbf{M\ddot{o}b}(\mathbb{C})$ with three fixed points.

A fractional linear transformation carries circles and lines to circles and lines. We're really working on the Riemann sphere, in which case it looks like circles to circles. It suffices to show that each of the three basic transforms takes circles to circles. Let $|-a| = r$, $w = 1/z$.

- $|\alpha - (a + \alpha)| = r$.
- $|k^2 z - k^2 a| = |k|^2 r$.

inversion $|1/w - a| = r$, so $|1/a - w| = r|w|/|a|$. Finally, $w\overline{w} = 2/\overline{a} - \overline{w}/a + 1/|a|^2 = r^2/|a|^2 w\overline{w}$. So $r^2/|a|^2 = z\Re\frac{w}{\overline{a}} = \frac{1}{|a|^2}$.

Suppose $\frac{r^2}{|a|^2} > 1$. Then we have $(\frac{r^2}{|a|^2-1})w\overline{w} + \frac{w}{\overline{a}} + \frac{\overline{w}}{a} - \frac{1}{|a|^2} = 0$. Let $\rho = r$. We eventually have $\left|w + \frac{1}{\rho a}\right|^2 = \frac{2}{\rho|a|^2} = \frac{z}{r^2-|a|^2}$. So it's a circle of center $(-1/\rho a)$ and radius $\sqrt{\frac{2}{r^2-|a|^2}}$.

Exercise; work on the case $r^2 < |a|^2$.

Now, we haven't yet said anything about lines. But a line is a limit of circles. ◇

Let's go back to $S(z) = \frac{z-z_3}{z-zz_4}\frac{z_2-z_3}{z_2-z_4}$. Define $S(z_1) = [z_2, z_2, z_3, z_4]$ the cross ratio.

Proposition $[z_1, z_2, z_3, z_4]$ is real $\iff$ $\{z_1, \cdots, z_4\}$ lie on a circle.

Lemma If $T \in \mathbf{M\ddot{o}b}(\mathbb{C})$, then $[z_1, z_2, z_3, z_4] = [Tz_1, z_2, Tz_3, Tz_4]$.

Proof $S(z) = [z, z_2, z_3, z_4]$ is the Möbius transformation which takes $\{z_2, z_3, z_4\}$ to $\{1, 0, \infty\}$. Consider $\widetilde{S}(z) = [z, Tz_2, Tz_3, Tz_4]$. This takes $\{Tz_2, Tz_3, Tz_4\}$ to $\{1, 0, \infty\}$. If we look at $S \circ T^{-1}$, it also takes $\{Tz_2, Tz_3, Tz_4\}$ to $\{1, 0, \infty\}$. We have $[Tz_1, Tz_2, Tz_3, Tz_4] = ST^{-1}(Tz_1) = S(z_1)$.$\diamond$

It's not hard to see that if $f(z_1, z_2, z_3) = f(Tz_1, Tz_2, T_3)$ for all $T \in \mathbf{M\ddot{o}b}(\mathbb{C})$, then f is a constant.

One can also prove a converse; there's a $T \in \mathbf{M\ddot{o}b}(\mathbb{C})$ that carries $\{z_1, z_2, z_3, z_4\}$ to $\{w_1, w_2, w_3, W_4\}$ $\iff$ $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4] = \kappa$.

Consider $S_1(z) = [z_1, z_2, z_3, z_4]$ and $S_2(z) = [z_1, w_2, w_3, w_4]$. Then $S_1\{z_1, z_2, z_4, z_4\} \to \{\kappa, 1, 0, \infty\}$, and $S_2 : \{w_1, w_{2,3}, w_4\} \to \{\kappa, 1, 0, \infty\}$. $\diamond$

This tells us that if $f(Tz_1, Tz_2, Tz_3, Tz_4) = f(z_1, z_2, z_3, z_4)$, then $f = \psi([z_1, z_2, z_3, z_4])$; it's a function of the cross ratio.

Proof [of proposition] Let $\{z_1, z_2, z_3, z_4\}$ lie on a circle C. Then $S(z)$ carries C to $\mathbb{R}$. So $S(z_1) \in \mathbb{R}$, as well. Conversely, if the cross ratio is real, then $S(z_1) \in \mathbb{R}$, which is the image of the circle C. $\Rightarrow z_1 \in C$.$\diamond$

From now on, we're working with $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm 1\}$. We'd like to know what algebraic properties are invariant under conjugation. Certainly, the determinant and trace work; $\det(BAB^{-1}) = \det(A)$, and $\operatorname{tr}(BAB^{-1}) = \operatorname{tr}(A)$.

Given A and C in $SL_2(\mathbb{C})$, when is there a B so that $BAB^{-1} = C$? The eigenvalues are the roots of $\det(A - \lambda I) = 0$, that is, [in the dimension two case] $\det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + 1 = 0$. The roots are

$$\lambda_\pm = \frac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr} A)^2 - 4}}{2}.$$

If $(\operatorname{tr}(A))^2 \neq 4$, then $\operatorname{tr} A = \operatorname{tr} C$ is the necesary and sufficient condition for there to exist B so that $BAB^{-1} = C$.

What if $(\operatorname{tr}(A))^2 = 4$? Then the eigenvalues are either $\{1, 1\}$ or $\{-1, -1\}$. So either $A = \pm I$ or $A \sim B \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} B^{-1}$.

Let's consider fixed points. Look at solutions of $\frac{az+b}{cz+d} = z$. Then

$$\begin{aligned} az+b &= cz^2+dz \\ cz^2+(d-a)z-b &= 0. \end{aligned}$$

We've ignored $c=0$; in which case the transformation is $\frac{az+b}{d}$, and so we get $\frac{a}{d}z+\frac{b}{d}=z$ which has one soluiton, and another at infinity; unless $a/d =$, in which case we have $z \mapsto z+b/d$ which has one fixed point at ∞. If $a/d \neq 1$, there's one finite fixed point and one at ∞.

Otherwise, we can solve the damned equation, and get

$$\begin{aligned} z &= \frac{(a-d) \pm \sqrt{(d-a)^2+4bc}}{2c} \\ &= \frac{(a-d) \pm \sqrt{(d^2-2ad+a^2+4bc)}}{2c} \\ &= \frac{(a-d) \pm \sqrt{(a+d)^2-4}}{2c}. \end{aligned}$$

So if $(a+d)^2=4$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has one fixed point. Otherwise, it has two distinct fixed points.

A homothety has fixed points $\{0,\infty\}$, and a translation has $\{\infty\}$. For the moment, we'll ignore the inversion map [fixed points $z=\pm 1$].

Consider the lines through the origin. They're circles through 0 and ∞. A homothety must therefore take such a circle to another circle through zero and ∞. And in fact we see that it must act as a (constant) rotation. For we're dealing with a conformal map.

Consider the orthogonal circles, centered at the origin. They are mapped to other such circles, as well.

Now, suppose that $S(z)=\frac{az+b}{cz+d}$ has two fixed points, α and β¿ Let's choose an element $T \in \mathbf{M\ddot{o}b}(\mathbb{C})$ so that $T(\alpha)=0$, $T(\beta 0=\infty$. Look at the transformation $T \circ S \circ T^{-1}$. This transformation fixes 0 and ∞.

We have, as always, $f(z) = \frac{az+b}{cz+d}$. The fixed points turn out to be

$$z = \frac{(d-a) \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

If $(a+d)^2 = 4$, it's called a parabolic transformation, and there's one fixed point. If $(a+d)^2 \neq 4$, then there are two fixed points.

Let F_A be the fixed points of the transformation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. If we conjugate, what is $F_{BAB^{-1}}$? It's simply BF_A. For if $Az_1 = z_1$ and $Az_2 = z_2$, then $BAB^{-1}(Bz_1) = BA_1 = Bz_1$; and similarly for z_2.[13] Note that $\operatorname{tr} A = (a+d)$; and actually, $\operatorname{tr} BAB^{-1} = \operatorname{tr} A$. So conjugating changes the fixed points, but it doesn't change the number of fixed points.

Now, suppose that $Az_1 = z_1$ is the unique fixed point. Suppose $z_1 = \infty$. Then $f(z) = az+b$. But $az + b = z$ has another solution if $a \neq 1$, namely, $z = \frac{b}{a-1}$. So $a = 1$ if f has a unique fixed point; $f(z) = z + b$, and the matrix looks like $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Furthermore, if $b \neq 0$, then $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This thing maps horizontal lines into themselves, and vertical lines to vertical lines one unit over. (These are circles and orthogonal circles.) If we move the fixed point to some finite place, then the fixed circles are the ones tangent to some line through that point.

Oh, shit. Major picture action. *Must* get from Scott.

We label one class of circles horizontal, and the other one vertical. Basically, the transformation preserves the horizontal circles, and acts as some rotation.

Now, suppose that there are two fixed points. Through conjugation, we can take them to be zero and ∞. Then the fixed circles are circles around the origin, and the orthogonal circles are lines through the origin. The most general such matrix is $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. There are two possibilities:

1. $\operatorname{tr} A = \lambda + \lambda^{-1} \in \mathbb{R}$ and $|\operatorname{tr} A| < 2, \iff \lambda = e^{i\theta}$. This is just rotation by θ. These are called elliptic

2. $\operatorname{tr} A \neq \pm 2$ or as above. This is called a loxodromic tansformation. Then $|\lambda| \neq 1$. If $\lambda \in \mathbb{R}$ then we just have a scaling. This is called a hyperbolic transformation. If $\lambda = \rho e^{i\theta}$, then $A(z) = \rho^2 e^{i\theta} z$. These are called loxodromic.

[13] Remember that last time we proved that you can view the composition as either composition of fractional linear transformations or as multiplication of matrices.

What's weird about all of this is that it's determined by the trace. We have $\lambda^2 - \lambda \operatorname{tr} A + 1 = 0$, so

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{\operatorname{tr} A^2 - 4}}{2}.$$

So (up to a sign) λ is determined by the trace.

"Most" transformations are loxodromic.

Proposition If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is 1-1 onto and conformal everywhere, then $f(z) = \frac{az+b}{cz+d}$.

Proof [14] Suppose $f(\infty) = \infty$. Then $f(z) = a_n z^n + \cdots$. Then $g(z) = \frac{1}{f(\frac{1}{z})} = \alpha_n^n + O(z^{n+1})$. Now, f is 1-1 near ∞, so $n = 1$. Therefore, $f(z) = az + O(1)$ near ∞. Consider the function $\frac{f(z)=f(0)}{z}$. This function has a removable singularity at 0, and is analytic in the finite plane. Furthermore we know that $\left|\frac{f(z)-f(0)}{z}\right| \leq a + O(\frac{1}{|z|})$. By Liouville's theorem, $\frac{f(z)=f(0)}{z} = a$ a constant, or $f(z) = az + f(0)$. So far, we know that the theorem is true for functions which take ∞ to ∞.

Now, suppose $f(\infty) = w_0 \neq \infty$. Take $g(z) = \frac{1}{f(z)-w_0} = f \circ (w \mapsto \frac{1}{w-w_0})$. Then g is a 1-1 onto map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $g(\infty) = \infty$. Thus, $g(z) = az + b$, and $\frac{1}{f(z)-w_0} = az + b$, or $f(z) = \frac{1}{az+b} + w_0 = \frac{aw_0 z + bw_0}{az+b}$. $\diamond$

Proposition If $f : \mathbb{C} \to \mathbb{C}$ is 1-1 onto and conformal everywhere then $f(z) = az + b$.

Proof Consider $g(w) = f(\frac{1}{w})$. This is analytic in $D_1(0) - \{0\}$ a punctured disk. If $\lim_{w\to 0} g(w) = \infty$, then we are reduced to the previous case. For g has a pole, $g(w) = \frac{a_n}{w^n} + \frac{a_{n-1}}{w^{n-1}} + \cdots$. Since g is 1-1 in a neighborhood of 0, it maps some neighborhood of 0 onto the exterior of some disk. Due to some reasoning which I tuned out, we know that $n = 1$.

Suppose that $\lim_{w\to 0} g(w)$ does not exist. Then it follows from the Casorati-Weierstrass theorem that we can find a sequence $w_n \to 0$ so that $g(w_n) \to 1$. There is some point w^* so that $g(w^*) = 1$, a contradiction. For the local mapping theorem asserts that for a $\delta > 0$ there's an $\epsilon > 0$ so that $g(D_\epsilon(w^*)) \supset D_\delta(1)$.[15] Choose ϵ small enough that $0 \notin \overline{D_\epsilon(w^*)}$. Then for large n, $w_n \notin D_\epsilon(w^*)$. And for really large n, $g(w_n) \in D_\delta(1)$, contradicting the fact that f and g are 1-1. And it turns out that this also covers the case where $\lim_{w\to 0}(w)$ is finite. $\diamond$

[14] Recall Liouville's theorem, that a bounded holomorphic function in $\mathbb{C}$ is constant.

[15] This is also called the open mapping theorem.

Note that this argument used compactification; we threw the point at infinity back in.

What are the 1-1 onto conformal maps of $D_1(0)$? We can't use Liouville's theorem, since we don't yet know that the map extends to all of $\mathbb{C}$. Instead we use

Schwartz's Lemma If $f : D_1 \to D_1$ and $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality holds $\iff f(z) = e^{i\theta}z$.

So now suppose we've got $f : D_1 \to D_1$ is 1-1 and onto. The maximum principle implies that $|f(z)| < 1$ if $|z| < 1$. In particular , $|f(0)| < 1$. Define $g(z) = \frac{f(z)-f(0)}{1-f(z)\overline{f(0)}}$.

From the homework, we know that a degree one rational function $R(z)$ with $|R(z)| = 1$ if $|z| = 1$ is of the form $R(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$ where $|a| < 1$. So $g : D_1 \to D_1$ is 1-1 and onto, and $g(0) = 0$. Schwartz's Lemma says that $|g'(0)| \leq 1$, and if $|g'(0)| = 1$ then $g(z) = e^{i\theta}z$. Now, $g^{-1}(z) : D_1 \to D_1$ is also 1-1 onto and conformal. $(g^{-1})'(0)) = \frac{1}{g'(0)}$. But both g and g'^{-1} must satisfy the conclusion of Schwartz, so $|g'(0)| = |(g^{-1})'(0)| = 1$, and g is a rotation. Therefore, $\frac{f(z)-f(0)}{1-f(z)\overline{f(0)}} = e^{i\theta}z$. Solve for f, and we get[16]

$$f(z) = \frac{e^{i\theta}z + f(0)}{1 + \overline{f(0)}e^{i\theta}z}.$$

$\diamond$

We've got a homework problem to show that all 1-1 onto conformal maps from $\Im > 0$ to $\Im z > 0$ are of the form $z \mapsto \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

Hyperbolic geometry The points are points, and the lines are circles which meet the real axis at right angles.

The parallel postulate doesn't hold for this geometry.

Suppose that z_1 and z_2 are two points in $\mathbb{H} = \{z : \Im z > 0\}$, and w_1, w_2 are two other points in $\mathbb{H}$. Is there a transformation $T \in SL_2(\mathbb{R})$ so that $Tz_i = w_i$?

Now, there is a line (looks like a semicircle) connecting z_1 and z_2. Call its endpoints [on the real line] x_1 and x_2. The analogous points for w_i are y_i. We know that T must carry the line x_i onto y_i. This condition may be expressed with the cross ratio:

$$[x_1, z_1, z_2, x_2] = [y_1, w_1, w_2, y_2]^{\pm 1}.$$

[16]Maybe; I made a typo.

Of course, both cross ratios must be real. Anyways, this is a necessary condition for T to exist. It turns out that it's sufficient, as well. We use the same trick as always; use $A = \frac{z-x_1}{z-x_2}$. It carries the first line to the imaginary line. Then Az_i are hanging out somewhere along the imaginary line. If you normalize [what?] correctly, the cross ratio is $\frac{Az_1}{Az_2} = \frac{A'w_2}{A'w_1}$ where $A'z = \frac{z-y_1}{z-y_2}$. This says that $\frac{Az_2}{A'w_2} = \frac{Az_1}{A'w_1} = \lambda \in \mathbb{R}$. From this, we conclude that $Az_i = \lambda A'w_i$. Compose with A^{-1}, and we're done.

Now, we can clean up to show that there's a line [circle] between any two points. Consider the circles through a point orthogonal to a given circle. *Blah. I prefer a more concrete approach.*

We're on hyperbolic geometry. Recall that a line is a semicircle sitting on the real axis. There are a couple notions of parallel lines; ultraparallel and otherwise.

There are various models for the hyperbolic plane. One is the upper half-plane, $\mathbb{H}$; and the other is the unit disk.

Last time, we showed that if you have a pair of pairs of points, then there's a transformation of $\mathbb{H}$ carrying one pair to the other $\iff$ $[x_1, z_1, z_2, x_2] = [y_1, w_1, w_2, y_2]^{\pm 1}$. For pairs of points on the imaginary axis, this is obvious; use $z \mapsto \lambda^2 z$.

A triangle is the interior of any three lines, more or less. It makes sense to talk about the interior angles; look at the tangents to the circles about the points of intersections.

Theorem If T_1 and T_2 are two triangles with angles $\alpha_i, \beta_i, \gamma_i$, then there's a transformation $A \in \mathrm{Aut}(\mathbb{H}^2) \iff (\alpha_1, \beta_1, \gamma_1)$ is some cyclic permutation of $(\alpha_2, \beta_2, \gamma_2)$.

($\Rightarrow$) Trivial, since A is conformal.

($\Leftarrow$) We won't actually go all the way through with this. This is a little easier with the unit disk model of the hyperbolic plane. It's a highly visual proof, and I think I'm going to blow it off.

In general, the sum of the interior angles of a hyperbolic triangle is strictly less than π radians. And indeed, for any $\alpha + \beta + \gamma$, one can find a triangle wih those angles, provided α, β, γ non-negative. Note that it's possible to have a hyperbolic triangles with a zero interior angle. In particular, there's a triangle all of whose angles are zero.

Theorem All triangles with all angles equal to zero are equivalent.

Proof Highly visual; bummer! Start off with one triangle having vertices $(0, 1, \infty)$. If the other triangle is (a, b, ∞), then the proposition is a triviality; use $z \mapsto \frac{z-a}{z-b}$. If there's no vertex at ∞ we have (x_1, x_2, x_3). So we just have to map those onto $(0, 1, \infty)$. But we already know that $z \mapsto \frac{z-x_1}{z-x_3}\frac{x_2-x_3}{x_2-x_1}$ is an element of $\mathrm{Aut}(\mathbb{H}^2)$ which carries $(x_1, x_2, x_3) \to (0, 1, \infty)$.

Back to analysis. Recall Schwartz's lemma; if $f : D_1 \to D_1$ with $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$; and equality holds $\iff$ $f(z) = e^{i\theta z}$.

What if $f(0) = w_0 \neq 0$? We can take $g(z) = \frac{f(z)-w_0}{1-\overline{w_0}f(z)}$ in order to use the Schwartz. This tells us that $\left|\frac{f(z)-w_0}{1-\overline{w_0}f(z)}\right| \leq |z|$.

Now, suppose that $f(z_0) = w_0$. Take some Möbius transformation carrying z_0 to 0, say $\zeta \to \frac{\zeta+z_0}{1+\overline{z_0}\zeta} =$. Then $g(\zeta) = f(\frac{\zeta+z_0}{1+\overline{z_0}\zeta})$ satisfies $g(0) = w_0$. The conclusion is that $\left|\frac{g(\zeta)-w_0}{1+\overline{w_0}g(\zeta)}\right| \leq |\zeta|$, and

$$\left|\frac{f(\frac{\zeta+z+-}{1-\zeta\overline{z_0}})-w_0}{1-\overline{w_0}f(\frac{\zeta+z_0}{1-\zeta\overline{z_0}})}\right|\leq|\zeta|.$$

Ultimately, we obtain

$$\left|\frac{f(z)-w_0}{1-\overline{w_0}f(z)}\right|\leq\left|\frac{z-z_0}{1-\overline{z_0}z}\right|.$$

If we have equality at any point, then $\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}}f(z)=e^{i\theta}(\frac{z-z_0}{1-\overline{z_0}z})$. If you lean on this for a while, you find out that equality means that you're working with a Möbius transformation.

We'll divide through, and get $\left|\frac{f(z)-f(z_0)}{z-z_0}\right|\frac{1}{|1-f(z)|\overline{f(z_0)}}\leq\frac{1}{|1-z\overline{z_0}|}$. Letting $z_0\to 0$, we conclude that $\frac{f'(z)^2}{1-|f(z_0)|^2}\leq\frac{1}{1-|z_0|^2}$. This is true for any $z_0\in\mathcal{D}_1$.

If $f:\mathcal{D}_1\to\mathcal{D}_1$ is a Möbius transformation, then actually $\frac{|f'(z)|}{1-|f(z)|^2}=\frac{1}{1-|z|^2}$ for all $z\in\mathcal{D}_1$.

Let γ be a smooth curve in $\overset{\circ}{\mathcal{D}}_1$. Define the length of the curve by $L(\gamma)=\int_\gamma\frac{|dz|}{1-|z|^2}$. If $\widetilde{\gamma}=f(\gamma)$, then we can compute $L(\widetilde{\gamma})=\int_\gamma\frac{|f'(z)||dz|}{1-|f(z)|^2}\leq\int_\gamma\frac{|dz|}{1-|z|^2}$. This tells us that f is a contraction, when distance is measured this way. If $f\in\textbf{Möb}$ then equality holds. Since we have a way to measure $L(\gamma)$ for γ any piecewise smooth curve, it follows that we can define a distance $d_{\mathbb{H}^2}(p,q)=\inf_{\gamma:\gamma_0=p,\gamma_1=q}L(\gamma)$.

Theorem If $p,q\in\mathcal{D}_1$, then $d_{\mathbb{H}^2}(p,q)=L(C_{pq})$, where C_{pq} is the arc of the circle orthogonal to the boundary of the unit disk which passes through p and q.

Proof Picture time. We've shown that if γ is a curve and $f\in\textbf{Möb}$, then $L(\gamma)=L(f(\gamma))$. Choose f with $f(p)=0$. So it suffices to show that $d_{\mathbb{H}^2}(f(p),f(q))=\int_{l_{f(p),f(q)}}\frac{|dz|}{1-|z|^2}$. Write $\gamma=r(t)e^{i\theta(t)}$. There is no loss of generality in assuming that γ does not pass [again] through the origin; for that would just make the length longer.

Now, $\dot{z}=\dot{r}e^{i\theta}+i\dot{\theta}re^{i\theta}$. So $|\dot{z}|=(\dot{r}^2+\dot{\theta}^2r^2)^{1/2}$. Then

$$L(\gamma)\quad=\quad\int_0^1\frac{\left|\dot{r}^2+r^2\dot{\theta}^2\right|^{1/2}dt}{1-r^2}$$

$$\begin{aligned} &\geq \int_0^1 \frac{|\dot{r}|dt}{1-r^2} \\ &\geq= \sum_{i=0}^{N} \int_{t_{i-1}}^{t_i} \frac{|\dot{r}|dt}{1-r^2} \end{aligned}$$

where $r(t)$ is increasing on each of the intervals $[t_{I-1}, t_i]$, and $[0,1] = \cup_{i=0}^{N} [r_{t_{i-1}}, r_{t_i}]$. This is basically just picking out the places where r is monotone.

$$\begin{aligned} L(\gamma) &= \sum_{i=0}^{N} \int_{t_{i-1}}^{t_i} \frac{\dot{r}(t)}{1-r^2} \\ &= \sum \int_{r(t_{i-1})}^{r(t_i)} \frac{dr}{1-r^2} \\ &= \int_0^1 \frac{dr}{1-r^2} \\ &= \frac{1}{2}\log(\frac{1+|f(q)|}{1-|f(q)|}). \end{aligned}$$

So we now know that $d_{\mathbb{H}^2} = \frac{1}{2}\log(\frac{1+|z|}{1-|z|})$. Otherwise, we can use the transformation $z \mapsto \frac{z-z_1}{1-\overline{z_1}z}$. In general,

Get this from someone else

Note that taking the length to be the infimum sort of builds the triangle inequality right in.

Remember our original inequality, namely, $\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \leq abs\frac{z-_0}{1-\overline{z_0}z}$.

Theorem [Schwarz-Pick] If $f : \mathcal{D}_1 \to \mathcal{D}_1$ holomorphic, then for any pair of points $p, q \in \mathcal{D}_1$ we have $d_{\mathbb{H}^2}(f(p), f(q)) \leq d_{\mathbb{H}^2}(p, q)$. If we have equality for any pair $p \neq q$, then $f \in \mathrm{Aut}(\mathcal{D}_1)$.

Define a map from a semicircle onto a circle. Start off by sending 1 to ∞ and -1 to 0. Try $-\frac{z+1}{z-1}$. Then the semicircular arc connecting -1 and 1 gets sent to the imaginary axis; we have a map from the semicircle to the upper right-hand quadrant.

Now we have to get it back to the disk. Using $z \mapsto z^2$ we can get it to the upper half-plane.

Finally, $z \mapsto \frac{z-i}{+i}$ gets us to the circle. Putting it all together, the composite map is

$$z \mapsto \frac{(\frac{1+z}{1-z})^2 - i}{(\frac{1+z}{1-z})^2 + i}.$$

Neat trick, that. Try it with something else. Let two circles intersect transversely; let's play with $A - (A \cap B)$. Can use a similar method to map that region to a wedge in the upper-right-hand quadrant; and then map onto the upper half-plane, and then back down onto the unit circle.

Suppose we wanted to map the complement of the segment $[-1, 1]$ to the unit disk. The trick here is to map the interval onto the half-line, via $z \mapsto \frac{z-1}{z+1}$. Sends it onto the negative real axis. Then $z \mapsto \sqrt{z}$ sends it to the imaginary axis. Finally, use $\frac{z-1}{z+1}$ to get to the unit circle. All told, the map is

Warning: this may not be quite right.

$$\begin{aligned}
z &\mapsto \frac{\sqrt{\frac{z-1}{z+1}} - 1}{\sqrt{\frac{z-1}{z+1}} + 1} = \zeta \\
\frac{z-1}{z+1} &= \frac{(\zeta+1)^2}{(\zeta-1)^2} \\
(z-1) &= \alpha^2(z+1 \\
\text{blah} & \\
\frac{1(1+\zeta^2)}{-4\zeta} &= -\frac{1}{2}(\zeta + 1/\zeta).
\end{aligned}$$

Suppose f is analytic in a disk D. We know, for γ closed curve in D, that $\int_\gamma f(z)dz = 0$. We defined a winding number; if γ a closed curve and $a \notin \gamma$, then $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$. $n(\gamma, a) \in \mathbb{Z}$, and $n(\gamma, a_1) = n(\gamma, a_2)$ if a_1 and a_2 are in the same connected component of $\mathbb{C} - \gamma$. Finally, $n(\gamma, a) = 0$ if a is in the noncompact [nonbounded?] component of $\mathbb{C} - \gamma$.

Consider the funciton $f(z0 = \frac{1}{z}$. It's analytic on $\mathbb{C}\ \{0\}$. We know that $\int_{|z|=1} f(z)dz = 2\pi i$. So removing just one point was enough to make Cauchy's theorem fail. Maybe we can separate it out; for

$$\int_{|z|=1} f(z)dz - \int_{|z|=\epsilon} f(z)dz = 0.$$

Let C_1 and C_ϵ be the appropriate curves. Define $\int_{C_1 - C_2} f(z)dz \stackrel{\text{def}}{=} \int_{C_1} f(z)dz - \int_{C_\epsilon} f(z)dz$.

Suppose f analytic in a set $\Omega \subset \mathbb{C}$ with Ω open. If $\{\gamma_1, \cdots, \gamma_n\}$ are curve inΩ then we can define the [formal] sum $\gamma_1 + \cdots + \gamma_n$. Also, let $m\gamma_1 = \gamma_1 + \cdots + \gamma_1$, and$-\gamma_1 = \gamma_1$ with the opposite orientation. Remember, a curve is parameterized as $\gamma \leftrightarrow \{z(t) | t \in [0,1]\}$, and $\int_\gamma f(z)dz \stackrel{\text{def}}{=} \int_0^1 f(z(t))z'(t)dt$. Then $-\gamma \leftrightarrow \{z(1-t)\}$.

You can take a closed curve and bust it into arcs; write $\gamma = \gamma_1 + \cdots + \gamma_n$. Then $\int_\gamma f(z)dz = \sum^n \int_{\gamma_i} f(z)dz$. Suppose $\gamma_i = z_i(t)$. Then the sets $\{z_i(0)\}$ and $\{z_i(1)\}$ are in 1-1 correspondence $\iff$ the γ_i piece together to form some closed curve γ. If $\{\gamma_1, \cdots, \gamma_n\}$ a collection of arcs and $\{a_i\} \in \mathbb{Z}$, then $a_1\gamma_1 + \cdots + a_n\gamma_n$ is another formal sum; and $\int_{\sum a_i\gamma_i} f(z)dz = \sum a_i \int_{\gamma_i} f(z)dz$.

Definition

- A chain in Ω is a formal sum of arcs with integer coefficients.
- A cycle is a formal sum of closed curves with integral coefficients.

Fact If $pdx + qdy = F_x dx + F_y dy$, then $\int_C pdx + qdy = 0$ for any cycle $C \in \Omega$.

If $a \notin \gamma_1$ or γ_2, then $n(a, \gamma_1 + \gamma_2) = n(a, \gamma_1) + n(a, \gamma_2)$.

Definition A region $\Omega \subset \mathbb{C}$ is simply connected if $\mathbb{C} - \Omega$ is connected.

Incidentally, the standard definition is Ω is simply connected if every closed curve in Ω can be contracted to a point in Ω.

Theorem A [bounded] set $\Omega \subset \mathbb{C}$ is simply connected $\iff$ $n(\gamma, a) = 0$ for every γ a cycle in Ω and $a \notin \Omega$.

Proof ($\Rightarrow$) Easy. If Ω is simply connected, then $\mathbb{C} - \Omega$ has a single component; so if $\gamma \subset \Omega$ is a cycle, and $a \in \mathbb{C} - \Omega$, then a lies in the unbounded component of $\mathbb{C} - \gamma$. As such, $n(a, \gamma) = 0$.

($\Leftarrow$) Suppose $\mathbb{C} - \Omega \subset A \cup B$ where B is the unbounded component, and A is some other [nonempty] component of $\mathbb{C} - \Omega$. A is closed and bounded, and thus compact; B is closed; and $A \cap B = \emptyset$. So there's a minimum distance between points in A and points in B. Choose $\delta > 0$ so that $d(x, y) > \delta$ if $x \in A$, $y \in B$. Choose a point $a \in A$, and cover the plane with squares of side $\frac{\delta}{\sqrt{2}}$. Call these Q_i, and suppose that a is at the center of Q_0. Let J be the set of indices such that $Q_j \cap A \neq \emptyset$. Let $\gamma = \sum_{j \in J} \partial Q_j$.[17] We'd like to see that $\int_\gamma \frac{dz}{z-a} = \int_{\widetilde{\gamma}} \frac{dz}{z-a}$, where $\widetilde{\gamma} \subset \Omega$.

Suppose an edge of ∂Q_j is not contained in Ω. Then it's hit by another $\partial Q_{j'}$ in the other direction, i.e., with opposite orientation.[18] Let $\widetilde{\gamma}$ be the edges of ∂Q_j that do not meet A. By construction, any Q_j is disjoint from B. So $\widetilde{\gamma} \subset \Omega$, and $\int_\gamma \frac{dz}{z-a} = \int_{\widetilde{\gamma}} \frac{dz}{z-a} = \sum_{j \in J} \int_{\partial Q_j} \frac{dz}{z-a} = \int_{\partial Q_0} \frac{dz}{z-a} = 2\pi i$.

Definition We'll say that a cycle $\gamma \subset \Omega$ is homologous to zero if $n(\gamma, a) = 0$ for all $a \notin \Omega$. Notationally, $\gamma \sim 0$. We'll say that $\gamma_1 \sim \gamma_2$ if $\gamma_1 - \gamma_2 \sim 0$, i.e., $n(\gamma_1, a) = n(\gamma_2, a)$ for all $a \in \mathbb{C} - \Omega$.

Let $\Omega = D(1, 0) - \{\frac{1}{n} | n \in \mathbb{N}\}$. The complement has infinitely many components. Then $\gamma_1 \sim \gamma_2$ if and only if infinitely many conditions are satisfied; we have to pick a test a from each component.

Theorem [Cauchy] If $f(z)$ is analytic in an open set Ω, then $\int_\gamma f(z)dz = 0$ for all cycles $\gamma \subset \Omega$ homologous to zero.

Corollary If Ω is simply connected, then $\int_\gamma f(z)dz = 0$ for all cycles γ.

Proof Let Ω be open and bounded and let γ be a cycle in Ω. We'll let $\{Q_j\}$ be a collection of squares of side ϵ covering the plane. Now define $J_\epsilon = \{j | Q_j \subset \Omega\}$. Since γ is a compact subset

[17] The sum is finite by compactness.
[18] That's assuming it's just on an edge; but if it's a vertex then it's even more true.

of Ω we can choose ϵ sufficiently small so that $\gamma \subset \cup_{j\in J_\epsilon} Q_j = \Omega_\epsilon$.[19] Let $\Gamma_\epsilon = \sum_{j\in J_\epsilon} \partial Q_j$. We choose a point $\zeta_0 \notin \Omega_\epsilon$. Then

$$\int_{\Gamma_\epsilon} \frac{dz}{z-\zeta_0} = \sum_{j\in J_\epsilon} \int_{\partial Q_j} \frac{dz}{z-\zeta_0} = 0.$$

As before, we can select a subset of $\sum_{j\in J_\epsilon} \partial Q_j$ which consists of edges that belong only to a single square in Ω_ϵ. Want to show that $\int_\gamma \frac{dz}{z-\zeta_0} = 0$ for $\zeta_0 \in \mathbb{C} - \Omega_\epsilon = \Omega_\epsilon^C$. This isn't too bad; for such $\zeta_0 \in \Omega_\epsilon^C$ there is some Q_j, $j \notin J_\epsilon$ so that $\zeta_0 \in Q_j$. We know that, for $\zeta \in \Omega^C$, $\int_\gamma \frac{dz}{z-\zeta_0} = 0$. By our choice of ϵ, $Q_j \cap \gamma = \emptyset$. This Q_j lies in a single component of γ^C, and so $\int_\gamma \frac{dz}{z-\zeta_0} = 0$ as well.

Since γ is a compact subset of Ω_ϵ, it follows that $\int_\gamma \frac{dz}{z-\zeta} = 0$ for all $\zeta \in \Gamma_\epsilon$.

At this point, we have a curve $\Gamma_\epsilon \sim \sum_{j\in J_\epsilon} \partial Q_j$ so that if $\zeta \in \Gamma_\epsilon$, then $\int_\gamma \frac{dz}{z-\zeta} = 0$. We can use the Cauchy integal formula as follows:

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(z)dz}{z-\zeta} = \begin{cases} f(\zeta) & \zeta \in Q_j \\ 0 & \zeta \notin Q_j \end{cases} \text{ for } j \in J_\epsilon\ .$$

So

$$\sum_{j\in J_\epsilon} \frac{1}{2\pi i} \int_{\partial_j} \frac{f(z)dz}{z-\zeta} = f(\zeta)$$

for $\zeta \in \Omega_\epsilon - \cup \partial Q_j$. And in fact, this integral is equal to

$$\frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{f(z)dz}{z-\zeta} = f(\zeta)$$

for $\zeta \in \Omega_\epsilon$. Thus, Fubini:

$$\begin{aligned} \int_\gamma f(\zeta)d\zeta &= \int_\gamma \frac{1}{2\pi i}\left(\int_{\gamma_\epsilon} \frac{f(z)dz}{z-\zeta}\right)d\zeta \\ &= \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \int_\gamma \frac{f(z)d\zeta}{z-\zeta} dz \\ &= 0. \end{aligned}$$

[19] Ω_ϵ is some approximation to our region which isn't too complicated.

◇

Now, $n(\gamma,\zeta)=0 \iff \int_\gamma \frac{dz}{z-\zeta}=0$. But we also know that any function can be represented by its boundary values integrated against a particular kernel. Γ_ϵ is a sort of polygonal approximation to γ.

If Ω^C is a finite union $A_1 \cup \cdots \cup A_n$ where A_n is the unbounded component, then the statement that $\gamma \sim 0 \iff \int_\gamma \frac{dz}{z-a_j}=0$ for a_j a point in A_j for each $j=1,\cdots,n-1$; a finite number of conditions. We can show the existence of curves $C_1,\cdots,C_{n-1}$ so that $\int_{C_j}\frac{dz}{z-\zeta}=0$ if $\zeta \in \cup_{i\neq j}A_i$, and $2\pi i$ if $\zeta \in A_j$.

If γ is a cycle, then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\zeta}=p_j$. But

$$\int_{\gamma-\sum_{j=1}^{n-1}p_jC_j}\frac{dz}{z-\zeta}=0$$

for all $\zeta \in \Omega^C$; every cycle γ can be written $\gamma \sim \sum_{j=1}^{n-1}p_jC_j$. The C_j are called a homology basis.

If f is analytic in Ω, we define $P_j=\frac{1}{2\pi i}\int_{C_j}f(z)dz$. The $\{P_j\}$ are [called] the periods of f.

Theorem If f is analytic in Ω, then $f=F'$ for some F analytic in $\Omega \iff P_j=0$ for all j.

To see this, define $F(z)=\int_{z_0}^z f(z)dz$ along any arc; it doesn't make a difference what path we pick.

Last time, we described a class of domains, the simply connected ones; $n(\gamma, a) = 0$ for all $\gamma \subset \Omega$ and $a \notin \Omega$. The basic fact is that if $f(z)$ is analytic in a simply connected set, then $\int_\gamma f(z)dz = 0$ for any γ a cycle in Ω. ($\gamma \sim 0$ in Ω if $n(\gamma, a) = 0$ for all $a \notin \Omega$.)

Corollary If $f(z)$ is defined in all of Ω a simply connected set, and $f(z) \neq 0$ for all $z \in \Omega$, then there's a function $F(z)$ analytic in Ω so that $e^{F(z)} = f(z)$. Loosely, $\log f = F$.

Proof Let $\widetilde{F}(z) = \int_{z_0}^{z} \frac{f'(s)}{f(s)} ds$. $\widetilde{F}(z)$ is clearly analytic wherever it is defined. If γ_1, γ_2 are two paths from z_0 to z in Ω, then $\int_{\gamma_1} \frac{f'(s)}{f(s)} ds - \int_{\gamma_2} \frac{f'(s)}{f(s)} ds = \int_{\gamma_1 - \gamma_2} = \frac{f'(s)}{f(s)} ds = 0$, since $\gamma_1 - \gamma_2$ is a cycle. Consider

$$\begin{aligned}
\frac{\partial}{\partial z} e^{-\widetilde{F}(z)} f(z) &= f' e^{-\widetilde{F}} - f \widetilde{F}' e^{-\widetilde{F}} \\
&= (f' - f \frac{f'}{f}) e^{-\widetilde{F}} \\
&= 0.
\end{aligned}$$

So $f(z)e^{-\widetilde{F}(z)}$ is a constant; but we know $\widetilde{F}(z_0) = 0$. So $f(z)e^{-\widetilde{F}(z)} = f(z_0)$. Let $F(z) = \log f(z_0) + \widetilde{F}(z)$. Then $e^{F(z)} = f(z)$.$\diamond$

If γ is a cycle in Ω homologous to zero, then of course $\int_\gamma \frac{f(z) - f(a)}{z - a} dz = 0$ for any $a \in \Omega - \gamma$. We can rewrite this as

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz = n(\gamma, a) f(a).$$

This is a perfectly general Cauchy integral formula.

There's a picture which I really can't capture; but if γ is chosen appropriately we have $f(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z - \zeta}$; take γ to be the boundary of the whole domain. Note that for a non-simply-connected domain, this isn't the same as integrating around the outside.

For each connected component of the complement, pick a cycle around it with winding number one. These cycles form something called a homology basis.

For each [such] curve C_i we define the period of f around C_i, $P_i = \int_{C_i} f(z) \frac{dz}{2\pi i}$. In general, the P_i won't be zero; it's zero only if f is analytic on the area bounded by C_i.

Question: Is there a function F defined in Ω so that $F' = f$? The answer is yes $\iff$ all the periods of f are zero.

As an example, take a square containing the [integer] points 1, 2, 3, 4 and 5. Define $f(z) = \sum_{j=1}^{5} \frac{1}{z-j}$. Each of the periods is 1. Now, suppose $f(z) = \sum \frac{a_i}{z-i}$; then the periods are a_i. If $f(z) = \sum \frac{a_i}{(z-i)^2}$, then the periods are zero.

Suppose μ is a measure with support on the components of the complement. Then $f(\zeta) = \int\int \frac{d\mu(z,\overline{z})}{z-\zeta}$ is analytic in the complement of the collection of closed, compact sets. Let the components be A_i. Then

$$\begin{aligned} \int_{C_i} f(\zeta)d\zeta &= \int_{C_i} (\int\int \frac{d\mu(z,\overline{z})}{z-\zeta})d\zeta \\ &= \int\int (\int_{C_i} \frac{d\zeta}{z-\zeta})d\mu(z,\overline{z}) \\ &= -2\pi i \int\int_{A_i} d\mu(z,\overline{z}). \end{aligned}$$

If $z \in C_i$, we get something nonzero; and it's zero otherwise.

Whatever. It could happen that $f(z)$ is analytic on $0 < |z-a| < \delta$, e.g. $f(z) = \frac{B_n}{(z-a)^n} + \cdots + \frac{B_1}{z-a} + f^a(z)$ where $f^a(z)$ is analytic in $|z-a| < \delta$. Let $0 < \epsilon < \delta$. Look at

$$\begin{aligned} \int_{|z-a|=\epsilon} \frac{f(z)dz}{2\pi i} &= \frac{1}{2\pi i} \int_{|-a|=\epsilon} \frac{B_n dz}{(z-a)^n} + \cdots + B_1 dz/(z-a) + f^a(z)dz \\ &= B_1. \end{aligned}$$

For a function with an isolated singularity at a, we define the residue of f at a to be the unique complex number r so that $f(z) = \frac{R}{z-a} = F'(z)$ for some analytic function. In the example above, the residue is B_1. In any case we know that $R = \int_{|z-a|=\epsilon} f(z)\frac{dz}{2\pi i}$ for ϵ sufficiently small.

Suppose that $f(z)$ is analytic in a set Ω, and γ is a cycle homologous to zero. Try to integrate $\int_\gamma f(z)dz$. Let $\{a_i\}$ denote the singularities of f. Only finitely many of these singularities can satisfy $n(\gamma, a_i) \neq 0$. Let R_i denote the residue of f at a_i. Let $\{C_i\}$ be small circles around the a_i. If the curve $\gamma \sim 0$ in Ω, then take $\gamma - \sum n(\gamma, a_i)C_i$. This is homologous to zero in $\Omega - \{a_1, \cdots, a_N\}$. We need to know that $\int_{C_i} \frac{dz}{z-a} = 0$ for a in one of these components. We now that

$$\begin{aligned} 0 = \int_{\gamma - \sum n(\gamma,a_i)C_i} f(z)dz &= \int_\gamma f(z)dz - \sum n(\gamma, a_i) \int_{C_i} f(z)dz \\ \int_\gamma f(z)dz &= 2\pi i \sum n(\gamma, a_i) R_i. \end{aligned}$$

That's the residue theorem;

Theorem If $f(z)$ is analytic in $\Omega-\{a_i\}$ where a_i have no point of accumulation in Ω, then for any $\gamma \sim 0$ in Ω we have

$$\int_\gamma \frac{f(z)dz}{2\pi i} = \sum_{i=1}^{\infty} n(\gamma, a_i)R_i$$

where R_i is the residue at a_i.

Note that $f(z)$ need not have nice poles. For example, $f(z) = e^{1/z}$. Then $\int_{|z|=1} f(z)dz = \sum \int_{|z|=1} \frac{1}{n!}(\frac{1}{z})^n dz = 2\pi i$. [We only get a contribution from the term $n = 1$.]

How to compute residues

1. If $f(z)$ has a simple pole at $z = a$, then $\operatorname{res}_{z=a} f = \lim_{z\to a}(z-a)f(z)$.
2. If f has a pole of order n at $z = a$, then we need to compute the first n terms of the Taylor series of $(z-a)^n f(z) = a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + (z-a)^n\widetilde{f}(z)$. Then $f(z) = \frac{a_0}{(z-a)^n} + \cdots + a_{n-1}/(z-a) + \widetilde{f}(z)$.

Consider $\int_\gamma \frac{f(z)dz}{(z-\zeta)2\pi i}$ with $\gamma \sim 0$. The integrand is holomorphic except at $z = \zeta$. So the integral is $\operatorname{res}_{z=\zeta}(\frac{f}{z-\zeta}) = f(\zeta)$.

A couple of weeks ago, we showed that if $f(z)$ is analytic in Ω and $n(\gamma, a) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right.$ for all $a \notin \gamma$ with $\gamma \sim 0$ in Ω. Then $\int_\gamma \frac{f'(z)}{f(z)}\frac{dz}{\pi i}$ is the number of zeros of f so that $n(\gamma, a) \neq 0$.

We'll say that an open D is bounded by a cycle γ if $n(\gamma, a) = \begin{cases} 1 & a \in D \\ 0 & a \notin \overline{D} \end{cases}$ We'll additionally denote $\gamma = \partial D$. *Be careful!* This implies an orientation. The orientation should be such that the domain is always on your left as you walk around the chain. If f is analytic in $\overline{D}$, and f does not vanish on ∂D, then $\int_{\partial D} \frac{f'(z)}{f(z)}dz$ is the number of zeros of f in D.

Suppose that $f(z)$ has poles at $b_1, \cdots, b_m$ in D of order $n_1, \cdots, n_m$. We can choose small circles $C_1, \cdots, C_m$ around the points $b_1, \cdots, b_m$. Consider $\partial D - (C_1 + \cdots + C_m)$. This bounds a domain in which f is analytic. By choosing the circles small enough we can arrange that all the zeros of f in D lie in this domain. By the argument principle,

$$\int_{\gamma-(C_1+\cdots+C_m)} \frac{f'(z)dz}{f(z)2\pi}$$

is the number of zeros of f in D. Can we evaluate $\int_{C_i} \frac{f'}{f} dz$? Sure! $f(z) = \frac{B_n}{(z-a)^n} + \cdots + \frac{B_1}{z-a} + \widetilde{f}(z)$. $(z-a)^n f(z) = B_n + B_{n-1}(-a) + \cdots + (z-a)^n \widetilde{f}(z) = g(z)$, with $B_n \neq 0$. Take the logarithmic derivative. Then $\frac{n}{z-a} + \frac{f'}{f} = \frac{g'}{g}$. By choosing C_i small enough, we obtain

$$\int_{C_i} \frac{f'dz}{f2\pi i} = -n.$$

Furthermore, $\int_{\partial D} \frac{f'(z)dz}{f(z)2\pi i} = N(D) - P(D)$, the number of zeros minus the number of poles [in D]. That's called the argument principle.

$\frac{1}{2\pi i}\int_{\partial D} \frac{f'}{f} dz = \frac{1}{2\pi i}\int_{f(\partial D)} \frac{dw}{w}$, the number of times $f(\partial D)$ wraps around 0.

Corollary Suppose that f, g analytic in a connected domain $\overline{D}$ with ∂D smooth, and that on ∂D, $|f(z) - g(z)| < |f(z)| \neq 0$. Then $f(z)$ and $g(z)$ have the same number of zeros in D.

Proof Divide by f; then $\left|1 - \frac{g}{f}\right| < 1$ on ∂D. This implies that $\int_{\partial D} \frac{(g/f)'}{g/f}\frac{dz}{2\pi i} = 0$. But this integral is $N_g(D) - N_f(D)$.

Example Consider $z^4 - 6z + 3 = 0$. How many roots in $|z| < 1$, $|z| < 2$? We have to look at the boundary.

On $|z| = 1$, we have $1 = |z|^4 = |-6z + 3 - (z^4 - 6z + 3)| < |-6z + 3|$ on $|z| = 1$. So $|-6z + 3| \geq 6|z| - 3 \geq 3$. So these two polynomials have the same number of roots in $|z| < 1$, i.e., 1.

Now try $|z| = 2$; then $|z|^4 = 16$, and $|-6z + 3| \leq 6\cdot 2+3 = 15$. So we have $|z^4 - 6z + 3 - z^4| < |z^4|$ on $|z| = 2$. So the two polynomials have the same number of roots, i.e., 4. So there are three roots of $z^4 - 6z + 3$ between 1 and 2, and there's one less than 1.

Suppose $f(z)$ is analytic in $\overline{D}$ compact, and ∂D is smooth and $f(z)$ is real on ∂D. Then $f(z)$ is constant.

Proof Look at $a = \alpha + i\beta$ for $\beta > 0$; consider $f(z) - a$. $\Im(f(z) - a) = -\beta$ on ∂D. $\Rightarrow$ $\int_{\partial D} \frac{f'(z)dz}{f(z)-a} = 0$ for all a with $\Im a > 0$; can do the same thing for $\Im a < 0$. Therefore, $f(z) \neq a$ for $z \in D$ for all a with $\Im a \neq 0$. So f is real on an open set, and f is constant. $\diamond$

Define a function $F(z) = \int_{-1}^{1} \frac{p(x)dx}{x-z}$. [For now, pretend p is a polynomial.] This should certainly be analytic off the interval [-1,1]. Can we extend F analytically?

Let's try to extend in a neighborhood of some point. Consider the semicircle Scott's drawing, γ. We know that $\int_\gamma \frac{p(x)dx}{x-z} = 2\pi i p(z)$. Now, $\gamma = \gamma_1 - \gamma_2$, where γ_1 is the piece on $[0,1]$ and γ_2 is the arc in a clockwise direction. Then

$$\int_{\gamma_1} \frac{p(x)dx}{x-z} = \int_{\gamma_2} \frac{p(x)dx}{x-z} + 2\pi i p(z).$$

So we can replace the contour integral by deleting γ_1 and going on the arc instead. Call the resulting curve Ξ. Then $\int_{-1}^{1} \frac{p(x)dx}{x-z} = \int_\Xi \frac{p(x)dx}{x-z} + 2\pi i p(z)$. These are both analytic inside $\gamma_1\gamma_2$. So we can use this to extend F to $z \in [-1,1]$. We say that $F(z)$ can be extended across the arc. Gotta show that it's the same thing you get if you extend from below.

Well, actually, it's not quite the same. Try computing the difference.

Continuing from last time, we know that $\lim_{\epsilon\downarrow 0}\int_{-1}^{1}\frac{p(x)}{x-(\xi+i\epsilon)}dx$ exists; and the limit is actually given by the principal value. More about that in a bit.

Laruent Expansions If $f(z)$ is analytic in $|z-a|<r$, then $f(z)=\sum_{j=0}^{\infty}a_j(z-a)^j$. The series converges absolutely and uniformly in $|z-a|<r-\epsilon$ for all $\epsilon>0$. That's a representation theorem for functions analytic on a disk.

Now we'll work with annular regions; $A_{rR}=\{z|r<|z|<R\}$. Consider a function $\sum_{j=-\infty}^{\infty}b_jz^j=\sum_{j=0}^{\infty}b_jz^j+\sum_{j=-\infty}^{-1}b_jz^j$. Rewrite the second series, and get

$$\sum_{j=0}^{\infty}b_jz^j+\sum_{j=1}^{\infty}\frac{b_{-j}}{z^j}.$$

The thing on the right is a power series in $1/z$, and converges on $\frac{1}{|z|}<r$, i.e., $\frac{1}{r}<|z|$.

If $f(z)$ is analytic in A_{rR}, then

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon}\frac{f(\zeta)}{\zeta-z}-\frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon}\int\frac{f(\zeta)}{\zeta-z}d\zeta.$$

If $|z|<R-\epsilon$, then $\frac{1}{\zeta-z}=\frac{1}{\zeta}(\frac{1}{1-z\zeta})=\frac{1}{\zeta}\sum_{j=0}^{\infty}(\frac{z}{\zeta})^j$. Similarly, $\frac{1}{\zeta-z}=-\frac{1}{z}(\frac{1}{1-\zeta/z})=-\frac{1}{z}\sum_{j=0}^{\infty}(\frac{\zeta}{z})^j$. So we now have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial A_{r+\epsilon,R-\epsilon}}\frac{f(\zeta)d\zeta}{\zeta-z} &= \frac{1}{2\pi i}\int_{|\zeta|=R-\epsilon}\sum_{0}^{\infty}(\frac{z}{\zeta}^j f(\zeta)\frac{d\zeta}{\zeta}+\frac{1}{2\pi i}\int_{|\zeta|=r+\epsilon}\frac{1}{z}\sum_{0}^{\infty}(\frac{\zeta}{z})^j f(\zeta)d\zeta \\
&= \frac{1}{2\pi i}\left(\sum_{0}^{\infty}\int(\frac{z}{\zeta})^j f(\zeta)\frac{d\zeta}{\zeta}+\sum_{0}^{\infty}\frac{1}{z}\int_{|\zeta|=r+\epsilon}(\frac{\zeta}{z})^j f(\zeta)d\zeta\right) \\
\int_{|\zeta|=R-\epsilon}(\frac{z}{\zeta})^j f(\zeta)\frac{d\zeta}{\zeta} &= z^j\int_{|z|=R-\epsilon}\frac{f(\zeta)d\zeta}{\zeta^{j+1}2\pi i} \\
&= z^j a_j \\
\int(\frac{\zeta}{z})^j\frac{f(\zeta)d\zeta}{2\pi i} &= \frac{b}{z^j}.
\end{aligned}$$

In conclusion,

$$\begin{aligned} a_j &= \int_{|\zeta|=R-\epsilon} \frac{f(\zeta)d\zeta}{\zeta^{j+1}(2\pi i)} \\ b_j &= \int_{|\zeta|=r+\epsilon} \int \frac{f(\zeta)\zeta^j d\zeta}{2\pi i}. \end{aligned}$$

Some stuff with pictures. Let γ_s be a cycle running around γ_t, and both contained in A_{rR}. By the Cauchy theorem, $\int_{\gamma_s-\gamma_t} \frac{f(\zeta)d\zeta}{\zeta^{j+1}} = 0$ since the curve of integration is homologous to zero. So $\int_{\gamma_s} \frac{f(\zeta)}{\zeta^{j+1}} d\zeta$ does not depend on s.

If $f(z)$ is analytic in A_{rR}, then the Laurent expansion for $f(z)$ in this domain is unique. Suppose $f(z) = \sum_{-\infty}^{\infty} a_j z^j$. Observe that

$$\int_0^{2\pi} f(\rho e^{i\theta})e^{im\theta}d\theta = \int_0^{\pi} \sum_{-\infty}^{\infty} a_j \rho^j e^{ij\theta} e^{im\theta} d\theta$$

where $r < \rho < R$. By uniform convergence[?] we have

$$\begin{aligned} \int_0^{2\pi} f(\rho e^{i\theta})e^{im\theta}d\theta &= \sum_{-\infty}^{\infty} \int_0^{\pi} a_j \rho^j e^{i(m-j)\theta} d\theta \\ &= 2\pi a_{-m}\rho^{-m} \\ a_{-m} &= \frac{1}{2\pi i}\rho^m \int_0^{2\pi} f(\rho e^{i\theta})e^{im\theta}d\theta \end{aligned}$$

and the coefficients are uniquely determined.

If $f(z)$ is analytic in A_{rR}, then $f(z) = g(z) + h(z)$ where $g(z)$ is analytic in $|z| > r$ and $h(z)$ is analytic in $|z| < R$.

Let's suppose that $f(e^{i\theta}) \in L^2(S^1)$. Then $f(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$.[20] Define $g(z) = \sum_{-\infty}^{-1} a_n z^n$, $h(z) = \sum_0^{\infty} a_n z^n$. Then $g(z)$ is analytic in $|z| > 1$, and h is analytic in $|z| < 1$. Note that $f(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a)ne^{in\theta} \iff \|f\|_{L^2} = (\sum_{-\infty}^{\infty} |a_n|^2)^{1/2}$.

If we look at $g(re^{i\theta})$ for $r > 1$, this is an L^2 function. $l.i.m._{r\to 1} g(re^{i\theta}) = \sum_{-\infty}^{-1} a_n e^{in\theta}$. For

[20] We have to write $\sim$ instead of $=$; this means that $\lim \left\| f(e^{i\theta} = \sum_{n=-M}^{N} a_n e^{in\theta} \right\|_{L^2(S^1)} = 0$.

$$\begin{aligned}
\left\| g(re^{i\theta}) - \sum_{-\infty}^{-1} a_n e^{in\theta} \right\|_{L^2} &\leq \left\| g(re^{i\theta}) - \sum_{-N}^{-1} a_n e^{in\theta} \right\|_{L^2} + \left\| \sum_{-\infty}^{-N-1} a_n e^{in\theta} \right\|_{L^2} \\
\left\| g(rei\theta) - \sum_{N}^{-1} a_n e^{in\theta} \right\|_{L^2} &\leq \left\| \sum_{-N}^{-1} (r^n - 1) a_n e^{in\theta} \right\| + \sqrt{\sum_{-\infty}^{N-1} r^{2n} |a_n|^2} \\
&\leq \sqrt{\sum_{-N}^{-1} (r^n - 1)^2 |a_n|^2} + \sqrt{\sum_{-\infty}^{-(N+1)} |a_n|^2} \\
\left\| g(re^{i\theta}) - \sum_{-\infty}^{-1} - \sum_{-\infty}^{-1} a_n e^{in\theta} \right\| &\leq 2\sqrt{\sum_{-\infty}^{-N-1} |a_n|^2} + \sqrt{\sum_{-N}^{-1} (r^n - 1)^2 |a_n|^2} \\
&< \frac{\epsilon}{2} + \sqrt{\sum_{-N}^{-1} (r^n - 1) |a_n|^2} \\
&< \epsilon.
\end{aligned}$$

For ϵ given, can choose N large enough [if r is close enough to 1]. We're not proving that $\lim_{\to 1} g(re^{i\theta})$ exists. We've just shown that the limit in the mean exists. Conclusion: If $f = g|_{S^1} + h|_{S^1}$ where g is holomorphic in $|z| > 1$ and h is holomorphic in $|h| < 1$.

Now, suppose we have $f(z) = \frac{1}{z-1} + \frac{1}{z-2}$. It has a pole at one and a pole at zero. It's holomorphic in three annular regions; $|z| < 1$, $1 < |z| < 2$¡ and $z < |z|$. We'll see that it has a unique representation in each of these regions.

We know that $\frac{1}{z-1} = -\sum_0^\infty z^j$, and $\frac{1}{z-2} = -\frac{1}{2}\sum_0^\infty (\frac{z}{2})^j$. So $f(z) = \sum_0^\infty (-1 - \frac{1}{2^{j+1}} z^j$ for $|z| < 1$.

In the second region, $\frac{1}{z-1} = \frac{1}{z}(\frac{1}{1-\frac{1}{z}}) = \frac{1}{z}\sum_0^\infty (1/z)^j$. So $f(z) = \sum_0^\infty \frac{1}{2^{j+1}} z^j + \sum_{-\infty}^{-1} z^j$.

Moving right along....

Definition A function $f(z)$ is meromorphic on an open set $\Omega \subset \mathbb{C}$ if $f(z)$ has only a countable collection of singularities $\{a_j\} \subset \Omega$ with no points of accumulation in Ω, and at worst a pole at each of the a_j.

Residue Theorem If $f(z)$ is meromorphic in a set Ω, and $\gamma \subset \Omega$ is homologous to zero, then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz \sum_0^\infty n(\gamma, a_j)\operatorname{res}_{a_j} f.$$

Consider $\int_{-\infty}^{\infty}\frac{e^{ix}dx}{1+x^2}$. It might seem that the theorem doesn't apply, as we're not integrating over a finite set. Consider Γ_R, the semicircle of radius R centered at $(0,0)$, along with the arc $[-R,R]$. Then the theorem applies to this;

$$\int \Gamma_R \frac{e^{iz}}{1+z^2}dz = 2\pi i \sum \operatorname{res}(\frac{e^{iz}}{1+z^2}).$$

This contour only encloses the pole at i. But $\operatorname{res}_{z=i}\frac{e^{iz}}{1+z^2} = \lim_{z\to i}(z-i)\frac{e^{iz}}{z^2+1} = \frac{e^{-1}}{2i}$. So

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma_R}\frac{e^{iz}}{z^2+1}dz &= \frac{1}{2ei} \\
\left|\int_{|z|=R, 0<\arg z\pi}\frac{e^{iz}}{1+z^2}dz\right| &= \left|\int_0^\pi \frac{e^{i(R\cos\theta+iR\sin\theta)}iRe^{I\theta}d\theta}{1+R^2e^{2i\theta}}\right| \\
&= \int_0^\pi \frac{e^{-R\sin\theta}Rd\theta}{R^2-1} \\
&\to 0.
\end{aligned}$$

So

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}dz}{(1_z^2)2\pi i} = \int_{-\infty}^{\infty}\frac{e^{ix}dx}{(1+x^2)(2\pi i)} = \frac{1}{2\pi i}.$$

The conclusion is that

$$\int_{-\infty}^{\infty}\frac{e^{ix}dx}{1+x^2} = \frac{\pi}{e}.$$

We took an upper contour since, in the upper half-plane, $|e^{iz}| < 1$.

Consider $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$, where $R(x,y)$ is a rational function in x and y. The trick is, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. So $\cos\theta = \frac{z+\frac{1}{z}}{2}|_{|z|=1}$, and $\sin\theta = \frac{z-\frac{1}{z}}{2i}|_{|z|=1}$. So $d\theta = \frac{de^{i\theta}}{ie^{i\theta}} = \frac{dz}{iz}$ if $|z|=1$.

So we write the real integral as a complex line integral;

$$\begin{aligned}
\int_0^{2\pi} R(\cos\theta,\sin\theta)d\theta &= \int_{|z|=1} R(\frac{1}{2}(z+\frac{1}{z}),\frac{1}{2i}(z-\frac{1}{z}))\frac{dz}{i} \\
&= \int_{|z|=1} r(z)dz \\
&= \pi i \sum_{z\in|z|<1} \operatorname{res} r(z).
\end{aligned}$$

As an example, try $\int_0^\pi \frac{d\theta}{a+\cos\theta}$ with $a>1$. Note that this is just an integral up to π; but we may be able to finesse this. The function cos is symmetric about π, so this integral is just $\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}$. So go to work as above.

$$\begin{aligned}
&= \frac{1}{2}\int_{|z|=1}\frac{dz}{iz}\frac{1}{a+\frac{1}{2}(z+\frac{1}{z})} \\
&= \frac{1}{2i}\int_{|z|=1}\frac{dz}{\frac{1}{2}+az+\frac{z^2}{2}}.
\end{aligned}$$

Now we have to find the roots of the polynomial in the denominator; get

$$z = -a \pm \sqrt{a^2-1}.$$

We only use the root which lies inside the unit circle, namely, $-a+\sqrt{a^2-1}$. That's the only place where we have to compute the residue.

$$\begin{aligned}
\int_{|z|=1}\frac{dz}{1/2+az+z^2/2} &= 2\pi i \operatorname{res}_{z=-a+\sqrt{a^2-1}}\frac{1}{\frac{1}{2}+az+\frac{z^2}{2}} \\
\frac{1}{2}+az+\frac{z^2}{2} &= \frac{1}{2}(z^2+2az+1) \\
&= \frac{1}{2}(z-(-a+\sqrt{a^2-1}))(z-(-a-\sqrt{a^2-1})) \\
&= \frac{1}{2}(-a+\sqrt{a^2-1})(-a+\sqrt{a^2-1}+a+\sqrt{a^2-1}) \\
&= \sqrt{a^2-1}
\end{aligned}$$

In conclusion,

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{1}{2i} 2\pi i \frac{1}{\sqrt{a^2-1}} = \frac{\pi}{\sqrt{a^2-1}}.$$

Let's try another; $\int_{-\infty}^{\infty} R(x)dx$ where $F(x)$ is a rational function of x. We'll try doing the semicircle trick again. [Assume $R(x)$ has no poles on the real axis.] Furthermore, insist that the degree of the numerator is at least 2 smaller than that of the denominator. Then

$$\begin{aligned} \int_{-\infty}^{\infty} R(x)dx &= \lim_{R\to\infty} \int_{\Gamma_R} R(z)dz \\ &= 2\pi i \sum \operatorname{res}_{\Im z>0} R(z). \end{aligned}$$

Use $|R(z)| \leq \frac{c}{|z|^2}$ if $|z|$ is large enough.

One last example; $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$. Could try

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx &= \int_{-\infty}^{\infty} \Re(\frac{e^{ix}}{x})dx \\ &= \Re \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx. \end{aligned}$$

Except that this doesn't converge. Ah, well. We'll integrate over a funky contour; but we'll do that next time.

We'd had a theorem saying: if $f(z)$ is analytic in $0 < |z| < 1$, and $\lim_{z\to 0} |zf(z)| = 0$, then $f(z)$ actually extends to $|z| < 1$ as an analytic function.

Suppose $f(z)$ is analytic in $0 < |z| < 1$, and the L^2 norm $\int\int_{|z|\leq 1} |f(z)|^2 dxdy < \infty$. Then $f(z)$ extends to $|z| < 1$; it has a removable singularity.

Write the Laurent expansion $f(z) = \sum_{-\infty}^{\infty} a_n z^n$. We can integrate:

$$\begin{aligned}
\int\int_{\rho<|z|<1} |f(z)|^2 dxdy &= \int_\rho^1 \int_0^{2\pi} \sum_{-\infty}^{\infty} |a_n z^n|^2 r d\theta dr \\
&= \int \rho^1 \int_0^{2\pi} \sum_{n,m} a_n \overline{a_m} z^n \overline{z}^m r d\theta dr \\
&= \sum_{n=-\infty}^{\infty} \int_\rho^1 |a_n|^2 r^{2n} r dr \\
&= \sum_{n\neq -1} |a_n| \frac{(1-\rho^{2n+2})}{2n+2} + |a_{-1}|^1 \log(\frac{1}{\rho}).
\end{aligned}$$

This is a sum of nonnegative terms. If any of the negative coefficients is < 0, the thing will blow up. In other words, if $\rho < 1$ then each term $\frac{|a_n|^2(1-\rho^{2n+2})}{2n+2} > 0$ for $n < -1$; and therefore, if $|a_n| \neq 0$ for all $n < 0$ then we obtain a contradiction. $\diamond$

Remix. Use the representation $f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{|z|=r} \frac{f(\zeta)d\zeta}{\zeta - z}$. We'll estimate:

$$\begin{aligned}
\left|\int \frac{f(\zeta)d\zeta}{\zeta - z}\right| &\leq C\int_{|\zeta=r|} |f(\zeta)||d\zeta| \\
&\leq C\sqrt{\int_{|\zeta|=r} |f(\zeta)|^2 |d\zeta| 2\pi r}.
\end{aligned}$$

Claim that $\lim_{r\to 0}\inf \int |f(\zeta)|^2 |d\zeta| 2\pi r = 0$.

If not, then there's a $\delta > 0$ and a constant $c > 0$ so that $\int_{|\zeta|=r} |f(\zeta)|^2 |d\zeta| > \frac{C}{r}$ for $r < \delta$. But this integral is $\int_{|\zeta|=r} |f(\zeta)|^2 r d\theta dr > \int_\rho^\delta \frac{C}{r} dr = C\log(\frac{\delta}{\rho})$, a contradiction.

So $\lim\inf = 0$, and there is a sequence $\{r_n\}$ with $r_n \to 0$ so that $\lim_{n\to\infty} \int_{|\zeta|=r_n} |f(\zeta)|^2 |d\zeta| r_n = 0$. So

$$\left|\int_{|\zeta|=r_n}\frac{f(\zeta)d\zeta}{\zeta-z}\right|\leq C\sqrt{r_n\int_{|\zeta|=r_n}|f(\zeta)|^2|d\zeta|}\to 0.$$

We conclude that

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)d\zeta}{\zeta-z}.$$

$\diamond$

We were talking about $\int_0^\infty \frac{\sin x}{x}dx = \int \frac{\Im e^{ix}dx}{x} = \Im\int_0^\infty \frac{e^{ix}dx}{x}$, which unfortunately doesn't exist. So instead, we take

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}dx &= \lim_{r\to 0, R\to\infty}\int_r^R \frac{\Im e^{ix}}{x}dx\\ &= \lim \frac{1}{2\pi i}\int_r^R \frac{e^{ix}-e^{-ix}}{x}dx\\ &= \lim\left[\frac{1}{2\pi i}\int_{-R}^{-r}\frac{e^{ix}dx}{x}+\int_r^R\frac{e^{ix}}{x}dx\right].\end{aligned}$$

This is part of a contour, Γ_{rR}. We know that $\Gamma_{rR}\frac{e^{iz}dz}{z}=0$.

Consider

$$\begin{aligned}\left|\int_{|z|=R}\frac{e^{iz}dz}{z}\right| &\leq \left|\int_0^\pi \frac{e^{-Rsin\theta}Rd\theta}{R}\right|\\ &= \left|\int_0^\pi e^{-R\sin\theta}d\theta\right|\\ &= \int_0^{\pi/2}e^{-R\sin\theta}d\theta\end{aligned}$$

Can choose a constant $0<c<1$ for which $\sin\theta > c\theta$ for $0\leq\theta\leq\frac{\pi}{2}$

$$\begin{aligned} &\leq 2\int_0^{\pi/2} e^{-Rc\theta}d\theta \\ \frac{e^{-Rc\theta}}{-Rc}|_0^{\pi/2} &\leq \frac{1}{Rc}. \end{aligned}$$

So as $\to \infty$, $\lim_{R\to\infty} \int_{|z|=R,\Im z\geq 0} \frac{e^{iz}dz}{z} = 0$.

Going back to the original integral. It remains to compute

$$\begin{aligned} \int_{|z|=r,\Im z>0} \frac{e^{iz}dz}{z} &= \int_0^{\pi} \frac{e^{ire^{i\theta}}ire^{i\theta}d\theta}{re^{i\theta}} \\ &= i\int_0^{\pi} e^{ire^{i\theta}}d\theta \\ e^{ire^{i\theta}} &= 1 + O(r) \\ i\int_0^{\pi} e^{ire^{i\theta}}d\theta &\to \int_0^{\pi} id\theta \\ &= \pi i. \end{aligned}$$

So $\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$. $\diamond$

Suppose $f(x)$ is not integrable at $x = 0$. It is sometimes possible to assign a definite value to a "regularized integral of f." Define the Cauchy principal value by

$$P.V. \int_{-1}^{1} f(x)dx = \lim_{\epsilon\to 0} \int_{-1}^{-\epsilon} + \int_{\epsilon}^{1} f(x)dx$$

whenever this limit exists.

Suppose $f(z)$ is a meromorphic function with a simple pole at $z = 0$. Then the principal value $P.V. \int_{-}^{1} f(x)dx$ exists, and can use the residue theorem to give a formula. Let $\Gamma_{r\epsilon}$ be a curve as in Scott's notes. $\Gamma_{r\epsilon} = \Gamma_{r\epsilon}^{+} \cup [-1, -r] \cup$ semicircle $\cup [r, 1]$. So

$$\int_{-1}^{-r} f(x)dx + \int_{r}^{f} (x)dx = -\int_{\Gamma_{r\epsilon}^{+}} f(z)dz - \int_{|z|=r,\Im z>0} f(z)dz$$

Our assumption is that $f(z) = \frac{B}{z} + \widetilde{f}(z)$ where $\widetilde{f}$ is analytic in a neighborhood of zero. Go compute.

$$\int_{|z|=r,\Im z>0} f(z)dz = \int_{|z|=r,\Im z>0} \frac{Bdz}{z} + \int_{|z|=r,\Im\epsilon>0} \widetilde{f}(z)dz.$$

The rightmost term dies as $r \to 0$. So

$$\begin{aligned}\int f(z)dz &+ \int_0^\pi \frac{Bire^{I\theta}d\theta}{re^{i\theta}} \\ &= i\pi B.P.V. \int_{-1}^1 f(x)dx \\ &= i\pi \operatorname{res}_{z=0} f - \int_{\Gamma_\epsilon^+} f(z)dz.\end{aligned}$$

Let's compute $\int_0^\infty \frac{\log x}{1+x^2}dx$. All the trickiness comes from determining the logarithm correctly. Let's pick the branch with the negative imaginary axis deleted; possible arguments range fro $-\pi/2$ to $3\pi/2$. Let Γ_r be the usual.

$$\begin{aligned}\int_{\Gamma_r R} \frac{\log z}{1+z^2}dz &= 2\pi i \operatorname{res}_{z=i} \frac{\log z}{z^2+1} \\ \lim_{z\to i}(z-i)\frac{\log z}{(z+i)(z-i)} &= \frac{\log e^{\pi i/2}}{2i} \\ &= \pi/4. \\ \int_{\Gamma_r R}\int \frac{\log dz}{1+z^2} &= 2\pi i\frac{\pi}{4} \\ &= \pi^i \int_r^R \frac{\log xdx}{1+x^2} + \int_R^r \frac{\log(xe^{\pi i})e^{\pi i}dx}{1+(xe^{\pi i})^2} \\ &= \int_r^R \frac{\log xdx}{1+x^2} - \int_R^r \frac{\log x+\pi i}{1+x^2}dx \\ &= \int_r^R \frac{\log}{1+x^2}dx + \pi i \int_r^R \frac{dx}{1+x^2}\end{aligned}$$

Thus far, we know that $\int_{\Gamma_r R} \frac{\log zdz}{1+z^2} = \int_{C_R} \frac{\log zdz}{1+z^2} + 2_r^R \frac{\log xdx}{1_x^2} + \pi i \int_r^R \frac{dx}{1+x^2} + \int_{C_r} \frac{\log zdz}{1+z^2}$.

But

$$
\begin{aligned}
\left|\int_{C_r} \frac{\log z}{1+z^2}dz\right| &\le \int_0^\pi \frac{\sqrt{\log^2 r + \pi^2} r\,d\theta}{1-r^2} \\
&\to 0 \text{ as } r \to 0. \\
\left|\int_{|z|=R, \Im z > R} \frac{\log z}{1+z^2}dz\right| &\le \int_0^\pi \frac{\sqrt{\log^2 R + \pi^2} R\,d\theta}{R^2-1} \\
&\to 0 \text{ as } R \to \infty \text{ like } \frac{\log R}{R}. \\
\frac{\pi^2 i}{2} &= 2\int^\infty \frac{\log x}{1+x^2}dx + \pi i \int_0^\infty \frac{dx}{1+x^2} \\
\int_0^\infty \frac{dx}{1+x^2} &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2} \\
&= \frac{1}{2} 2\pi i \operatorname{res} \frac{1}{1+z^2} \\
\operatorname{res}_{z=i} \frac{1}{(z+i)(z-i)} &= \frac{1}{2i} \\
\frac{1}{2} 2\pi \operatorname{res} \frac{1}{1+z^2} &= \frac{\pi^2 i}{2}\frac{\pi^2 i}{2} \\
&= 2\int_0^\infty \frac{\log x}{1+x^2}dx + \pi^2 i/2 \\
\int_0^\infty \frac{\log x}{1+x^2}dx &= 0.
\end{aligned}
$$

◇

One last integral; $\int_0^\infty \frac{x^{-\alpha}}{1+x}dx$ with $0 < \alpha < 1$; that way, the integral is absolutely convergent. Pick a contour which looks like a big circle. The problem is, $x^{-\alpha}$ isn't single-valued all the way around a circle. So as a domain we'll use the complement of the positive real axis.

$$
\lim_{\epsilon \downarrow 0} \int_{r+i\epsilon}^{R+i\epsilon} \frac{z^{-\alpha}dz}{1+z}
$$

is what we want.

I blinked a bit; may be in a bit of trouble.

$$
\lim_{\epsilon \downarrow 0} \int_{r+i\epsilon}^{+i\epsilon} \frac{z^{-\alpha}dz}{1+z} = \int_r^R \frac{x^{-\alpha}dx}{1+x}
$$

Along some contour, $\lim_{\epsilon\uparrow 0}^{-\alpha} = (xe^{2\pi i})^{-\alpha} = x^{-\alpha}e^{\pi i\alpha}$. Some contour or other is being called Γ^{ϵ}_{rR}. Be careful of orientations to keep plus and minus signs accurate.

$$\begin{aligned}
\int_{\Gamma^{\epsilon}_{rR}} \frac{z^{-\alpha}dz}{1+z} &= 2\pi i \operatorname{res}_{z=-1}^{-\alpha} 1+z \\
&= 2\pi i(-2\pi i e^{-\pi i\alpha}) \\
\lim_{\epsilon\downarrow 0} \int_{\Gamma^{\epsilon}_{rR}} \frac{z^{-\alpha}dz}{1+z} &= \int_{C_R} \frac{z^{\alpha}dz}{1+z} + \int_r^R \frac{x^{-\alpha}dx}{1+x} - e^{-2\pi i\alpha}\int_r^R \frac{x^{-\alpha}dx}{1+x} - \int_{C_r} \frac{z^{-\alpha}dz}{1+z} \\
-2\pi i e^{-\pi i\alpha} &= (1-e^{-2\pi i\alpha})\int_r R\frac{x^{-\alpha}dx}{1+x} + \int_{C_R}\frac{z^{-\alpha}dz}{1+z} - \int_{C_r}\frac{z^{-\alpha}dz}{1+z} \\
\lim_{R\to\infty} \frac{R^{-\alpha}R}{R-1} &= 0 \\
\lim_{r\to 0} \frac{r^{-\alpha}rd\theta}{1-r} &= 0 \\
\int_0^{\infty} \frac{x^{-\alpha}}{1+x}dx &= \frac{-2\pi i e^{-\pi i\alpha}}{1-e^{-2\pi i\alpha}} \\
&= \frac{-2\pi i}{e^{\pi i\alpha}-e^{-\pi i\alpha}} \\
&= -\frac{2\pi i}{2i\sin\pi\alpha} \\
&= -\frac{\pi}{sin\pi\alpha}
\end{aligned}$$

[Except that the sign, somewhere, is wrong. The residue should have been positive.]

Harmonic Functions A twice differentiable function is harmonic if $\Delta u = u_{xx} + u_{yy} = 0$; $\Delta u = 4\partial_z \partial_{\overline{z}} u$. If $u(w, \overline{w})$ is harmonic and $f(z)$ is holomorphic, then $v(z, \overline{z}) \stackrel{\text{def}}{=} u(f(z), \overline{f(z)})$ is harmonic. For $\partial_{\overline{z}} v = u_w f'$, $\partial_{\overline{z}} \partial_z v = u_{w\overline{w}} f' \overline{f'}$. This is a pretty useful fact.

The function $\frac{1}{z}$ is pretty important.

In polar coordinates, $\Delta = \frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\partial_\theta^2$. The $d\theta$ vector field is globally defined. We want to find solutions $\Delta w = 0$ where $u = f(r)$ some function of the radius. Then

$$\frac{1}{r}\partial_r r \partial_r f(r) = 0$$

and

$$f(r) = \alpha + \beta \log r.$$

The $\beta \log r$ term is like $\frac{1}{z}$, in terms of its utility. $\partial_z \log r = \partial_z \frac{1}{z} \log z\overline{z} = \frac{1}{2z}$.

Proposition Suppose that $u(z)$ is a harmonic function in $r_1 < |z| < r_2$ such that $u(e^{i\theta} z) = u(z)$. Then $u(z) = \alpha + \beta \log |z|$.[21]

That's essentially what we just proved.

If $u(z)$ is harmonic, then so is the function $u_\theta(z) = u(e^{i\theta} z)$. Define $v(z) = \int_0^{2\pi} u_\theta(z) \frac{d\theta}{2\pi}$, then

$$\begin{aligned} v(e^{i\phi} z) &= \int_0^{2\pi} u_\theta(e^{i\phi} z) \frac{d\theta}{2\pi} \\ &= \int u(e^{i\theta + \phi} z) \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} u(e^{i\theta} z) \frac{d\theta}{2\pi} \\ &= v(z). \end{aligned}$$

Corollary $v(z) = \alpha + \beta \log |z|$.

Note that u must be defined in some annular region, or we couldn't average it over the whole circle.

Suppose that $u(z)$ is harmonic in $D(0, r)$. Then

[21] Our functions are no longer necessarily holomorphic, so they're really functions of z and $\overline{z}$.

$$\begin{aligned} \int_0^{2\pi} u(e^{i\theta}z)\frac{d\theta}{2\pi} &= \alpha \\ u(0) &= \alpha. \end{aligned}$$

We see that harmonic functions satisfy the mean value property, to wit: if u is harmonic in the disk $D(z_0, r)$, then $u(z_0) = \int u(z_0 + \rho e^{i\theta})\frac{d\theta}{2\pi}$. The value at the center is the average of the values along the circle.

We define a weakly harmonic function as a function u which is continuous and satisfies the mean value property.

Hmmm. Time passes, and Charlie says that

$$u(z_0) = \frac{\int_{r_1}^{r_2}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\frac{d\theta}{2\pi}\rho d\rho}{r_2^2/2 - r_1^2/2}.$$

Theorem [Weyl's lemma] A weakly harmonic function is actually a C^∞ function.

Proof Choose a C^∞ function of compact support $\psi(x)$ such that $\int_0^\infty \psi(r^2)rdr = \frac{1}{2\pi}$.[22] Define $U_\epsilon(z) = \int\int_{\mathbb{C}} u(z - re^{i\theta})\psi(\frac{r^2}{\epsilon^2})\frac{1}{\epsilon^2}rdrd\theta$. This is an approximate identity. This is defined for points sufficiently far from the boundary. Observe that

$$\begin{aligned} u_\epsilon(z) &= \int\int_0^{2\pi} u(z - re^{i\theta})d\theta\psi(\frac{r^2}{\epsilon^2})\frac{rdr}{\epsilon^2} \\ &= \int_0^\infty 2\pi u(z)\psi(\frac{r^2}{\epsilon^2})\frac{rdr}{\epsilon^2} \\ &= u(z). \end{aligned}$$

So $u_\epsilon(z) = u(z)$. We're looking at

$$\int\int_{\mathbb{C}} u(re^{i\theta})\psi(\frac{|z - re^{I\theta}|^2}{\epsilon^2})\frac{rdrd\theta}{\epsilon^2}.$$

Since u is bounded and ψ is C^∞, it follows that for all $\epsilon > 0$, $u_\epsilon(z)$ is a smooth function of z. $\diamond$

[22] Equivalently, $\int \psi(\frac{r^2}{\epsilon^2})\frac{rdr}{\epsilon^2} = \frac{1}{2\pi}$.

Exercise If $u(z)$ is C^2 and satisfies the mean value property, then $\Delta u = 0$.

Assume u is harmonic in $D_1(0)$ and continuousin $\overline{D_1(0)}$. Then $u(0) = \frac{1}{\pi}\int_0^\pi u(re^{i\theta})d\theta$ for all $r < 1$. And actually, we can by continuity take $r = 1$. Let

$$Sz = \frac{z+a}{1+\overline{a}z}$$

for some $|a| < 1$. Define $v_a(z) = u(\frac{z+a}{1+\overline{a}z})$. Then $v_a(z)$ is harmonic in $D_1(0)$ and continuous in $\overline{D_1(0)}$; and $v_a(0) = u(a)$.

$$\begin{aligned}
v_a(0) &= \int_0^{2\pi} v_a(e^{i\theta})\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} u(\frac{e^{i\theta}+a}{1+e^{i\theta}\overline{a}})\frac{d\theta}{2\pi} \\
\text{Let } e^{i\phi} &= \frac{e^{i\theta}+a}{1+\overline{a}e^{i\theta}} \\
e^{i\theta} &= \frac{e^{i\phi}-a}{1-\overline{a}e^{i\phi}} \\
ie^{i\theta}\frac{d\theta}{d\phi} &= \frac{ie^{i\phi}(1-\overline{a}e^{i\phi})+\overline{a}ie^{i\phi}(e^{i\phi}-a)}{(1-\overline{a}e^{i\phi})^2} \\
&= \frac{ie^{i\theta}(1-|a|^2)}{(1-\overline{a}e^{i\phi})^2} \\
\left|\frac{d\theta}{d\phi}\right| &= \frac{1-|a|^2}{|1-\overline{a}e^{i\phi}|^2} \\
\text{So the integral is} & \\
&= \int_0^{2\pi} u(e^{i\phi})\frac{1-|a|^2}{|1-\overline{a}e^{i\phi}|^2}\frac{d\phi}{2\pi} \\
&= \int_0^{\pi} u(e^{i\phi})\frac{1-|a|^2}{|e^{i\phi}-a|^2}\frac{d\phi}{2\pi}.
\end{aligned}$$

This is alternately called the Poisson or Schwarz integral formula. The kernel $P(a,\phi) = \frac{1-|a|^2}{\left|e^{i\phi}-a\right|^2}$ is the Poisson kernel. Some people write it by substituting $a = re^{i\phi}$ and crunching it out.

This formula is to harmonic analysis what the Cauchy integral formula is to complex analysis.

Since $P(a, e^{I\phi}) > 0$ and we have the special case of $u = 1$, yielding

$$1 = \int_0^{2\pi} P(a, e^{i\phi}) \frac{d\phi}{2\pi}$$

for all $a \in D_1(0)$. Let $M = \max u(e^{i\phi})$¡ $m = \min u(e^{i\phi})$. Then

$$\begin{aligned} M - u(e^{i\phi}) &\geq 0 \\ 0 \leq \int (M - u(e^{i\phi}))P(a, e^{i\phi}) &= M - u(a). \end{aligned}$$

So $u(a) < M$ unles $u \equiv M$; a similar argument shows that $u(a) > m$ unless $u \equiv m$.

This is the maximum principle for harmonic functions: A harmonic function u in a connected set Ω satisfies $u(a) \leq \max_{z \in \partial\Omega} u(z)$, with equality $\iff$ u is constant.

Let $P(z, e^{i\phi}) = \frac{1-|z|^2}{\left|z-e^{i\phi}\right|^2}$. Then $P(z, e^{i\phi}) = \Re(\frac{e^{i\phi}+z}{e^{i\phi}-z})$. This is the real part of an analytic function on $|z| < 1$; so for $e^{i\phi} \in \partial D_1(0)$ and $z \in D_1(0)$, $P(z, e^{i\phi})$ is a harmonic function of z. This means that if $u(e^{i\phi})$ is any L^1 function on $\partial D_1(0)$, then $U(z) = U(re^{i\phi}) \stackrel{\text{def}}{=} \int u(e^{i\phi})P(z, e^{i\phi})\frac{d\phi}{\pi}$ is harmonic in $D_1(0)$.

Theorem [H.A. Schwarz] If $u(e^{i\theta}) \in L^1(S^1)$, then $U(z) \to u(e^{i\phi_0})$ as $\to e^{i\phi_0}$ if $e^{i\phi_0}$ is a point of continuity for $u(e^{i\phi})$.

Proof Suppose that we are given an $\epsilon > 0$ and a $\delta > 0$. Then there's an $\eta > 0$ such that if $\left|z - e^{i\phi_0}\right| < \eta$ and $\left|e^{i\phi} - e^{i\phi_0}\right| > \delta$¡ then $P(z, e^{i\phi}) < \epsilon$. *Blah.* $\left|z - e^{i\phi}\right| > \delta - \eta$. On the other hand, the numerator $1 - |z|^2 < 2\eta$; it's basically the distance to the boundary, which is necessarily smaller than the distance to any given point on the boundary. So

$$\frac{1 - |z|^2}{|z - e^{i\phi}|^2} \leq \frac{2\eta}{(\delta - \eta)^2} < \epsilon.$$

We were able to pick an appropriate η. Go to work.

$$\begin{aligned} \left|U(z) - u(e^{i\phi_0})\right| &= \left|\int u(e^{i\phi})P(z, e^{i\phi})d\phi - \int u(e^{i\phi_0})P(z, e^{i\phi})d\phi\right| \\ &= \left|\int [u(ei\phi) - u(e^{i\phi_0})]P(z, e^{i\phi})d\phi\right|. \end{aligned}$$

For all ϵ there' a δ so that $\left|u(e^{i\phi}) - u(e^{i\phi_0})\right| < \delta$, because u is continuous at $e^{i\phi_0}$. Then

$$\begin{aligned}
\left|U(z) - u(e^{i\phi_0})\right| &\leq \int_{\left|e^{i\phi}-e^{i\phi_0}\right|<\delta} \left|u(e^{i\phi}) - u(e^{i\phi_0})\right| P(z, e^{i\phi}) d\phi + \\
&\quad \int_{\left|e^{i\phi}-e^{i\phi_0}\right|>\delta} \left|u(e^{i\phi}) - u(e^{i\phi_0})\right| P(z, e^{i\phi}) d\phi \\
&\leq \int_0^{\pi} \epsilon P(z, e^{i\phi}) d\phi + \left|u(e^{i\phi}) + u(e^{i\phi_0})\right| \epsilon d\phi \\
&\quad \text{Since } \left|z - e^{i\phi_0}\right| < \eta \\
\left|U(z) - u(e^{i\phi_0})\right| &\leq \epsilon(1 + \|U\|_{L^1(S^1)} + \left|u(e^{i\phi_0})\right|) \\
&\quad \text{for } \left|z - e^{i\phi_0}\right| < \eta.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $\lim_{z \to e^{i\phi_0}} U(z) = u(e^{i\phi_0})$. $\diamond$

Corollary If $u(e^{i\phi})$ is in $C^0(S^1)$, then $U(z)$ extends continuously to $\overline{D_1(0)}$ by setting $U(e^{i\phi}) = u(e^{i\phi})$.

The Dirichlet problem on the unit disk is solvable for any continuous function u defined on $\partial D_1(0)$: given $u \in C^0(S^1)$, find a function $U(z)$ continuous on $\overline{D_1(0)}$ such that

1. $U|_{\partial D_1} = u$.
2. $\Delta U = 0$ in $D_1(0)$.

Let Ω be a simply connected set. If $u \in C^0(\partial\Omega)$, is there U in $C^0(\overline{\Omega})$ which is harmonic and agrees with u on the boundary?

Suppose the Dirichlet problem has two solutions U_1 and U_2. Well, $U_1 - U_2$ is a harmonic function in Ω¡ continuous in $\overline{\Omega}$, so that $U_1 - U_2|_{\partial\Omega} = 0$. By the maximum principle, $U_1 - U_2 \leq 0$. On the other hand, $U_2 - U_1 \leq 0$, as well; $U_1 = U_2$. So if there's a solution, it must be unique.

Moving right along. Consider $\int_{|z-a|=R} \log r d\theta$. This integral is not $-\infty$, which is $\log 0$. So the mean value property sort of doesn't hold. That's cause log isn't harmonic on the whole disk. It's actually a subharmonic function. Such a function is one which has the property that $\int_0^{2\pi} u(z + \rho e^{i\theta}) \frac{d\theta}{2\pi} \geq u(z)$. From this, one can [easily?] show that a subharmonic function satisfies the maximum principle. A subharmonic function must be bounded above; $u(z) \in [-\infty, \infty)$. $\log|f(z)|$ is harmonic only in domains where f is nonvanishing; otherwise, it's only subharmonic. [Provided f is holomorphic, of course.]

Last time we were jamming with the Poisson integral formula. It works in any disk $|z| < r$;

$$U(z) = \int_0^{2\pi} u(re^{i\theta}) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \frac{d\theta}{2\pi}.$$

This defines a harmonic function in $D_r(0)$; and if $u(re^{i\theta})$ is a continuous function then U is continuous in $\overline{D_r(0)}$ with $U|_{\partial D_r(0)} = u$.

If U is harmonic in Ω, Ω bounded with reasonable boundary, and U harmonic in Ω, continuous in $\overline{\Omega}$ with $U|_{\partial\Omega} = 0$, then $U \equiv 0$. A harmonic function is determined by its boundary values.

The bounded hypothesis is necessary.

Given a continuous function f on $\partial\Omega$ there's a harmonic function U in ω continuous on $\overline{\Omega}$ such that $U|_{\partial\Omega} = f$.

We draw a picture of Ω, separated into Ω^+ and Ω^-, and $\sigma = \Omega \cap \Im z = 0$. Suppose $f(z)$ is holomorphic in $\Omega^+ \cup \Omega^- \cup \sigma$, and that $f(z)$ is real on the real axis. Then $f(z) - \overline{f(\overline{z})} \equiv 0$; $f(z) = \overline{f(\overline{z})}$.

Now, $f(z) = u + iv$; and we're assuming that v vanishes as $z \to \sigma$. This leads us to the

Schwarz reflection principle Let Ω be symmetric relative to the real axis, $\overline{\Omega} = \Omega$, connected, open. Let $\sigma = \Omega \cap \mathbb{R}$. Suppose that $v(z)$ is harmonic in $\Omega^+ = \Omega \cap \{\Im z > 0\}$, and that v vanishes along σ. Then the function $V(z) = \begin{cases} v(z) & z \in \Omega^+ \\ -v(\overline{z}) & z \in \Omega^- \end{cases}$ is harmonic in Ω.

This is clearly harmonic on Ω^-; we just have to worry about σ. Let $z_0 \in \sigma$; consider $D(z_0, r) \overset{\text{cpt}}{\subset} \Omega$. Let $u(z_0 + re^{i\theta}) = V(z_0 + re^{i\theta})$. Then u is defined on $\partial D(z_0, r)$, and it is clearly a continuous function. Define a harmonic function with boundary values $u(z_0 + re^{i\theta})$ on $D(z_0, r)$, by setting

$$U(z_0 + z) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

Check out the symmetry of the kernel; u is an odd function, so integrating over the top half is minus integrating the bottom half. $U(z_0 + x) = 0$ for $z_0 + x \in \sigma$. Consider $v(z_0+z) - U(z_0+z)$. These two functions agree on the upper semicircle; and they both die on the real axis. So $v(z_0 + z) - U(z_0 + z) \equiv 0$, by the uniqueness of such things. Similarly play with the bottom. We've thus shown that $V(z_0 + z)$ agrees with a harmonic function in the whole disk. This establishes that $V(z)$ actually defines a harmonic function in all of Ω. ◇

Something about v on the upper semidisk, $-v$ on the bottom; $u(z)$ is the conjugate holomorphic function. So $\partial_x u = \partial_y v$ and $-\partial_y u = \partial_x v$. Want to show that, since v is odd, u is even. We'll work on $u(z) - u(\overline{z}) = U_0(z)$, which is certainly a harmonic function. We know that $\partial_z U_0 = \frac{1}{two}(\partial_x - i\partial_y)(U_0$ is a holomorphic function, since $\partial_{\overline{z}}\partial_z U_0 = 0$. Well, $\partial_x U_0 = \partial_x u(z) - \partial_x u(\overline{z}) = 0$, where $z = x \in \mathbb{R}$. The other derivative is a little trickier.

$$\begin{aligned} \partial_y U_0 &= \partial_y u(z) - \partial_y u(\overline{z}) \\ &= \partial_x v(\overline{z}) + \partial_x v(\overline{z} \\ &= 0 \text{ where } z = x. \end{aligned}$$

So $\partial_z U_0 \equiv 0$, because $\partial_z U_0$ is an analytic function that vanishes on a real arc. So

$$\begin{aligned} \partial_z U_0 &= \frac{1}{2}(\partial_x U_0 - \partial_y U_0) \\ &= 0. \end{aligned}$$

We conclude that U_0 is constant; but U_0 vanishes on the real axis, so in fact $U_0 \equiv 0$, and $u(z) = u(\overline{z})$.

Theorem If $f(z)$ is analytic in Ω^+, and $\Im f(z)$ vanishes on σ, then the function $F(z) = \begin{cases} f(z) & z \in \Omega^+ \\ \overline{f(z)} & z \in \Omega^- \end{cases}$ is analytic in $\Omega = \Omega^+ \cup \sigma \cup \Omega^-$.

Proof Write $f = u + iv$. The basic plan is this.

1. v extends to a harmonic funciton in Ω.

2. The conjugate function to v, that is, u¡ also extends to a neighborhood of σ.

3. $u(z) = u(\overline{z})$.

There's no great trick to showing that it's holomorphic in $\Omega^+ \cup \Omega^-$, since that's not connected. What's tricky is to patch things together across σ.

Oh, shit. Major picture time. If you have a region whose boundary is pieces of circles and gets mapped to such a region, then you can use a Möbius transformation for each such piece to flatten the circle into a line. Then you get a similar reflection principle; but instead of doing the regular-old-reflection around a line, you reflect it about a circle. If R_1 is reflection around the first piece of circle, and R_2 that aruond the second, then the Schwarz reflection principle yields $f(R_1 z) = R_2(f(z))$.

We're switching gears now. Say $f : \Omega \to D_1$, $\Omega \subset \mathbb{C}$ simply connected, f analytic, 1-1, onto $D_1(0)$.

Normal families

Arzela-Ascoli theorem Consider continuous mappings $f : \Omega \to \mathcal{S}$ with $\mathcal{S}$ a metric space, $\Omega \subset \mathbb{C}$, that is, $C^0(\Omega, \mathcal{S})$. $C^0(\Omega, \mathcal{S})$ has a natural topology; that of locally uniform convergence. There is an exhaustion of Ω by compact subsets $\{K_j\}$ so that $f_n \to f$ provided that $f_n|_{K_j} \to f|_{K_j}$ uniformly for each j. So given $\epsilon > 0$ there's an N_j so that $\sup_{z \in K_j} \|f_n(z) - f(z)\| < \epsilon$ if $n > N_j$. This is the weakest topology you can put on so that the limit of analytic functions is analytic.

Equicontinuity: A family of functions $\mathcal{F} \subset C^0(\Omega, \mathcal{S})$ is said to be equicontinuous on a set $E \subset \Omega$ if, for every $\epsilon > 0$ there's a $\delta > 0$ so that $\|f(z) - f(z')\| < \epsilon$ for all $f \in \mathcal{F}$, all $z, z' \in E$ so that $|z - z'| < \delta$. So all the functions are uniformly continuous, and the same δ works for all the functions in the family.

Normal family: $F \subset C^0(\Omega, \mathcal{S})$ is a normal family provided every sequence $\{f_n\}$ has a convergent subsequence. [Locally uniform convergence.] Such a set is sometimes called a precompact set; its closure is certainly compact.

Arzela's Theorem A family $\mathcal{F} \subset C^0(\Omega, \mathcal{S})$ is normal $\iff$

1. $\mathcal{F}$ is equicontinuous on every compact subset $E \subset \Omega$.

2. For any $z \in \Omega$ the values $\{f(z) | f \in \mathcal{F}\}$ lie in a compact subset of $\mathcal{S}$.

Proof Suppose that $\mathcal{F}$ is normal We'll show that $\mathcal{S}_z = \{f(z) : f \in \mathcal{F}\}$ lies in a compact subset of $\mathcal{S}$. Let $\{w_n\}$ is a sequence of points in $\overline{\mathcal{S}_z}$. For each n we can find $f_n \in \mathcal{F}$ so that $d(f_n(z), w_n) < \frac{1}{n}$. Now, $\{f_n\}$ has a convergent subsequence say $\{f_{n_j}\}$. Then clearly $\lim_{j\to\infty} f_{n_j}(z) = \lim_{j\to\infty} w_{n_j}$ exists.

Suppose $\mathcal{F}$ is *not* equicontinuous on some compact subset $E \subset \Omega$. This implies that there's a sequence $\{f_n(z)\}$ and points $\{z_n\}$, $\{z'_n\} \subset E$ and an ϵ so that $d(f_n(z_n), f_n(z'_n)) > \epsilon$, but $\lim_{n\to\infty} |z_n - z'_n| = 0$. We can assume that $f_{n_j} \to f_n$, $z'_{n_j} \to z^*$. Now,

$$\begin{aligned}\lim_{j\to\infty} d(f_{n_j}(z_{n_j}), f_{n_j}(z'_{n_j})) &= d(f(z^*), f(z^*))\\ &= 0\end{aligned}$$

a contradiction.

Now, we want to show that $\mathcal{F}$ is normal. Let $\{\zeta_n\}$ be a countable dense subset of Ω. Let $\{f_n\}$ be any sequence. $f_n(\zeta_1)$ lies in $\overline{\mathcal{S}_{\zeta_1}}$ a compact set, so we can select a subsequence n_{1j} so that $\lim_{j\to\infty} f_{n_{1j}}(\zeta_1)$ exists.

Now, consider $f_{n_{1j}}(\zeta_2)$. We can select a subsequence n_{2j} of n_{1j} so that $\lim j \to \infty f_{n_{2j}}(\zeta_2)$ exists. We proceed inductively to construct subsequences n_{kj}a subsequence of $n_{(k-1)j}$ so that $\lim_{j\to\infty} f_{n_{kj}}(\zeta_k)$ exists.

Cantor realized the following; n_{jj} is a subsequence of n_{kj} for all k when j is large enough.[23] Set $n_j = n_{jj}$. We now know that $\lim_{j\to\infty} f_{n_j}(\zeta_m)$ exists for every m.

We need to use equicontinuity to show that $f_{n_j}(z)$ converges for every z. Fix an $\epsilon > 0$. Then there's a δ so that if $|z - z'| < \delta$, then $d(f(z), f(z')) < \epsilon$ fo rall $f \in \mathcal{F}$. So choose ξ_m so that $|\xi_m - z| < \delta$. Then

$$d(f_{n_j}(z), f_{n_k}(z)) \leq d(f_{n_j}(z), f_{n_j}(\zeta_m)) + d(f_{n_j}(\zeta_m), f_{n_k}(\zeta_m)) + d(f_{n_k}(\zeta_m), f_{n_k}(z)).$$

So for some J, if $j, k > J$ then this distance $d(f_{n_j}(z), f_{n_k}(z)) < 3\epsilon$. Since $\overline{\mathcal{S}_z}$ is compact, $\lim_{j\to\infty} f_{n_j}(z)$ exists for all $\in \Omega$. And actually, we've shown that the limit exists uniformly; given $\epsilon > 0$, there's a J such that $\sup_{z\in E} d(f_{n_j}(z), f^*(z)) < 3\epsilon$ if $j > J$.

By compactness, we can pick M balls $B(\zeta_{m_l}, \delta)$ to cover. $\diamond$

[23] i.e., when $j > k$.

Last time we wre talking about $\{f : \Omega \to S\} \supset \mathcal{F}$; and for now, $S = \mathbb{C}$.

The Arzela-Ascoli theorem says that $\mathcal{F}$ is precompact $\iff$

1. $\mathcal{F}$ is equicontinuous.
2. $\{f(z) | f \in \mathcal{F}\}$ is bounded for each $z \in \Omega$.

Let's say we give an ϵ. Then for each point $\in \Omega$ there's a δ, so that if $|z-w| < \delta$ then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$. There's some number $M_z = \max\{|f(z)| : f \in \mathcal{F}\}$. In the set $|z-w| < \delta$, $|f(w)| \leq |f(w) - f(z)| + |f(z)| \leq M_z + \epsilon$. So we can replace condition (2) with the following:

2' For every compact set $K \overset{\text{cpt}}{\subset} \Omega$ there's a constant M_K so that $|f(z)| < M_K$ for all $z \in K$, $f \in \mathcal{F}$.

Theorem [Montel] If $\mathcal{F}$ is a family of analytic functions on ω an open subset of $\mathbb{C}$ so that $\mathcal{F}$ is locally uniformly bounded, then $\mathcal{F}$ is precompact.

$$
\begin{aligned}
|f(z) - f(w)| &= \left| \int_w^f (\zeta) d\zeta \right| \\
&\leq \int_w^z |f'(\zeta)||d\zeta| \\
&\leq \max_{s \in (w,z)} |f'(z)||z-w|. \\
f'(z) &= \frac{1}{2\pi i} \int_{|\zeta - \eta_0| = r} \frac{f(\zeta) d\zeta}{(\zeta - z)^2} \\
& \qquad \text{where we assume that } |\zeta - \zeta_0| \leq r \overset{\text{cpt}}{\subset} \Omega
\end{aligned}
$$

For $|z - \zeta_0| < \frac{r}{2}$ we have the estimate

$$
|f'(z)| \leq 4 \frac{c \max_{|\zeta - \zeta_0| = r} |f(\zeta)|}{r}.
$$

So there's an M such that $|f'(z)| \leq \frac{M}{r}$ for all $f \in \mathcal{F}$, $|z - \zeta_0| < \frac{r}{2}$.$\diamond$[?]

Hurwitz Theorem Suppose $\{f_n(z)\}$ is a sequence of holomorphic functions in Ω (a connected open set) that vanish once in Ω, and $f = \lim_{n\to\infty} f_{n_k}(z)$ is the limit of a subsequence (in the locally uniform topology). Then either

1. $f \equiv 0$.
2. f has at most one zero.

Proof Simple consequence of argument principle.$\diamond$

Recall that a domain $\Omega \subset \mathbb{C}$ is simply connected provided $n(\gamma, a) = 0$ for all $a \in \Omega^c$, $\Gamma \overset{\text{cpt}}{\subset} \Omega$. If $\phi(z)$ is analytic and nowhere vanishing in Ω, then $\log\phi(z)$ can bedefined as a holomorphic function; for $\int_{\zeta_0}^{\zeta} \frac{\phi'(z)}{\phi(z)}dz$ is well-defined.

If the logarithm of a function, then the n^{th} root, $\exp\frac{1}{n}\log\phi$ is also well-defined for all n.

Riemann Mapping Theorem If $\Omega \subset \mathbb{C}$ is simply conected and $\Omega \neq \mathbb{C}$, then there exists a holomorphic map $f : \Omega \to D_1(0)$ which is 1-1 and onto.

Suppose there exists such a function $f : \Omega \to D_1(0)$, and another one f'. Then $f \circ (f')^{-1} \in$ **Möb**, and thus it's $e^{i\theta}\frac{z-a}{1-z\overline{a}}$. So if we have one such mapping f, then all others can be written as $f'(z) = e^{i\theta}\frac{f(z)-a}{1-\overline{a}f(z)}$.

The first thing we want to do is normalize; choose $f(z_0) = 0$ in $D_1(0)$.[24] The proof that we're working on is due to Koebe.[25]

Consider the class of maps $\mathcal{F}$ such that

1. $f : \Omega \to D_1(0)$.
2. f is 1-1.
3. $f(z_0) = 0$.
4. $f'(z_0) > 0$.

We will show:

[24] This stops the map from creeping out to the boundary and beyond.

[25] It would be difficult to overestimate the importance of Koebe's work, but he was consistently able to do it.

1. $\mathcal{F} \neq \emptyset$.
2. If $m = \max\{f'(z_0) : f \in \mathcal{F}\}$, then there's a function $f_0 \in \mathcal{F}$ such that $f'(z_0) = m$.
3. $f_0 : \Omega \to D_1(0)$ is onto.

Let $a \in \Omega^c$. Well, $h(z) = \sqrt{z-a}$ is well-defined in Ω. This is 1-1. Furthermore, $\sqrt{z-a} = -\sqrt{w-a}$ never occurs; for we square, and get $z = w$, and then you get garbage.

There exists a $\rho > 0$ so that $|h(z_0 0 - w| < \rho$ lies in the image of $h(\Omega)$. Then $|h(z) + h(z_0)| < \rho$ for all $z \in \Omega$.[26] So $2|h(z_0)| > \rho$. Define

$$g_0(z) = \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \frac{h(z_0)}{h'(z_0)} \frac{h(z) - h(z_0)}{h(z) + h(z_0)}.$$

Actually, $g_0(z) = c\frac{w-h(z_0)}{w+h(z_0)} \circ h$. Then $g_0 \in \mathcal{F}$, and $\mathcal{F} \neq \emptyset$.

Observe that $l(f) = f'(z_0)$ is a continuous function on $\mathcal{F}$; for $f'(z_0) = \int \frac{f(z)dz}{(z-z_0)^2}\frac{1}{2\pi i}$ where $\{|-z_0| \leq r\} \stackrel{\text{cpt}}{\subset} \Omega$. The topology on $\mathcal{F}$ is that of locally uniform convergence. $\mathcal{F}$ is a precompact set, so there is a sequence $\{f_n\}$ so that $\lim_{n\to\infty} l(f_n) = \sup_{f \in \mathcal{F}} l(f)$. Since $\mathcal{F}$ is normal, $\{f_n\}$ has a convergent subsequence $f_{n_k} \to f$ locally uniformly. Then $f'(z_0) = \sup_{f\in\mathcal{F}} l(f) > 0$. So $f \not\equiv 0$. $f_{n_k}(z)$ is 1-1, so for every $z_1 \in \Omega$, $f_{n_k}(z) - f_{n_k}(z_1)$ is nonvanishing on $\Omega - \{z_1\}$. The function f is nonconstant; by Hurwitz, $f(z) - f(z_1) \neq 0$ at any pont of $\Omega - \{z_1\}$. Since z_1 arbitrary, f is 1-1.

Suppose there's a w_0 so that $f(z) - w_0 \neq 0$ for all $z \in \Omega$. Then we can take

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w_0} f(z)}}.$$

This function is defined in Ω, because Ω is simply connected and $\frac{f(z)-w_0}{1-\overline{w_0}f(z)}$ does not vanish.[27] Define

$$G(z) = \frac{|F'(z_0)|}{F(z_0)} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}.$$

[26] If a number is in the image, then minus that number is not in the image.

[27] The idea is, you know the function should have a greater-than-zero derivative; by taking the square root [of a number < 1], you get something bigger than 1.

G is 1-1, $|G| < 1$, $G(z_0) = 0$. Of course,

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{|w_0|}} f'(z_0) > f'(z_0).$$

So we've constructed a function with a larger derivative at z_0, a contradiction; since f was where the derivative was maximized. The map f is uniquely determined by these properties; for if there were two, say, f_1 and f_2 then $f_2 \circ f_1^{-1}(z) : D_1 \to D_1$ is 1-1 and onto such that $f_2 \circ f_1^{-1}(0)0$, or $f_2 \circ f_1^{-1}(z) = e^{i\theta} z$, and $\theta = 0.\diamond$

If $\partial\Omega$ is a simple closed cuve ($\exists f : S^1 \to \mathbb{C}$ with f continuous and 1-1, $\partial\Omega = f(S^1)$), then the mapping function extends coninuously to $\partial\Omega$.

Proposition If $f : \Omega \to D_1(0)$ is a Riemann mapping function, then $\lim_{z\to\partial\Omega} |f(z)| = 1$.

Proof Think of it as $z_n \to \partial\Omega$. Given any compact set $K \overset{\text{cpt}}{\subset} \Omega$, there's an N so that $z_\in \Omega - K$ for $n > N$. For each $r < 1$ let $f^{-1}(\overline{D_r}) = K_r \overset{\text{cpt}}{\subset} \Omega$. For any $r < 1$ there's an N such that ${}_n \in \Omega - K_r$ for $n > N$. So $\liminf_{n\to\infty} |f(z_n)| \geq r$; so $\lim_{n\to\infty} |f(z_n)| = 1$.[28]$\diamond$

Let's now think of the map $f : \Omega \to D_1(0)$. Suppose $\partial\Omega$ contains a straight line segment l. We'll assume that, for each $p \in l$, there's an $r > 0$ so that $D_p(r) \cap \partial\Omega = l \cap D_p(r)$; this rules out a lot of really gross cases, like the dragon's teeth thing. This is a free boundary arc.

There are two possibilities; either Ω is just on one side of l, or on both. For now assume that we're working with a 1-sided [free] boundary arc. Further assume that $l \subset \mathbb{R}$. We know that as $z \to l$, $|f(z)| \to 1$. Look at $\log f(z)$. If we pick a disk around $p \in l$ small enough, f doesn't vanish anywhere on the circle around p [at least, the part of it inside Ω]; so we can write $i \log f(z) = \log |f(z)| + \arg f$. We can reflect across the arc. So $f(z)$ has an analytic extenison to the lower half-disk.

The function gives a monotone parameterization of the disk. If $x_0 \in l$, then $f'(x_0) \neq 0$; for if it were zero, then $f(z) = (z - x_0)^n h(z)$ near $z = x_0$. Some geometric argument, and $n \leq 1$. We know that $\frac{\partial}{\partial y} \log |f| = -\frac{\partial}{\partial x} \arg f$. *Blah.* $\partial_x \arg f > 0$; $f|_l$ gives a strictly monotone parameterization of an arc of S^1.

[28]How do we know that the inverse image of a compact set is compact?

If $\Omega \subset \mathbb{C}$ is simply connected, $\partial\Omega \neq \emptyset$, then there's a holomorphic map $f : \Omega \to D_1(0)$ which is 1-1 and onto.

Simple connectivity is on the Riemann sphere. For example, the map $z \mapsto (\frac{1}{z} + z)$ maps the unit disk onto the complement of a [real] line segment. So the image Ω can be thought of as a simply connected subset *of the sphere.* We say that if $\Omega \subset \hat{\mathbb{C}}$, Ω is simply connected if $\partial\Omega \neq \{p\}$.[?] There's actually no map $f : \mathbb{C} \to D_1(0)$. It's plausible that, on the Riemann sphere, the boundary is connected.

The proof of the Riemann mapping theorem isn't particularly effective. This is actually a current site of research.

Today we'll play with polygonal regions. Suppose we have a finite number of vertices $p_1, \cdots, p_n$. At each vertex we have an [interior] angle, $\alpha_i\pi$ with $0 \leq \alpha_i \leq 2$. We'll map it to the upper half plane, but the principle is the same. We know that at the boundary, the function has an analytic continuation across the arc. We don't know what happens to a vertex. Let f : poly $\to$ upper half plane. We can translate so that the vertex is at the origin. Look at the map $\zeta \mapsto e^{i\theta}\zeta^{\alpha_1}$. It maps a neighborhood N of the origin to a region with the proper angle. So we have $g(\zeta) = f(e^{i\theta}z(\alpha_1)) : N \to \mathbb{H}^2_+$. Note that $\Im g(\zeta) \to 0$ as $\Im\zeta \to 0$.

The Schwarz reflection principle tells us that $g(\zeta)$ has an extension as an analytic function to $N^- = \{z : \overline{z} \in N\}$. Can say $N^+ = N$. Let $G(\zeta)$ denote this extension; then $G : N^+ \to \mathbb{H}^2_+$, $G : N^- \to \mathbb{H}^2_-$. Call this condition $(*)$.

This implies that $G'(x) \neq 0$ on $\overline{N^+} \cap \mathbb{R}$. Suppose not; so $G'(x_0) = 0 \Rightarrow G(z) = (z - x_0)^n h(z) + K(x_0)$, where $h(x_0) \neq 0$. If $n > 1$, then $(*)$ cannot hold.

From this, we conclude that $G'(x) \neq 0$ on $\mathbb{R} \cap N^+$. This implies that $f(z)$ restricted to ∂P near the vertex P_1 gives a strictly monotone map to $\mathbb{R}$. This tells us that at least locally the vertex P_i corresponds to a unique point on $\partial\mathbb{H}^2_+$.

Arguing in this fashion we easily show that f has a continuous extension to $P \cup \partial P$and that $f : \partial P \to \partial\mathbb{H}^2_+ \cup \{\infty\}$ in a strictly monotone fashion. As you move around the boundary, you move in the same direction on the real axis.

Thus far we only know that it's locally 1-1; gotta show that nothing bad happens, e.g., going around the disk twice. So compute

$$\int_{\partial P} \frac{f'(z)}{f(z)} \frac{dz}{2\pi i}.$$

The nervous student can integrate on something within ϵ of the boundary; call it ∂P_ϵ. This integral is the number of points where $f = 0$. On the other hand, it's the number of times $f(\partial P)$ goes around the circle. We know that the number of zeros is one, and so it goes around exactly once.

By computing the winding number of $f|_{\partial P}$, we conclude that $f : \partial P \to \partial\mathbb{H}^2_+ \cup \{\infty\}$ is actually a continuous, 1-1 onto map.

Each vertex of the polygon corresponds to exactly one point on the boundary of $\mathbb{H}^2_+$. So there's a sequence of points $\{x_1, \cdots, x_m\}$ with $f(Q_i) = x_m$, where the Q_i are the vertices of the polygon. Let's normalize so that all of the x_i's are finite.

Let $F = f^{-1} : \mathbb{H}^2_+ \to$ poly. We know that $(F(z))^{1/\alpha_i}$ has an analytic extension to the lower half plane, $\mathbb{H}^2_-$. This means that $(F(z))^{1/\alpha_i} = (z - x_i)h_i(z)$ with $h_i(0) \neq 0$ and $h_1(z)$ is analytic in a neighborhood of x_i. If we restrict to the upper half plane – that is, select a branch – then $F(z) = (z - x_i)^{\alpha_i}\widetilde{h}_i(z)$ near x_i, where $\widetilde{h}_i$ is an analytic function. Recap: locally,

$$F(z) = (z - x - i)^{\alpha_i}\widetilde{h}_i(z) + P_i.$$

Differentiate to get rid of some of the localized stuff;

$$\begin{aligned}
F'(z) &= \alpha_i(z - x_i)^{\alpha_i - 1}\widetilde{h}_i(z) + (z - x_i)^{\alpha_i}\widetilde{h}_i{}'(z) \\
(\log F'(z))' &= \frac{\alpha_i - 1}{z - x_i} + \frac{d}{dz}\log[\widetilde{h}_i(z) + (z - x_i)\widetilde{h}_i{}'(z)] \\
\text{But } [\log F'(z)]' &= \frac{F''}{F'} \\
\frac{F''}{F'} &= \frac{\alpha_i - 1}{z - x_i} + \text{ analytic near to } z = x_i. \\
\frac{F''}{F'} &= \frac{\widetilde{F}''}{\widetilde{F}'}
\end{aligned}$$

where $\widetilde{F'}$ is F after a double reflection, i.e., $\widetilde{F} = c_1F + c_2$. Why the ratio? You look at the function, and you have to look at the action of Euclidean motions on the image. You want to find some quantity which is the same for all those functions; F''/F' is the simplest such function for a mapping which is independent of which particular domain you map onto if you look at all possible rotations and translates.

This argument shows that $\frac{F''}{F'}$ is actually an analytic function in all of $\mathbb{C}$, where we compute it in $\mathbb{H}^2_-$ using any Schwarz reflection of F. So we know that

$$G(z) = \frac{F''}{F'} - \sum_{i=1}^{m}\frac{\alpha_i - 1}{z - x_i}$$

is actually analytic on $\mathbb{C}$. Subtracting off these terms leaves an analytic function. [We took out the singular part in every neighborhood.]

Let's investigate the behavior at ∞. Let $\phi(z) = F(-\frac{1}{z})$ for $z \in \mathbb{H}^2_+$. We've assumed that $x_i \neq \infty$ for all i. There's an i with $x_i < 0 < x_{i+1}$. Clearly [?] $\phi(0)$ is some point on ∂P, and $\phi(z)$ has an analytic extension by reflection to a neighborhood of zero. Again, $\phi'(0) \neq 0$. We can therefore write

$$\begin{aligned} \phi(z) &= \phi(0) + \sum_{I=1}^{n} a_i z^i \text{ where } a_i \neq 0 \\ F(z) &= \phi(0) + \sum a_i (-\frac{1}{z})^n. \end{aligned}$$

We obtain by a simple computation that

$$\frac{F''}{F'}(z) = -\frac{2}{z} + O(\frac{1}{z^2})$$

as $z \to \infty$. So

$$C = \frac{F''}{F'} - \sum_{I=1}^{m} \frac{\alpha_i 1}{z - x_i}$$

is analytic in $\hat{\mathbb{C}}$, and as such is constant; can eyeball it and see that the constant is zero.

We've determined that

$$\begin{aligned} \frac{F''}{F'}(z) &= \sum_{i=1}^{m} \frac{\alpha_i - 1}{z - x_i} \\ \partial_z \log F' &= \partial_z \sum (\alpha_i - 1) \log(z - x_i). \end{aligned}$$

Since we're about to integrate log instead of differentiating, we have to pick a branch; restrict $z \in \mathbb{H}^2_+$ and choose some branch of the logarithm. Then

$$\begin{aligned} \log F' &= \sum (\alpha_i - 1) \log(z - x_i) + C_1. \\ F'(z) &= C_1 \prod (z - x_i)^{\alpha_i - 1} \\ F(z) &= C_1 \int_{z_0}^{z} \frac{dw}{(w - x_1)^{1+\alpha_1} \cdots (w - x_m)^{1-\alpha_m}} + C_2. \end{aligned}$$

This last formula is what we were looking for; it's called the Schwarz-Christoffel transformation.◇

Unfortunately, some choices of $x_1, \cdots, x_m$, you get mappings on to crossing domains [pentagrams]. Figuring out the relation in general is unsolved; this is the accessory parameter problem.

Suppose we have $F : \overline{D_1}(0) \to \overline{\Omega}$ If (z_1, z_2, z_3), (w_1, w_2, w_3) are tuples of points on $\partial D_1(0)$ then there is a Möbius transformation T so that $Tz_i = w_i$. It's clear that $T(\partial D_1) = \partial D_1$, and the interior goes to the interior.[29] Now pick three points in the image $F(w_i)$. Then $F \circ T : z_i \mapsto F(w_i)$. In general, we can specify arbitrarily the image of three distinct points on the boundary. And in general, this is the best you can do.

So suppose our polygon is a triangle, and we want to send the vertices to 0, 1 and ∞. Then the function is

$$f(z) = c_1 \int_0^z z^{\alpha_0 1}(1-z)^{\beta-1} + C_2$$

where the angles are α, β and γ. Since this is a triangle, $\alpha+\beta+\gamma = 1$. Note that in general, $\sum(1-\alpha_i)\pi = 2\pi$.

We can determine the length of the side that the interval $[0,1]$ is mapped onto. This length, c, is

$$\begin{aligned}
\int_0^1 |f'(z)|dz &= \int_0^1 \left|z^{\alpha-1}(1-z)^{\beta-1}\right|dz \\
&= \int_0^1 \rho^{\alpha-1}(1-\rho)^{\beta-1}d\rho \\
&= B(\alpha,\beta) \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
\text{But } \Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin \pi x} \\
\text{and } \alpha+\beta+\gamma = 1 & \\
c &= \frac{1}{\pi} \sin \pi\gamma \Gamma(|alpha \Gamma(\beta)\Gamma(\gamma)
\end{aligned}$$

Of course, there's the law of sines,

$$\frac{a}{\sin \pi\alpha} = \frac{b}{\sin \pi\beta} = \frac{c}{\sin \pi\gamma}.$$

Can use this to compute the lengths of the sides in terms of the angles.

[29]Provided the w_i's and z_i's are in the same cyclic order.

To date we've concentrated on holomorphic functions, the solutions to the equation $\overline{\partial} u = 0$. Now we'll take a look at $\overline{\partial} u = f$. Let $\mathcal{H}(\Omega)$, for Ω an open set in $\mathbb{C}$, be the holomorphic functions on Ω with the topology of locally uniform convergence. Let $\mathcal{O}_K$ be holomorphic functions defined on a neighborhood of K. Consider $\mathcal{H}(D_1(0))$. We know that any $u \in \mathcal{H}(D_1(0))$ has a power series expansion $u(z) = \sum_0^\infty a_n z^n$. Let $\{r_n\} \nearrow 1$, i.e., an increasing sequence going to 1. Let $\|u\|_n = \sup_{D_{r_n}(0)} |u(z)|$. Then we can put on a metric

$$d(u, v) = \sum_0^\infty u^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n}.$$

And this metric does indeed induce the desired topology.

Given a function $u \in \mathcal{H}(D_1(0))$ and an $\epsilon > 0$, can we find a polynomial $p_\epsilon(z)$ so that $d(u, p_\epsilon) < \epsilon$?

The answer is yes. Let $p_\epsilon(z) = \sum_0^{N_\epsilon} a_n z^n$. The grungy term in $d(u, v)$ is no bigger than 1. So $\|p_\epsilon - u\|_{D_{r_{N_\epsilon}}(0)} < \epsilon$.

Another way of phrasing this is, if we fix any compact subset of the unit disk, we can find a function which uniformly approximates the function to any desired degree of accuracy on that compact subset.

What if we look at functions in an annular region, $\mathcal{H}(A_{rR})$? Well, $u \in \mathcal{H}(A_{rR})$ is given by a Laurent series; $u(z) = \sum_{-\infty}^\infty a_n z^n$. Such a function won't be polynomial approximable. F'rinstance, suppose that for every $\epsilon \exists p_\epsilon$ so that $\|u - p_\epsilon\|_{A_{r+\delta, R-\delta}} < \epsilon$. It should be clear that $p_\epsilon(z) \to u(z)$ uniformly on $\partial D_{r+2\delta}$. Now, p_ϵ is holomorphic everywhere, including in the hole in the middle of the annulus. Using, say, the Cauchy integral formula – or even just the maximum principle – the p_ϵ must converge uniformly to u on the whole disk. So u is holomorphic on the whole disk.

So maybe polynomials aren't the right thing to do. Try $q_\epsilon(z) = \sum_{n=-N_\epsilon}^{N_\epsilon} a_n z^n$. Can choose N_ϵ so that $\|u - q_\epsilon\|_{A_{r+\delta, R-\delta}} < \epsilon$. Can approximate uniformly in an annulus with so-called Laurent polynomials. Note that the q_ϵ are holomorphic everywhere off the origin.

Partitions of unity Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of a set $\Omega \subset \mathbb{C}$. A partition of unity relative to Ω is a family of functions $\{\phi_i\}_{i \in I}$ such that

1. Each ϕ_i is smooth, nonnegative, compactly supported.
2. $\operatorname{supp} \phi_i \subset U_i$.

3. For any compact set $K \subset \Omega$, the set $I_K = \{i | K \cap \operatorname{supp} \phi_i \neq \emptyset\}$ is finite.
   item $\sum_{i \in I} \phi_i(x) = 1$ for all $x \in \Omega$.

Consider the function

$$f(t) = \begin{cases} e^{\frac{-1}{1-t}} & t < 1 \\ 0 & t \geq 1 \end{cases}.$$

This is infinitely differentiable. The real reason is that $e^{-x} \to 0$ as $x \to \infty$ faster than $x^n \to \infty$ for all n.

We define $\phi(x) = f(|x|^2)$. Then $\phi(\frac{x-a}{\delta})$ is smooth and supported in the ball of radius δ centered at $x = a$, and $\phi \geq 0$.

Let $\{V_j\}_{j \in J}$ cover by a countable collection of balls. We can assume without loss of generality that at most finitely many of the V_j's intersect a given V_{j_0}. Then we define a map $\tau : J \to I$ so that $V_j \subset U_{\tau(j)}$. Let ϕ_j be a function as above so that $\phi_j \geq 0$, $\operatorname{supp} \phi_j = V_j$. Now, $\sum_{j \in J} \phi_j(x)$ is finite by assumption.[30] So $\sum_{j \in J} \phi_j(x) = \phi(x) > 0$ for $x \in \Omega$. Set $\chi_j(x) = \frac{\phi_j(x)}{\phi(x)}$. So we define $\psi_i(s) = \sum_{j \in \tau^{-1}(i)} \chi_j(x)$. And it should be clear that $\sum \psi_i = \sum \chi_j = 1$ for all points in Ω. That's how you construct a partition of unity.

As an application, let $X \subset \mathbb{R}^n$ be a closed set, $U \supset X$ an open set. Then there is a function ϕ so that ϕ is smooth, $\phi = 1$ on X, $\pi = 0$ on $\mathbb{R}^n - U$. For let our cover of $\mathbb{R}^n$ be U and $V = \mathbb{R}^n - X$. So there's a partition of unity $\{\phi_U, \phi_V\}$ satisfying $\phi_U(x) + \phi_v(x) = 1$. For $x \in X$, $\phi_U(x) = 1$.

Suppose that $X_1, X_2 \subset \Omega$ closed and disjoint. Let $\phi_i \in C^\infty(\Omega)$. Then there's a function $\phi \in C^\infty(\Omega)$ so that $\phi|_{X_i} = \phi_i$. Choose an open set U so that $X_1 \overset{\text{cpt}}{\subset} U$ and $X_2 \cap U = \emptyset$. Then choose a partition of unity as above. Let $\alpha = \phi_U(x)$. Let $\phi = \alpha\phi_1 + (1-\alpha)\phi_2$.

Let R and R' be two closed rectangles in $\mathbb{C}$, $R' \subset \overset{\circ}{R}$. Let U be an open set containing $R - R'$, and $\phi \in C^\infty(U)$. Then

$$2i \int\int \frac{\partial \phi}{\partial \overline{z}} dxdy = \int_{\partial R} \phi dz - \int_{\partial R} \phi dz.$$

For let $K = R - R'$ and α a function in $C_0^\infty(U)$, such that $\alpha = 1$ on K. Then $\psi(z) = \alpha(z)\phi(z)$ is defined on a neighbohood of R. Then

[30] Only finitely many of the V_j's cover any particular point.

$$\begin{aligned} 2i\int\int_R \frac{\partial\psi}{\partial\bar{z}}dxdy &= \int_{\partial R}\psi(z)dz \\ &= \int_{\partial R}\phi(z)dz. \end{aligned}$$

Similarly, $2i\int\int_{R'}\frac{\partial\psi}{\partial\bar{z}}dxdy = \int_{\partial R'}\phi(z)dz$. So then

$$2i\int\int_{R-R'}\frac{\partial\phi}{\partial\bar{z}}dxdy = \int_{\partial R}\phi(z)dz - \int_{\partial R'}\phi(z)dz.$$

Lemma Let R be a rectangle. Then

$$\int\int_R \frac{dxdy}{|z|} < \infty.$$

Why is this true? Well,

$$\int\int_R \frac{dxdy}{|z|} = \int\int_R \frac{rdrd\theta}{r}.$$

Theorem Let $\phi \in C_0\infty(\mathbb{C})$. Then for any number $w \in \mathbb{C}$, we have

$$\int\int_{\mathbb{C}} \frac{\partial\phi}{\partial\bar{z}}\frac{dxdy}{z-w} = -\pi\phi(w).$$

Equivalently, the linear operator given by integrating against $\frac{1}{\pi(w-z)}$ is the inverse to $\frac{\partial}{\partial\bar{z}}$.

Proof

$$\int\int_{\mathbb{C}} \frac{\partial\phi}{\partial\bar{z}}\frac{1}{z-w}dxdy \quad = \quad \int_{\mathbb{C}}\frac{\partial\phi}{\partial\bar{z}}\frac{1}{z}dxdy$$

Choose a large rectangle R so that $\operatorname{supp}\phi(z+w) \overset{\text{cpt}}{\subset} R$. Let $R_\epsilon = [-\epsilon,\epsilon]\times[-\epsilon,\epsilon]$. Then

$$\begin{aligned}
\frac{\partial\phi}{\partial\overline{z}}(z+w)\frac{1}{z} &= \frac{\partial}{\partial\overline{z}}(\frac{\phi(z+w)}{z}) \text{ in } R-R_\epsilon \\
2i\int\int_{R-R_\epsilon}\frac{\partial\phi}{\partial\overline{z}}(z+w)\frac{dxdy}{z} &= -\int_{\partial R_\epsilon}\frac{\phi(z+w)}{z}dz \\
&= -\int_{\partial R_\epsilon}\phi(w)\frac{dz}{z}+\int_{\partial R_\epsilon}O(\epsilon)\frac{dz}{z} \\
&= -2\pi i\phi(w)+O(\epsilon).
\end{aligned}$$

Let $\epsilon \to 0$ using the fact that $\frac{1}{z}$ is locally integrable to conclude that

$$\int\int\frac{\partial\phi}{\partial\overline{z}}\frac{dxdy}{z-w} = -\pi\phi(w).$$

◇

Consider this from a functional point of view. Let $D\phi = \frac{\partial\phi}{\partial\overline{z}}$. Let $L\psi = -\frac{1}{\pi}\int\psi(z)\frac{dxdy}{z-w}$. Clearly, $LD = I$. Is it also true that $DL = I$? Maybe. We certainly have a big kernel; but in infinite-dimensional spaces that's not necessarily so bad.

Theorem Let $\phi \in C_0^\infty(\mathbb{C})$, and define $u(w) = -\frac{1}{\pi}\int\int\frac{\phi(\zeta)d\xi\eta}{\zeta-w}$. Here, $\zeta = \xi + i\eta$. Then $u \in C^\infty(\mathbb{C})$, and $\frac{\partial u}{\partial\overline{z}} = \phi$.

Proof We can start by rewriting

$$u(z) = \frac{1}{\pi}\int\int\frac{\phi(\zeta+z)}{\zeta}d\xi d\eta.$$

$\frac{1}{\zeta}$ integrable $\Rightarrow u(z)$ is continuous.[31] We can look at difference quotients.

$$\begin{aligned}
\frac{u(z+h)-u(z)}{h} &= -\frac{1}{\pi}\int\int_{\mathbb{C}}\frac{\phi(\zeta+z-h)-\phi(\zeta+z)}{h}\frac{1}{\eta}d\xi d\eta \\
\text{Use Lebesgue dominated convergence} & \\
\text{to pass inside integral} & \\
h &\in \mathbb{R} \\
\frac{\partial u}{\partial x} &= -\frac{1}{\pi}\int\int_{\mathbb{C}}\frac{\partial\phi}{\partial\xi}(z+\zeta)\frac{1}{\zeta}d\xi d\eta.
\end{aligned}$$

[31] A continuous fuction of compact support, multiplied by a locally integrable function, then the integral of this composition is continuous.

Similarly, one shows that

$$\frac{\partial u}{\partial y} = -\frac{1}{\pi}\int\int_{\mathbb{C}} \frac{\partial \phi}{\partial \eta}(z+\zeta)\frac{1}{\zeta}d\xi d\eta.$$

We can iterate this process to pick off any partial derivative. So u is a smooth function. Now, $\frac{\partial u}{\partial \bar{z}} = -\frac{1}{\pi}\int\int \frac{\partial \phi}{\partial \zeta}(z+\zeta)\frac{d\xi d\eta}{\zeta}$. The previous theorem tells us that this is actually equal to $\phi(z)$.$\diamond$

We now see that $DL = I$ the identity.

Let Ω be an open subset of $\mathbb{C}$, K a compact subset of Ω. Let $\alpha \in C_0^\infty(\Omega)$ with $\alpha = 1$ on K. Then for any $f \in \mathcal{H}(\Omega)$, we have

$$f(z) = -\frac{1}{\pi}\int\int_{\Omega} \frac{\partial \alpha}{\partial \bar{\zeta}} f(\zeta)\frac{1}{\zeta - z}d\xi d\eta$$

for all $z \in K$.

This is a little like the Cauchy integral formula.

The proof is all but immediate; let $\phi(z) = \begin{cases} \alpha(z)f(z) & z \in \Omega \\ 0 & z \in \mathbb{C} - \Omega \end{cases}$. The result just proved says that for any $z \in \mathbb{C}$,

$$\begin{aligned}\phi(z) &= -\frac{1}{\pi}\int\int \frac{\partial}{\partial \bar{\zeta}}\phi(\eta)\frac{d\xi d\eta}{\zeta - z} \\ &= -\frac{1}{\pi}\int\int \frac{\partial \alpha}{\partial \bar{\zeta}}(\zeta)f(\zeta)\frac{d\xi d\eta}{\zeta - z}.\end{aligned}$$

And if $z \in K$ then $\phi(z) = f(z)$. $\diamond$

Note that we've got this nice formula without any winding numbers; we just need some C^∞ function. The integral takes place – if you work things out – somewhere between K and the boundary of the support of α.

Let $(X, \|\cdot\|)$ be a normed, complete vector space; that is, a Banach space. If $Y \subset X$ is a closed subspace then $l : Y \to \mathbb{C}$ is a bounded linear functional provided that

1. Linearity $l(\alpha x + \beta y) = \alpha l(x) + \beta l(y)$ for $\alpha, \beta \in \mathbb{C}$, $x, y \in Y$.
2. Boundedness $\sup_{x \in Y} \frac{|l(x)|}{\|x\|} = \|l\| < \infty$.

Remember that, in infinite-dimensional spaces, the unit ball isn't compact. The only reason Hahn-Banach isn't a triviality is because of the infinite dimensionality.

There are a number of formulations of the Hahn-Banach theorem. We'll use this one.

Hahn-Banach Theorem There is a linear functional $L : X \to \mathbb{C}$ so that

1. $L|_Y = l$.
2. $|||L|||_X = \|l\|_Y$.

Corollary Suppose we have $Y_1 \subset Y_2 \subset X$, Y_i closed. Then $Y_1 = Y_2 \iff$ every linear functional l on X which vanishes on Y_1 also vanishes on Y_2.

($\Rightarrow$) A triviality.

($\Leftarrow$) This is tantamount to showing that the Y_i can be separated with a linear functional. If $Y_1 \neq Y_2$ then there's a $v \in Y_2 - Y$. Define $\widetilde{Y_1} = Y_1 + \mathbb{C}\{v\}$. $\widetilde{Y_1}$ is closed. If $y_1 + c_1 v = y_2 + c_2 v$ then $y_1 = y_2$ and $c_1 = c_2$. We thus have $Y_1 \ni y_1 - y_{=}(c_2 - c_1)v \in Y_2 - Y_1$, and everything is zero. Define $l(y + cv) = c$. It's a bounded linear functional defined on $\widetilde{Y_1}$. So there's an L defined on all of X so that $L|_{\widetilde{Y_1}} = l$, But $L|_{Y_1} = 0$. On the other hand, $L|_{Y_1} \neq 0$, since $L(v) \neq 0$. $\diamond$

Last time we were talking about the Hahn-Banach theorem. You can tell whether two spaces $L \subset M$ actually agree by looking at whether all linear functionals vanishing on one vanish on the other. Recall that if $Y \subset X \subset B$ a Banach space, then $Y = X \iff \{\lambda \in B^+ : \lambda|_Y = 0\} = \{\lambda \in B^+ : \lambda|_X = 0\}$. We'll apply this with $K \subset \mathbb{C}$ compact. $\mathcal{C}(K)$ is the continuous functions on K with the sup-norm topology. We describe the dual with

Riesz-Markov Theorem $\mathcal{C}(K)^*$ is the set of finite signed Baire measures.

The Baire sets is the smallest σ-algebra of sets so that every continuous function is measurable with respect to that σ-algebra. A Baire measure is a measure defined on the Baire sets. A signed measure can be written as $d\mu = d\mu_+ - d\mu_-$ where $d\mu_\pm$ are ordinary Baire measures. And its finite if $\int_K d\mu_+ + d\mu_- < \infty$. So the theorem says that for every $l \in \mathcal{C}(K)^*$, there's a finite signed Baire measure $d\mu$ so that $l(f) = \int_K f(x)d\mu(x)$.

Theorem [Runge] Let $\Omega \subset \mathbb{C}$ be an open set, $K \subset \Omega$ compact. Then the following conditions on K and Ω are equivalent.

1. Every function analytic in a neighborhood of K can be uniformly approximated on K by functions in $\mathcal{H}(\Omega)$.
2. The open set $\Omega - K$ has no component which is relatively compact in Ω.
3. For every $z \in \Omega - K$, there's a function $f \in \mathcal{H}(\Omega)$ so that $|f(z)| > \sup_K |f|$.

If $K \subset \mathbb{C}$ is compact, we define

$$\widehat{K}_{\text{convex}} = \bigcap_{C | C \supset K, C \text{convex}} C.$$

Can also define $\widehat{K}_{\text{cvx}}$ by

$$\widehat{K}_{\text{cvx}} = \{z : \Re az \leq \sup_{z \in K} \Re az \forall a \in \mathbb{C}\}.$$

This says take all the points in the plane where linear functions may be estimated by their values on K. We've thus obtained an analytic definition of the convex hull. If $z \notin \widehat{K}_{\text{cvx}}$, then there's an a so that $\Re az > \sup_{z \in K} \Re az$. That's the connection with the third part of the Runge theorem.

Proof Runge's original theorem can be proved just using power series. But we're not going to do that.

$(iii) \Rightarrow (ii)$. Assume (ii) is false. Then $\Omega - K$ has a component $\mathcal{O}$ so that $\overline{\mathcal{O}} \subset \Omega$ is relatively compact. Therefore, $\partial\mathcal{O} \subset K$.[32] We know that $\sup_{\mathcal{O}} |f| = \sup_{\partial\mathcal{O}} |f|$ for any $f \in \mathcal{H}(\Omega)$. But $\sup_{\partial\mathcal{O}} |f| \le \sup_K |f|$.

$(i) \Rightarrow (ii)$. If (i) holds then for every f analytic in a neighborhood of K, we can find sequence $\{f_n\} \subset \mathcal{H}(\Omega)$ so that $f_n \to f$ uniformly on K. Suppose that there were some $\mathcal{O}$ a relatively compact component of $\Omega - K$. Notice that $\sup_{\overline{\mathcal{O}}} |f_n - f_m| \le \sup_K |f_n - f_m|$. So f_n converges to some function $F(z)$ on $\overline{\mathcal{O}}$ as well. If we chosoe $f(z) = \frac{1}{z-\zeta}$ where $\zeta \in \mathcal{O}$, thne we can find a sequence $\{f_n\} \in \mathcal{H}(\Omega)$ so that $f_n(z) \to \frac{1}{z-\zeta}$ uniformly on K. So if we look at $(z-\zeta)f_n(z)$ on $\mathcal{O}$, it's going to converge again. Since $f_n \to \frac{1}{z-\zeta}$ on $\partial\mathcal{O}$, it follows that $(z-\zeta)F = 1$ on $\partial\mathcal{O}$, and therefore on all of $\mathcal{O}$. A contradiction, sinc $\zeta \in \mathcal{O}$.

$(ii) \Rightarrow (i)$. We'll use the Hahn-Banach theorem. $(ii) \Rightarrow (i)$ follows if we can show that any finite signed Baire measure $d\mu$ wih support on K such that $\int f(z) d\mu(z) = 0$ for all $f \in \mathcal{H}(\Omega)$ also satisfies $\int f(z) d\mu(z) = 0$ for all $f \in O_K$. What we're showing is

$$\overline{\mathcal{H}(\Omega)|_K} = \overline{\mathcal{O}_K|_K}.$$

Define a function $\phi(\zeta) = \int \frac{d\mu(z)}{\zeta - z}$ for all $\zeta \in \mathbb{C} - K$. Some facts are at hand.

1. $\phi(\zeta)$ is an analytic function for $\zeta \in \mathbb{C} - K$. Can prove this with Morera's theorem, and interchange the order of operations.

2. $\zeta \in \mathbb{C} - \Omega \Rightarrow$

$$\phi^{[k]}(\zeta) = k! \int \frac{d\mu(z)}{(-\zeta)^{k+1}};$$

   and this is zero for all $k \ge 0$. For if $\zeta \in \mathbb{C} - \Omega$, then $\frac{1}{z-\zeta}^{k+1} \in \mathcal{H}(\Omega)$.

   Note that this implies $\phi(\zeta) = 0$ on any component of $\mathbb{C} - K$ which intersects $\mathbb{C} - \Omega$. Since $\Omega - K$ has no relatively compact components, the closure of every bounded component of $\Omega - K$ must intersect a component of $\mathbb{C} - \Omega$. For otherwise, the closure would be compact in Ω. Thus, $\phi(z) = 0$ on every bounded component of $\mathbb{C} - K$.

[32] Prove this using the fact that $\mathcal{O}$ is a *maximal* connected set.

$\mathbb{C} - K$ has a unique unbounded component, which may or may not intersect a component of $\mathbb{C} - \Omega$. Since K is compact, if we choose ζ sufficiently large then

$$\begin{aligned} \frac{1}{z-\zeta} &= \frac{1}{\zeta}\frac{1}{\frac{z}{\zeta}-1} \\ &= -\frac{1}{\zeta}\sum_0^\infty (\frac{z}{\zeta})^j \end{aligned}$$

with uniform convergence of the sum on K. For large $|\zeta|$,

$$\phi(\zeta) = -\sum_0^\infty \int \frac{z^j d\mu(z)}{\zeta^{j+1}} = 0.$$

Choose a function $f \in \mathcal{O}_K$. There's an open set W with $\Omega \supset W \supset K$, so that f is actually in $\mathcal{H}(W)$. Can choose a function $\psi \in C_0^\infty(\Omega)$ so that $\psi \equiv 1$ on K.

Try out the Cauchy integral formula, which said that

$$f(z) = \psi(z)f(z) = \int\int f(\zeta)\frac{d\psi}{d\bar{\zeta}}\frac{1}{\zeta - z}dxdy$$

for $z \in K$. So we have

$$\int f(z)d\mu(z) = \int -\frac{1}{\pi}\int\int \frac{f(\zeta)d\psi}{d\bar{\zeta}}\frac{1}{\zeta - z}dxdy$$

Since $\frac{\partial\psi}{\partial\bar{\zeta}} = 0$ on a neighborhood of K, it follows from the Fubini theorem that we can interchange the integrals:

$$\begin{aligned} \int f(z)d\mu(z) &= -\frac{1}{\pi}\int\int \frac{f(\zeta)d\psi}{d\bar{\zeta}}\phi(\zeta)dxdy \\ &= 0. \end{aligned}$$

And this is precisely what we were trying to see. $\diamond$

The advantage of the Cauchy representation is that it let us get away from the shape of K.

In general, to prove something about functions using a representation formula, you only have to prove it about the kernel.

$(ii) \Rightarrow (iii)$. Want to show that if $\Omega - K$ has no relatively compact components, then for all $z \in \Omega - K$, there is an analytic function $f \in \mathcal{H}(\Omega)$ so that $|f(z)| > \sup_{z \in K} |f(z)|$. Let $\Omega - K = \cup_{\alpha \in A} U_\alpha$, where the U_α are components of the complement. $z \in \Omega - K \Rightarrow z \in U_{\alpha_0}$ for some α_0. Let $L = K \cup \{z\}$. Then $\Omega - L = \cup_{\alpha \neq \alpha_0} \cup U_\alpha \cup U_{\alpha_0} - \{z\}$.

Let $f(z)$ be zero on a neighborhood of K disjoint from z, and 1 on a neighborhood of z disjoint from K. f can be approximated on L by functions $\{f_n\} \subset \mathcal{H}(\Omega)$. We can choose $f_n(z)$ so that $\sup_L |f - f_n| < \frac{1}{2}$. Thus, $|f_n(z)| > \frac{1}{2} > \sup_K |f_n|$. $\diamond$

Thus endeth the Runge theorem.

We define the holomorphic convex hull as follows. If $K \subset \Omega$ is compact, then

$$\widehat{K}_\Omega \stackrel{\text{def}}{=} \{z \in \Omega : |f(z)| \leq \sup_{w \in K} |f(w)| \; \forall f \in \mathcal{H}(\Omega)\}.$$

For example, if Ω is some simply connected domain, and K is a curve, then $\widehat{K}_\Omega$ is just the interior of the curve.

Choose $\zeta \in \Omega^C$. Then $\frac{1}{z-\zeta} \in \mathcal{H}(\Omega)$. This means that if $z \in \widehat{K}_\Omega$, then $\frac{1}{|z-\zeta|} \leq \sup_{w \in K} \frac{1}{|w-\zeta|}$.

$$\begin{aligned}
\sup_{\zeta \in \Omega} \frac{1}{|z-\zeta|} &\leq \sup_{\zeta \in \Omega^C} \sup_{w \in K} \frac{1}{|w-\zeta|} \\
&= \frac{1}{d(K, \Omega^C)} \\
d(\widehat{K}_\Omega, \Omega^C) &= d(K, \Omega^C).
\end{aligned}$$

This isn't just an idle curiosity. We know that $|e^{az}| = e^{\Re az}$. So we can set

$$\widehat{K}_{\text{cvx}} = \{z : |e^{az}| \leq \sup_{z \in K} |e^{az}| \forall a \in \mathbb{C}\}.$$

We throw on more test functions to define the holomorphic convex hull; thus,

$$\widehat{K}_\Omega \subset \widehat{K}_{\text{cvx}}.$$

This tells us that $\widehat{K}_\Omega$ is a relatively compact subset of Ω. The distance from the boundary is the same as the distance from the boundary of K.

Theorem $\widehat{K}_\Omega = K \cup$ relatively compact components of $\Omega - K$.

Proof Suppose $\mathcal{O}$ is a relatively compact subset of $\Omega - K$. Then $\partial\mathcal{O} \subset K$, and thus

$$\begin{aligned} |f(z)| &\leq \sup_{\partial\mathcal{O}} |f(z)| \text{ for } z \in \mathcal{O} \\ &\leq \sup_{K} |f(z)|. \end{aligned}$$

If we define $K_1 = K \cup$ relatively compact components of $\Omega - K$, then we've thus far shown $K_1 \subset \widehat{K}_\Omega$.

To go the other way, $\Omega - K_1$ has no relatively compact components. So part (iii) of the Runge theorem tells us that $K_1 \supset \widehat{K}_{1\Omega} \supset \widehat{K}_\Omega$. So $K_1 = \widehat{K}_\Omega$. $\diamond$

Proposition $\mathbb{C} - \widehat{K}_\Omega$ has finitely many components.

Proof $\mathbb{C} - \widehat{K}_\Omega = W_0 \cup_{j=1}^{\infty} W_j$ where these are the components, and W_0 is the non-compact component. Well,

- $W_j \cap W_k = \emptyset$ for $j \neq k$.
- $W_j \subset B(0, R)$ for some large enough R.
- $W_j \not\subset \Omega$ for all j. That's because $\Omega - \widehat{K}_\Omega$ has no relatively compact components.

From 1 and 2, we know that $\sum_{j=1}^{\infty} |W_j| < \infty$. Let $r_j = \sup_r \{r : D(a, r) \subset W_j \text{ for } a \in W_j - \Omega\}$. Now, $\sum \pi r_j^2 < \sum |W_j| < \infty$. If the sum is infinite, hen $r_j \to 0$. So $d(\widehat{K}_\Omega, \Omega^C) = 0$. We have a contradiction,since $\widehat{K}_\Omega$ is a compact subset of Ω.

Classical Runge Theorem Let $\Omega \subset \mathbb{C}$ be an open set, and let $\mathbb{C} - \Omega = \cup_{\alpha \in A} C_\alpha$. Here, C_α are components of $\mathbb{C} - \Omega$. Let $A' \subset A = \{\alpha : C_\alpha \text{compact}\}$. For each $\alpha \in A'$ choose a point $a_\alpha \in C_\alpha$. Then any $f \in \mathcal{H}(\Omega)$ can be uniformly approximated on compact subsets of Ω by rational functions whose poles are contained in $\{a_\alpha\}$.

Corollary In order for the restriction of polynomials to be dense in $\mathcal{H}(\Omega)$, it is necessary and sufficient that $\mathbb{C} - \Omega$ have no compact components.

Classical Runge theorem $\Omega \subset \mathbb{C}$ open. $\mathbb{C}-\Omega = \cup_{\alpha\in A} C_\alpha$.[33] Let $A' = \{\alpha | C_\alpha \text{ compact}\} \subset A$. We choose points $a_\alpha \in C_\alpha$, $\alpha \in A'$. Then any holomorphic function $f \in \mathcal{H}(\Omega)$ can be approximated locally uniformly on Ω by rational functions whose poles lie in $\{a_\alpha\}_{\alpha\in A'}$.

Proof Choose some $K \overset{\text{cpt}}{\subset} \Omega$ and an $\epsilon > 0$. Let $L = \widehat{K}_\Omega$, the holomorphic convex hull; we took K, and threw in the relatively compact components of the complement of K. We showed last time that $\mathbb{C} - L$ has a unique unbounded component U, and finitely many bounded components $W_1, \cdots, W_p$. Since $\Omega - L$ has no relatively compact components, every component of $\Omega - L$ meets one of the W_i. Suppose $\zeta \notin \Omega$ but $\zeta \in W_i$ for some i. Then it also belongs to C_{α_i} for some i, and $C_{\alpha_i} \subset W_i$. $\Rightarrow C_{\alpha_i}$ is compact. For each i, choose α_i so that $C_{\alpha_i} \subset W_i$. In each of these sets we have selected a point a_{α_i}. Let $\Omega_0 = \mathbb{C} - \{\alpha_i\}_{i=1,\cdots,p}$. Look at $\Omega_0 - L$. Well, $\mathbb{C}-L = U \cup W_1 \cup \cdots \cup W_p$; $\Omega - L = U \cup (W_1 - \{a_{\alpha_1}\}) \cup \cdots \cup (W_p - \{a_{\alpha_p}\})$. None of these components is relativley compact. Now we choose $f \in \mathcal{H}(\Omega)$; f is holomorphic on a neighborhood of L. By the theorem proved last time, given an $\epsilon > 0$, there's an $F \in \mathcal{H}(\Omega_0)$ so that $\|F - f\|_{L^\infty(L)} < \epsilon$. At each point, F has a principle part. At a_{α_i}, we have $p_i(z) = \sum_{j=-\infty}^{1} b_{ij}(z - a_{\alpha_i})^j$. Observe that $F = h + \sum_{i=1}^{p} p_i(z)$; something holomorphic plus the sum of the principal parts. For each i, can choose an N_i so that $\left\| p_i - \sum_1^{N_i} \frac{b_{ij}}{(z-a_{\alpha_i})^j} \right\|_{L^\infty(L)} < \epsilon$.[34] We can also choose a finite part of the Taylor series of $h(z)$, call it $h^0(z)$, so that $\|h - h^0\|_{L^\infty(L)} < \epsilon$, $\|F_0 - f\|_{L^\infty(L)} \leq \|F_0 - F\|_{L^\infty(L)} + \|F - f\|_{L^\infty(L)} \leq (p+2)\epsilon$, where $F^0 = h^0 + \sum_{i]1}^{p} \sum_{j=1}^{N_i} \frac{b_{ij}}{(z-a_{\alpha_i})^j}$. $\diamond$

Corollary The restriction of polynomials to Ω is dense in $\mathcal{H}(\Omega)$ $\iff$ $\mathbb{C} - \Omega$ has no compact component.

This isn't at all how Runge did it. Play with $\frac{1}{z} = \frac{1}{z-\epsilon}(1 - \frac{-\epsilon}{z-\epsilon})^{-1}$.

Let $E \subset \mathbb{C}$ be a discrete set; no finite points of accumulation. Let $p_a(z)$ be a function holomorphic in $\mathbb{C} - \{a\}$ for each $a \in E$. Does there exist a function $f \in \mathcal{H}(\mathbb{C} - E)$ so that $f - p_a$ is holomorphic in a neighborhood of a for each $a \in E$? Might as well asssume that the p_a is a Laurent series with just negative terms.

More generally, let Ω be an open set in $\mathbb{C}$, $E \subset \Omega$ a discrete subset. $p_a(z)$ as above. Does there exist a function $f \in \mathcal{H}(\Omega - E)$ so that $f - p_a$ is analytic in a neighborhood of a for each $a \in E$?

Well, yes.

[33] note that this need not be a countable index; e.g., the deletion of the Cantor set has an uncountable number of components.

[34] Go back and adjust notation for b_{ij}.

Theorem [Mittag-Leffler] Yes.

1. We choose an exhaustion of Ω by compact subsets $K_j \overset{\text{cpt}}{\subset} K_{j+1} \overset{\text{cpt}}{\subset} \cdots$. We've shown that $\widehat{K}_\Omega \overset{\text{cpt}}{\subset} \Omega$ if $K \overset{\text{cpt}}{\subset} \Omega$. Thus, we can replace the K_j by $\widehat{K}_{j\Omega}$. We'll denote them by $\{K_j\}$. We also assume that $E \cap \partial K_j = \emptyset$ for all j.[35]

2. Define the functions $\widetilde{f}_j(z) = \sum_{a \in K_j} p_a(z)$. There's no reason, *a priori*, to believe that this sum actually converges.

3. By induction, assume that we've found $h_1, \cdots, h_{k-1},\ h_i \in \mathcal{H}(\Omega)$, so that if we set $f_j = \widetilde{f}_j + h_j$ then

$$\|f_j - f_{j-1}\|_{L^\infty(K_{j-1})} < 2^{-j}.$$

   Look at $\widetilde{f}_k - f_{k-1}$ on K_{k-1}. By the construction of f_j and $\widetilde{f}_j$, this function is holomorphic on a neighborhood of K_{k-1}. The Runge theorem applies to show that we can find $h_k \in \mathcal{H}(\Omega)$ so that

$$\left\|\widetilde{f}_k - f_{k-1} + h_k\right\|_{L^\infty(K_{k-1})} < 2^{-k}.$$

   So let $\widetilde{f}_k + h_j = f_k$; this completes the induction.

4. We claim that $\lim_{k\to\infty} f_k$ exists locally uniformly on $\Omega - E$ and has the desired properties.

   Fix an m. Let $j, k \gg m$; consider $\|f_j - f_k\|_{L^\infty(K_m)}$. We write this as

$$\begin{aligned} f_j - f_k &= \sum_{l=k}^{j-1} f_{l+1} - f_l \\ \|f_j - f_k\|_{L^\infty(K_m)} &\leq \sum_{l=k}^{j-1} \|f_{l+1} - f_l\|_{L^\infty(K_m)} \\ &\leq \sum_{l=k}^{j-1} 2^{-l} \\ &< 2^{j-k} \end{aligned}$$

[35] This is no problem, since E discrete; can wiggle the K_j a bit.

since $K_l \supset K_m$. Finally, we consider

$$\begin{aligned} \left\| (f_j - \widetilde{f_m}) - (f_k - \widetilde{f_m}) \right\|_{L^\infty(K_m)} &\leq 2^{l-k} \\ \lim_{j\to\infty} f_j - \widetilde{f_m} &= g \end{aligned}$$

exists on K_m and is analytic $\Rightarrow \lim_{j\to\infty} f_j$ exists locally uniformly on $\Omega - E$.

We now know that $\lim_{j\to\infty} f_j = \widetilde{f_m} + g$ on K_m. Call this thing F. F is an analytic function plus a sum whose principal parts are exactly what we wanted. $\diamond$

If $\{p_a(z) | a \in E\}$ are all rational functions, then so is $F(z)$.

If we try doing this on $\mathbb{C}$, list $\{a_j\}$, $\{p_j(z)\}$. E is a discrete set. We only need locally uniform convergence. So we can fix some ball, and there are only finitely many E-points inside it. Taylor expand $p_j(z)$ at the origin. Choose a finite part of the Taylor series $\phi_j(z)$ so that $\|p_j - \phi_j\|_{L^\infty(|z| \frac{|a_j|}{2})} < 2^{-j}$.

Claim: $\sum p_j - \phi_j$ converges locally uniformly on $\mathbb{C} - E$.

The argument is basically trivial. Fix an R. Then there's a J so that $\|a_j\| > 2R$ if $j > J$. Write the sum as

$$\sum_{j<J} p_j - \phi_j + \sum_{j>J} p_j - \phi_j$$

The point is that we have a Weierstrass M-test sort of argument;

$$\begin{aligned} \left| \sum_{j=J}^{L} p_j - \phi_j \right| &\leq \sum_{j=J}^{L} |p_j - \phi_j| \\ &\leq \sum_{j=J}^{L} 2^{-j} \end{aligned}$$

if $z \in D(0, R)$. By the Weierstrass M-test, the sum converges uniformly on this disk to give a holomorphic function. $\diamond$

Recall that we had a partial fractions expansion; $F(z) = P(z) + \sum^m P_j(\frac{1}{z-a_j})$. What we have on our hands is a generalization of this; $h(z) + \sum^\infty (p_j - \phi_j)$.

We'd like to try [for some reason – I spaced out a bit] $\sum \frac{1}{z-n}$; but this doesn't converge. But $\frac{1}{z} + \sum_{z\neq 1} \frac{1}{z-n} + \frac{1}{n}$ converges. $= \frac{1}{z}\sum \frac{n+-n}{n(z-n)} = \frac{1}{z} + \sum \frac{z}{n(z-n)}$. This converges uniformly, so we can reorder this;

$$\frac{1}{z} + \sum_{n=1}^{\infty}(\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n}) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = f(z).$$

$f(z+1) = f(z)$. If we could write $f(z) = \sum_{-\infty}^{\infty} \frac{1}{z-n}$, this would be obvious; we'd just be changing the index of summation. For look at $(f(z+1) - f(z))'$. Well,

$$\begin{aligned}
f'(z) &= -[\frac{1}{z^2} + \sum_{1}^{\infty} \frac{1}{(z-n)^2} + \frac{1}{(z+n)^2}] \\
&= -\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \text{ obviously periodic.} \\
f'(z) - f'(z+1) &= 0 \\
f(z) - f(z+1) &= C
\end{aligned}$$

Observe that $f(-z) = -f(z)$;

$$\begin{aligned}
\lim_{z\nearrow i\infty} f(z) = f(z+1) &= \lim_{z\to i\infty} \sum(\frac{2z}{z^2-n^2} - \frac{2(z+1)}{(z+1)^2-n^2}) \\
&= 0.
\end{aligned}$$

The last step isn't obvious, but it's not too bad, either. The integral test fails, but we're all adults here.

So we know the following:

1. f is periodic of period 1.
2. f has poles at the integers with residue 1.

Consider $\pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z}$. This is actually has the right properties. So look at $f(z) - \pi \cot \pi z$; this is analytic in the whole plane. Now, $\lim_{y\to\pm\infty} f(x+iy)$ is bounded; we only have to check this on $[0, 1]$. But $\pi \cot \pi z = \pi i \frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}$. Letting $z = x + iy$, shouldn't be hard to see that $\lim_{y\to\infty} |\pi \cot \pi z|$ is bounded. So $f(z) - \pi \cot \pi$ is a constant; their difference is a bounded holomorphic function on the entire plane. And it's not hard to see that the constant is zero; subtract $\frac{1}{z}$ from both terms; $(f(z)-\frac{1}{z})-(\pi\cot\pi-\frac{1}{z})|_{z=0}=0$. So we now have this formula,

$$\pi\cot\pi z=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{^2-n^2}.$$

Equivalently, we have

$$\pi\cot\pi z=\frac{1}{z}+\sum_{n=1}^{\infty}(\frac{1}{z-n}+\frac{1}{z_n}).$$

By differentiating term by term, we have

$$(\pi\cot\pi z-\frac{1}{z})^{[2k-1]}(0)=\sum_{n=1}^{\infty}\frac{2(-1)(2k-1)!}{n^{2k}}.$$

On the other hand,

$$\pi\cot\pi z=\pi i[\frac{2}{e^{2\pi iz}-1}+1].$$

We can work out the Laurent series for this at the origin.

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\sum\frac{(-1)^{k-1}B_kz^{2k-1}}{(2k)!}.$$

Here, B_k is a Bernoulli number. Substituting everything in, we have

$$\pi\cot\pi z-\frac{1}{z} = \sum_{1}^{\infty}\frac{(-1)^{k-1}B_k(2\pi i)^{2k-1}2\pi i}{(2k)!}$$

By equating coefficients, we have

$$\sum_{k=1}^{\infty}\frac{1}{n^{2k}}=\frac{2^{2k-1}\pi^{2k}B_k}{(2k)!}.$$

It was only proved about ten years ago that $\sum\frac{1}{n^3}$ is transcendental; nothing is known about n^5. But we have closed forms for even exponents.

Just for purposes of reference, $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$, $B_6=\frac{691}{2730}$, $B_7=\frac{7}{6}$.

Whereas the Mittag-Leffler theorem was in a sense additive, today we'll be looking at the Weierstrass theorem, which is essentially a multiplicative statement.

For starters, consider the infinite product $\prod_{n=1}^{\infty} p_n$. We have the following conventions.

1. Only finitely many of the p_n are allowed to vanish.

2. We say that the product converges if $\lim_{N\to\infty}\prod_{n=1}^{N} p'_n = P$ exists where $p'_n = \begin{cases} 1 & p_n = 0 \\ p_n & p_n \neq 0 \end{cases}$. We also insist that $P \neq 0$; we'll say that the product diverges if the limit is zero.

Proposition If $\lim_{N\to\infty}\prod_{n=1}^{N} p_n$ exists, then $\lim_{n\to\infty} p_n = 1$.

We assume wlog that $p_n \neq 0$ for all n. Set P_N to be the N^{th} partial product, $P_N = \prod_{n=1}^{N} p_n$. Then $\lim_{N\to\infty} P_n = P \neq 0$. This says that given $\eta > 0$ there's a M so that $|P_N - P| < \eta$ if $M < N$. And actually, we can assume that $|P_{N_1} - P_{N_2}| < \eta$ for $N_i > M$. Suppose that $\lim_{n\to\infty} p_n \neq 1$. Then there's a sequence $n_i \to \infty$ so that $|p_{n_i} - 1| > \epsilon > 0$. Consider $\left|P_{N_{n_i}-1} - P_{N_{n_i}}\right| = \left|P_{N_{n_i-1}}(p_{n_i} - 1)\right|$. We've simply factored out the common factor. But by assumptions, $\left|P_{N_{n_i-1}}(p_{n_i} - 1)\right| > (|P| - \eta)(\epsilon)$. Choose $\eta \ll \epsilon$, and i very large to derive a contradiction.

Theorem $\prod_{n1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty}\log(1 + a_n)$ conerges, where we take log to be the principle branch.

Proof If $S_n = \sum_{k=1}^{n}\log(1+a_k)$, then $P_n = e^{S_n}$. If $\lim_{n\to\infty} S_n$ exists, then $\lim P_n = e^{\lim S_n} \neq 0$.

To go the other way, fix an $\epsilon \ll 1$. Then there exists an N so that $|a_n| < \epsilon$ if $n > N$. Clearly it suffices to show that $\lim_{M\to\infty}\prod_{n=N}^{M}(1 + a_n)$ exists $\Rightarrow \sum_{n=N}^{\infty}\log(1 = a_n)$ exists. We know that $\log\frac{P_M}{P_{N-1}} = \sum_{l=N}^{M}\log(1+a_l) + 2\pi i h_M$ where $h_M \in \mathbb{Z}$. The point is this. When M is quite large, the $2\pi i h_M$ term can't change. For if we take $\log\frac{P_{M+1}}{P_{N-1}} = \log\frac{P_{M+1}}{P_M}\frac{P_M}{P_{N-1}} = \log\frac{P_{M+1}}{P_M} + \log\frac{P_M}{P_{N-1}} = \sum_{n=1}^{M+1}\log(1 + a_n) + 2\pi i h_{M+1}$. If we compute the difference, we have $\log\frac{P_{M+1}}{P_M} = \log(1+a_{M+1}) + 2\pi i(h_{M+1} - h_M)$. So that integer had damned well better be zero. The upshot of this is that for M sufficiently large we have a constant h so that $\log\frac{P_M}{P_{N-1}} = \sum_N^M \log(1 + a_n) + 2\pi i h$. Since $\lim_{M\to\infty} P_M = P$, it follows that $\lim_{M\to\infty}\sum_N^M\log(1 + a_n)$ also exists.

We know that $(1-\epsilon)|a_n| < |\log(1+a_n)| < (1+\epsilon)|a_n|$ if $|a_n| << 1$, since $\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$. This gives us a criterion for absolute convergence; $\prod(1+a_n)$ converges absolutely $\iff$ $\sum |a_n|$ converges.[36]

Consider $a_n = \frac{(-1)^n}{\sqrt{n}}$. Then $\sum a_n$ converges – not absolutely, but it converges nonetheless. What about $\prod_{n=1}^{\infty}(1+\frac{(-1)^n}{\sqrt{n}})$? Well, it diverges. Consider $\sum \log(1+\frac{(-1)^n}{\sqrt{n}})$. Well, $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{4}{4} + \cdots$. So The aforementioned sum is $\sum \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n} + O(\frac{1}{n^{3/2}})$. The first and last terms converge, but the middle one doesn't converge. So the product doesn't converge.

Now consider the sequence $\frac{1}{n}, \sqrt{\frac{2}{n}}, \frac{1}{n+1}, -\sqrt{\frac{2}{n+1}}, \frac{1}{n+2}, \sqrt{\frac{2}{n+2}} \cdots$. This diverges; the harmonic part diverges, while the other one converges. Let these things be a_n. Consider $\sum \log(a_m + 1) = \sum a_m - \frac{a_m^2}{2} + O(a_m^3)$. The $O(a_m^3)$ part converges; so we only worry about the first two terms. We get $\frac{1}{n} - \frac{1}{2}(\frac{1}{n})^2 + \sqrt{\frac{2}{n}} - \frac{1}{2}(\frac{\sqrt{2}n}{)}^2 + \frac{1}{n+1} - \frac{1}{2}(\frac{1}{n+1})^2 - \sqrt{\frac{2}{n+1}} - \frac{1}{2}(\sqrt{\frac{2}{n+1}})^2 + \cdots$. Then a lot of terms cancel out, and we're left with $\sum_1^{\infty}\sqrt{\frac{2}{n}}(-1)^n$, which converges. Thus, $\sum \log(1+a_n)$ converges, even though $\sum a_n$ diverges. In the absence of absolute convergence, there's no implication one way or the other.

Consider $\prod_{n=2}^{\infty}(1-\frac{1}{n^2}) = \frac{1}{2}$. How do you prove that? It's

$$\begin{aligned} \prod_{n=2}^{\infty}(\frac{n^2-1}{n^2}) &= \prod_{n=2}^{\infty}\frac{(n-1)(n+1)}{n^2} \\ \prod_{n=2}^{N}\frac{(n-1)(n+1)}{n^2} &= \cdots = \frac{1}{2}\frac{N+1}{N}. \end{aligned}$$

How about $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \frac{1}{1-z}$. This is for $|z| < 1$. This isn't too hard to show; use binary representation.

$\Omega \subset \mathbb{C}$ an open set. Let $E = \{z_j\}$ be a discrete subset of Ω. Let $\{n_j\}$ be a sequence of integers.

Weierstrass Theorem There is a function $f \in \mathcal{H}(\Omega - E)$ so that $(z-z_j)^{-n_j} f$ is holomorphic and nonvanishing in a neighborhood of z_j for each j.

Proof Choose compact sets $K_j \subset \Omega$ so that $K_j \subset K_{j+1}$, and $\widehat{K_j} = K_j$; in other words, $\Omega - K_j$ has no relatively compact components. Finally, we insist that $\cup_{j=1}^{\infty} K_j = \Omega$. We proceed inductively.

[36] That is, $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.

Choose $\epsilon_j L \epsilon_{j+1} > \cdots$ so that $\sum \epsilon_j < \infty$. For the inductive hypothesis, suppose that we can find $f_1, \cdots, f_j$ rational functions with the poles and zeros of f_i on K_i as specified, and also functions $g_1, \cdots, g_{j-1} \in \mathcal{H}(\Omega)$ so that $\left|\frac{f_i}{f_{i-1}} e^{g_{i-1}} - 1\right| < \epsilon_i$ on K_{i-1}.[37] There's no real need to do a base case. Anyway, choose a rational function f so that the poles and zeros of f are as specified on K_{j+1}; f'rinstance, $\prod^J (z-z_j)^{n_j}$. Then $\frac{f}{f_j} = c\prod(z-w_\nu)^{m_\nu}$ for some $w_\nu \in \Omega - K_j$. Since $\Omega - K_j$ has no relatively compact components, I can choose $w'_\nu \in \Omega - K_{j+1}$ so that w_ν, w'_ν belong to the same component of $\Omega - K_j$. Look at $f\prod(z-w'_\nu)^{-m_\nu} = f_{j+1}$. Then

$$\frac{f_{j+1}}{f_j} = c\prod\left(\frac{z-w_\nu}{z-w'_\nu}\right)^{m_\nu}.$$

We can draw a line from w_ν to w'_ν. Then $\log(\frac{z-w_\nu}{z-w'_\nu})$ is an analytic function on K_j. Well, $\log \frac{f_{j+1}}{f_j} = \log c + \sum m_\nu \log \frac{z-w_\nu}{z-w'_\nu}$ is analytic in a neighborhood of K_j. We needed a device so as not to introduce any new zeros and poles. The obvious choice is to take logarithms and then exponentiates. The only possible fly in the ointment is that the logarithm of the ratio might not be single-valued. That's why we diddle the f_j's a bit. By the Runge theorem there is an analytic function g_j so that $\left|\log \frac{f_{j+1}}{f_j} - g_j\right| < \epsilon_j$ on K_j. We're approximating on a compact subset. If we exponentiate this, we have $\left|\frac{f_{j+1}}{f_j} e^{-g_j} - 1\right| < c\epsilon_j$.

This completes the induction step. We have $f = \lim_{J\to\infty} f_{J+1} \prod^J e^{-g_i} = \lim_{J\to\infty} f_N \prod_{j=N}^J \frac{f_{j+1}}{f_j} e^{-g_i}$ (times some finite bunch of other terms) exists locally uniformly on Ω. Note that the term $\frac{f_{j+1}}{f_j} e^{-g_j}$ does not vanish on K_N. So f has precisely the right zeros and poles on a subset K_N. So f has the right ones at all points of E. The limit is of the form $f_N h_N$ where $h_N \in \mathcal{H}(K_N)$, and $h_N(z) \neq 0$ for any $z \in K_N$. $\diamond$

Recall that a rational function is a quotient of polynomials. We've defined meromorphic functions in terms of a local property; it's analytic in the complement of a discrete set, and if you look at the Laurent expansion at any point in the discrete set, that expansion has a finite number of nonzero coefficients of negative powers.

Let's suppose that $E = \{z_j\}$ and $f \in \mathcal{H}(\Omega - E)$, meromorphic on Ω. This means that at each z_j there's an integer n_j so that $(z-z_j)^{n_j} f(z)$ is analytic near to z_j. Let h be the holomorphic function in $\mathcal{H}(\Omega)$ with zeros of order n_j at z_j, whose existence follows from the Weierstrass theorem. Set $g(z) = h(z)f(z)$. Then $g(z) \in \mathcal{H}(\Omega)$, by construction; it has removable singularities at each point E, and we removed 'em. Thus we have a representation

[37] The point is that $f_{J+1}\prod_{i=1}^J e^{g_i} = f_1 \prod_{I=1}^J \frac{f_{i+1}}{f_i} e^{g_i}$. Now, the left-hand term clearly has the proper zeros and poles.

$f(z) = \frac{g(z)}{h(z)}$, where $g, h \in \mathcal{H}(\Omega)$. This is an analogue of a rational function. We've proved an algebraic statement; the set of meromorphic functions is the fraction field of the ring of holomorphic functions. Note that this is a global condition, *not* a local one.

Let $\Omega \subset \mathbb{C}$ be given, and $\{z_j\} = E \subset \Omega$ a discrete subset. Let $\{a_j\}$ be a set of complex numbers. Is there a function $f \in \mathcal{H}(\Omega)$ so that $f(z_j) = a_j$? Let h be a function that vanishes to order 1 at each point $\{z_j\}$.[38] If f existed, we'd know that $\frac{f-a_j}{h}$ should be analytic near $z = z_j$. We can choose a function g meromorphic in Ω with principle parts the same as $\frac{a_j}{h}$ at $z = z_j$. So $g - \frac{a_j}{h}$ is analytic near to z_j. Let $f = gh$.Then $g - \frac{a_j}{h} = \frac{f}{h} - \frac{a_j}{h}$; $f(z_j) = a_j$. Thus we see that $f = gh$ solves the problem. We used the Mittag-Leffler theorem to find a theorem with the right principle part, $\frac{a_j c_j}{z-z_j}$.

[38] This function is courtesy the Weierstrass theorem.

Let $\Omega \subset \mathbb{C}$ open, $E \subset \Omega$ discrete. For each $a \in E$ choose ϕ_a a meromorphic function defined near a, and an integer k_a. Then there's a function $f \in \mathcal{H}(\Omega - E)$ so that $f - \phi_a$ is $O((z-a)^{k_a})$.

If you look at the Laurent expansion $\sum_{-m}^{\infty} a_j(z-a)^j$, then $\sum_{-m}^{k_a} a_j(z-a)^j$ is the same as the Laurent expansion of ϕ_a up to the k_a'th term. So you can interpret arbitrary pieces of meromorphic functions, not just values.

Proof Choos $h \in \mathcal{H}(\Omega)$ so that h vanishes at a to order $k_a + 1$ [for each a], and is otherwise nonzero. If g is meromorphic and has principle parts given by $\frac{\phi_a}{h}$ at $a \in E$, then $f = gh$ solves the interpolation problem. For $\frac{gh-\phi_a}{h} = g - \frac{\phi_a}{h}$ *blah.* $\diamond$

If polynomials p, q relatively prime[39] then there are polynomials r and s so that $rp + sq = 1$.

For if $z \in Z_q$, we want $r(z)p(z) = 1$; we get $\deg q$ such constraints. Similarly, $s(z)q(z) = 1$ for $z \in Z_p$. Then $r(z)p(z) + s(z)q(z) - 1 = 0$ for $z \in Z_p \cup_q$. The degree of the left-hand term is one less than $\deg p + \deg q$. But it has $\deg p + \deg q$ roots. So it must be identically zero.

Suppose $f, g \in \mathcal{H}(\Omega)$ and ${}_f \cap Z_g = \emptyset$. Then there are $\alpha, \beta \in \mathcal{H}(\Omega)$ so that $\alpha f + \beta g = 1$.

Proof We need to find β so that $\frac{1-\beta g}{f}$ is holomorphic. We need to find $\beta \in \mathcal{H}(\Omega)$ so that $\frac{1}{g} - \beta$ vanishes to the same order as f at zeros of f. Then $\frac{\frac{1}{g}-\beta}{f}$ has removeable singularities at Z_f, and $\frac{1-g\beta}{f}$ is therefore holomorphic in Ω; set this thing equal to α.

Suppose Ω is an open set in $\mathbb{C}$ and $f \in \mathcal{H}(\Omega)$. We say that f has an extension across $\partial\Omega$ near a point $z \in \partial\Omega$ if there's an $r > 0$ and a function $\phi \in \mathcal{H}(D(z,r))$ so that $f = \phi$ in $D(z,r) \cap \Omega$.

Theorem If $\Omega \subset \mathbb{C}$ is open, then there's a function $f \in \mathcal{H}(\Omega)$ which cannot be extended across any boundary point

Proof

1. List the rational points in Ω with each appearing infinitely often $\{z_j\}$.
2. Choose an exhaustion of Ω by compact subsets K_j. Assume without loss of generality that $\widehat{K}_j = K_j$.
3. Let $r_j = d(z_j, \Omega^C)$.

[39] $Z_p = \{z : p(z) = 0\}$, then $Z_p \cap Z_q = \emptyset$.

For each j choose a point $w_j \in \Omega - K_j$ so that $d(z_j, w_j) < r_j$. Choose a function $f \in \mathcal{H}(\Omega)$ which vanishes at w_j for all j. f cannot be extended across any boundary point. If $a \in \Omega$ irrational and $r = d(a, \Omega^C)$, then $D(a, r)$ contains infinitely many points where f vanishes, since a appears in the list $\{z_j\}$ infinitely often. $\Rightarrow f$ cannot be extended across any point in $\partial\Omega$. $\diamond$

Weierstrass showed [directly] that $\sum z^{n!}$ can't be extended across the unit disk.

Now we'll just review some problems. Suppose $f(z) = [f_n(z)]^n$ for some $f_n(z) \in \mathcal{H}(\Omega)$ for all n¿ Show that $f(z) = e^F$ for some $F \in \mathcal{H}(\Omega)$.

- f does not vanish. If it did at some z_0, then $f_n(z_0) = 0$ for all n¿ So $f(z) = (z - z_0)^n(\widetilde{f}_n(z))^n$ for every n, and the zero would have infinite order, a contradiction.

- o show that F exists it is necessary and sufficient to show that $\int_{w_0}^{w} \frac{f'(z)}{f(z)} dz \stackrel{\text{def}}{=} \widetilde{F}(w)$ is well-defined in Ω.

  Note that $f'/f = nf_n'/f_n$. Let γ be a closd path in Ω. Then $\int_\gamma \frac{f'}{f} dz = n \int_\gamma \frac{f_n'}{f_n} dz$. Interpret this geometrically. It's a winding number, and thus an integral multiple of 2π. So the integral over γ is $< \infty$, and thus $\int_\gamma \frac{f'}{f} dz = 0$. [If it had been $2\pi j$, we would have had things zooming off to ∞.]$\diamond$

We're doing number 7 on 5. Show $\int\int_\Omega |f|^2 dxdy < M$. The most direct way is to show that by taking the Taylor expanison at $D(z, r)$, use the square norm to estimate the coefficients in the Taylor expansion. If $f(z) = \sum a_n(z - z_0)^n$, then $\int_{D(z_0,r)} |f(z)|^2 = \sum_0^\infty |a_n|^2 r^{2n} < M$, more or less. So each term we have $|a_n| < \frac{M}{r^n}$. Therefore, f is uniformly bounded on a smaller disk centered at the same point.

Alternatively, use $f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(\zeta)}{\zeta - z} d\zeta$. So integrating wrt r yields $\int_R^{R+\epsilon} f(z) dr = \frac{1}{2\pi i} \int_R^{R+r} \int_{\partial D(z_0,r)} \frac{f(\zeta) d\zeta}{(\zeta - z)} dr$. Then $\epsilon |f(z)| \leq \frac{1}{2\pi} \int_R^{R+\epsilon} \int_0^{2\pi} \frac{|f| r dr d\theta}{|(|\zeta - z)|} \leq C_\epsilon M$ if $|z - z_0| < R - \epsilon$. Rule of thumb; you can use the Cauchy integral formula and integrate, to relate areas to pointwise estimates. Keep this in mind. Any estimate whatsoever on holomorphic functions implies pointwise estimates. Estimate it in any reasonable toplogy, and it gives pointwise esimaes on relatively compact subsets. That's sort of the basic lesson. For a family to be normal, it suffices that they be locally uniformly bounded, because then the dervatives are, and it's equicontinuous. What we just saw was another example of this principle.

Moving on. Define $I(r) = \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right| d\theta$, $f \in \mathcal{H}(D(0, R))$. Want to show

1. $I(r)$ is strictly increasing.

2. $I(r)$ is log-convex.

The idea is this. For almost every r there's a continuous function $\phi_r(\theta)$ so that $|\phi_r(\theta)| = 1$, and $\phi_r(\theta) f(re^{i\theta}) = \left|f(re^{i\theta})\right|$. The hint says consider the function $F(z) = \int_0^{2\pi} \phi_r(e^{i\theta}) F(ze^{i\theta}) d\theta/2\pi$. This is analytic. Notice that $F(r) = I(r)$. Also, $|F(z)| \le \int \left|\phi_r(e^{i\theta}) f(ze^{i\theta})\right| d\theta/2\pi = I(|z|)$. So this tells us that

1. $I(r)$ is increasing, by the maximum modulus principle. In particular, it says that $I(r) = \max_{|z|=r} |F(z)| \le \max_{|z|=R} |F(z)| \le I(R)$ if $R > r$. To get strictly increasing, use the sharp version of the maximum modulus principle.

2. We want to show its log convex. So show that $I(r) \le I(r_1)^\alpha I(r_2)^{1-\alpha}$ where $\alpha = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}$ where $r_1 < r < r_2$. Now apply the Hadamard three-circle theorem. Let $M(r) = \sup_{|z|=r} |f(z)|$. Then $I(r) = M(r) \le M(r_1)^\alpha M(r_2)^{1-\alpha}$ by Hadamard. But $M(R) \le I(R)$, so the last term is $\le I(r_1)^\alpha I(r_2)^{1-\alpha}$.

For the next problem, study $I_2(r) = \int \left|f(re^{i\theta})\right|^2 d\theta/2\pi$. Define $F_2(r) = \int [\phi_r(e^{i\theta}) f(ze^{i\theta})]^2 d\theta/2\pi$. Then $F_2(r) = I_2(r)$, and we get the same argument more or less.

Theorem If $S \hookrightarrow \mathbb{R}^n$ is immersed locally convex and compact, then it is embedded and the boundary of a convex region if $n \ge 3$.

This is way false in the plane, e.g., the pentagram.

Consider $\frac{1}{(z^2-2xz+1)} = \sum_0^\infty z^n P_n(x)$ the generating series for the Legendre polynomials. $P_n(x) = \frac{1}{\pi}\int_{-1}^1 \frac{(x+\alpha\sqrt{x^2-1})^n}{\sqrt{1-\alpha^2}} d\alpha$. We know from the Cauchy integral formula that $P_n(x) = \frac{1}{2\pi i}\int_{|z|=r} \frac{1}{(z^2-2xz+1)^{1/2}} \frac{1}{z^{n+1}} dz$ for r small enough. The roots of the denominator are $\frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm i\sqrt{1-x^2}$. Assume $x \in [-1,1]$. These roots are conjugate points on the unit circle. So if you integrate around a little circle, you don't care about them. Now, how shall we define the square root? We want to define a domain in which sqrt is well-defined. Draw a line between the two roots; take its image under the map $z \mapsto \frac{1}{z}$. The image is a circle; the complex plane minus that circle is where we'll define square root. The reason to do this is to change variables; $z = \frac{1}{w}$. We then have $\frac{1}{2\pi i}\int_{|w|=\frac{1}{r}} \frac{w^n}{(\frac{1}{w^2} - \frac{2x}{w} + 1)^{1/2}} \frac{dw}{w}$. In the w variable, the square root is defined off the line connecting $x \pm \sqrt{x^2-1}$. Factor the denominator. $(\frac{1}{w^2} - \frac{2x}{w} + 1)^{1/2} = [\frac{1}{w^2}(1 - 2xw + w^2)]^{1/2} = \frac{1}{w}(1-2xw+w^2)^{1/2}$. So our integral becomes

$$\frac{1}{2\pi i}\int \frac{w^n dw}{(1-2xw+w^2)^{1/2}}.$$

Cauchy's intgral formula says that if we take an integral just around the cut, we get zero. Now, w^n takes the same value on either side of the cut. But the denominator has opposite sides on either side of the cut; but we're going up one side and down the other. So as the contour shrinks down to the cut we get twice the integral along one side. So the integral is

$$\frac{1}{2\pi i}\int_{x-i\sqrt{1-x^2}}^{x+i\sqrt{1-x^2}} \frac{w^n dw}{(1-2xw+w^2)^{1/2}}.$$

Paramterize this curve, and get

$$\frac{1}{\pi i}\int_{-1}^{1} \frac{x+i\alpha\sqrt{1-x^2})i\sqrt{1-x^2}d\alpha}{Blah.^{1/2}} = \frac{1}{\pi} int_{-1}^{1} \frac{x+i\alpha\sqrt{1-x^2})d\alpha}{\sqrt{1-\alpha^2}}.$$

For the other formula, the trick starts the same; invert. Our goal is $P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})tdt}{\sqrt{2(\cos t - \cos\theta)}}$. Somewhere along the way, you should observe that the integral is real; the imaginary part integrates out to zero.

Moving along. If $f(z)$ is analytic in $|z| < 1$ and has roots $\{a_i\}$, then if $|f(z)| < M$ then $\sum_{i=1}^{\infty} \log\frac{1}{|a_i|} < \infty$. How do you do it?

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta = \log|f(0)| + \sum_{a_i<r} \log\frac{r}{|a_i|}.$$

Hypothesis says that this thing is $< \log M$. Note that we're always working with positive numbers. Here's the trick. This is

$$\geq \quad \log|f(0)| + \sum_{a_i<R} \log\frac{r}{|a_i|} \text{ if } R < r$$

Let $r \to 1$. Then $\log\frac{M}{|f(0)|} \geq \sum_{a_i<R}\log\frac{1}{|a_i|}$, and thus $\log\frac{M}{|(0)|} \geq \lim_{R\to 1}\sum_{|a_i|<R}\log\frac{1}{|a_i|}$. We're using $\overline{\lim}_{r\to 1}\int \log\left|f(re^{i\theta})\right| d\theta < \infty$. This has something to do with Nevanlinna theory.