Gauge groups of E_0-semigroups

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Definition

We say a semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ of $*$-endomorphisms of $B(H)$ is an **$\mathbf{E}_0$-semigroup** *if:*

- *For each $f, g \in H$ and $A \in B(H)$, the inner product $(f, \alpha_t(A)g)$ is continuous in t;*
- *$\alpha_t(I) = I$ for all $t \geq 0$ (i.e. α is unital).*

A *unit* for α is a strongly continuous semigroup $W = \{W_t\}_{t \geq 0}$ of operators in $B(H)$ such that

$$\alpha_t(A) W_t = W_t A \qquad \forall A \in B(H),\ t \geq 0.$$

Types of E_0-semigroups

Type I: α's units can reconstruct H for every $t > 0$.

Type II: α has at least one unit, but is not type I.

Type III: α has no units.

If α is of type I or II, it is assigned an index $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which is invariant under "equivalence."

Equivalence

Definition

Let α and β be E_0-semigroups acting on $B(H_1)$ and $B(H_2)$, respectively.

α and β are **conjugate** *if there is an onto $*$-isomorphism $\theta : B(H_1) \to B(H_2)$ such that $\alpha_t = \theta^{-1} \circ \beta_t \circ \theta$ for all $t \geq 0$.*

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*A β-***cocycle** *is a strongly continuous family of bounded operators $\mathcal{W} = \{W_t\}_{t \geq 0}$ acting on H_2 such that $W_t \beta_t(W_s) = W_{t+s}$ for all $t, s \geq 0$.*

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α and β are **cocycle conjugate** *if there exists a unitary β-cocycle $\{W_t\}_{t \geq 0}$ such that $\beta'_t(A) := W_t \beta_t(A) W_t^*$ is conjugate to α.*

Bhat's dilation theorem and boundary weight maps

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Powers ([8]):

1. All spatial E_0-semigroups are (cocycle conjugate to) dilations of CP-flows acting on $H = K \otimes L^2(0, \infty)$.
2. Every CP-flow is determined by a q-weight map $\omega : B(K)_* \to \mathfrak{A}(H)_*$, where $\Lambda(f)(x) = e^{-x} f(x)$ and

$$\mathfrak{A}(H) = (I - \Lambda)^{1/2} B(H)(I - \Lambda)^{1/2}.$$

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The q-weight map is c.p., and its truncated maps $\omega_t(\rho) := \omega(\rho)(E_{(t,\infty)} A E_{(t,\infty)})$ map $B(K)_*$ into $B(H)_*$.

Functionals $\omega(\rho) \in \mathfrak{A}(H)_*$ are called *boundary weights*.

If $K = \mathbb{C}$, q-weight maps are just boundary weights

A positive boundary weight $\omega \in \mathfrak{A}(L^2(0,\infty))_*$ has the form

$$\omega\big((I-\Lambda)^{1/2}A(I-\Lambda)^{1/2}\big) = \sum_{i=1}^{n}(f_i, Af_i)$$

for some orthogonal vectors f_i. Suppose $\omega(I-\Lambda)=1$. Then

$$\omega \longrightarrow \quad E_0\text{-semigroup } \alpha^d : \begin{cases} \text{Type I}_n \text{ if } \omega \text{ bounded} \\ \text{Type II}_0 \text{ if } \omega \text{ unbounded} \end{cases}$$

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For this reason, we say ω is a type I Powers weight if it is bounded and a **type II Powers weight** if it is unbounded.

E_0-semigroups obtained from CP-flows over $K = \mathbb{C}$

Powers [9] constructed uncountably many non-cocycle conjugate type II_0 E_0-semigroups using type II Powers weights.

Given two E_0-semigroups dilated from CP-flows over $\mathbb{C}$, it is often difficult to tell when they are cocycle conjugate. Basically, boundary weights $\nu \in \mathfrak{A}(L^2(0,\infty))_*$ are complicated.

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For example, if

$$\nu\big((I-\Lambda)^{1/2}A(I-\Lambda)^{1/2}\big) = (f, Af) + (g, Ag),$$

and if no linear combination of f and g is in the range of $(I-\Lambda)^{1/2}$, then ν gives rise to a *q-pure* CP-flow.

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We build a q-weight map $\omega : B(K)_* \to \mathfrak{A}(K \otimes L^2(0, \infty))_*$ by

1. Taking $\nu \in \mathfrak{A}(L^2(0, \infty))_*$ and
2. Combining it a with a linear map $\phi : B(K) \to B(K)$.

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If ν is as simple as possible, this takes all the complexity out of the $L^2(0,\infty)$ component and shifts our attention to $M_n(\mathbb{C})$.

Question: What are the candidates ϕ for such a q-weight map?

q-positive maps

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Definition

A completely positive map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with no negative eigenvalues is **q-positive** *if*

$$\phi(I + t\phi)^{-1} \text{ is c.p. for all } t \geq 0.$$

The condition for q-positivity condition is demanding, since

- Completely positive maps with negative eigenvalues exist in abundance;
- Given $s \geq 0$, we can find a c.p. map ϕ (with no negative eigenvalues) such that $\phi(I + t\phi)^{-1}$ is c.p. if and only if $t \leq s$.

Constructing E_0-semigroups from boundary weight doubles

When $K = \mathbb{C}^n$ and we want to get E_0-semigroups using CP-flows over K, we can combine type II Powers weights with q-positive maps in a natural way.

Proposition (J, [4])

Let ν be a type II Powers weight, and let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital and q-positive.

Then the boundary weight map $\omega : M_n(\mathbb{C})_* \to \mathfrak{A}\big(\mathbb{C}^n \otimes L^2(0,\infty)\big)$ defined by

$$\omega'(A) = \phi\big[(I \otimes \nu)(A)\big]$$

induces a type II_0 E_0-semigroup. We call (ϕ, ν) a boundary weight double.

Order structure for q-positive maps

If ϕ and ψ are q-positive, we say $\phi \geq_q \psi$ if

$$\phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1} \text{ is c.p. } \forall t \geq 0.$$

Fact: $\phi \geq_q 0 \implies \phi \geq_q \phi(I + s\phi)^{-1} \geq_q 0 \quad \forall s \geq 0.$
If these are its *only* nonzero subordinates, we say ϕ is **q-pure**.

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Proposition (J, [4])

If ϕ is unital and q-pure and ν is totally pure, then (ϕ, ν) induces a q-pure CP-flow α.

The set of non-trivial flow subordinates of α is $\{\alpha^{(s)}\}_{s \geq 0}$, where $\alpha^{(s)}$ is the CP-flow induced by $(\phi(I + s\phi)^{-1}, \nu)$. Furthermore, $\alpha^{(s)} \geq \alpha^{(t)}$ if $s \leq t$.

Comparison theory

If (ϕ, ν) and (ψ, η) are boundary weight doubles, how can we tell if they induce cocycle conjugate E_0-semigroups? We have a partial answer, and it involves:

Definition

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be q-positive. We say a linear map $\gamma : M_{n\times k} \to M_{n\times k}(\mathbb{C})$ is a **corner** *from ϕ to ψ if $\phi \oplus_\gamma \psi$ is completely positive.*

We say γ is a **hyper-maximal q-corner** *from ϕ to ψ if*

1. $\phi \oplus_\gamma \psi \geq_q 0$ *and*
2. $\phi \oplus_\gamma \psi \geq_q \phi' \oplus_\gamma \psi' \geq_q 0 \implies \phi = \phi', \psi = \psi'$.

Proposition (J, [4])

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ be unital q-positive maps, and let ν be a type II Powers weight.

If ν is **totally pure**, *which is to say that it has the form*

$$\nu(\sqrt{I - \Lambda(1)} B \sqrt{I - \Lambda(1)}) = (f, Bf),$$

then (ϕ, ν) and (ψ, ν) induce cocycle conjugate E_0-semigroups if and only if there is a hyper maximal q-corner from ϕ to ψ.

Boundary Expectations

J-Markiewicz-Powers, [7]: If $\omega : B(K)_* \to \mathfrak{A}(H)_*$ is a q-weight map with finite range rank and gives rise to a type II_0 E_0-semigroup, then it has a boundary expectation L.

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Boundary weight double case: $(\phi, \nu) \longrightarrow$ *unique* L. Moreover,

1. $L = \lim_{t \to \infty} t\phi(I + t\phi)^{-1}$;
2. $L^2 = L$ and $||L|| = 1$;
3. $\text{range}(L) = \text{range}(\phi)$;
4. $\text{nullspace}(L) = \text{nullspace}(\phi)$.

Proposition (J, [5])

If ν and η are type II Powers weights and (ϕ, ν) and (ψ, η) induce cocycle conjugate E_0-semigroups, then there is a norm one corner from L_ϕ to L_ψ.

The Rank One Case

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From general theory, every state $\rho \in M_n(\mathbb{C})_*$ has the form

$$\rho(A) = \sum_{i=1}^{k} \lambda_i (g_i, A g_i)$$

for some orthonormal vectors $\{g_i\}_{i=1}^{k}$ and numbers $\lambda_1 \geq \cdots \geq \lambda_k > 0$ with $\sum_{i=1}^{k} \lambda_i = 1$.

We call $\{\lambda_i\}_{i=1}^{k}$ the **eigenvalue list** of ρ.

We have the following:

Proposition (J, [5])

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be unital rank one q-positive maps, so

$$\phi(A) = \ell(A) I_n, \quad \psi(D) = \ell'(D) I_{n'}.$$

Let ν and η be type II Powers weights.

If (ϕ, ν) and (ψ, η) induce cocycle conjugate E_0-semigroups, then ℓ and ℓ' have identical eigenvalue lists.

Furthermore:

Theorem (J-Markiewicz, [6])

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be rank one unital q-positive maps, and let ν be a totally pure type II Powers weight.

Let α and β be the E_0-semigroups induced by (ϕ, ν) and (ψ, ν).

The following are equivalent:

1. *α and β are cocycle conjugate;*
2. *$n = n'$ and for some unitary $U \in M_n(\mathbb{C})$, $\phi(A) = \psi(UAU^*)$.*
3. *α and β are* **conjugate**.

Gauge Groups

Let α be an E_0-semigroup. Recall that a strongly continuous semigroup $W = \{W_t\}$ of bounded operators is an α-cocycle if $W_t \alpha_t(W_s) = W_{t+s}$ for all $s, t \geq 0$.

Definition

An α-cocycle is **local** *if*

$$W_t \alpha_t(A) = \alpha_t(A) W_t \qquad \forall A \in B(H), \ t \geq 0.$$

The set of all local unitary α-cocycles forms a multiplicative group with respect to the pointwise operation $(\mathcal{W} \cdot \mathcal{W}')_t = W_t W_t'$. This is called the **gauge group** *of α which we denote by $G(\alpha)$.*

Known Gauge Groups

Theorem (Arveson, [2])

If α is of type I_n and K is the Hilbert space of (complex) dimension n, then $G(\alpha) \simeq \mathcal{H}_n \rtimes \mathcal{U}(K)$.

- *$\mathcal{H}_n$ is the Heisenberg group of real dimension $2n+1$ (homeomorphic to $\mathbb{R} \times K$);*
- *$\mathcal{U}(K)$ is the unitary group of K.*

Multiplication is given by

$$(r, \xi, U) \cdot (s, \eta, V) = \big(r + s + Im\langle \xi, U\eta \rangle,\ \xi + U\eta,\ UV\big).$$

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Type II: Alevras, Powers, and Price [1] found a description of the gauge groups for type II_0 E_0-semigroups arising from CP-flows over $\mathbb{C}$. However, these gauge groups are often very difficult to compute explicitly.

Flow Cocycles

Every unital CP-flow α has a minimal dilation to an E_0-semigroup α^d which is also a CP-flow.

Definition

Let K be a separable Hilbert space and denote by $\{S_t\}_{t \geq 0}$ the right-shift semigroup acting on $K \otimes L^2(0, \infty)$.

Suppose α^d is both a CP-flow over K and an E_0-semigroup. A contractive α^d-cocycle $\{W_t : t \geq 0\}$ is called a *flow cocycle* if $W_t S_t = S_t$ for all $t \geq 0$. We denote by $G_{flow}(\alpha^d)$ the subgroup of $G(\alpha^d)$ consisting of all local unitary flow α-cocycles.

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Fact: If α^d has type II_0, then

$$G(\alpha^d) = \{e^{irt} C_t : r \in \mathbb{R},\ C \in G_{flow}(\alpha^d)\}.$$

The elements of $G_{flow}(\alpha^d)$ are in one-to-one correspondence with the hyper-maximal flow corners from α to α.

In the case of boundary weight doubles (ϕ, ν) for $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ rank one and ν totally q-pure, these are in bijection with the h.m.q.c's from ϕ to ϕ, which were computed explicitly in [5].

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Since $rank(\phi) = 1$, we have $\phi(A) = \text{tr}(A\Omega)I$ for some positive matrix Ω. Let $\mathcal{U}_\Omega = \{W \in \mathcal{U}(n) : W\Omega = \Omega W\}$, and let G_Ω be the group

$$G_\Omega = \mathbb{R} \times \mathcal{U}_\rho / \mathbb{T}$$

with the usual product, where $\mathbb{T} = \{cI : c \in \mathbb{C}, |c| = 1\}$.

In light of the bijection between h.m.f.c's from α to α and h.m.q.c's from ϕ to ϕ, the theorem below gives us a bijection between G_Ω and $G_{flow}(\alpha^d)$.

Theorem (J-Markiewicz, [6])

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a rank one unital q-positive map, so $\phi(A) = \mathrm{tr}(A\Omega)I$. If $\{x, X\} \in G_\Omega$, then the map

$$\gamma_{\{x,X\}}(A) = \frac{1}{1+ix}\mathrm{tr}(X^* A\Omega)X$$

is a well-defined h.m.q.c. from ϕ to ϕ. Conversely, if γ is a h.m.q.c. from ϕ to ϕ, then $\gamma = \gamma_{\{x,X\}}$ for some $\{x, X\} \in G_\Omega$. Furthermore, if $g, h \in G_\rho$ and $\gamma_g = \gamma_h$, then $g = h$.

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Let $C_g, C_h \in G_{flow}(\alpha^d)$. Their product $C_g C_h$ is another unitary local flow cocycle C_s. What is s? As it turns out, $s = gh$.

Cocycle Multiplication

Theorem

Let ν be a totally q-pure type II Powers weight, and let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital rank one q-positive map, so $\phi(A) = \text{tr}(A\Omega)I$ for all $A \in M_n(\mathbb{C})$.

Let α^d be the minimal flow dilation of the CP-flow α induced by the boundary weight double (ϕ, ν). Then the map $g \mapsto C_g$ is an isomorphism from G_Ω onto $G_{flow}(\alpha^d)$, thus $G(\alpha^d) \simeq \mathbb{R} \times G_\Omega$.

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In other words, $G(\alpha^d) \simeq \mathbb{R} \times \mathbb{R} \times (\mathcal{U}_\Omega / \mathbb{T})$ under the multiplication $(r, x, X) \cdot (s, y, Y) = (r + s, x + y, XY)$.

Some Examples

1. $\phi(A) = \frac{1}{n} tr(A) I$;

$$G(\alpha^d) \simeq \mathbb{R} \times \mathbb{R} \times \mathcal{U}(n)/\mathbb{T},$$

2. $\phi(A) = (\sum_{i=1}^{n} \lambda_i a_{ii}) I$ acting on $M_n(\mathbb{C})$, where $\sum \lambda_i = 1$ and $\lambda_i \neq \lambda_j$ whenever $i \neq j$;

$$G(\alpha^d) \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{T}^{n-1}.$$

It all comes down to the eigenvalue list and the dimension of the matrix algebra.

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