Properties of q-positive maps and their relation to E_0-semigroups

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Definition

*A completely positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-**positive** (we write $\phi \geq_q 0$) if ϕ has no negative eigenvalues and $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$.*

It is certainly possible for a completely postitive map ϕ to have negative eigenvalues. Just take

$$\phi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}.$$

Suppose ϕ has no negative eigenvalues. Is there a "slowest rate of failure" for q-positivity? Absolutely not. To demonstrate, let $s > 0$ be arbitrary, so for some $r \in (1, \sqrt{2})$ we have $s = (2 - r^2)/(2r^2 - 2r)$. Define a completely positive map by

$$\phi \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & \frac{r(1+i)a_{12}}{2} \\ \frac{r(1-i)a_{21}}{2} & a_{22} \end{pmatrix}.$$

Then $\phi(I + t\phi)^{-1}$ is completely positive if and only if $t \leq s$.

Why do we make this definition? Because (as we will see) if ϕ is unital and q-positive, we can naturally pair it with a boundary weight ν over $L^2(0,\infty)$ to form an E_0-semigroup.

Definition

We say a family $\alpha = \{\alpha_t\}_{t \geq 0}$ *of* $*$*-endomorphisms of* $B(H)$ *is an* **E$_0$-semigroup** *if:*

- $\alpha_s \circ \alpha_t = \alpha_{s+t}$ *for all* $s, t \geq 0$ *and* $\alpha_0(A) = A$ *for all* $A \in B(H)$*;*
- *For each* $f, g \in H$ *and* $A \in B(H)$*, the inner product* $(f, \alpha_t(A)g)$ *is continuous in* t*;*
- $\alpha_t(I) = I$ *for all* $t \geq 0$*.*

We have two notions of equivalence for E_0-semigroups, namely conjugacy and cocycle conjugacy. The latter will be our main focus.

Definition

Let α *and* β *be* E_0*-semigroups acting on* $B(H_1)$ *and* $B(H_2)$*, respectively, are said to be* **conjugate** *if there is a* $*$*-isomorphism* θ *from* $B(H_1)$ *onto* $B(H_2)$ *such that* $\theta \circ \alpha_t = \beta_t \circ \theta$ *for all* $t \geq 0$*.*

We say α *and* β *are* **cocycle conjugate** *if, for some collection* $\{\gamma_t\}_{t\geq 0}$ *of maps from* $B(H_2, H_1)$ *into itself, the family* $\Theta = \{\Theta_t\}_{t\geq 0}$ *defined below is an* E_0*-semigroup on* $B(H_1 \oplus H_2)$*:*

$$\Theta_t \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(A_{11}) & \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) & \beta_t(A_{22}) \end{pmatrix}.$$

A *boundary weight over* $L^2(0,\infty)$ is a positive linear functional ν defined on

$$\mathfrak{A}(L^2(0,\infty)) = \sqrt{I-\Lambda}B(L^2(0,\infty))\sqrt{I-\Lambda}$$

such that $\tilde{\nu}$ below is in $B(L^2(0,\infty))_*$:

$$\tilde{\nu}(A) = \nu(\sqrt{I-\Lambda}A\sqrt{I-\Lambda}).$$

In other words, $\nu(\sqrt{I-\Lambda}A\sqrt{I-\Lambda}) = \sum_{i=1}^k (f_i, Af_i)$ for some orthogonal vectors $\{f_i\}_{i=1}^k$ with $\sum ||f_i||^2 = 1$. We say ν is bounded if there exists some $r > 0$ such that $|\nu(B)| \leq r||B||$ for all $B \in \mathfrak{A}(H)$. Otherwise, we say ν is unbounded.

From the work of Powers, each boundary weight ν over $L^2(0,\infty)$ satisfying $\tilde{\nu}(I) = 1$ defines a unital CP-flow α which (by Bhat's Theorem) dilates to an E_0-semigroup α^d. If ν is bounded, then α^d is of type I_k, while if ν is unbounded, α^d is of type II_0.

Boundary weight doubles

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a q-positive map. If ν is an unbounded and normalized ($\nu(I - \Lambda) = 1$) boundary weight over $L^2(0, \infty)$, then the boundary weight map $\rho \to \omega(\rho)$ from $M_n(\mathbb{C})^*$ to weights on $\mathfrak{A}(\mathbb{C}^n \otimes L^2(0, \infty)) = M_n(\mathfrak{A}(L^2(0, \infty)))$ defined by

$$\omega(\rho)\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \rho\left(\phi\begin{pmatrix} \nu(A_{11}) & \cdots & \nu(A_{1n}) \\ \vdots & \ddots & \vdots \\ \nu(A_{n1}) & \cdots & \nu(A_{nn}) \end{pmatrix}\right)$$

induces a (unique up to conjugacy) type II_0 E_0-semigroup α^d. We call this the E_0-semigroup induced by the boundary weight double (ϕ, ν). In this terminology, the "E_0-semigroup induced by ν" in the sense of the previous page is the one induced by $(\imath_{\mathbb{C}}, \nu)$.

There is a natural order structure for q-positive maps. If $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are q-positive, we say that ϕ q-dominates ψ ($\phi \geq_q \psi$) if $\phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1}$ is completely positive for all $t \geq 0$. As it turns out, every q-positive ϕ is guaranteed to have at least a one-parameter family of subordinates: For every $s \geq 0$, the map $\phi(I + s\phi)^{-1}$ is q-positive and $\phi \geq_q \phi(I + s\phi)^{-1}$.

Definition

*A q-positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-**pure** if its set of q-subordinates is $\{\phi(I + s\phi)^{-1}\}_{s \geq 0} \cup \{0\}$.*

So, what are all the q-positive maps? In the least, what are the q-pure maps? We note that we only care about a q-positive map up to something we call conjugacy. If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive and $U \in M_n(\mathbb{C})$ is unitary, we can define another q-positive map by $\phi_U(A) = U^* \phi(UAU^*)U$. We say ϕ is **conjugate** to ψ if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$. For reasons we will discuss later, we only care about q-positive maps up to conjugacy.

We have a classification for all q-pure maps which are rank one or invertible.

A rank one unital map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-pure if and only if $\phi(A) = \rho(A)I$ for some *faithful state* ρ.

An invertible unital $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-pure if and only if, for some unitary $U \in M_n(\mathbb{C})$, the map $\phi_U(A) := U^*\phi(UAU^*)U$ has the form

$$\phi_U(a_{jk}e_{jk}) = \begin{cases} \frac{a_{jk}}{1+i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j < k \\ a_{jk}e_{jk} & \text{if } j = k \\ \frac{a_{jk}}{1-i(\lambda_j-\lambda_k)}e_{jk} & \text{if } j > k, \end{cases}$$

for all $j, k = 1, \ldots, n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\lambda_1 + \ldots + \lambda_n = 0$.

We have a criterion for judging when certain E_0-semigroups arising from boundary weight doubles are cocycle conjugate. It involves the following definition:

Definition

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *and* $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ *be q-positive maps. A linear map* $\gamma : M_{n,k}(\mathbb{C}) \to M_{n,k}(\mathbb{C})$ *is a* **corner** *from* ϕ *to* ψ *if*

$$\Theta \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \psi(A_{22}) \end{pmatrix}$$

is completely positive. If $\Theta \geq_q 0$*, then* γ *is a q-***corner** *from* ϕ *to* ψ*. A q-corner is* **hyper maximal** *if, whenever*

$$\Theta \geq_q \Theta' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi' \end{pmatrix},$$

we have $\Theta = \Theta'$*.*

Proposition

Let ν be a normalized unbounded boundary weight over $L^2(0,\infty)$ which has the form

$$(*)\ \nu(\sqrt{I-\Lambda(1)}B\sqrt{I-\Lambda(1)}) = (f, Bf).$$

Let ϕ and ψ be unital q-positive maps on $M_n(\mathbb{C})$ and $M_k(\mathbb{C})$, respectively.

Then (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups if and only if there is a hyper maximal q-corner from ϕ to ψ.

Consequently, (ϕ, ν) and (ϕ_U, ν) induce cocycle conjugate E_0-semigroups for every unitary $U \in M_n(\mathbb{C})$.

Fact: the last sentence of this proposition holds if ν is any normalized boundary weight over $L^2(0,\infty)$ (joint work with Daniel Markiewicz) and cocycle conjugacy is replaced by *conjugacy* (an observation of Powers).

Assume ν has the form $(*)$:

If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ are rank one and q-pure, then (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups if and only if $n = k$ and ϕ is conjugate to ψ.

If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is invertible and q-pure, then (ϕ, ν) induces the same E_0-semigroup as the single boundary weight ν.

A limit for q-positive maps

Suppose $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive and

$$||t\phi(I + t\phi)^{-1}|| < 1$$

for all $t > 0$ (in particular, this holds if ϕ is unital). Then there is a unique limit

$$L_\phi = \lim_{t\to\infty} t\phi(I + t\phi)^{-1}.$$

Moreover, L_ϕ satisfies:

(i) $L_\phi \circ \phi = \phi \circ L_\phi = \phi$,
(ii) $L_\phi^2 = L_\phi$, and
(iii) $\text{range}(L_\phi) = \text{range}(\phi)$.

q-positive and q-pure maps on $M_2(\mathbb{C})$

Suppose $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is unital and q-positive. If $rank(\phi) \geq 2$, then from the equality $L_\phi^2 = L_\phi$ and the fact that L_ϕ is completely positive, we see that L_ϕ is conjugate to a Schur map, whereby we observe that (this conjugate version of) L_ϕ is either the diagonal map or the identity map.

In other words, ϕ cannot have rank 3, and if ϕ has rank 2 then (a conjugate version of) ϕ must satisfy

$$\phi(A) = \rho_1(A)e_{11} + \rho_2(A)e_{22}$$

for all $A \in M_2(\mathbb{C})$, where ρ_1 and ρ_2 are diagonal states $(\rho_j(e_{12}) = \rho_j(e_{21}) = 0)$.

The previous page and some additional work give us the following:

Theorem

Let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a unital linear map. Then ϕ is q-positive if and only if it satisfies one of the following:

(i) $\phi(A) = \rho(A)I$ *for all* $A \in M_2(\mathbb{C})$, *where* $\rho \in M_2(\mathbb{C})^*$ *is a state.*

(ii) ϕ *is conjugate to the map* ψ *defined by* $\psi(A) = \rho_1(A)e_{11} + \rho_2(A)e_{22}$ *for all* $A \in M_2(\mathbb{C})$, *where* ρ_1 *and* ρ_2 *are states satisfying* $\rho_j(e_{12}) = \rho_j(e_{21}) = 0$ *for each* j *and*

$$\rho_1(e_{11})\rho_2(e_{22}) - \rho_1(e_{22})\rho_2(e_{11}) > 0.$$

(iii) $\phi = \psi^{-1}$, *where* $\psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ *is a unital conditionally negative map.*

Furthermore, a non-trivial calculations shows that none of the rank two q-positive maps acting on $M_2(\mathbb{C})$ are q-pure, so:

Theorem

A unital linear map $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is $q-$pure if and only if it has one of the following forms:

(i) $\phi(A) = \rho(A)I$ *for all* $A \in M_2(\mathbb{C})$, *where* $\rho \in M_2(\mathbb{C})^*$ *is a faithful state.*

(ii) *For some unitary* $U \in M_2(\mathbb{C})$ *and* $\lambda \in \mathbb{R}$, ϕ_U *is the Schur map*

$$\phi_U(A) = \left\{ \begin{array}{cc} a_{11} & \frac{a_{12}}{1+i\lambda} \\ \frac{a_{21}}{1-i\lambda} & a_{22} \end{array} \right)$$

for all $A \in M_2(\mathbb{C})$.

Onward to the 3×3 matrices

For each $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the set of all unital completely positive maps $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $\Phi^2 = \Phi$. We see that

$$\mathcal{E}_n = \{L_\phi \mid \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}),\ \phi(I) = I,\ \text{and } \phi \geq_q 0\}.$$

Let $\Phi \in \mathcal{E}_3$. Solely using linear algebra and the fact that $\Phi^2 = \Phi$, we can show that if Φ does not annihilate any nonzero projections, then Φ must fix a projection. Replacing Φ with a conjugate version of itself, we may assume $\Phi(e_{11}) = 0$ or $\Phi(e_{11}) = e_{11}$. We have a full classification of the maps in $\mathcal{E}_3$:

Lemma

Let $\Phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ be a unital map such that $\Phi(E) = 0$ for some projection E. Then $\Phi \in \mathcal{E}_3$ if and only if, up to conjugacy, Φ has one of the following forms (for some $\lambda \in [0,1]$):

$$\Phi(A) = \begin{pmatrix} \lambda a_{22} + (1-\lambda)a_{33} & 0 & 0 \\ 0 & \lambda a_{22} + (1-\lambda)a_{33} & 0 \\ 0 & 0 & \lambda a_{22} + (1-\lambda)a_{33} \end{pmatrix}$$

$$\Phi(A) = \begin{pmatrix} \lambda a_{22} + (1-\lambda)a_{33} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix};$$

$$\Phi(A) = \begin{pmatrix} \lambda a_{22} + (1-\lambda)a_{33} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix};$$

Lemma

Suppose $\Phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ is a linear map which does not annihilate any projections and satisfies rank$(\Phi) > 1$. Then $\Phi \in \mathcal{E}_3$ if and only if, up to conjugacy, it has one of the following forms for all $A \in M_3(\mathbb{C})$:

$$\Phi(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix};$$

$$\Phi(A) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix};$$

$$\Phi(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \lambda a_{22} + (1-\lambda)a_{33} & 0 \\ 0 & 0 & \lambda a_{22} + (1-\lambda)a_{33} \end{pmatrix}, \quad \lambda \in (0,1);$$

$$\Phi(A) = A.$$

Corollary

Let $\phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ *be unital and q-positive. Then* ϕ *has rank* $1, 2, 3, 4, 5,$ *or* 9*.*

Unfortunately, we do not have a classification theorem for all unital q-pure maps $\phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$. However, we can say the following:

Theorem

If $\phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ *is a unital q-positive map and* $\phi(R) = 0$ *for some* $R \gneq 0$*, then* ϕ *is not q-pure.*

Cocycle conjugacy results

Proposition

Let ν be a normalized unbounded boundary weight over $L^2(0,\infty)$ of the form $\nu(\sqrt{I-\Lambda(1)}B\sqrt{I-\Lambda(1)}) = (f, Bf)$. Let ϕ_2 be the diagonal map $\phi_1(A) = a_{11}e_{11} + a_{22}e_{22}$ for all $A \in M_2(\mathbb{C})$, and let $\phi_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be any unital invertible q-positive Schur map.

The E_0-semigroups induced by (ϕ_1, ν) and (ϕ_2, ν) are non-cocycle conjugate.

We also have the following, which reaffirms what we already learned in the q-pure case, namely that the rank one q-positive maps are simply “different” than the invertible ones in terms of cocycle conjugacy:

Theorem

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *(*$n \geq 2$*) and* $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ *be unital q-positive maps. Suppose that* ϕ *has rank one and that* L_ψ *is a Schur map. Let* ν *be a normalized unbounded boundary weight over* $L^2(0,\infty)$ *of the form* $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$*. Then* (ϕ, ν) *and* (ψ, ν) *induce non-cocycle conjugate* E_0*-semigroups* α^d *and* β^d*.*