A family of non-cocycle conjugate E_0-semigroups obtained from boundary weight doubles

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January 9, 2011

Wigner, 1939

Every suitable group $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of $B(H)$ is implemented by a unitary group $W = \{W_t\}_{t\in\mathbb{R}}$ in the sense that

$$\alpha_t(A) = W_t A W_t^* \quad (\text{all } A \in B(H), t \in \mathbb{R}).$$

Replace $*$-*automorphism* with $*$-*endomorphism* and group with semigroup. Do we get a result analogous to Wigner's Theorem?

For example, is there always a semigroup $S = \{S_t\}_{t\geq 0}$ of isometries such that $\alpha_t(A) = S_t A S_t^*$ for all $t \geq 0$?

Definition

We say a semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ of $*$-endomorphisms of $B(H)$ is an **$\mathbf{E}_0$-semigroup** *if:*

- *For each $f, g \in H$ and $A \in B(H)$, the inner product $(f, \alpha_t(A)g)$ is continuous in t;*
- *$\alpha_t(I) = I$ for all $t \geq 0$ (i.e. α is unital).*

E_0-semigroups acting on $M_n(\mathbb{C})$ are automatically $*$-automorphism semigroups (so Wigner's Theorem applies).

CAR / CCR flows

Let $H = L^2((0,\infty);K)$, where $dim(K) = n$, and let $U = \{U_t\}_{t\geq 0}$ be the right shift semigroup on H. Form $F_-(H) = \bigoplus_{k=0}^{\infty} H^{\wedge k}$. Each $f \in H$ yields a creation operator $c(f)$:

$$c(f)(f_1 \wedge \cdots \wedge f_n) = f \wedge f_1 \cdots \wedge f_n.$$

CAR flow of rank n: The E_0-semigroup on $B(F_-(H))$ defined by

$$\alpha_t(c(f)) = c(U_t f) \ \ (f \in H, \ t \geq 0).$$

A similar construction for $F_+(H)$ gives us the CCR flow of rank n.

What might another analogue of Wigner's result be?

Units: A unit is a strongly continuous semigroup $\{V_t\}_{t \geq 0}$ of bounded operators such that $\alpha_t(A)V_t = V_t A$ for all A, $t \geq 0$.

Every one-parameter group of $*$-automorphisms has a unique (up to exponential scaling) unit.

Q: If α is an E_0-semigroup, is there necessarily a unique unit for α?

A: No! α need not have a unique unit. In fact, α can have infinitely many units, or none at all.

We divide E_0-semigroups into three types, depending on the structure of their units.

Type I: α has enough units to (roughly speaking) reconstruct H.

Type II: α has at least one unit, but is not type I.

Type III: α has no units.

If α is of type I or II, it is assigned an index $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which is invariant under "equivalence."

We have two notions of equivalence for E_0-semigroups, conjugacy and cocycle conjugacy.

Definition

Let α and β be E_0-semigroups acting on $B(H_1)$ and $B(H_2)$, respectively. We say α and β are **conjugate** *if there is a $*$-isomorphism θ from $B(H_1)$ onto $B(H_2)$ such that $\alpha_t = \theta^{-1} \circ \beta_t \circ \theta$ for all $t \geq 0$.*

We say α' is a cocycle perturbation of α if there is a strongly continuous family of unitaries $V = \{V_t\}_{t \geq 0}$ such that $V_t \alpha_t(V_s) = V_{t+s}$ and $\alpha'_t(A) = V_t \alpha_t(A) V_t^*$ for all $s, t \geq 0$ and $A \in B(H)$.

We say α and β are **cocycle conjugate** *(i.e. $\alpha \simeq \beta$) if β is conjugate to a cocycle perturbation of α.*

Type I

The type I E_0-semigroups are completely determined (in terms of cocycle conjugacy) by their index ([1]): If α is of type I_n ($n \neq 0$), then α is cocycle conjugate to the CAR/CCR flow of rank n.

The CAR flow of rank n is conjugate to the CCR flow of rank n for each $n \in \mathbb{N} \cup \{\infty\}$ ([10]).

Types II and III

Powers ([11], [9]): There are type II and type III E_0-semigroups.

Arveson, [1]: Every E_0-semigroup α has a *product system* $\mathcal{E}_\alpha$, and $\alpha \simeq \beta$ if and only if $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic.

Tsirelson, [14]: There are uncountably many non-cocycle conjugate type II and type III E_0-semigroups.

Another way of getting type II's

Bhat, [3]: Unital CP-semigroups dilate to E_0-semigroups.
Powers, [12]: Every type I or II E_0-semigroup can be obtained by dilating CP-flows, i.e. CP-semigroups acting on $L^2((0,\infty);K)$ which are intertwined by right translation.

Every CP-flow corresponds to a *boundary weight map* $\rho \to \omega(\rho)$ from $B(K)_*$ to (possibly unbounded) linear functionals acting on

$$\mathfrak{A}(H) = (I - \Lambda)^{\frac{1}{2}} B(H)(I - \Lambda)^{\frac{1}{2}},$$

where $\Lambda \in B(H)$ is the operator $\Lambda(f)(x) = e^{-x} f(x)$.

Powers investigated the case where $K = \mathbb{C}$ in [13]. In this case, the boundary weight map is a single "boundary weight," i.e. a linear functional ω defined on $\mathfrak{A}(L^2(0,\infty))$ of the form

$$\omega\Big((I-\Lambda)^{\frac{1}{2}}A(I-\Lambda)^{\frac{1}{2}}\Big) = \sum_{i=1}^{n}(f_i, Af_i)$$

for some orthogonal nonzero vectors $\{f_i\} \subset L^2(0,\infty)$. So:

$$\omega \quad \longrightarrow \quad \text{CP-flow } \alpha \quad \longrightarrow \quad E_0\text{-semigroup } \alpha^d.$$

If ω is unbounded (i.e. is a *type II Powers weight*), then it induces a type II_0 E_0-semigroup.

Being type II_0 means:

1. α has exactly one unit (up to exponential scaling) and
2. α_t is a proper $*$-endomorphism for all $t > 0$.

Powers constructed uncountably many non-cocycle conjugate E_0-semigroups in the $K = \mathbb{C}$ case using type II Powers weights.

We can naturally generalize this method to construct type II_0 examples for $K = \mathbb{C}^n$ by combining type II Powers weights with q-positive maps.

q-positive maps

We say $\phi : B(H_1) \to B(H_2)$ is completely positive if the maps $\phi_n : M_n(B(H_1)) \to M_n(B(H_2))$ defined by

$$\phi_n \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \begin{pmatrix} \phi(A_{11}) & \cdots & \phi(A_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(A_{n1}) & \cdots & \phi(A_{nn}) \end{pmatrix}$$

are positive for all $n \in \mathbb{N}$.

We will need more:

Definition

A completely positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with no negative eigenvalues is said to be **q-positive** *if $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$.*

Proposition (J, [5])

Let ν be a type II Powers weight, and let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital and q-positive.

Then the boundary weight map $\rho \to \omega(\rho)$ from $M_n(\mathbb{C})^*$ into $\mathfrak{A}(L^2((0,\infty);\mathbb{C}^n))$ defined by

$$\omega(\rho)\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \rho\left(\phi\begin{pmatrix} \nu(A_{11}) & \cdots & \nu(A_{1n}) \\ \vdots & \ddots & \vdots \\ \nu(A_{n1}) & \cdots & \nu(A_{nn}) \end{pmatrix}\right)$$

induces a type II_0 E_0-semigroup. We call (ϕ, ν) a boundary weight double.

To summarize:

$$(\phi,\nu) \quad\longrightarrow\quad \text{CP-flow } \alpha \quad\longrightarrow\quad \text{type II}_0\ E_0\text{-semigroup } \alpha^d.$$

Order structure for q-positive maps

If $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are q-positive, we say ϕ q-dominates ψ ($\phi \geq_q \psi$) if

$$\phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1} \text{ is c.p. for all } t \geq 0.$$

If $\phi \geq_q 0$, it automatically has a 1-parameter family of q-subordinates: for every $s \geq 0$,

$$\phi \geq_q \phi(I + s\phi)^{-1} \geq_q 0.$$

If these are its *only* nonzero subordinates, we say ϕ is **q-pure**.

Some examples of unital q-positive maps

1. **Rank one**: ϕ is q-positive if and only if it has the form $\phi(A) = \rho(A)I$ for some state $\rho \in M_n(\mathbb{C})^*$, and ϕ is q-pure if and only if ρ is *faithful*.

2. **Invertible**: $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is q-positive if and only if ϕ^{-1} is conditionally negative, and is q-pure if and only if it is (up to change of basis) a Schur map of a particular form.

3. Some other basic completely positive maps are q-positive, such as $\phi(A) \equiv A$ and ϕ the map which preserves the diagonal entries and erases the rest. Any idempotent completely positive map is q-positive.

Comparison theory

If (ϕ, ν) and (ψ, η) are boundary weight doubles, how can we tell if they induce cocycle conjugate E_0-semigroups? We have a partial answer, and it involves:

Definition

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be q-positive. We say a linear map $\gamma : M_{n\times k} \to M_{n\times k}(\mathbb{C})$ is a **hyper-maximal q-corner** *from ϕ to ψ if*

1. $\Theta : M_{n+k}(\mathbb{C}) \to M_{n+k}(\mathbb{C})$ defined by

$$\Theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \phi(A) & \gamma(B) \\ \gamma^*(C) & \psi(D) \end{pmatrix}$$

is q-positive;

2.

$$\Theta \geq_q \Theta' = \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \psi' \end{pmatrix} \geq_q 0 \Longrightarrow \Theta = \Theta'.$$

Proposition (J, [5])

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be unital q-positive maps, and let ν be a type II Powers weight.

If ν is **totally pure**, *which is to say that it has the form*

$$\nu(\sqrt{I - \Lambda(1)} B \sqrt{I - \Lambda(1)}) = (f, Bf),$$

then (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups if and only if there is a hyper maximal q-corner from ϕ to ψ.

The rank one case

We recall that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital map of rank one, then $\phi \geq_q 0$ if and only if $\phi(A) = \rho(A)I$ for some state $\rho \in M_n(\mathbb{C})^*$. From general theory, ρ has the form

$$\rho(A) = \sum_{i=1}^{k} \lambda_i (g_i, Ag_i)$$

for some orthonormal vectors $\{g_i\}_{i=1}^k$ and numbers $\lambda_1 \geq \cdots \geq \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$.

We call $\{\lambda_i\}_{i=1}^k$ the **eigenvalue list** of ρ.

We have the following:

Proposition

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be unital rank one q-positive maps, so

$$\phi(A) = \ell(A)I_n, \quad \psi(D) = \ell'(D)I_{n'}.$$

Let ν and η be type II Powers weights.

If (ϕ, ν) and (ψ, η) induce cocycle conjugate E_0-semigroups, then ℓ and ℓ' have identical eigenvalue lists.

Furthermore:

Theorem (J, [6])

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be rank one unital q-positive maps, and let ν be a totally pure type II Powers weight.

Let α and β be the E_0-semigroups induced by (ϕ, ν) and (ψ, ν).
TFAE:
(1) α and β are cocycle conjugate;
(2) $n = n'$ and for some unitary $U \in M_n(\mathbb{C})$, $\phi(A) = \psi(UAU^*)$.

(3) ((J-Markiewicz, [8]) α and β are **conjugate***!)*

Some further questions:

Q: What are the gauge groups for these E_0-semigroups induced by (ϕ, ν) if ϕ has rank one and ν is totally pure?

A: $\mathbb{R} \times \mathbb{R} \times U_\phi / S^1$, where U_ϕ is the group of unitary matrices that commute with the state that implements ϕ.

Q: How do these E_0-semigroups compare to those induced by single boundary weights ω in the $K = \mathbb{C}$ case?

A: None of these are cocycle conjugate to any of those from the $K = \mathbb{C}$ case.

References

- W.B. Arveson, *Continuous Analogues of Fock space*, Memoirs Amer. Math. Soc. **80**, no. 409 (1989).
- W.B. Arveson, *The Index of a Quantum Dynamical Semigroup*, J. Funct. An. **146** (1997), 557-588.
- B.V.R. Bhat, *An index theory for quantum dynamical semigroups*, Trans. A.M.S. **348** (1996), no. 2, 561-583.
- M. Choi, *Completely positive linear maps on complex matrices*, Lin. Alg. Appl. **10** (1975), 285-290.
- C. Jankowski, *On type II_0 E_0-semigroups induced by boundary weight doubles*, J. Func. Anal. **258** (2010), no. 10, 3413-3451.
- C. Jankowski, *A family of non-cocycle conjugate E_0-semigroups obtained from boundary weight doubles*, J. Operator Theory, to appear.
- C. Jankowski, *Unital q-positive maps on $M_2(\mathbb{C})$ and a related*