SUSTAINED p-DIVISIBLE GROUPS

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In this chapter we develop and explain the notion of *sustained p-divisible groups*, which provides a scheme-theoretic definition of the notion of *central leaves* introduced in [21].

1. Introduction: what is a sustained p-divisible group

1.1. In a nutshell, a p-divisible group $X \to S$ over a base scheme S in equi-characteristic p is said to be *sustained* if its p^n-torsion subgroup $X[p^n] \to S$ is constant locally in the flat topology of S, for every natural number n; see 1.3 for the precise definition. In music a sustained (*sostenuto* in Italian) tone is constant, but as the underlying harmony may change it does not feel constant at all. Hence we use the adjective "sustained" to describe a familiy of "the same objects" whereas the whole family need not be constant. This concept enriches the notion of *geometrically fiberwise constant* family of p-divisible groups. Proofs of statements in this introduction are given in later sections.

Careful readers are likely to have zeroed in on an important point which was glossed over in the previous paragraph: being "constant" over a base scheme is fundatmentally a *relative* concept. There is no satisfactory definition of a "constant family $Y \to S$ of algebraic varieties" over a general base scheme S, while for a scheme S over a field κ, a κ-constant family over S is a morphism $Y \to S$ which is S-isomorphic to $Y_0 \times_{\mathrm{Spec}(\kappa)} S$ for some κ-scheme Y_0. Similarly a base field κ needs to be fixed, before one can properly define and study sustained p-divisible groups relative to the chosen base field κ. Not surprisingly, the meaning of sustainedness changes with the base field: if S is a scheme over a field $\kappa \supset \mathbb{F}_p$ and $X \to S$ is a p-divisible group over S, being sustained relative to a subfield δ of κ is a stronger condition than being sustained relative to κ itself.

Perhaps the most familiar example of sustained p-divisible groups is the p-divisible group $A[p^\infty] \to S$ attached to an *ordinary* abelian scheme $A \to S$ over a scheme S in characteristic p of constant relative dimension. Note that a p-divisible group $A[p^\infty] \to S$ is in general *not constant*, in the sense that $A[p^\infty] \to S$ is not S-isomorphic to $f^*(X_0)$, for any field $\kappa \supset \mathbb{F}_p$, any p-divisible group over κ and any morphism $f : S \to \mathrm{Spec}\,\kappa$. However $A[p^\infty] \to S$ has the following property: for any algebraically closed field $k \supset \mathbb{F}_p$ and any two geometric points $s_1, s_2 : \mathrm{Spec}\,k \to S$, the two p-divisible groups $s_1^*A[p^\infty]$ and $s_2^*A[p^\infty]$ are isomorphic over k. It is natural to say that the p-divisible group $A[p^\infty] \to S$ attached to the ordinary abelian scheme $A \to S$ is "geometrically fiberwise constant" (relative to $\mathbb{F}_p$).

A basic fact underly the above example is the following: if $k \supset \mathbb{F}_p$ is an algebraically closed field and X_0 is a p-divisible group over k which is *ordinary* in the sense that its slopes are contained in $\{0, 1\}$, then X_0 is isomorphic to $\mu_{p^\infty}^d \times (\mathbb{Q}_p/\mathbb{Z}_p)^c$ for some $c, d \in \mathbb{N}$. In particular for any field endomorphism $\tau : k \to k$, the τ-twist $X_0 \times_{\mathrm{Spec}(k),\tau} \mathrm{Spec}(k)$ of the p-divisible group X_0 is k-isomorphic to X_0 itself. This property does not hold for all p-divisible groups over algebraically closed base field. So a reasonable definition of "geometrically fiberwise constant" p-divisible group has to include a "reference base field".

1.2. Definition. Let $\kappa \supset \mathbb{F}_p$ be a field, and let S be a reduced scheme over κ. A p-divisible group $X \to S$ is *geometrically fiberwise constant relative to* κ if for any algebraically closed field k which contains κ and any two points $s_1, s_2 \in S(k)$, there exists some k-isomorphism from $s_1^* X$ to $s_2^* X$.

Remark. Working over an algebraically closed based field $k \supset \mathbb{F}_p$, and let $n \geq 3$ be an positive integer which is prime to p, a *central leaf* $\mathscr{C}$ over k in $\mathscr{A}_{g,n}$ is defined in [21] to be a reduced locally closed k-subscheme of $\mathscr{A}_{g,n}$ which is maximal among all reduced locally closed k-subschemes S of $\mathscr{A}_{g,n}$ such that the restriction to S of the universal p-divisible group is geometrically fiberwise constant. This "point-wise" definition was the best available at the time. The geometric picture that the family of central leaves and the family of isogeny leaves together provided new insight on the structure of the moduli spaces $\mathscr{A}_{g,n}$.

Admittedly the point-wise definition of central leaves is a bit awkward to work with. For instance it makes studying differential properties of central leaves a bit of a challenge. As another example it can be shown, with some effort, that the restriction to a central leaf of the universal p-divisible group over $\mathscr{A}_{g,n}$ admits a slope filtration. This is an important property of central leaves; in contrast this statement does not hold for Newton polygon strata of $\mathscr{A}_{g,n}$. However the proof does not provide a transparent explanation of the existence of slope filtration on central leaves.

The definition below resolves the above plights in one stroke.

1.3. Definition. Let $\kappa \supset \mathbb{F}_p$ be a field, and let S be a κ-scheme. Let X_0 be a p-divisible group over κ.

(a1) A p-divisible group over $X \to S$ is *strongly κ-sustained modeled on* X_0 if the S-scheme
$$\mathscr{I}\!\mathit{som}_S\big(X_0[p^n]_S, X[p^n]\big) \to S$$
of isomorphisms between the truncated Barsotti-Tate groups $X_0[p^n]$ and $X[p^n]$ is faithfully flat over S for every $n \in \mathbb{N}$. Here $X_0[p^n]_S$ is short for $X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S$; we will use similar abbreviation for base change in the rest of this article if no confusion is likely to arise.

(a2) A p-divisible group $X \to S$ is *strongly κ-sustained* if there exists a p-divisible group X_0 over κ such that $X \to S$ is κ-sustained modeled on X_0. In this case we say that X_0 is a κ-model of X.

(b1) A p-divisible group over $X \to S$ is *κ-sustained* if the $S \times_{\operatorname{Spec}\kappa} S$-scheme

$$\mathscr{I}\!\mathit{som}_{S\times_{\operatorname{Spec}\kappa}S}\big(\mathrm{pr}_1^*X[p^n], \mathrm{pr}_2^*X[p^n]\big) \to S \times_{\operatorname{Spec}\kappa} S$$

is faithfully flat over $S \times_{\operatorname{Spec}\kappa} S$ for every $n \in \mathbb{N}$, where $\mathrm{pr}_1, \mathrm{pr}_2 : S \times_{\operatorname{Spec}\kappa} S \to S$ are the two projections from $S \times_{\operatorname{Spec}\kappa} S$ to S.

(b2) Let $X \to S$ be a κ-sustained p-divisible group over S as in (b1). Let K be a field containing the base field κ. A κ-sustained p-divisible group X_2 over K is said to be a *K/κ-model* of $X \to S$ if the structural morphism

$$\mathscr{I}\!\mathit{som}_{S_K}\big(X_1[p^n] \times_{\operatorname{Spec}(K)} S_K, X[p^n] \times_S S_K\big) \longrightarrow S_K$$

of the above Isom-scheme is faithfully flat, for every positive integer n, where $S_K := S \times_{\operatorname{Spec}(\kappa)} \operatorname{Spec}(K)$.

(c) The above definitions extend to polarized p-divisible groups, giving us two more notions:

- a polarized p-divisible group $(X \to S, \mu : X \to X^t)$ is *strongly κ-sustained modeled on a polarized p-divisible group* (X_0, μ_0) over κ if the S-scheme

$$\mathscr{I}\!\mathit{som}_S\big((X_0[p^n]_S, \mu_0[p^n]_S), (X[p^n], \mu[p^n])\big) \to S$$

is faithfully flat over S for every $n \in \mathbb{N}$.

- a polarized p-divisible group $(X \to S, \mu : X \to X^t)$ is *κ-sustained* if

$$\mathscr{I}\!\mathit{som}_{S\times_{\operatorname{Spec}\kappa}S}\big(\mathrm{pr}_1^*(X[p^n], \mu[p^n]), \mathrm{pr}_2^*(X[p^n], \mu[p^n])\big) \to S \times_{\operatorname{Spec}\kappa} S$$

is faithfully flat for every $n \in \mathbb{N}$.

(d) Just as in the unpolarized case, given a over-field K of κ, a *K/κ-model* of a κ-sustained polarized p-divisible group (X, μ) over a κ-scheme S is a κ-sustained polarized p-divisible group (X_3, μ_3) over K such that the Isom-scheme

$$\mathscr{I}\!\mathit{som}_{S_K}\big((X_2[p^n], \mu_2[p^n]) \times_{\operatorname{Spec}(K)} S_K, (X[p^n], \mu[p^n]) \times_S S_K\big) \longrightarrow S_K$$

is faithfully flat over S_K, for every positive integer n.

1.3.1. Remark. (a) Suppose that $X \to S$ is κ-sustained and $S(\kappa) \neq \emptyset$, then $X \to S$ is strongly κ-sustained with X_s as a model, for any κ-rational point s of S. This fact follows immediately from the definition. Similarly for any point $s \in S$, the the fiber X_s above s of a κ-sustained p-divisible group X is a κ-sustained p-divisible group over the residue field $\kappa(s)$ of s, and X_s is a $\kappa(s)/\kappa$ model of $X \to S$.

(b) Clearly a p-divisible group X over a κ-scheme S is strongly κ-sustained if and only if it is κ-sustained and admits a κ/κ-model. We will abbreviate "κ/κ-model" to "κ-model" when no confusion is likely to arise. We would like to consider the notion "κ-sustained" as more fundamental, and "strongly κ-sustained" as the special case when there exists a κ-model for the κ-sustained p-divisible group in question.

(d) Being sustained is weaker than being *quasi-isotrivial.* Every ordinary family of p-divisible group $X \to S$ of constant relative dimension and height is strong $\mathbb{F}_p$-sustained, but there are many examples of ordineary p-divisible groups over a κ-scheme S which is not κ-quasi-isotrivial. For instance p-divisible group

$$\mathcal{E}[p^\infty]|_{\mathcal{M}_n^{\mathrm{ord}}} \longrightarrow \mathcal{M}_n^{\mathrm{ord}}$$

attached to the restriction to the ordinary locus $\mathcal{M}_n^{\mathrm{ord}}$ of the universal elliptic curve

$$\mathcal{E} \longrightarrow \mathcal{M}_n$$

is not $\overline{\mathbb{F}_p}$ -quasi-isotrivial. Here $\mathcal{M}_n$ is the modular curve over $\overline{\mathbb{F}_p}$, with level-structure n, where $n \geq 3$ and $\gcd(n,p) = 1$. Moreover for every positive integer $m \geq 1$, the BT_n-group $\mathcal{E}[p^m] \to \mathcal{M}_n^{\mathrm{ord}}$ is not $\overline{\mathbb{F}_p}$ -quasi-isotrivial either.

To clarify what we mean: we say that a p-divisible group $X \to S$ is *quasi-κ-isotrivial* (respectively *locally κ-isotrivial*) if there exist

- a Zariski open cover $\big(U_i\big)_{i\in I}$ of S,
- etale surjective morphisms (respectively finite etale morphisms)

$$\big(f_i : T_i \to U_i\big)_{i\in I},$$

- a p-divisible groups X_0 over κ, and
- T_i-isomorphisms $X_0 \times_{\mathrm{Spec}(\kappa)} T_i \xrightarrow{\sim} X \times_S T_i$ of p-divisible groups for all $i \in I$.

The above terminology is based on the notion of "quasi-isotrivial" and "locally isotrivial" for torsors, as in SGA3 exposé IV, 6.5, p. 249. For instance for a group scheme $G \to S$, a G-torsor $\mathcal{P} \to S$ is quasi-isotrivial if there exists a Zariski open cover $\big(U_i\big)_{i\in I}$, etale surjective morphisms $\big(f_i : T_i \to U_i\big)_{i\in I}$ and sections $e_i : T_i \to \mathcal{P} \times_S T_i$ of the $G \times_S T_i$-torsor $\mathcal{P} \times_S T_i$ for all $i \in I$.

1.3.2. Let S be a κ-scheme. Let $X \to S$ be a p-divisible group. The following properties are easily verified from the definition.

(1) If $X \to S$ is strongly κ-sustained over S modeled on a p-divisible group X_0, then $X \to S$ is κ-sustained.
(2) If $X \to S$ is a κ-sustained and $s_0 \in S(\kappa)$ is a κ-rational point of S, then $X \to S$ is strongly κ-sustained modeled on s_0^*X.
(3) Let $f : T \to S$ be a morphism of κ-schemes. If $X \to S$ is κ-sustained (respectively strongly κ-sustained modeled on a p-divisible group X_0 over κ), then $f^*X \to T$ is also κ-sustained (respectively strongly κ-sustained modeled on X_0).
(4) Suppose that $f : T \to S$ is a faithfully flat morphism and $f^*X \to T$ is κ-sustained (respectively strongly κ-sustained modeled on a p-divisible group) X_0 over κ, then $X \to S$ is also κ-sustained (respectively strongly κ-sustained modeled on X_0).
(5) Let U, V be open subschemes of S.
   – If both restrictions $X_U \to U$ and $X_V \to V$ are both strongly κ-sustained modeled on a p-divisible group X_0 over κ, then $X_{U\cup V} \to U \cup V$ is also strongly κ-sustained modeled on X_0.

– If both $X_U \to U$ and $X_V \to V$ are κ-sustained and $U \cap V \neq \emptyset$, then the p-divisible group $X_{U\cup V} \to U \cup V$ is also κ-sustained.

It is instructive to examine the case when the base scheme S is the spectrum of a field K containing the base field κ. The following properties, although not immediately obvious from the definition, make it clear that κ-sustainedness is a relative concept and depends crucially on the base field κ.

(6A) Let X_0 be a p-divisible group over κ. A p-divisible group X over K is strongly κ-sustained modeled on X_0 if and only if one of the following equivalent conditions hold.
- There exists an algebraically closed field L containing κ and a κ-linear embedding $\tau : K \to L$ such that $X \times_{\mathrm{Spec}(K),\tau} \mathrm{Spec}(L)$ is L-isomorphic to $X_0 \times_{Spec(\kappa)} \mathrm{Spec}(L)$.
- The base extension $X \times_{\mathrm{Spec}(K),\tau} \mathrm{Spec}(L)$ of X from K to L is L-isomorphic to $X_0 \times_{Spec(\kappa)} \mathrm{Spec}(L)$ for every an algebraically closed field L containing κ and every κ-linear embedding $\tau : K \to L$.

(6B) Suppose that X is a κ-sustained p-divisible group over K.
  (i) For any algebraically closed field L containing κ and any two κ-linear homomorphism $\tau_1, \tau_2 : K \to L$, the p-divisible groups $\tau_1^{\,*}X$, $\tau_2^{\,*}X$ are isomorphic over L.
  (ii) There exists a finite extension field κ_1 of κ, a κ-linear embedding $\tau : \kappa_1 \hookrightarrow L$ to an algebraic closure L of K, and a p-divisible group X_1 over κ_1 such that the p-divisible group $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is strongly κ_1-sustained modeled on X_1.

(6C) Assume that the base field $\kappa \supset \mathbb{F}_p$ is *perfect*. A p-divisible group X over K is strongly κ-sustained if and only if the statement (6B) (i) hold.

1.4. (a schematic definition of central leaves). We are now in a position to give a scheme-theoretic definition of central leaves in terms of sustained p-divisible groups. Let $n \geq 3$ be a positive integer relatively prime to p. Let $\kappa \supset \mathbb{F}_p$ be a field.

(i) Let $x_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,n}(\kappa)$ be a κ-rational point of $\mathscr{A}_{g,n}$. The *central leaf* $\mathscr{C}(x_0)$ *in* $\mathscr{A}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z}[\mu_n,1/n])} \mathrm{Spec}(\kappa)$ *passing through* x_0 is the maximal member among the family of all locally closed subscheme of $\mathscr{A}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z}[\mu_n,1/n])} \mathrm{Spec}(\kappa)$ such that the principally polarized p-divisible group attached to the restriction to $\mathscr{C}(x_0)$ of the universal abelian scheme is strongly κ-sustained modeled on $(A_0[p^\infty], \lambda_0[p^\infty])$. (This definition depends on the fact that there exist a member $\mathscr{C}(x_0)$ in this family which contains every member of this family.)

(ii) A *central leaf over* κ is a maximal member in the family of all all locally closed subscheme of $\mathscr{A}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z}[\mu_n,1/n])} \mathrm{Spec}(\kappa)$ such that the principally polarized p-divisible group attached to the restriction to $\mathscr{C}(x_0)$ of the universal abelian scheme is κ-sustained.

Of course there is a backward compatibility issue: we have to show that the above definitions coincide with the definition in [21] of central leaves when the field κ is algebraically closed. In particular $\mathscr{C}(x_0)$ is reduced (and smooth over κ).

An analysis of the Isom-schemes of the truncated Barsotti-Tate groups of a sustained p-divisible group reveals two basic properties. Propositions 1.5 says that every strongly sustained isoclinic κ-sustained p-divisible group is a Galois twist of a constant p-divisible group. Proposition 1.6 asserts that there exists a functorial slope filtration on every κ-sustained p-divisible group.

1.5. Proposition. *Let $\kappa \supset \mathbb{F}_p$ be a field and let X_0 be a p-divisible group over κ. Let $X \to S$ be an* isoclinic *strongly κ-sustained p-divisible group over S modeled on X_0. There exist*

- *a projective family of finite groups G_n indexed by $\mathbb{N}$ such that all transition maps $h_{n,n+1} : G_{n+1} \to G_n$ are surjective,*
- *a projective family of right etale torsors $(T_n)_{n\in\mathbb{N}}$ for constant groups G_n such that the transition maps $\pi_{n,n+1} : T_{n+1} \to T_n$ are finite etale and compatible with the transition maps $h_{n,n+1} : G_{n+1} \to G_n$,*
- *a family of homomorphisms $\rho_n : G_n \to \mathrm{Aut}(X_0[p^n])$ such that $\circ\rho_n\circ h_{n,n+1}$ is equal to the composition of ρ_{n+1} with the natural homomorphism*
$$\mathrm{Aut}(X_0[p^{n+1}]) \to \mathrm{Aut}(X_0[p^n]$$
for every $n \in \mathbb{N}$, and
- *a compatible family of isomorphisms*
$$\alpha_n : T_n \times^{(G_n,\rho_n)} X_0[p^n] \xrightarrow{\sim} X[p^n], \quad n \in \mathbb{N},$$
where $T_n \times^{(G_n,\rho_n)} X_0[p^n] = T_n \times_S X_0[p^n]/G_n$ is the contraction product of the G_n-torsor with $X_0[p^n]$, defined by the action ρ_n of G_n on $X_0[p^n]$.

1.5.1. Corollary. *Let $\kappa \supset \mathbb{F}_p$ be a field. Let R be an artinian κ-algebra, and let $\epsilon : R \twoheadrightarrow \kappa$ be a surjective κ-linear ring homomorphism. Suppose that $Y \to \mathrm{Spec}(R)$ is an isoclinic κ-sustained p-divisible group. Then Y is naturally isomorphic to the constant p-divisible group*
$$(Y \times_{(\mathrm{Spec}(R),\epsilon^{\cdot})} \mathrm{Spec}(\kappa)) \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R).$$

1.5.2. Corollary. *Let $\kappa \supset \mathbb{F}_p$ be a field. Let K be a purely inseparable algebraic extension field of κ. Suppose that X is an isoclinic p-divisible group over K which is κ-sustained. Then X descends to κ, i.e. there exists a p-divisible group X_0 over κ and an isomorphism $X \xrightarrow{\sim} X_0 \times_{\mathrm{Spec}\kappa} \mathrm{Spec}(K)$. In particular X is strongly κ-sustained.*

1.6. Proposition. *Let S be a scheme over a field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a sustained p-divisible group over S.*

(1) *There exists a natural number r, rational numbers $1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$ and sustained p-divisible subgroups $X_i \to S$, $0 \leq i \leq r$, with*
$$0 = X_0 \subsetneqq X_1 \subset \cdots \subsetneqq X_r = X$$
such that $X_i/X_{i-1} \to S$ is an isoclinic sustained p-divisible group of slope λ_i, for each $i = 1, \ldots, r$. Moreover the numbers $r, \lambda_1, \ldots, \lambda_r$ and the p-divisible subgroups $X_i \to S$ of X in (1) are uniquely determined by the p-divisible group $X \to S$.

(2) *Let $Y \to S$ be another sustained p-divisible group over S. Let $\mathrm{Fil}^{\bullet}_{\mathrm{slope}}X$ and $\mathrm{Fil}^{\bullet}_{\mathrm{slope}}Y$ be the slope filtration on X and Y defined below. Then*

$$\alpha(\mathrm{Fil}^{\lambda}_{\mathrm{slope}}X) \subseteq \mathrm{Fil}^{\lambda}_{\mathrm{slope}}Y \quad \forall\, \lambda \in [0,1]$$

for every S-homomorphism $\alpha : X \to Y$.

Definition. The *slope filtration* $\mathrm{Fil}^{\bullet}_{\mathrm{slope}}X$ on a κ-sustained p-divisible group $X \to S$ is the filtration

$$0 = X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_r = X$$

as in 1.6 (1), reindexed to become a decreasing filtration with jumps at $\lambda_1, \ldots, \lambda_r$; i.e.

$$\mathrm{Fil}^{\lambda}_{\mathrm{slope}}X/\mathrm{Fil}^{\lambda+}_{\mathrm{slope}}X \cong \begin{cases} X_i/X_{i-1} & \text{if } \lambda = \lambda_i \text{ for some } i = 1, \ldots, r \\ 0 & \text{otherwise} \end{cases}$$

To be more explicit,

$$\mathrm{Fil}^{\lambda}_{\mathrm{slope}}X := \begin{cases} X_0 = 0 & \text{if } \lambda > \lambda_1 \\ X_i & \text{if } \lambda_{i+1} < \lambda \leq \lambda_i,\ 1 \leq i \leq r-1 \\ X_r = X & \text{if } \lambda \leq \lambda_r \end{cases}$$

for any real number $\lambda \in [0,1]$.

1.7. The existence of the slope filtration on a sustained p-divisible group immediately implies the following:

> *A κ-sustained deformations of a p-divisible group X_κ over κ is determined by the deformation of the slope filtration of X_κ.*

To elaborate, suppose that $\kappa \supset \mathbb{F}_p$ is field and $R \twoheadrightarrow \kappa$ is an augmented artinian κ-algebra. Let X_κ be a p-divisible group over κ and let X be a κ-sustained p-divisible group over R whose closed fiber is X_κ. Let $X_1 \subsetneqq X_2 \subsetneqq \cdots X_r$ be the slope filtration of X as in 1.6 (1). Let $Y_i = (X_i/X_{i-1}) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$ for $i = 1, \ldots, r$. Then X_i/X_{i-1} is an isoclinic strongly κ-sustained p-divisible group over the artinian local ring R modeled on Y_i for each i. Through the natural isomorphisms $\psi_i : X_i/X_{i-1} \xrightarrow{\sim} Y_i \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$ given by 1.5.1, we can regard X as a successive extension of the constant p-divisible groups $Y_i \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$, $i = 1, \ldots, r$. So in a precise sense every sustained deformation of X_κ arises from a deformation of its slope filtration.

1.8. Here is a natural follow-up question.
Question. *Which ones among all deformations of the slope filtration of X_κ give rise to κ-sustained deformations of X_κ? Is there some structure on the sustained locus $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ of the local deformation space $\mathcal{D}ef(X_0)$, based on the consideration of the slope filtration, which will provide better understanding of sustained deformations?*

The first non-trivial case when X_κ has exactly two slopes, to be explained in proposition 1.12, reveals a new phenomenon, that the sustained locus in the local deformation space of a p-divisible group are “built up” from p-divisible formal groups through successive fibrations. Before we get to that, it is comforting to confirm that the deformation problem for sustained p-divisible groups is formally smooth.

1.9. We set up notation for proposition 1.10.

- Let $k \supset \mathbb{F}_p$ be a perfect field. Let X_0 be a p-divisible group over a perfect field k.
- Let $\mathfrak{art}_k^+$ be the category whose objects are pairs (R, ϵ) where R is an artinian commutative local k-algebra and $\epsilon : R \twoheadrightarrow k$ is a k-linear surjective ring homomorphism, and a morphism $(R_1, \epsilon_1) \to (R_2, \epsilon_2)$ is a k-linear ring homomorphism $h : R_1 \to R_2$ such that $\epsilon_2 \circ h = \epsilon_1$.
- Let $\mathcal{D}ef(X_0)$ be the deformation functor of X_0, which associates to each object (R, ϵ) of $\mathfrak{art}_k^+$ the set of all isomorphism classes of pairs (X_R, δ), where X_R is a p-divisible group over R and
$$\delta : X_R \times_{(\mathrm{Spec}(R), \epsilon^\cdot)} \mathrm{Spec}(k) \xrightarrow{\sim} X_0$$
is an isomorphism of p-divisible groups over k.
- It is well-known that the functor $\mathcal{D}ef(X_0)$ is representable, as the formal spectrum of a complete local ring over k, non-canonically isomorphic to a formal power series ring over k in $c \cdot d$ variables, where
$$c = \mathrm{height}(X_0) - \dim(X_0) = \dim(X_0^t) \quad \text{and} \quad d = \dim(X_0)$$
are the codimensiion and dimension of X_0 respectively. Let
$$(\mathscr{X}_{\mathrm{univ}} \to \mathcal{D}ef(X_0), \delta_{\mathrm{univ}} : \mathscr{X}_{\mathrm{univ}} \times_{\mathcal{D}ef X_0} \mathrm{Spec}(k) \xrightarrow{\sim} X_0)$$
be the tautological deformation of X_0 over $\mathcal{D}ef(X_0)$ characterized by the following universal property: for every object (R, ϵ) of $\mathfrak{art}_k^+$ and every deformation $(X \to \mathrm{Spec}(R), \delta)$ of X_0 over R, there exists a unique k-morphism $f : \mathrm{Spec}(R) \to \mathcal{D}ef(X_0)$ and a unique isomorphism from $(X \to \mathrm{Spec}(R), \delta)$ to $f^*(\mathscr{X}_{\mathrm{univ}}, \delta_{\mathrm{univ}})$.

1.10. Proposition. *Notation as in the preceding subsection* 1.9.

(1) *There exists a closed formal subscheme $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ of $\mathcal{D}ef(X_0)$, uniquely characterized by the following properties.*
 - *The p-divisible group*
$$\mathscr{X}_{\mathrm{univ}} \times_{\mathcal{D}ef(X_0)} \mathcal{D}ef(X_0)_{\mathrm{sus}} \to \mathcal{D}ef(X_0)_{\mathrm{sus}}$$
is strongly k-sustained modeled on X_0.
 - *If Z is a closed formal subscheme of $\mathcal{D}ef(X_0$ such the p-divisible group*
$$\mathscr{X}_{\mathrm{univ}} \times_{\mathcal{D}ef(X_0)} Z \longrightarrow Z$$
is strongly k-sustained modeled on X_0, then Z is a subscheme of $\mathcal{D}ef(X_0)_{\mathrm{sus}}$.

(2) *The formal scheme $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ is formally smooth over k.*

1.11. We turn to the problem of sustained deformations of a p-divisible with two slopes, a very special case of the general question in 1.8. We will use the following notations.

- Let $k \supset \mathbb{F}_p$ be a perfect field. Let Y and Z be isoclinic p-divisible groups over k such that $\text{slope}(Y) < \text{slope}(Z)$. Let $X_0 = Y \times_{\text{Spec}(k)} Z$.
- Let $\mathcal{E}xt_{\text{def}}(Y, Z)$ be the set-valued functor on $\mathfrak{art}_k^+$ which associates to every object (R, ϵ) of $\mathfrak{art}_k^+$ the set of all isomorphism classes of pairs (E, ψ), where
  - E is a short exact sequence
    $$0 \to Y \times_{\text{Spec}(k)} \text{Spec}(R) \to E_R \to Z_{\text{Spec}(k)}\text{Spec}(R) \to 0$$
    of p-divisible groups over R,
  - ψ is an isomorphism of short exact sequences over k, from the closed fiber $E \times_{(\text{Spec}(R), \epsilon^{\cdot})} \text{Spec}(k)$ of E $0 \to Z \to Y \times_{\text{Spec}(k)} Z \to Y \to 0$.

  The natural arrow from the functor $\mathcal{E}xt_{\text{def}}(Y, Z)$ on $\mathfrak{art}_k^+$ to the deformation functor $\mathcal{D}ef(X_0)$ gives $\mathcal{E}xt_{\text{def}}(Y, Z)$ a natural structure as sub-functor of $\mathcal{D}ef(X_0)$, because every homomorphism from $Z \times_{\text{Spec}(k)} \text{Spec}(R)$ to $Y \times_{\text{Spec}(k)} \text{Spec}(R)$ is 0.
- The Baer construction gives $\mathcal{E}xt_{\text{def}}(Y, Z)$ a natural structure as a functor from $\mathfrak{art}_k^+$ to the category of commutative groups, such that the zero element in $\mathcal{E}xt_{\text{def}}(Y, Z)(R, \epsilon)$ given by the class of the split extension

$$0 \to Y \times_{\text{Spec}(k)} \text{Spec}(R) \to X_0 \times_{\text{Spec}(k)} \text{Spec}(R) \to Z_{\text{Spec}(k)}\text{Spec}(R) \to 0.$$

- Let $\mathrm{M}(Y)$, $\mathrm{M}(Z)$ be the covariant Dieudonné modules of Y and Z respectively. Both are finite free modules over $W(k)$, plus semi-linear actions F and V. (The covariant theory is normalized so that $\mathrm{M}(Z)$ is the Cartier module for Z, and same for $\mathrm{M}(Y)$ if $\text{slope}(Y) > 0$.)

  The operator V corresponds to the relative Frobenius homomorphisms $Y \to Y^{(p)}$ and $Z \to Z^{(p)}$ over k, while F corresponds to the Verschiebung homomorphisms $Y^{(p)} \to Y$ and $Z^{(p)} \to Z$ over k.
- On $\text{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))[1/p]$ we have natural actions of F and V defined by
  $$(F \cdot h)(x) := F(h(Vx)), \quad (V \cdot h)(x) := V(h(V^{-1}x))$$
  for all $h \in \text{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))[1/p]$ and all $x \in \mathrm{M}(X)$. Note that the fee $W(k)$-module
  $$\text{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))$$
  is stable under F but not necessarily under V.

1.12. Proposition. *Notation as in subsection* 1.11*.*

(1) *The group-valued functor $\mathcal{E}xt_{\text{def}}(X_0)$ is representable by a commutative smooth formal group over k.*

(2) *There commutative smooth formal group $\mathcal{E}xt_{\text{def}}(X_0)$ over k has a largest p-divisible subgroup over k, denoted by $\mathcal{E}xt_{\text{def}}(X_0)_{\text{div}}$. Similarly the formal group $\mathcal{E}xt_{\text{def}}(X_0)$ has a largest unipotent smooth formal subgroup over k, denoted by*
$$\mathcal{E}xt_{\text{def}}(X_0)_{\text{unip}};$$

it is killed by p^N for some $N \in \mathbb{N}$.

(3) *The smooth formal group $\mathcal{E}xt_{\mathrm{def}}(Y,Z)$ over k is generated by the two smooth formal subgroups $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{div}}$ and $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{unip}}$ over k. The intersection $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{div}} \cap \mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{unip}}$ of these two smooth formal subgroups is a commutative finite group scheme over k.*

(4) *The p-divisible subgroup $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{unip}}$ is equal to the k-sustained locus in $\mathcal{D}ef(X_0)$, in the following sense.*
   - *The restriction to $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{div}}$ of the universal p-divisible group*
   $$\mathscr{X}_{\mathrm{univ}} \to \mathcal{D}ef(X_0)$$
   over $\mathcal{D}ef(X_0)$ is strictly κ-sustained modeled on X_0.
   - *If $V \subseteq \mathcal{D}ef(X_0)$ is a closed formal subscheme of $\mathcal{D}ef(X_0)$ such that the restriction to V of the universal p-divisible group over $\mathcal{D}ef(X_0)$ is strictly κ-sustained modeled on X_0, then $V \subseteq \mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{div}}$.*

(5) *The p-divisible formal group $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{div}}$ is isoclinic; its slope is equal to* $\mathrm{slope}(Z) - \mathrm{slope}(Y)$, *and its height is equal to* $\mathrm{height}(Y)\cdot\mathrm{height}(Z)$.

(6) *The covariant Dieudonné module of $\mathcal{E}xt_{\mathrm{def}}(Y,Z)_{\mathrm{div}}$ is the largest $W(k)$-submodule of* $\mathrm{Hom}_{W(k)}(\mathrm{M}(Y),\mathrm{M}(Z))$ *which is stable under the actions of both F and V.*

Proposition 1.12 provides a very satisfactory answer on the structure of the k-sustained locus in the deformation space of a p-divisible group X_0 over a perfect field $k \supset \mathbb{F}_p$ with exactly two slopes. Optimists will naturally expect that a general structural result for the sustained deformation space $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ when X_0 has more than two slopes. This is indeed the case, but the structure of $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ is a bit involved and will be discussed in a later section §???.

1.13. Remark. (**completely slope divisible** vs. **sustained**) Recall that a p-divisible group $X \to S$ over a base scheme S in characteristic p is said to be *completely slope divisible* if $X \to S$ admits a filtration

$$X = X_0 \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_{m-1} \supsetneqq X_m = 0$$

by p-divisible subgroups $X_i \to S$, and there exist natural numbers $r_0 < r_1 < \cdots < r_{m-1}$ and a positive integer s such that X_i/X_{i+1} is isoclinic with slope $\frac{r_i}{s}$ for $i = 0, 1, \ldots, m-1$, and

$$\mathrm{Ker}(\mathrm{Fr}^s_{X_i/S}) \supseteq X_i[p^{r_i}]$$

for $i = 0, 1, \ldots, m-1$; see [23, 1.1].

It is shown in [23, §1] that the following statements hold. for $X \to S$.

- If the base scheme S of a completely slope divisible p-divisible group $X \to S$ is *perfect*, i.e. the absolute Frobenius map for S is a bijection on $\mathscr{O}_S$), then X is isomorphic to $\oplus_{i=0}^{m-1} X_i/X_{i+1}$.
- A p-divisible group over an algebraically closed field k of characteristic p is completely slope divisible if and only if it is isomorphic to a direct sum of isoclinic p-divisible groups Y_j over k such that each Y_j can be descended to a finite subfield of k.

Now we are in position to explain the relation of the two notions.

A. If we are willing to be sloppy about the base field κ in the notion of sustained p-divisible group, we can say that over reduced base schemes
 (a) completely slope divisible p-divisible groups are sustained, and
 (b) a sustained p-divisible group is completely slope divisible if and only if it admits a model which is a direct sum of isoclinic p-divisible groups over finite fields.

So that the notion of "completely slope divisible" is stronger than "sustained" over reduced base schemes. To be more precise, suppose that κ is a field of characteristic p and S is an integral κ-scheme such that $S(\kappa) \neq \emptyset$, then the following hold.
 – Every completely slope-divisible p-divisible group over S is strongly κ-sustained.
 – A κ-sustained p-divisible group $X \to S$ is completely slope divisible if and only if there exists a finite subfield E of an algebraic closure of k of κ and isoclinic p-divisible groups $Y_1, \ldots, Y_m$ over E such that $X \times_S S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathsf{k})$ is strongly $S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathsf{k})$-sustained modeled on $Y_1 \times \cdots \times Y_m$.

B. However the two notions diverge over non-reduced base schemes. There are plenty of examples of completely slope divisible p-divisible groups $X \to S$ with two slopes, where the base scheme S is the spectrum of an augmented artinian κ-algebra with perfect residue field κ, such that $X \to S$ is not κ-sustained.

The next proposition sums up some properties about Sustained p-divisible groups over fields. The proof depends on some basic finiteness properties for p-divisible groups and is not a formal consequence of the definitions.

1.14. Proposition. *Let K be an extension field of κ, and let X be a p-divislbe group over K.*

(1) *The p-divisible group X over K is strongly κ-sustained with a κ-model X_0 if and only if there exists a κ-linear ring homomorphism $\sigma : K \to \Omega$ to an algebraically closed field Ω such that the base extension of X_0 to Ω is isomorphic to $\sigma_\bullet X$ over Ω. Moreover if this condition holds for one such embedding σ, then it hold for every κ-linear embedding of K into an algebraically closed field.*

(2) *Suppose that K is purely inseparable over κ and X is an* isoclic *p-divisible group over K. If X is κ-sustained, then X descends to κ, hence it is κ-sustained.* (This is 1.5.2.)

(3) *If X is κ-sustained, then the following condition $(*)$ is satisified:*

 $(*)$ *For any algebraically closed field Ω and any two κ-linear ring homomorphisms $\sigma, \tau : K \to \Omega$, the p-divisible groups $\sigma_\bullet X$ and $\tau_\bullet X$ over Ω, obtained from X by extension of base fields via σ and τ respectively, are isomorphic over Ω.*

(4) *Suppose that K is a separable (but not necessarily finite) algebraic extension of κ and the condition $(*)$ holds. Then X is κ-sustained.*

(5) *Let Ω_1 be an algebraic closure of K, and let κ^{a} be the algebraic closure of κ in Ω_1. If the condition $(*)$ in* (3) *holds for X, then there exists a*

p-divisible group X_1 over κ^{a} such that $X_1 \times_{\mathrm{Spec}\kappa_1} \mathrm{Spec}(\Omega_1)$ is isomorphic to $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\Omega_1)$. In particular if X is κ-sustained, then X has a $\kappa^{\mathrm{a}}/\kappa$-model.

2. Stabilized Hom schemes for truncations of p-divisible groups

This section may seem a bit technical at the beginning, but is of critical importance for the theory of sustained p-divisible groups. Among other things, we will discuss a procedure which produces, from two input p-divisible groups X, Y over a field $\kappa \supset \mathbb{F}_p$, a commutative smooth formal group $\mathscr{H}\!om'(X,Y)$ and a smooth p-divisible formal subgroup $\mathscr{H}\!om'_{\mathrm{div}}(X,Y)$ of $\mathscr{H}\!om(X,Y)$.

2.1. Definition. Let $\kappa \supset \mathbb{F}_p$ be a field and let X, Y be p-divisible groups over κ.

(1) For every $n \in \mathbb{N}$ we have a commutative affine algebraic group
$$\mathscr{H}\!om(X[p^n], Y[p^n])$$
of finite type over κ, which represents the functor
$$S \mapsto \mathrm{Hom}_S(X[p^n] \times_{\mathrm{Spec}(\kappa)} S, Y[p^n] \times_{\mathrm{Spec}(\kappa)} S)$$
on the category of all κ-schemes S. We often shorten $\mathscr{H}\!om(X[p^n], Y[p^n])$ to $H_n(X,Y)$, and sometimes we will shorten it further to H_n if there is no danger of confusion.

(2) For any $n, i \in \mathbb{N}$, let $j_{X,n+i,n} : X[p^n] \to X[p^{n+i}]$ be the inclusion homomorphism, and let $\pi_{X,n,n+i} : X[p^{n+i}] \to X[p^n]$ be the faithfully flat homomorphism such that $[p^i]_{X[p^{n+i}]} = j_{X,n+i,n} \circ \pi_{X,n,n+i}$. Define $\pi_{Y,n,n+i}$ and $j_{Y,n+i,n}$ similarly.

(3) For all $n, i \in \mathbb{N}$, denote by
$$r_{n,n+i} : H_{n+i} \to H_n$$
the homomorphism such that
$$\alpha \circ j_{X,n+i,n} = j_{Y,n+i,n} \circ r_{n,n+i}(\alpha)$$
for any κ-scheme S and any homomorphism $\alpha : X[p^{n+i}]_S \to Y[p^{n+i}]_S$. In other words $r_{n,n+i}$ is defined by restricting homomorphisms from $X[p^{n+i}]$ to $Y[p^{n+i}]$ to the subgroup scheme $X[p^n] \subset X[p^{n+i}]$.

(4) For all $n, i \in \mathbb{N}$, denote by
$$\iota_{n+i,n} : H_n \to H_{n+i}$$
the homomorphism such that
$$\iota_{n+i,n}(\beta) = j_{Y,n+i,n} \circ \beta \circ \pi_{X,n,n+i}$$
for any κ-scheme S and any homomorphism $\beta : X[p^n]_S \to Y[p^n]_S$. This homomorphism $\iota_{n+i,n}$ is a closed embedding.

2.2. Lemma. *The maps $\iota_{n+i,n}$ and $r_{i,n+i}$ defined in 2.1 satisfy the following properties.*

(1) *$\iota_{n+i+j,n+i} \circ \iota_{n+i,n} = \iota_{n+i+j,n}$ and $r_{n,n+i} \circ r_{n+i,n+i+j} = r_{n,n+i+j}$ for all $n, i, j \in \mathbb{N}$.*

(2) $r_{n,n+i} \circ \iota_{n+i,n} = [p^i]_{H_n}$ *and* $\iota_{n+i,n} \circ r_{n,n+i} = [p^i]_{H_{n+i}}$ *for all* $n, i \in \mathbb{N}$, *where* $[p^i]_{H_m}$ *denotes the endomorphism "multiplication by* p^i*" on* H_m.

(3) $\iota_{n+j,n} \circ r_{n,n+i} = r_{n+j,n+i+j} \circ \iota_{n+i+j,n+i}$ *for all* $n, i, j \in \mathbb{N}$.

PROOF. Statements (1), (2) are easily verified from the definition, and the statement (3) is an immediate consequence of (2). $\square$

2.3. We introduce some notation and recall some basic facts about commutative affine group schemes of finite type over a field. In the following κ is a field and G is a commutative group scheme of finite type over κ.

(1) Commutative affine groups schemes of finite type κ form an abelian category $\mathbf{CAGS}_\kappa$ Commutative finite group schemes form an abelian subcategory $\mathbf{CFGS}_\kappa$ of $\mathbf{CAGS}_\kappa$ which is stable under extensions.

(2) The neutral component G^0 of G is a subgroup schme of G. The quotient group scheme G/G^0 is finite étale over κ.

(3) Suppose that $\kappa \supset \mathbb{F}_p$ and G is connected. There exists a unique greatest subgroup G_{mult} of multiplicative type in G, and $G^{\mathrm{unip}} := G/G_{\mathrm{toric}}$ is unipotent.
   - If G is smooth, then both G_{mult} and G^{unip} are connected and smooth.
   - If the field κ is perfect, there exists a unique splitting of the short exact sequence $0 \to G_{\mathrm{mult}} \to G \to G^{\mathrm{unip}} \to 0$.

(4) Suppose that $\kappa \supset \mathbb{F}_p$. Let G is a unipotent commutative group scheme of finite types over κ
   - If $\dim(G) > 0$, then $\dim(\mathrm{Ker}([p]_G)) > 0$.
   - If moreover G is connected and smooth over κ then $[p^{\dim(G)}]_G = 0$.

(5) Suppose that $\kappa \supset \mathbb{F}_p$. Let G be a connected unipotent commutative group scheme smooth of finite type over κ. Then $\dim(\mathrm{Image}([p^i]_G)) = \dim(G_{\mathrm{mult}})$ for all $i \geq \dim(G)$. In particular if $\dim(\mathrm{Image}([p^i]_G)) = 0$ for some $i \in \mathbb{N}$, then G is unipotent.

(6) Let G_{red} be the closed subscheme of G with the same underlying topological space, whose structure sheaf is the quotient of $\mathscr{O}_G$ by the radical of $\mathscr{O}_G$. If the κ is perfect, then G_{red} is a subgroup scheme of G, and G_{red} is smooth over κ.

(7) Suppose that κ is a perfect field. Then $(G^0)_{\mathrm{red}} = (G_{\mathrm{red}})^0$, $(G_{\mathrm{mult}})_{\mathrm{red}} = (G_{\mathrm{red}})_{\mathrm{mult}}$, and we have a short exact sequence

$$0 \to (G_{\mathrm{mult}}))_{\mathrm{red}} \to G_{\mathrm{red}} \to (G^{\mathrm{unip}})_{\mathrm{red}} \to 0.$$

(8) Suppose that G is a commutative finite group scheme over κ, and $\kappa \supset \mathbb{F}_p$.
   - We have canonical short exact sequences

   $$0 \to G^0 \to G \to G/G^0 =: G_{\mathrm{et}} \to 0$$

   and

   $$0 \to G_{\mathrm{mult}} \to G^0 \to (G^0)^{\mathrm{unip}} \to 0$$

   of finite group schemes over κ, where G_{et} is étale, G_{mult} is of multiplicative type, and $(G^0)^{\mathrm{unip}}$ is unipotent. When κ is perfect, each of the above two short exact sequences splits uniquely.
   - Let G^{D} be the Cartier dual of G.

* The orthogonal complement of $G_{\rm mult}$ in $G^{\rm D}$ is $(G^{\rm D})^0$.
* The orthogonal complement of G^0 in $G^{\rm D}$ is $(G^{\rm D})_{\rm mult}$.

(9) For any commutative finite group scheme H over κ, define order(H) to $\dim_\kappa(\kappa[H])$, where $\kappa[H]$ is the coordinate ring of H. The equality
$$\mathrm{order}(G_3) = \mathrm{order}(G_1)\cdot \mathrm{order}(G_2)$$
holds for every short exact sequence $0 \to G_1 \to G_3 \to G_2 \to 0$ of finite group schemes over κ.

(10) Let $\alpha : G_1 \to G_2$ be a homomorphism between commutative finite group schemes over κ with $\mathrm{order}(G_1) = \mathrm{order}(G_2)$. If α is either a monomorphism or an epimorphism, then α is an isomorphism.

2.4. Corollary. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. For every $n \in \mathbb{N}$, the connected smooth group scheme $(H_n)(X,Y)^0_{\rm red}$ over κ is unipotent.*

2.5. Lemma. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$.*

(i) *For every $n, i \in \mathbb{N}$, the homomorphism*
$$\iota_{n+i,i} : H_i = H_i(X,Y) \to H_{n+i}(X,Y) = H_{n+i}$$
is the kernel of the homomorphism $r_{n,n+i} : H_{n+i} \to H_n$. In particular $\iota_{n+i,i}$ is a monomorphism and $r_{n,n+i}$ induces a monomorphism
$$\nu_{n;i} \colon H_{n+i}/\iota_{n+i,i}(H_i) \rightarrowtail H_n$$
for all $n, i \in \mathbb{N}$.

(ii) *For every $n, i, j \in \mathbb{N}$, the homomorphism*
$$\nu_{n;i,i+j} \colon H_{n+i+j}/\iota_{n+i+j}(H_{i+j}) \longrightarrow H_{n+i}/\iota_{n+i,i}(H_i)$$
induced by $r_{n+i,n+i+j} : H_{n+i+j} \to H_{n+i}$ is a monomorphism.

PROOF. The proof of (i) is straight forward and omitted. The statement (ii) follows from (i), because the composition of the homomorphism $\nu_{n;i,i+j}$ with the monomorphism
$$\nu_{n;i} \colon H_{n+i}/\iota_{n+i,i}(H_i) \rightarrowtail H_n$$
is equal to the monomorphism $\nu_{n;i+j} \colon H_{n+i+j}/\iota_{n+i+j}(H_{i+j}) \rightarrowtail H_n$. $\square$

2.6. Lemma. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. Let*
$$H_n = H_n(X,Y) = \mathscr{H}\!om(X[p^n], Y[p^n]),$$
and let $r_{i,n+i} : H_{n+i} \to H_n$ and $\iota_{n+i,i} : H_i \to H_{n+i}$ be the homomorphisms defined in 2.1. Let h_X, h_Y be the heights of X and Y. For every $m \geq h_X \cdot h_Y$, the quotient $H_{n+m}/\iota_{n+m,m}(H_m)$ is a commutative finite group scheme over κ.

PROOF. From deformation theory for truncated Barsotti-Tate groups we know that
$$\dim(H_i) \leq h_X \cdot h_Y \qquad \forall\, i \in \mathbb{N}.$$
If follws from 2.3 and 2.4 that $\mathrm{Ker}([p^m]_{H_{n+m}}) \supseteq (H_{n+m})^0_{\rm red}$ for all $m \geq h_X \cdot h_Y$. Because $\iota_{n+m,n} \circ r_{n,n+m} = [p^m]_{H_{n+i}}$ and $\iota_{n+m,n}$ is a monomorphism,
$$\mathrm{Image}(\iota_{n+m,m}) = \mathrm{Ker}(r_{n,n+m}) = \mathrm{Ker}[p^m]_{H_{n+m}} \supseteq (H_{n+m})^0_{\rm red}$$
for all $m \geq h_X \cdot h_Y$. $\square$

2.7. Lemma. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. Let*

$$H_n = H_n(X,Y) = \mathscr{H}\!om(X[p^n], Y[p^n]),$$

and let $r_{i,n+i} : H_{n+i} \to H_n$ and $\iota_{n+i,i} : H_i \to H_{n+i}$ be the homomorphisms defined in 2.1. There exists a positive integer n_0 such that the monomorphism

$$\nu_{n;1,2} \colon H_{n+2}/\iota_{n+2,n+1}(H_{n+1}) \rightarrowtail H_{n+1}/\iota_{n+1,n}(H_n)$$

is an isomorphism of commutative finite group schemes over κ for every integer $n \geq n_0$.

Proof. We know from 2.6 that $H_{n+1}/\iota_{n+1,n}(H_n)$ is a finite group scheme over κ for all natural numbers n with $n \geq h_X \cdot h_Y$. Because $\nu_{n;1,2}$ is a monomorphism for all n, the order of $H_{n+1}/\iota_{n+1,n}(H_n)$ for $n \geq h_X \cdot h_Y$ is non-increasing as n increases. Let n_0 be the smallest natural number such that $n_0 \geq h_X \cdot h_Y$ and

$$\mathrm{order}(H_{n+1}/\iota_{n+1,n}(H_n)) = \mathrm{order}(H_{n_0+1}/\iota_{n_0+1,n_0}(H_{n_0}))$$

for all $n \geq n_0$. It follows from 2.3 (10) that $\nu_{n;1,2}$ is an isomorphism for all $n \geq n_0$. $\square$

2.8. Definition. Let n_0 be a positive integer such that 2.7 holds, i.e. the monomorphism $\nu_{n;1,2}$ is an isomorophism for every $n \geq n_0$.

(1) Define commutative finite group schemes $G_n = G_n(X,Y)$ over κ, $n \in \mathbb{N}$, by

$$G_n := H_{n+n_0}/\iota_{n+n_0,n_0}(H_{n_0}).$$

(2) Define monomorphisms $\nu_n \colon G_n \rightarrowtail H_n$ by $\nu_n = \nu_{n;i_0}$ for all $n \in \mathbb{N}$, where $\nu_{n;i_0}$ is defined in 2.5(ii).

(3) For every $n, i \in \mathbb{N}$, denote by

$$j_{n+i,n} \colon G_n \longrightarrow G_{n+i}$$

the homomorphisms over κ induced by $\iota_{n+i+n_0} \colon H_{n+n_0} \longrightarrow H_{n+n_0+i}$.

(4) For every $n, i \in \mathbb{N}$, denote by

$$\pi_{n,n+i} : G_{n+i} \longrightarrow G_n$$

the homomorphism induced by $r_{n+n_0,n+n_0+i} \colon H_{n+n_0+i} \longrightarrow H_{n+n_0}$.

Remark. The monomorphisms $\nu_n \colon G_n \rightarrowtail H_n$ is independent of the choice of n_0 up to unique isomorphism: For every $i \in \mathbb{N}$, the homomorphism $r_{n+n_0,n+n_0+i} : H_{n+n_0+i} \to H_{n+n_0}$ induces an isomorphism

$$\alpha \colon (\nu'_n \colon H_{n+n_0+i}/\iota_{n+n_0+i,n_0+i}(H_{n_0+i}) \rightarrowtail H_n) \xrightarrow{\sim} (\nu_n \colon H_{n+n_0}/\iota_{n+n_0,n_0}(H_{n_0}) \rightarrowtail H_n)$$

where ν'_n is the monomorphism when n_0 is replaced by $n_0 + i$ in 2.8. This assertion is a consequence of 2.9. If we identify $H_{n+n_0+i}/\iota_{n+n_0+i,n_0+i}(H_{n_0+i})$ with $G_{n+j}/j_{n+j,j}(G_j)$, then the isomorphism α $H_{n+n_0+i}/\iota_{n+n_0+i,n_0+i}(H_{n_0+i})$ to $H_{n+n_0}/\iota_{n+n_0,n_0}(H_{n_0}) = G_n$ is induced by the epimorphism $\pi_j : G_{n+j} \to G_j$.

2.9. Proposition. *Supppose that* $\mathrm{order}(H_{n+1}/\iota_{n+1,n}(H_n)) = p^h$ *for all $n \geq n_0$ in the notation of 2.8. The system of commutative finite group schemes G_n over κ together with the homomorphisms $j_{n+m,n}$ and $\pi_{n,n+m}$ form a p-divisible group over κ of height h. In other words the following statements hold.*

(1) *The commutative finite group scheme G_n over κ has order p^{nh} for each $n \geq 0$.*

(2) *For all $n, m \in \mathbb{N}$, the homorphism $\pi_{n,n+m}$ is an epimorphism and the homomorphism $j_{n+m,n}$ is a monomorphism.*

(3) $\pi_{n,n+m} \circ j_{n+m,n} = [p^m]_{G_n}$ *and* $j_{n+m,m} \circ \pi_{n,n+m} = [p^m]_{G_{n+m}}$ *for all* $n, i \in \mathbb{N}$.

(4) *The monomorphisms $j_{n+m,n} \colon G_n \to G_{n+m}$ is the kernel of the epimorphism*

$$\pi_{m,n+m} \colon G_{n+m} \to G_m$$

for all $n, m \in \mathbb{N}$.

Proof. The statement (1) follows from the stabilization assumption and 2.3 (9).

The statement (3) is a consequence of 2.2(2). The homomorphism

$$j_{n+i,n} \colon H_{n+n_0}/\iota_{n+n_0,n_0}(H_{n_0}) = G_n \longrightarrow G_{n+i} = H_{n+j+n_0}/\iota_{n+j+n_0,n_0}(H_{n_0})$$

satisfies

$$\nu_{n+j;n_0} \circ j_{n+i,n} = \iota_{n+j,n} \circ \nu_{n;n_0} .$$

So $j_{n+i,n}$ is a monomorphism because $\nu_{n;n_0}$, $\nu_{n+j;n_0}$ and $\iota_{n+j,n}$ all are.

For every $m, n \in \mathbb{N}$, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{n+n_0} \arrow[r, "\iota_{n+m+n_0,n+n_0}"] \arrow[d] & H_{n+m+n_0} \arrow[r, "\pi_{m,n+m+n_0}"] \arrow[d] & H_m \arrow[r] & 0 \\
0 \arrow[r] & G_n \arrow[r, "j_{n+m,n}"'] & G_{n+m} \arrow[r, "\pi_{m,n+m}"'] & G_m \arrow[u, hook, "\nu_{m;n_0}"'] &
\end{tikzcd}$$

We know that the horizontal rows are exact except possibly at G_{n+m}. A simple diagram chase shows that the lower row is indeed exact at G_{n+m}. Because the order of G_{n+m} is equal to the product of the order of G_n and the order of G_m, we conclude from 2.3 (9), (10) that $\pi_{m,n+m}$ is an epimorphism. We have proved statements (2) and (4). □

2.10. Definition. (1) Denote by $\mathscr{H}\!om'(X,Y)$ the *inductive* system of commutative group schemes $H_n(X,Y)$ of finite type over κ, with transition maps

$$\iota_{n+m,n} \colon H_n(X,Y) \to H_{n+m}(X,Y).$$

(The supersript $'$ in $\mathscr{H}\!om'(X,Y)$ is meant to indicated that the arrows in the projective system $\mathscr{H}\!om'(X,Y)$ of $\mathscr{H}\!om$-schemes $H_i(X,Y) = \mathscr{H}\!om(X[p^i], Y[p^i])$ are reversed, giving rise to to an inductive system instead.)

(2) We will write $\mathscr{H}\!om^{\rm st}(X,Y)_n$ for the group scheme $G_n(X,Y)$ over the base field κ. We will call it the stabilized $\mathscr{H}\!om$ scheme at truncation level n.

(3) Denote by $\mathscr{H}\!om'_{\rm div}(X,Y)$ the p-divisible group

$$\big(\, \mathscr{H}\!om^{\rm st}(X,Y)_n,\ j_{n+m,n},\ \pi_{n,n+m} \,\big)_{n,m\in\mathbb{N}} .$$

In other words $\mathscr{H}\!om'_{\rm div}(X,Y)[p^n] = \mathscr{H}\!om^{\rm st}(X,Y)_n$ for all $n \in \mathbb{N}$.

(4) Let $\nu : \mathscr{H}\!om'_{\rm div}(X,Y) \rightarrowtail \mathscr{H}\!om'(X,Y)$ be the monomorphism defined by the compatible compatible family of monomorphisms

$$\nu_n : \mathscr{H}\!om'_{\rm div}(X,Y)[p^n] \rightarrowtail G_n(X,Y) \to H_n(X,Y) = \mathscr{H}\!om'(X,Y)[p^n]$$

Remark. The limit of the inductive system $\mathscr{H}\!om'(X,Y)$ is canonically identified with $\mathscr{H}\!om(\mathbf{T}_p(X),Y)$, the sheafified $\mathscr{H}\!om$ (or the internal Hom) in the category of sheaves of abelian groups, from the *projective system*

$$\mathbf{T}_p(X) := \big(X[p^n], \pi_{X,n,n+m} : X[p^{m+n}] \to X[p^n]\big)$$

to the *inductive system*

$$\big(Y[p^n], j_{Y,n+m,n} : Y[p^n] \to Y[p^{n+m}]\big)\,.$$

The group scheme $\mathscr{H}\!om(X[p^i],Y[p^i])$ is the kernel of the endomorphism "multiplication by p^n" of this inductive limit:

$$\mathscr{H}\!om(X[p^i],Y[p^i]) = \mathscr{H}\!om(\mathbf{T}_p(X),Y)[p^n].$$

2.11. Corollary. (a) *The formation of $\mathscr{H}\!om'(X,Y)$ and $\mathscr{H}\!om'_{\rm div}(X,Y)$ commute with extension of base fields: for every homomorphism of fields $\kappa \to \kappa'$, the natural maps*

$$\mathscr{H}\!om'(X,Y)_{\kappa'} \longrightarrow \mathscr{H}\!om'(X_{\kappa'},Y_{\kappa'})$$

and

$$\mathscr{H}\!om'_{\rm div}(X,Y)_{\kappa'} \longrightarrow \mathscr{H}\!om'_{\rm div}(X_{\kappa'},Y_{\kappa'})$$

are isomorphisms.

(b) *The monomorphism $\nu : \mathscr{H}\!om'_{\rm div}(X,Y) \rightarrowtail \mathscr{H}\!om'(X,Y)$ identifies the inductive system $\mathscr{H}\!om'_{\rm div}(X,Y)$ as the maximal p-divisible subgroup of $\mathscr{H}\!om'(X,Y)$, which satisfies the following universal property: for every p-divisible group Z over a κ-scheme S and every S-homomorphism*

$$f : Z \to \mathscr{H}\!om'(X,Y),$$

there exists a unique homomorphism $g : Z \to \mathscr{H}\!om'_{\rm div}(X,Y)_S$ over S such that $f = \nu \circ g$.

2.12. Lemma. *Let $\kappa \supset \mathbb{F}_p$ be a field of characteristic p. Let $\mathfrak{BT}_\kappa$ be the category of p-divisible groups over κ*

(1) *The formation of $\mathscr{H}\!om'_{\rm div}(?_1,?_2)$ is a bi-additive functor*

$$\mathscr{H}\!om'_{\rm div}(?_1,?_2)\colon \mathfrak{BT}_\kappa^{\rm opp} \times \mathfrak{BT}_\kappa \longrightarrow \mathfrak{BT}_\kappa\,.$$

In other words $\mathscr{H}\!om'_{\rm div}(?_1,?_2)$ is in convariant in $?_2$ and contravariant in $?_1$.

(2) *Let X_1,X_2,Y_1,Y_2 be p-divisible groups over κ. If X_1 is isogenous to X_2 and Y_1 is isogenous to Y_2, then the p-divisible group $\mathscr{H}\!om'_{\rm div}(X_1,Y_1)$ over k is isogenous to $\mathscr{H}\!om'_{\rm div}(X_2,Y_2)$ over k.*

Proof. Statements (1) is immediate from the definition of $\mathscr{H}\!om'_{\rm div}(X,Y)$ and proposition 2.9. The statment (2) follows from (1). $\square$

2.13. Proposition. *Let X,Y be p-divisible groups over a field $\kappa\supset\mathbb{F}_p$. Suppose that X,Y are both isoclinic and let λ_X and λ_Y be their slopes.*

(a) *If $\lambda_X>\lambda_Y$, then the p-divisible group $\mathscr{H}\!om'_{\rm div}(X,Y)$ is isoclinic of slope $\lambda_Y-\lambda_X$.*
(b) *If $\lambda_X>\lambda_Y$, then $\mathscr{H}\!om'_{\rm div}(X,Y)=0$.*
(c) *If $\lambda_X=\lambda_Y$, then $\mathscr{H}\!om'_{\rm div}(X,Y)$ is an étale p-divisible group.*

XXXXX add remark that the above proposition generalizes??? XXXXX

2.13.1. We need to recall some standard notation for Frobenius twists and some facts about Frobenius twists of $\mathscr{H}\!om$ schemes before the proof of proposition 2.13 in 2.14.1.

(1) For any scheme S over κ and any $i\in\mathbb{N}$, $S^{(p^i)}$ stands for the the base change $S\times_{{\rm Spec}(\kappa),{\rm Fr}_\kappa^i}{\rm Spec}(\kappa)$ of S under the i-th power ${\rm Fr}_\kappa^i$ of the (absolute) Frobenius map ${\rm Fr}_\kappa:{\rm Spec}(\kappa)\to{\rm Spec}(\kappa)$ for ${\rm Spec}(\kappa)$. We have a Cartesian diagram

$$\begin{tikzcd} S^{(p^i)} \arrow[r, "W^i_{S/\kappa}"] \arrow[d] & S \arrow[d] \\ {\rm Spec}(\kappa) \arrow[r, "{\rm Fr}^i_\kappa"] & {\rm Spec}(\kappa) \end{tikzcd}$$

(2) For any κ-morphism $f:S_1\to S_2$ between κ-schemes, $f^{(p)}:S_1^{(p^i)}\to S_2^{(p^i)}$ is the base change of f by ${\rm Fr}^i_\kappa$.
(3) The relative Frobenius ${\rm Fr}^i_{S/\kappa}$ for p^i, or the i-th iterate of the relative Frobenius ${\rm Fr}_{S/\kappa}$, is the morphism

$${\rm Fr}_{p^i,S/\kappa}={\rm Fr}^i_{S/\kappa}={\rm Fr}^{(p^{i-1})}_{S/\kappa}\circ\cdots{\rm Fr}^{(p)}_{S/\kappa}\circ{\rm Fr}_{S/\kappa}\colon S\longrightarrow S^{(p^i)}$$

whose composition with the base change morphism

$$W^i_{S/\kappa}\colon S^{(p^i)}\longrightarrow S$$

is the i-th absolute Frobenius morphism ${\rm Fr}_{p^i,S}:S\to S$ for S. In the following commutative diagram

$$\begin{tikzcd} S \arrow[r, "{\rm Fr}^i_{S/\kappa}"] \arrow[d] & S^{(p^i)} \arrow[r, "W^i_{S/\kappa}"] \arrow[d] & S \arrow[r, "{\rm Fr}^i_{S/\kappa}"] \arrow[d] & S^{(p^i)} \arrow[d] \\ {\rm Spec}(\kappa) \arrow[r, "="] & {\rm Spec}(\kappa) \arrow[r, "{\rm Fr}^i_\kappa"] & {\rm Spec}(\kappa) \arrow[r, "="] & {\rm Spec}(\kappa) \end{tikzcd}$$

we have

$${\rm Fr}^i_{S/\kappa}\circ W^i_{S/\kappa}={\rm Fr}^i_S\,,\qquad W^i_{S/\kappa}\circ{\rm Fr}^i_{S/\kappa}={\rm Fr}^i_{S^{(p^i)}}\ .$$

(4) More generally for any κ-morphism $f\colon Z/S$ we have a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{Fr}_{Z/S}"] \arrow[d, "f"'] & \mathrm{Fr}_S^* Z \arrow[r, "W_{Z^{(p)}/S^{(p)}/\kappa}"] \arrow[d, "\mathrm{Fr}_S^* f"'] & Z^{(p)} \arrow[r, "W_{Z/\kappa}"] \arrow[d, "f^{(p)}"] & Z \arrow[d, "f"] \\
S \arrow[r, "="] \arrow[d] & S \arrow[r, "\mathrm{Fr}_{S/\kappa}"] \arrow[d] & S^{(p)} \arrow[r, "W_{S/\kappa}"] \arrow[d] & S \arrow[d] \\
\mathrm{Spec}(\kappa) \arrow[r, "="] & \mathrm{Spec}(\kappa) \arrow[r, "="] & \mathrm{Spec}(\kappa) \arrow[r, "\mathrm{Fr}_\kappa"] & \mathrm{Spec}(\kappa)
\end{tikzcd}$$

where the middle and right squares in the top two rows are Cartesian, and

$$W_{Z^{(p)}/S^{(p)}/\kappa} \circ \mathrm{Fr}_{Z/S} = \mathrm{Fr}_{Z/\kappa}, \quad W_{Z/\kappa} \circ W_{Z^{(p)}/S^{(p)}/\kappa} \circ \mathrm{Fr}_{Z/S} = \mathrm{Fr}_Z \ .$$

2.14. Lemma. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$.*

(1) *The natural map*

$$\mathrm{can} : \mathscr{HOM}'(X,Y)^{(p^i)} \longrightarrow \mathscr{HOM}(X^{(p^i)}, Y^{(p^i)})$$

is an isomorphism for all $i \in \mathbb{N}$. Similarly we have a natural isomorphisms

$$\mathrm{can} : \mathscr{HOM}'_{\mathrm{div}}(X,Y)^{(p^i)} \xrightarrow{\sim} \mathscr{HOM}'_{\mathrm{div}}(X^{(p^i)}, Y^{(p^i)}) \qquad \forall i \in \mathbb{N}.$$

(2) *The compsition*

$$\mathscr{HOM}'(X,Y) \xrightarrow{\mathrm{Fr}^{\,d}_{\mathscr{HOM}'(X,Y)/\kappa}} \mathscr{HOM}'(X,Y)^{(p^i)} \xrightarrow[\simeq]{\mathrm{can}} \mathscr{HOM}'(X^{(p^i)}, Y^{(p^i)})$$

of the relative Frobenius homomorphism $\mathrm{Fr}^i_{\mathscr{HOM}'(X,Y)/\kappa}$ for $\mathscr{HOM}'(X,Y)$ with the canonical isomorphism

$$\mathrm{can} : \mathscr{HOM}'(X,Y)^{(p^i)} \xrightarrow{\sim} \mathscr{HOM}'(X^{(p^i)}, Y^{(p^i)})$$

can be explicitly described as follows: it sends every homomorphism $h : X[n]_R \to Y[n]_R$ over a commutative κ-algebra R, regarded as an R-valued point of $\mathscr{HOM}'(X,Y)$, to the R-valued point of $\mathscr{HOM}'(X^{(p^i)}, Y^{(p^i)})$ represented by the homomorphism $h^{(p^i)} : X[p^n]^{(p^i)} \to Y[p^n]^{(p^i)}$, the base change of h by the i-th power of the absolute Frobenius for R. A similar statement holds for $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$.

(3) *The following diagram*

$$\begin{tikzcd}
\mathscr{HOM}'(X,Y) \arrow[r, "\mathrm{Fr}^{\,d}_{\mathscr{HOM}'(X,Y)/\kappa}"] & \mathscr{HOM}'(X,Y)^{(p^d)} \arrow[d, "\simeq"', "\mathrm{can}"] \\
\mathscr{HOM}'(X^{(p^d)}, Y) \arrow[u, "(\mathrm{Fr}^d_{X/\kappa}, 1_Y)^*"] \arrow[r, "(1_{X^{(p^d)}}, \mathrm{Fr}^d_{Y/\kappa})_*"] & \mathscr{HOM}'(X^{(p^d)}, Y^{(p^d)})
\end{tikzcd}$$

commutes, where

$$(\mathrm{Fr}^d_{X/\kappa}, 1_{Y^{(p^d)}})^* : \mathscr{HOM}'(X^{(p^d)}, Y^{(p^d)}) \longrightarrow \mathscr{HOM}'(X, Y^{(p^d)})$$

is the homomorphism induced by $\mathrm{Fr}^d_{X/\kappa}$ *and*

$$(1_X,\, \mathrm{Fr}^d_{Y/\kappa})_* : \mathscr{H}om'(X,Y) \longrightarrow \mathscr{H}om'(X, Y^{(p^d)})$$

is the homorphism induced by $\mathrm{Fr}^d_{Y/\kappa}$.

(4) *Similarly the diagram*

$$\begin{tikzcd}
\mathscr{H}om'_{\mathrm{div}}(X,Y) \arrow[r, "\mathrm{Fr}^{\,d}_{\mathscr{H}om'_{\mathrm{div}}(X,Y)/\kappa}"] & \mathscr{H}om'_{\mathrm{div}}(X,Y)^{(p^d)} \arrow[d, "\mathrm{can}", "\simeq"'] \\
\mathscr{H}om'_{\mathrm{div}}(X^{(p^d)},Y) \arrow[u, "(\mathrm{Fr}^d_{X/\kappa},\,1_Y)^*"] \arrow[r, "(1_{X^{(p^d)}},\,\mathrm{Fr}^d_{Y/\kappa})_*"] & \mathscr{H}om'_{\mathrm{div}}(X^{(p^d)},Y^{(p^d)})
\end{tikzcd}$$

commutes.

Proof. The formation of the $\mathscr{H}om$ scheme $\mathscr{H}om(X[p^n],Y[p^n])$ commutes with arbitrary base change. In particular the canonical map

$$\mathrm{can}\colon \mathscr{H}om(X[p^n],Y[p^n])^{(p^i)} \xrightarrow{\sim} \mathscr{H}om(X[p^n]^{(p^n)},Y[p^n]^{(p^i)})$$

is an isomorphism for all $n,i\in\mathbb{N}$. Taking the inductive limit, we get

$$\mathrm{can} : \mathscr{H}om'(X,Y)^{(p^i)} \xrightarrow{\sim} \mathscr{H}om(X^{(p^i)},Y^{(p^i)})\,.$$

Because the formation of kernel and cokernel for homomorphisms between commutative group schemes over κ commutes with change of base fields, the canonical map

$$\mathrm{can}\colon \mathscr{H}om'_{\mathrm{div}}(X,Y)[p^n]^{(p^i)} \xrightarrow{\sim} \mathscr{H}om'_{\mathrm{div}}(X^{(p^i)},Y^{(p^i)})[p^n]$$

is an isomorphism for all $n,i\in\mathbb{N}$. Therefore

$$\mathrm{can} : \mathscr{H}om'_{\mathrm{div}}(X,Y)^{(p^i)} \xrightarrow{\sim} \mathscr{H}om'_{\mathrm{div}}(X^{(p^i)},Y^{(p^i)}) \qquad \forall i\in\mathbb{N}.$$

We have proved the statement (1).

The statement (2) is not difficult to verify from the definition of the relative Frobenius morphisms $\mathrm{Fr}^i_{\mathscr{H}om'(X,Y)/\kappa}$ and $\mathrm{Fr}^i_{\mathscr{H}om'_{\mathrm{div}}(X,Y)/\kappa}$. The statement (3) follows from the explicit description of the relative Frobenius $\mathrm{Fr}^i_{\mathscr{H}om'(X,Y)/\kappa}$ in (2). The commutativity in the statement (4) follows from (3) because $\mathscr{H}om'_{\mathrm{div}}(X,Y)$ is embedded in $\mathscr{H}om'(X,Y)$. Details are omitted and left as an exercise. □

Remark. One can verify that the commutative diagrams

$$\begin{tikzcd}
\mathscr{H}om'(X,Y)^{(p^d)} \arrow[r, "W_{\mathscr{H}om'(X,Y)/\kappa}"] \arrow[d, "\mathrm{can}"', "\simeq"] & \mathscr{H}om'(X,Y) \arrow[d, "(W_{X/\kappa},\,1_Y)^*"] \\
\mathscr{H}om'(X^{(p^d)},Y^{(p^d)}) \arrow[r, "(1_{X^{(p^d)}},\,W^d_{Y/\kappa})_*"'] & \mathscr{H}om'(X^{(p^d)},Y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{HOM}'_{\mathrm{div}}(X,Y)^{(p^d)} \arrow[r, "W_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}"] \arrow[d, "\mathrm{can}"', "\simeq"] & \mathscr{HOM}'_{\mathrm{div}}(X,Y) \arrow[d, "(W_{X/\kappa},\, 1_Y)^*"] \\
\mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y^{(p^d)}) \arrow[r, "(1_{X^{(p^d)}},\, W^d_{Y/\kappa})_*"'] & \mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y)
\end{tikzcd}$$

similar to 2.14 (3), (4) also hold.

2.14.1. Proof of proposition 2.13. Let d be a positive integer such that $d\cdot\lambda_X =: a \in \mathbb{N}$ and also $d\cdot\lambda_Y =: b \in \mathbb{N}$. By 2.12(3), we may and do assume that κ is algebraically closed. By 2.12(2), we may replace X by a p-divisible group X_1 over κ isogeous to X and Y by an isogenous Y_1. So we may and do assume that

$$\mathrm{Ker}(\mathrm{Fr}^d_X) = \mathrm{Ker}(p^a) \quad \text{and} \quad \mathrm{Ker}(\mathrm{Fr}^d_Y) = \mathrm{Ker}(p^b).$$

This assumption means that there exists isomorphisms

$$\alpha\colon X^{(p^d)} \xrightarrow{\sim} X, \quad \beta\colon Y^{(p^d)} \xrightarrow{\sim} Y$$

such that

$$\alpha\circ\mathrm{Fr}^d_X = [p^a]_X \quad \text{and} \quad \beta\circ\mathrm{Fr}^d_Y = [p^b]_Y.$$

From the commutative diagram

$$\begin{tikzcd}
\mathscr{HOM}'_{\mathrm{div}}(X,Y) \arrow[r, "\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}"] & \mathscr{HOM}'_{\mathrm{div}}(X,Y)^{(p^d)} \arrow[d, "\simeq"', "\mathrm{can}"] \\
\mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y) \arrow[u, "(\mathrm{Fr}^d_{X/\kappa},\, 1_Y)^*"] \arrow[r, "(1_{X^{(p)}},\, \mathrm{Fr}^d_{Y/\kappa})_*"] & \mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y^{(p^d)})
\end{tikzcd}$$

in 2.14 (4) we get

$$\mathrm{Ker}\left((1_{X^{(p)}},\, \mathrm{Fr}^d_{Y/\kappa})_*\right) = \mathrm{Ker}\left(\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}\right) \times_{\hookrightarrow,\, \mathscr{HOM}'_{\mathrm{div}}(X,Y),\, (\mathrm{Fr}^d_{X/\kappa},1_Y)^*} \mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y).$$

In other words the kernel of $(1_{X^{(p)}},\, \mathrm{Fr}^d_{Y/\kappa})_*$ is equal to the scheme theoretic inverse image of the kernel of $\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}$ under $(\mathrm{Fr}^d_{X/\kappa}, 1_Y)^*$. The assumption that $\beta\circ\mathrm{Fr}^d_Y = [p^b]_Y$ and β is an isomorphism tells us that

$$\mathrm{Ker}\left((1_{X^{(p)}},\, \mathrm{Fr}^d_{Y/\kappa})_*\right) = \mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y)[p^b].$$

Pulling back the equality

$$\mathrm{Ker}\left((1_{X^{(p)}},\, \mathrm{Fr}^d_{Y/\kappa})_*\right) = \mathrm{Ker}\left(\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}\right) \times_{\hookrightarrow,\, \mathscr{HOM}'_{\mathrm{div}}(X,Y),\, (\mathrm{Fr}^d_{X/\kappa},1_Y)^*} \mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y).$$

by the isomorphism

$$(\alpha, 1_Y)^* : \mathscr{HOM}'_{\mathrm{div}}(X,Y) \xrightarrow{\sim} \mathscr{HOM}'_{\mathrm{div}}(X^{(p^d)},Y),$$

we get

$$\boxed{[p^a]^{-1}_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)}\left(\mathrm{Ker}(\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa})\right) = \mathscr{HOM}'_{\mathrm{div}}(X,Y)[p^b]}$$

where the left hand side is the pull-back of the kernel of $\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}$ under the endomorphism $[p^a]$ on $\mathrm{Hom}(X,Y)$.

- If $a > b$, the last displayed equality implies that the p-divisible group $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ is 0.
- In the case when $a \geq b$, the same equality implies that
$$\mathrm{Ker}(\mathrm{Fr}^d_{\mathscr{HOM}'_{\mathrm{div}}(X,Y)/\kappa}) = \mathscr{HOM}'_{\mathrm{div}}(X,Y)[p^{b-a}]\,,$$
so the p-divisible group $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ is isoclinic of slope $\frac{b-a}{d} = \lambda_Y - \lambda_X$.

We have finished the proof of proposition 2.13. $\square$

2.15. Corollary. *Let X, Y be p-divisible groups over a field κ. If every slope of X is strictly bigger than every slope of Y, then $\mathscr{HOM}'_{\mathrm{div}}(X,Y) = 0$.*

2.16. Warning. Proposition 2.13 might lead to overly optimistic expectations about the exactness of the functor $\mathscr{HOM}'_{\mathrm{div}}(\cdot\,,\,\cdot)$; for instance that when X is isoclinic the formation of $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ is exact in Y, and the graded pieces $\mathrm{gr}^\bullet\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ of the slope filtration of $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ are naturally isomorphic to $\mathscr{HOM}'_{\mathrm{div}}(X, \mathrm{gr}^j Y)$, where $\mathrm{gr}^j Y)$ are the graded pieces of the the slope filtration of Y. Such expectations do *not* hold in general.

To illustrate what may cause non-exactness, let us analyse the special case when Y sits in the middle of a short exact sequence $0 \to W \to Y \to Z \to 0$, with W and Z isoclinic and $\mathrm{slope}(Z) < \mathrm{slope}(W)$, and X is isoclinic with $\mathrm{slope}(X) \leq \mathrm{slope}(Y)$. Consider the sequence of homomorphisms
$$\mathscr{HOM}'_{\mathrm{div}}(X,W) \xrightarrow{j} \mathscr{HOM}'_{\mathrm{div}}(X,Y) \xrightarrow{h} \mathscr{HOM}'_{\mathrm{div}}(X,Z)$$
attached to the short exact sequence $0 \to W \to Y \to Z \to 0$, which is the inductive limit of sequences
$$\mathscr{HOM}^{\mathrm{st}}(X,W)_n \xrightarrow{j_n} \mathscr{HOM}^{\mathrm{st}}(X,Y)_n \xrightarrow{h_n} \mathscr{HOM}^{\mathrm{st}}(X,Z)_n\,.$$
The last sequence sits inside an exact sequence
$$0 \to \mathrm{Hom}(X[p^n], W[p^n]) \xrightarrow{\tilde{j}_n} \mathrm{Hom}(X[p^n], Y[p^n]) \xrightarrow{\tilde{h}_n} \mathrm{Hom}(X[p^n], Z[p^n]).$$
So $j_n : \mathscr{HOM}^{\mathrm{st}}(X,W)_n \xrightarrow{j_n} \mathscr{HOM}^{\mathrm{st}}(X,Y)_n$ is a monomorphism, and
$$\mathrm{Ker}(h_n) = \tilde{j}_n^{-1}(\mathscr{HOM}^{\mathrm{st}}(X,Y)_n) \supset \mathscr{HOM}^{\mathrm{st}}(X,W)_n,$$
where
$$\tilde{j}_n^{-1}(\mathscr{HOM}^{\mathrm{st}}(X,Y)_n) = \mathscr{HOM}^{\mathrm{st}}(X,Y)_n \times_{\mathscr{HOM}(X[p^n],Y[p^n]),\tilde{j}_n} \mathscr{HOM}(X[p^n], W[p^n])$$
is the intersection of the subschemes $\mathscr{HOM}^{\mathrm{st}}(X,Y)_n$ and $\mathscr{HOM}(X[p^n], W[p^n])$ inside the scheme $\mathscr{HOM}(X[p^n], Y[p^n])$. However the last intersection may not be equal to $\mathscr{HOM}^{\mathrm{st}}(X,W)_n$:

> a homomorphism $f : X[p^n]_T \to W[p^n]_T$ over a κ-scheme T with the property that for any $N > n$ there exists an fppf cover $T_N \to T$ such that $f \times_T T_N$ extends to a homomorphism $\tilde{f}_N : X[p^N]_{T_N} \to Y[p^N]_{T_N}$ may not admit any extension $f_{N,U} : X[p^N]_U \to W[p^N]_U$ with target in W instead of Y, for any fppf cover $U \to T$.

For a specific example, take κ to be an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$, let Z be a p-divisible group of height 3 and slope $1/3$, let W be a p-divisible group of height 2 and slope $1/3$, let Y be an extension of Z by W such that Y is not isomorphism to $Z \times W$, and let $X = Z$. Such a p-divisible group Y exists, and by the main result [22] we know that $Y[p^n]$ is not isomorphic to $Z[p^n] \times W[p^n]$, for every positive integer n. The identity map $\mathrm{id}_{Z[p^n]}$ on $Z[p^n]$ is obviously a $\overline{\mathbb{F}_p}$-rational point of $\mathscr{H}om^{\mathrm{st}}(Z,Z)_n$, for every positive integer n. However the homomorphism $\mathscr{H}om^{\mathrm{st}}(Z,Y)_n \to \mathscr{H}om^{\mathrm{st}}(Z,Z)_n$ is not faithfully flat for any $n > 0$. For otherwise there exists an $\overline{\mathbb{F}_p}$-rational point of $\mathscr{H}om^{\mathrm{st}}(Z,Y)_n$ which maps to $\mathrm{id}_{Z[p^n]}$, which implies that $Y[p^n]$ is isomorphic to $Z[p^n] \times W[p^n]$, a contradiction.

Here are a few further remarks on the above example. The p-divisible group $\mathscr{H}om'_{\mathrm{div}}(Z,Y)$ admits a slope filtation

$$0 \to P \to \mathscr{H}om'_{\mathrm{div}}(Z,Y) \to Q \to 0,$$

where P is an isoclinic p-divisible group with slope $1/6$ and height 6 and Q is an étale (in fact constant) p-divisible group of height 9. The subgroup scheme $j_n(\mathscr{H}om^{\mathrm{st}}(Z,W)_n))$ of $\mathscr{H}om^{\mathrm{st}}(Z,Y)_n$ coincides with $P[p^n]$, so the quotient

$$\mathscr{H}om^{\mathrm{st}}(Z,Y)_n / j_n(\mathscr{H}om^{\mathrm{st}}(Z,W)_n)$$

is identified with Q_n. However the natural homomorphism $\beta : Q \to \mathscr{H}om'_{\mathrm{div}}(Z,Z)$ is an isogeny with a non-trivial kernel contained in $Q[p]$.

2.17. Definition. Let $\kappa \supset \mathbb{F}_p$ be a perfect field of characteristic $p > 0$. Let X, Y be p-divisible groups over κ.

(a) Denote by $\mathfrak{Art}_\kappa^+$ the category of augmented commutative artinian local algebras over κ. Objects of $\mathfrak{Art}_\kappa^+$ are pairs

$$(R,\ j\colon \mathsf{k} \to R,\ \epsilon : R \to \mathsf{k})$$

where
- R is a commutative artinian local algebra over κ,
- k is a perfect field containing κ,
- j and ϵ are κ-linear ring homomorphisms such that $\epsilon \circ j = \mathrm{id}_{\mathsf{k}}$.

Morphisms from $(R_1,\ j_1\colon \mathsf{k}_1 \to R_1, \epsilon_1 : R_1 \to \mathsf{k}_1)$ to $(R_2,\ j_2\colon \mathsf{k}_2 \to R_2, \epsilon_2\colon R_2 \to \mathsf{k}_2)$ are κ-linear homomorphisms from R_1 to R_2 which are compatible with the augmentations ϵ_1 and ϵ_2.

(b) Define $\mathscr{E}xt_{\mathrm{def}}(X,Y) : \mathfrak{Art}_\kappa^+ \longrightarrow \mathfrak{Sets}$ to be the functor from $\mathfrak{Art}_\kappa^+$ to the category of sets which sends any object $(R,\ j\colon \mathsf{k} \to R,\ \epsilon\colon R \to \mathsf{k})$ of $\mathfrak{Art}_\kappa^+$ to the set of isomorphism classes of

$$\left(0 \to Y_R \to E \to X_R \to 0,\ \zeta : E_{\mathsf{k}} \xrightarrow{\sim} X_{\mathsf{k}} \times Y_{\mathsf{k}}\right),$$

where $0 \to Y_R \to E \to X_R \to 0$ is a short exact sequence of p-divisible groups over R, and ζ is an isomorphism from the closed fiber of the extension E to the split extension $0 \to Y_{\mathsf{k}} \to X_{\mathsf{k}} \times Y_{\mathsf{k}} \to X_{\mathsf{k}} \to 0$ which induces the identity maps on both X_{k} and Y_{k}. The Baer sum gives $sExt(X,Y)(R)$ a natural structure as an abelian group, for every object (R, ϵ) in $\mathfrak{Art}_\kappa^+$, so we can promote $\mathcal{E}xt_{\rm def}(X,Y)$ to a functor from $\mathfrak{Art}_\kappa^+$ to the category $\mathfrak{Ab}\,\mathfrak{Grp}$ of abelian groups.

(c) For each $n\in\mathbb{N}$, let $\mathcal{E}xt_{\rm def}(X,Y[p^n]):\mathfrak{Art}_k^+\to\mathfrak{Ab}\,\mathfrak{Grp}$ be the group-valued functor defined in a similar way as in (b), with Y replaced by $Y[p^n]$.

2.18. Proposition. *Let X,Y be p-divisible groups over a field $\kappa\supset\mathbb{F}_p$. The commutative formal group $\mathscr{H}\!om'(X,Y)$, canonically isomorphic to $\mathcal{E}xt_{\rm def}(X,Y)$ is formally smooth over κ of dimension $\dim(Y)\cdot\dim(X^t)$.*

===== Write down a proof using deformation theory of BTT in Illusie. =====

Proof. Easy consequence of Grothendieck-Messing deformation theory. ∎

2.19. Definition. Let X,Y be p-divisible groups over a field $\kappa\supset\mathbb{F}_p$.

(1) For each $n\in\mathbb{N}$, let

$$\delta_n\colon H_n=\mathscr{H}\!om(X[p^n],Y[p^n])\longrightarrow\mathscr{E}\!xt(X,Y[p^n])$$

be the homomorphism of group-valued functors on $\mathfrak{Art}_\kappa^+$ defined as follows. For every object $(R,\epsilon:\mathsf{k}\to R)$ in sends every homomorphism $h:X[p^n]_R\to Y^{[}p^n]$ in $H_n^\wedge(R)=\operatorname{Ker}(H_n(R)\to H_n(\mathsf{k}))$ to push-our of the short exact sequence $0\to X[p^n]_R\to X_R\to X_R\to 0$ by h.

(2) For every $m,n\in\mathbb{N}$, let $\jmath_{n+m,n}:\mathscr{E}\!xt(X,Y[p^n])\longrightarrow\mathscr{E}\!xt(X,Y[p^{n+m}])$ be the homomorphism induced by the inclusion map $j_{Y,n+m,n}:Y[p^n]\to Y[p^{n+m}]$.

(3) For every $n\in\mathbb{N}$, let $\jmath_n:\mathscr{E}\!xt(X,Y[p^n])\longrightarrow\mathscr{E}\!xt(X,Y)$ be the homomorphism induced by the inclusion map $j_{Y,n}:Y[p^n]\to Y$ between commutative smooth formal groups over κ.

XXXXX Need to systematically re-examine the system of notations. XXXXX

2.20. Proposition. *We follow the notation in 2.19.*

(1) *The homomorphism*

$$\delta_n\colon H_n^\wedge=\mathscr{H}\!om(X[p^n],Y[p^n])^\wedge\longrightarrow\mathscr{E}\!xt(X,Y[p^n])$$

is an isomorphism for all $n\in\mathbb{N}$.

(2) *The homomorphism*

$$\jmath_n:\mathscr{E}\!xt(X,Y[p^n])^\wedge\longrightarrow\mathscr{E}\!xt(X,Y)[p^n]$$

is an isomorphism. In other words the map

$$\jmath_n(R):\mathscr{E}\!xt(X,Y[p^n])(R)\longrightarrow\mathscr{E}\!xt(X,Y)[p^n](R)$$

is a bijection for every object $(R,\epsilon:\mathsf{k}\to R)$ in $\mathfrak{Art}_\kappa^+$.

(3) $\jmath_{n+m,n}\circ\delta_n=\delta_{n+m}\circ\iota^\wedge_{n+m,n}$ *for all $m,n\in\mathbb{N}$, where $\iota^\wedge_{n+m,n}\colon H_n^\wedge\rightarrowtail H^\wedge_{n+m}$ is the formal completion of the monomorphism $\iota_{n+m,n}$ defined in 2.1.*

(4) *The compatible family of isomorphisms* $\delta_n\colon H_n \xrightarrow{\sim} \mathscr{E}xt(Y, Y[p^n])$ *defines an isomorphism*

$$\delta\colon \mathscr{H}om'(X, Y) \xrightarrow{\sim} \mathscr{E}xt(X, Y).$$

PROOF. Let $(R, j : \mathsf{k} \to R, \epsilon : R \to \mathsf{k})$ be an object of $\mathfrak{Art}_\kappa^+$. The left exact functor $\mathrm{Hom}(?, Y[p^n]_R)$ applied to the short exact sequence

$$0 \longrightarrow X[p^n]_R \longrightarrow X_R \xrightarrow{[p^n]} X_R \longrightarrow 0$$

yields an isomorphism $\mathrm{Hom}(X[p^n]_R, Y[p^n]_R) \xrightarrow{\sim} \mathrm{Ext}^1(X_R, Y[p^n]_R)$. The statement (1) follows.

Consider the commutative diagram

$$\begin{tikzcd}
 & & & H_n^\wedge(R) \arrow[r, "\jmath_n\circ\delta_n"] \arrow[d, hook] & \mathscr{E}xt(X,Y)[p^n](R) \arrow[d, hook] \\
0 \arrow[r] & \mathrm{Hom}(X_R, Y_R) \arrow[r, "p^n"] \arrow[d, "\simeq"] & \mathrm{Hom}(X_R, Y_R) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}(X[p^n]_R, Y_R) \arrow[r, two heads] \arrow[d] & \mathrm{Ext}^1(X_R, Y_R)[p^n] \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(X_\mathsf{k}, Y_\mathsf{k}) \arrow[r, "p^n"] & \mathrm{Hom}(X_\mathsf{k}, Y_\mathsf{k}) \arrow[r] & \mathrm{Hom}(X[p^n]_\mathsf{k}, Y_\mathsf{k}) \arrow[r, two heads] & \mathrm{Ext}^1(X_\mathsf{k}, Y_\mathsf{k})[p^n]
\end{tikzcd}$$

whose second and third rows are exact, and the vertical arrows are given by "restriction to the closed fiber". The restriction map $\mathrm{Hom}(X_R, Y_R) \to \mathrm{Hom}(X, Y)$ is surjective because R is augmented; it is injective because the functor $\mathscr{H}om(X, Y)$ is unramified. An inspection of this diagram shows that the top horizontal arrow $\jmath_n \circ \delta_n$ is a bijection. Since δ_n is an isomorphism, so is $\jmath_n$. We have proved (2). The proof of statement (3) is straight forward from the definition and is omitted here. The statement (4) is a summary of (1)–(3). □

2.21. Corollary. *The isomorphism* $\delta : \mathscr{H}om'(X, Y) \xrightarrow{\sim} \mathscr{E}xt(X, Y)$ *in* 2.20 *induces an isomorphism*

$$\delta\colon \mathscr{H}om'_{\mathrm{div}}(X, Y) \xrightarrow{\sim} \mathscr{E}xt_{\mathrm{div}}(X, Y)$$

We record a general property about the push-out construction used in 2.19.

Let S be a scheme and let X, Y be p-divisible groups over S. Let n be a positive integer, and let $h : X[p^n] \to Y[p^n]$ be an S-homomorphism. Let E_h be the p-divisible group over S, defined as the push out of the short exact sequence

$$0 \to X[p^n] \to X \to X \to 0$$

by the composition $X[p^n] \xrightarrow{h} Y[p^n] \hookrightarrow Y$. The p-divisible group $E_h \to S$ sits in a short exact sequence

$$0 \to Y \to E_h \xrightarrow{\pi_h} X \to 0$$

of p-divisible groups over S. Moreover the short exact sequence

$$0 \to Y[p^n] \to E_h[p^n] \to X[p^n] \to 0$$

is the push-out by $h : X[p^n] \to Y[p^n]$ of the short exact sequence

$$0 \to X[p^n] \to X[p^{2n}] \xrightarrow{[p^n]_X} X[p^n] \to 0\,.$$

2.22. Lemma. *Let $X \to S$, $Y \to S$ be p-divisible groups over a scheme S. Let n be a positive integer, $h : X[p^n] \to Y[p^n]$ be an S-homomorphism, and let E_h be the extension of X by Y attached to h as in the previous paragraph.*

(a) *Let m be a positive integer. Let T be an S-scheme. Suppose that*

$$\tilde{h} : X[p^{n+m}]_T \to Y[p^{n+m}]_T$$

is a T-homomorphism whose restriction to $X[p^n]_T = X[p^n] \times_S T$ is equal to the base change of h to T. There exists an unique homomorphism

$$\zeta_{\tilde{h}} : X_0[p^m] \to E_h[p^m] \times_S T$$

with the following properties.

(i) *The composition $\pi_h \circ \zeta_{\tilde{h}}$ is equal to the identity map $\mathrm{id}_{X[p^m]_T}$ on $X[p^m]_T$. In other words $\zeta_{\tilde{h}}$ defines a splitting of the short exact sequence*

$$0 \longrightarrow Y[p^m]_T \longrightarrow E_h[p^m]_T \longrightarrow X[p^m]_T \longrightarrow 0\,.$$

(ii) *The composition*

$$X[p^{n+m}]_T \xrightarrow{[p^n]_X} X[p^n]_T \xrightarrow{\zeta_{\tilde{h}}} E_h[p^m]_T \hookrightarrow E_h[p^{m+n}]_T$$

is equal to the composition

$$X[p^{n+m}]_T \xrightarrow{(1_{X[p^{n+m}]_T},\, -j_{n+m}\circ\tilde{h})} X[p^m]_T \times_T Y_T \xrightarrow{\pi_h \times_S T} E_h \times_S T\ ,$$

where $j_{n+m} : Y[p^{n+m}]_T \hookrightarrow Y_T$ is the inclusion homomorphism, and $\pi_h \times_S T$ is the base change to T of the canonical faithfully flat homomorphism

$$\pi_h : X[p^m] \times_S Y \to E_h$$

whose kernel is the graph of the composition

$$X[p^n] \xrightarrow{-h} Y[p^n] \hookrightarrow Y\ .$$

(b) *Let T be an S-scheme and let m be a positive integer. For every splitting ζ of the short exact sequence*

$$0 \longrightarrow Y[p^m]_T \longrightarrow E_h[p^m]_T \longrightarrow X[p^m]_T \longrightarrow 0\ ,$$

there exists a unique T-homomorphism

$$\tilde{h} : X[p^{n+m}]_T \to Y[p^{n+m}]_T$$

whose restriction to $X[p^n]_T$ is equal to $h \times_S T$ such that ζ is equal to the splitting $\zeta_{\tilde{h}}$ attached to $\tilde{h}$ as in (a).

PROOF. Once formulated, the proofs of (a) and (b) are not difficult to check. They hold in the setting of p-divisible commutative groups in a topos. The version of (a) and (b) in the setting of p-divisible commutative group consisting of p-power torsion elements (in the naive sense, as understood by students in a college level algebra course) are easily verified. The statements (a), (b) follow from this "elementary case", by descent. ∎

The rest of this section is devoted to explicit descriptions of the Dieudonné modules of $\mathscr{HOM}'(X,Y)$ and $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$

2.23. Biextensions. We recall some general properties of biexsions.

=========to do: Recall the definition and general properties of biextensions.

2.24. Proposition. *Let X, Y, Z be p-divisible groups over a scheme S. There are natural isomorphism between the following three abelian groups.*

(a) *The group* $\mathrm{Biext}(X, Y; Z)$ *of all isomorphism classes of biextensions of (X, Y) by Z.*

(b) *The group of all families of bilinear maps*

$$\beta_n : X[p^n] \times_S Y[p^n] \longrightarrow Z[p^n], \qquad n \in \mathbb{N}$$

satisfying

$$\beta_n(\pi_{X,n,n+i}(x_{n+i}), \pi_{Y,n,n+i}(y_{n+i})) = \pi_{Z,n,n+i}(\beta_{n+i}(x_{n+i}, y_{n+i}))$$

for all $n, i \in \mathbb{N}$, all morphisms $S_1 \to S$, all S_1-points $x_{n+i} \in X[p^{n+i}](S_1)$ and S_1-points $y_{n+i} \in Y[p^{n+i}](S_1)$.

(c) *The cohomology group* $\mathrm{Ext}^1_S(X \otimes^{\mathbb{L}} Y, Z)$, *where X, Y, Z are regarded as fppf-sheaves of abelian groups on S.*

2.25. Remark. (1) The compatibility condition in (b) is equivalent to

$$j_{Z,n+i,n}(\beta_n(x_n, \pi_{Y,n,n+i}(y_{n+i}))) = \beta_{n+i}(j_{X,n+i,n}(x_n), y_{n+i})) \qquad \forall\, n, i \in \mathbb{N}$$

and also equivalent to

$$j_{Z,n+i,n}(\beta_n(\pi_{X,n,n+i}(x_{n+i}), y_n)) = \beta_{n+i}(x_{n+i}, j_{Y,n+i,n}(y_n))) \qquad \forall\, n, i \in \mathbb{N}.$$

(2) A compatible family of pairings β_n in (b) can be interpreted as a bilinear pairing

$$\beta : \mathbf{T}_p(X) \times \mathbf{T}_p(Y) \longrightarrow \mathbf{T}_p(Z)$$

between the *projective* systems $\mathbf{T}_p(X) := \varprojlim_n X[p^n]$ and $\mathbf{T}_p(Y) := \varprojlim_n Y[p^n]$ with values in $\mathbf{T}_p(Z) := \varprojlim_n Z[p^n]$. Note that the β_n's do *not* define a bilinear pairing

$$\varinjlim_n X[p^n] \times \varinjlim_n Y[p^n] \longrightarrow \varinjlim_n Z[p^n]$$

between the fppf-sheaves defined by X and Y.

2.26. Proposition. *Suppose that $\kappa \supset$ is a perfect field Let G_0, G_1, G_2 be finite group schemes over over κ. Let $\mathrm{M}_*(G_0), \mathrm{M}_*(G_1), \mathrm{M}_*(G_2)$ be their covariant Dieudonné modules.*[1] *There is natural bijection between the set of all bihomomorphisms from $G_1 \times G_2$ to G_0 and the set of all $W(\kappa)$-bilinear homomorphisms*

$$b : \mathrm{M}_*(G_1) \times \mathrm{M}_*(G_2) \longrightarrow \mathrm{M}_*(G_0)$$

[1] The definition of the covariant Dieudonne module used in [9] is:

$$\mathrm{M}_*(G) = \mathrm{Hom}(\varprojlim_{m,n} W_{m,n}, G)$$

where $W_{m,n} = W_m[F^n]\mathrm{Ker}(F^n : W_m \to W_m)$ and the transition maps in the projective system are generated by the standard projection maps $\pi_{m,m+1} : W_{m+1} \longrightarrow W_m$ and the Frobenius maps $F : W_{m,n+1} \longrightarrow W_{m,n}$.

satisfying

- $b(Vm_1, Vm_2) = Vb(m_1, m_2)\ \forall\, m_1 \in \mathrm{M}_*(G_1), \forall\, m_2 \in \mathrm{M}_*(G_2)$,
- $b(Fm_1, m_2) = Fb(m_1, Vm_2)$ *and* $b(m_1, Fm_2) = Fb(Vm_1, m_2)\ \forall\, m_1 \in \mathrm{M}_*(G_1)$ *and* $\forall\, m_2 \in \mathrm{M}_*(G_2)$.

PROOF. See [9, Cor 3.0.21] and/or [10, Cor. 1.23]. □

2.27. Corollary. *Let X, Y, Z be p-divisible groups over a perfect field $\kappa \supset \mathbb{F}_p$. Let $\mathrm{M}_*(X)$, $\mathrm{M}_*(Y)$ and $\mathrm{M}_*(Z)$ be their covariant Dieudonné modules. There are natural bijections between the following three sets.*

(a) *biextensions of $X \times Y$ by Z,*
(b) *families of bilinear pairings $\beta_n : X[p^n] \times Y[p^n] \longrightarrow Z[p^n]$ compatible with projections.*
(c) *$W(\kappa)$-bilinear pairings*

$$b : \mathrm{M}_*(X) \times \mathrm{M}_*(Y) \longrightarrow \mathrm{M}_*(Z)$$

satisfying the following conditions:

- $b(Vx, Vy) = Vb(x, y)\ \forall\, x \in \mathrm{M}_*(X), \forall\, y \in \mathrm{M}_*(Y)$,
- $b(Fx, y) = Fb(x, Vy)$ *and* $b(x, Fy) = Fb(Vx, y)\ \forall\, x \in \mathrm{M}_*(X)$ *and* $\forall\, y \in \mathrm{M}_*(Y)$.

2.28. Remark. It is not known whether an analog of the equivalence of (b) and (c) in 2.27 hold for more general base schemes such as a non-perfect field of characteristic $p > 0$.

2.29. Remark. The $W(\kappa)[1/p]$-vector space $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X), \mathrm{M}_*(Z))[1/p]$ has natural actions of F and V as follows:

$$(Vh)(x) = V \cdot h(V^{-1}x), \quad (Fh)(x) = F \cdot h(Vx) \quad \forall\, x \in \mathrm{M}_*(X).$$

Note that $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X), \mathrm{M}_*(Y))$ is stable under the action of F but not necessarily stable under the action of V. A $W(\kappa)$-bilinear pairing

$$b : \mathrm{M}_*(X) \times \mathrm{M}_*(Y) \longrightarrow \mathrm{M}_*(Z)$$

as in (c) defines a $W(\kappa)$-linear map $y \mapsto \Psi(y)$ from $\mathrm{M}_*(Y)$ to the space of $W(k)$-linear Hom's $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X), \mathrm{M}_*(Z))$. The two conditions for b in (c) mean that

$$\Psi(Vy)|_{V\mathrm{M}_*(X)} = (V \cdot \Psi(y))|_{V\mathrm{M}_*(X)}$$

and

$$(F \cdot \Psi(y))|_{\mathrm{M}_*(X)} = \Psi(Fx)|_{\mathrm{M}_*(X)}$$

for all $y \in \mathrm{M}_*(Y)$.

PROOF. See [9, Cor 5.5.6] and/or [10, Cor. 3.7]. □

Additional information on references: Reference for the bijection between bilinear pairing for finite group schemes (respectively p-divisible groups) and bilinear pairings for Dieudonné modules. (Documented for perfect fields. Not known for displays; this is the content of [9, Question 6.2.12].)

1. [9, Cor 3.0.21] and [9, Cor 5.5.6] in Hadi's thesis, the first for finite group schemes over perfect fields and the second for p-divisible groups over perfect fields.
2. [10, Cor. 1.23] and [10, Cor. 3.7] in Hadi's preprint, over perfect fields, for finite group schemes and p-divisible groups respectively.

2.30. Corollary. *Let X, Y be p-divisible groups over a perfect field $\kappa \supset \mathbb{F}_p$. The covariant Dieudonné module $\mathrm{M}_*(\mathscr{H}om'_{\mathrm{div}}(X,Y))$ of the p-divisible formal group $\mathscr{H}om'_{\mathrm{div}}(X,Y)$ is naturally isomorphic to the largest $W(\kappa)$-submodule of the $W(k)$-module $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X), \mathrm{M}_*(Y))$ which is stable under the actions of both operators F and V on $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X), \mathrm{M}_*(Y))[1/p]$.*

Proof. This is a consequence of basic Diedonné theory (including functoriality), the definition of the p-divisible group $\mathscr{H}om'_{\mathrm{div}}(X,Y)$ and the equivalence of (b) and (c) in corollary 2.27. $\square$

2.31. Remark. Over a nonperfect base field κ, we don't know how to describe the display of the p-divisible group $\mathscr{H}om'_{\mathrm{div}}(X,Y)$ in terms of the displays of X and Y.

3. Sustained p-divisible groups: definitions and basic properties

In this section κ is a field of characteristice $p > 0$, not necessarily perfect. We will define the notion of *sustained p-divisible groups* relative to the base field κ.

3.1. Definition. Let S be a scheme over κ. Let X_0 be a p-divisible group over κ.

(i) A p-divisible group $X \to S$ is *strongly κ-sustained* if there exists a p-divisible group X_0 over κ such that for every positive integer n, the structural morphism

$$\mathscr{I}som_S(X_0[p^n] \times_{\operatorname{Spec}\kappa} S, X_S[p^n]) \longrightarrow S$$

of the sheaf of all isomorphisms between the two truncated BT-groups is faithfully flat.

Clearly the last condition implies (and is equivalent to) that

$$\mathscr{I}som_S(X_0[p^n] \times_{\operatorname{Spec}\kappa} S, X_S[p^n])$$

is a right torsor for the S-group scheme $\mathscr{A}ut(X_0[p^n]) \times_{\operatorname{Spec}\kappa} S$ under composition of isomorphisms. Here $\mathscr{A}ut(X_0[p^n])$ is the κ-group scheme whose T-points is the set of all T-automorphisms of the truncated BT-group $X_0[p^n] \times_{\operatorname{Spec}\kappa} T$, for every κ-scheme T

(ii) If a p-divisible group X_0 over κ satisfied the condition (i) above for a strongly κ-sustained p-divisible group $X \to S$, we say that X_0 is a *model* (or a *κ-model*) of X, and that $X \to S$ is strongly κ-sustained *modeled on* X_0.

3.2. Lemma. *Let S be a κ-scheme and let X_0 be a p-divisible group over κ. A p-divisible group $X \to S$ is strongly κ-sustained modeled on X_0 if and only if for every positive integer n, there exists a faithfully flat morphism $\tilde{S} \to S$ and an isomorphism $\psi_n : X_0[p^n] \times_{\mathrm{Spec}(\kappa)} \tilde{S} \xrightarrow{\sim} X[p^n] \times_S \tilde{S}$.*

Proof. Suppose $X \to S$ is κ-sustained. Then for each $n \in \mathbb{N}_{>0}$, over the faithfully flat cover

$$\mathscr{I}(X_0, X)_n := \mathscr{I\!S\!O\!M}_S(X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S, X[p^n]) \to S$$

of S, we have a tautological isomorphism from (the base change to $\mathscr{I}(X_0, X)_n$ of) $X_0[p^n]$ to (the base change to $\mathscr{I}(X_0, X)_n$ of) $X[p^n]$. Note that the morphism $\mathscr{I}(X_0, X)_n \to S$ is a torsor for $\mathscr{A\!U\!T}_\kappa(X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S$, therefore it is of finite presentation.

Conversely assume that for each $n > 0$ there exists a faithfully flat morphism $\tilde{S} \to S$ and an isomorphism $\psi_n : X_0[p^n] \times_{\mathrm{Spec}(\kappa)} \tilde{S} \xrightarrow{\sim} X[p^n] \times_S \tilde{S}$. Then the base change of $\mathscr{I}(X_0, X)_n \to S$ to $\tilde{S}$ has a section, therefore $\mathscr{I}(X_0, X)_n \to S$ is a torsor for $\mathscr{A\!U\!T}(X_0[p^n]) \times_{\mathrm{Spec}(\kappa)} \tilde{S}$ and is faithfully flat over $\tilde{S}$. By descent we conclude that $\mathscr{I}(X_0, X)_n \to S$ is faithfully flat over S for every $n \in \mathbb{N}_{>0}$. □

3.3. Lemma. *Let S be a κ-scheme. Let $X \to S$ be a p-divisible group over S and let X_0 be a p-divisible group over κ. Let E be an extension field of κ. Let $S_E := S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(E)$, let $X_E := X \times_S S_E$, and let $X_{0E} := X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(E)$.*

(i) *If $X \to S$ is strongly κ-sustained, then $X_E \to S_E$ is strongly E-sustained.*

(ii) *$X \to S$ is strongly κ-sustained modeled on X_0 if and only if $X_E \to S_E$ is strongly E-sustained modeled on X_{0E}.*

The proof 3.3 is left to the readers.

Proof of the equivalence of 1.3 (1) and 1.3 (2). Assume that 1.3 (1) holds. Let $S_L := S \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\,L$ and let $X_L := X \times_S S_L$. It follows easily from the assumption 1.3 (1) that $\mathscr{I\!S\!O\!M}_{S_L \times_{\mathrm{Spec}\,L} S_L}(\mathrm{pr}_1^* X_L[p^n], \mathrm{pr}_2^* X_L[p^n])$ is faithfully flat over $S_L \times_{\mathrm{Spec}\,L} S_L$. On the other hand we know that

$$\mathscr{I\!S\!O\!M}_{S_L \times_{\mathrm{Spec}\,L} S_L}(\mathrm{pr}_1^* X_L[p^n], \mathrm{pr}_2^* X_L[p^n]) \cong \mathscr{I\!S\!O\!M}_{S \times_{\mathrm{Spec}\,\kappa} S}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]) \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\,L,$$

because the formation of the $\mathscr{I\!S\!O\!M}$ scheme commutes with arbitrary base change, therefore $\mathscr{I\!S\!O\!M}_{S \times_{\mathrm{Spec}\,\kappa} S}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n])$ is faithfully flat over $S \times_{\mathrm{Spec}\,\kappa} S$ by fpqc descent. So 1.3 (2) holds.

Conversely assume that 1.3 (2) holds. Let s be a point of S and let $L = \kappa(s)$ be the residue field of s. The pull-back of the faithfully flat morphism

$$\mathscr{I\!S\!O\!M}_{S \times_{\mathrm{Spec}\,\kappa} S}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]) \longrightarrow S \times_{\mathrm{Spec}\,\kappa} S$$

by the inclusion map $s \times_{\mathrm{Spec}\,\kappa} S \to S \times_{\mathrm{Spec}\,\kappa} S$ is of course faithfully flat. On the other hand this pull-back is nothing other than the $\mathscr{I\!S\!O\!M}$ scheme

$$\mathscr{I\!S\!O\!M}_{S_L}(X_s[p^n] \times_{\mathrm{Spec}\,L} S_L, X_L) \longrightarrow S_L,$$

so 1.3 (1) holds.

3.4. Lemma. *Let S be a κ-scheme. Let $X \to S$ be a p-divisible group, and let X_0 be a p-divisible group over κ. Let $f : T \to S$ be a κ-morphism, and let $f^*X \to T$ be the pull-back of $X \to S$ by f.*

(1) *If X is κ-sustained over S, then f^*X is κ-sustained over T. Similarly if $X \to S$ is strongly κ-sustained modeled on X_0, then $f^*X \to T$ is strongly κ-sustained modeled on X_0.*
(2) *Assume that $T \to S$ is fully faithful. If f^*X is κ-sustained over T, then X is κ-sustained over S. Similarly if $f^*X \to T$ is strongly κ-sustained modeled on X_0, then $X \to S$ is strongly κ-sustained modeled on X_0.*
(3) *Suppose that $\{U_i : i \in I\}$ is a familty of open subsets of S such that $\bigcup_{i\in I} U_i = S$ and $X \times_S U_i \to U_i$ is strong κ-sustained modeld on X_0 for each $i \in I$. Then $X \to S$ is strongly κ-sustained modeled on X_0. (Being strongly κ-sustained modeled on a fixed p-divisible group over κ is Zariski local on the base scheme.)*

Proof. The statement (1) follows from the fact that the formation of $\mathscr{I}\mathit{som}$ schemes commutes with base change. The statement (2) follows from flat descent. The statement (3) is obvious. □

3.5. Lemma. *Let S be a κ-scheme and let $X \to S$ be a p-divisible group. Let X_0 be a p-divisible group over κ. The following are equivalent.*

(1) *$X \to S$ is strongly κ-sustained modeled on X_0.*
(2) *$X \times_S \mathrm{Spec}(\mathscr{O}_{S,s}) \to \mathrm{Spec}(\mathscr{O}_{S,s})$ is strongly κ-sustained modeled on X_0 for every $s \in S$.*
(3) *$X \times_S \mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{S,s}) \to \mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{S,s})$ is strongly κ-sustained modeled on X_0 for every $s \in S$, where $\mathscr{O}^{\mathrm{h}}_{S,s}$ is the henselization of the local ring $\mathscr{O}^{\mathrm{h}}_{S,s}$.*
(4) *$X \times_S \mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{S,s}) \to \mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{S,s})$ is strongly κ-sustained modeled on X_0 for every $s \in S$, where $\mathscr{O}^{\mathrm{sh}}_{S,s}$ is the strict henselization of the local ring $\mathscr{O}^{\mathrm{sh}}_{S,s}$.*

Suppose that S is locally Noetherian. Then each of (1)–(4) above is also equivalent to

(5) *$X \times_S \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s}) \to \mathrm{Spec}(\mathscr{O}^{\wedge}_{S,s})$ is strongly κ-sustained modeled on X_0 for every $s \in S$, where $\mathscr{O}^{\wedge}_{S,s}$ is the formal completion of the Noetherian local ring $\mathscr{O}_{S,s}$.*

Proof. Obvious. □

3.6. Lemma. (1) *Every strongly κ-sustained p-divisible group is κ-sustained.*
(2) *Suppose that $X \to S$ is κ-sustained and $s_0 : \mathrm{Spec}\,\kappa \to S$ is a κ-rational point of S, then $X \to S$ is strongly κ-sustained modeled on s_0^*X.*
(3) *Let E be an extension field of κ and let S be an E-scheme. Every κ-sustained p-divisible group over S is E-sustained. Similarly every strongly κ-sustained p-divisible group is strongly E-sustained.*

Proof. The statements (1) and (3) are obvious. The statement two is a consequence of the fact the formation of the $\mathscr{I}\mathit{som}$ schemes commutes with arbitrary base change, and the formal fact that the composition of the natural morphism $S \times_{\mathrm{Spec}(E)} S \to S \times_{\mathrm{Spec}(\kappa)} S$ with the i-th projection $\mathrm{pr}_{i,\kappa} : S \times_{\mathrm{Spec}(\kappa)} S \to S$ is equal to the i-th projection $\mathrm{pr}_{i,} : S \times_{\mathrm{Spec}(\kappa)} S \to S$ for $i = 1, 2$. The details are left to the readers.

3.7. Lemma. *Let S be a κ-scheme. Let $X \to S$ be a p-divisible group over S. Let E be an extension field of κ. Let $S_E := S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(E)$ and let $X_E := X \times_S S_E$. The p-divisible group $X \to S$ is κ-sustained if and only if $X_E \to S_E$ is E-sustained.*

Proof. For every positive integer n, the morphism

$$\mathscr{I}\!\mathit{SOM}_{S_E \times_{\mathrm{Spec}(E)} S_E}(\mathrm{pr}_1^* X[p^n]_E, \mathrm{pr}_2^* X[p^n]_E) \to S_E \times_{\mathrm{Spec}(E)} S_E$$

is the base change of the morphism

$$\mathscr{I}\!\mathit{SOM}_{S \times_{\mathrm{Spec}(\kappa)} S}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]) \to S \times_{\mathrm{Spec}(\kappa)} S$$

by $\mathrm{Spec}(E) \to \mathrm{Spec}(\kappa)$. The assertion follows from descent. $\square$

3.8. Remark. The notion of sustained p-divisible groups depends essentially on the base field κ. For a κ-sustained p-divisible group $X \to S$ and a subfield $F \subset \kappa$, it may well happen that $X \to S$ is not F-sustained. This phenomenon already happens when $S = \mathrm{Spec}\,\kappa$ and $F \subsetneqq \kappa$.

3.9. Remark. Let S be a κ-scheme. The property for a p-divisible group $X \to S$ to be κ-sustained is *not* Zariski local in S. For instance if S is the disjoint union of two non-empty open subschemes U and V, $X_U \to U$ and $X_V \to V$ are two κ-sustained p-divisible groups, there is no reason that $X := (X_U \cup X_V) \to (U \cup V) = S$ is κ-sustained—even the heights of X_U and X_V may be different. However the following weaker statement holds.

> *If S is the union of two open subschemes U and V with $U \cap V \neq \emptyset$, and $X \to S$ is a p-divisible group such that $X|_U \to U$ and $X|_V \to V$ are both κ-sustained, then $X \to S$ is κ-sustained.*

Proof. Because sustainedness may be checked after arbitrary extension of the base field κ, therefore we may assume that $(U \cap V)(\kappa) \neq \emptyset$. So it suffices to show the following.

> *Under the same assumption on (S, U, V) as above, if there is a p-divisible group X_0 over κ such that both $X|_U \to U$ and $X|_V \to V$ are strongly κ-sustained modeled on X_0, then $X \to S$ is also strongly κ-sustained modeled on X_0.*

The assumptions imply that there exist faithfully flat κ-morphisms $u : \tilde{U} \to U$ and $v : \tilde{V} \to V$ and isomorphisms

$$\phi : X_{\tilde{U}} = X \times_{S,u} \tilde{U} \xrightarrow{\sim} X_0 \times_{\mathrm{Spec}(\kappa)} \tilde{V}$$

and

$$\psi : X_{\tilde{V}} = X \times_{S,v} \tilde{V} \xrightarrow{\sim} X_0 \times_{\mathrm{Spec}(\kappa)} \tilde{V}.$$

Let $\tilde{S}$ be the disjoint union of $\tilde{U}$ and $\tilde{V}$. Combining ϕ and ψ, we obtain a faithfully flat κ-morphism $\tilde{S} \to S$ and an isomorphism

$$X \times_S \tilde{S} \xrightarrow{\sim} X_0 \times_{\mathrm{Spec}(\kappa)} \tilde{S}.$$

We have shown that $X \to S$ is strongly κ-sustained. $\square$

3.10. Lemma. *Let S be a connected κ-scheme. Let $X \to S$ be a p-divisible group. Assume that $X \times_S \mathrm{Spec}(\mathscr{O}_{S,s})$ is κ-sustained for every $s \in S$. Then $X \to S$ is κ-sustained.*

PROOF. For any positive integer n, consider the $\mathscr{I}\!\mathit{SOM}$ scheme

$$\mathscr{I}_n := \mathscr{I}\!\mathit{SOM}_{S\times_{\mathrm{Spec}(\kappa)}S}\mathrm{pr}_1^*X[p^n], \mathrm{pr}_2^*X[p^n] \longrightarrow S \times_{\mathrm{Spec}(\kappa)} S = S_1,$$

which is of finite presentation over $S \times_{\mathrm{Spec}(\kappa)} S$. The assumption means that

$$\mathscr{I}_n \times_{S\times_{\mathrm{Spec}(\kappa)}S} (\mathrm{Spec}(\mathscr{O}_{S,s}) \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathscr{O}_{S,s}))$$

is faithfully flat for every $s \in S$.

Because $\mathscr{I}_n$ is of finite presentation over S_1, there exists an open neighborhood U_1 of the point $(s, s) \in S_1$ such that the structural morphism $\pi_n : \mathscr{I}_n \to S_1$ of $\mathscr{I}_n$ is faithfully flat above (s, s). Since U_1 contains an open subset of the form $U \times_{\mathrm{Spec}(\kappa):} U$, where U is an affine open neighborhood of S. We have seen that with n given, for each point $s \in S$, there exists an open neighborhood U of s such that U satisfies the property that

$$(\dagger) \qquad \mathscr{I}_n \times_{S_1} (U \times_\kappa U) \to U \times_\kappa U \quad \text{is faithfully flat.}$$

By Zorn's lemme, there exists an open subset V of S which is maximal with respect to the property ($\dagger$). The argument in 3.8 shows that if V_1 and V_2 are two open subsets of S both satisfying ($\dagger$), and $V_1 \cap V_2 \neq \emptyset$, then $V_1 \cup V_2$ also satisfies the property ($\dagger$). Therefore V is closed. Because S is connected, we conclude that $V = S$. We have proved that $\mathscr{I}_n$ is faithfully flat over $S \times_{\mathrm{Spec}(\kappa)} S$ for every positive integer n. □

3.11. Remark. In the first sentence of the second paragraph of the proof of 3.10, we have used the following fact.

> *Let A be a commutative ring and let B be an A-algebra of finite presentation over A. Suppose that $\mathfrak{q}$ is a prime ideal of B such that $B_\mathfrak{q}$ is flat over $\mathfrak{p} := \mathfrak{q} \cap A$. Then there exists an open neighborhood $U \subset \mathrm{Spec}(B)$ of $\mathfrak{q}$ such that $B_{\mathfrak{q}'}$ is flat over $A_{\mathfrak{q}'\cap A}$ for every point $\mathfrak{q}' \in U$ and the image of U in $\mathrm{Spec}(A)$ contains an open neighborhood of $\mathfrak{p}$.*

By EGA $\mathrm{IV}_{\mathrm{III}}$ 11.2.6.1 and 11.2.6.2, there exists a commutative $\mathbb{Z}$-algebra A_0 which is finitely generated over $\mathbb{Z}$, an A_0-algebra B_0 of finite type, a ring homorphism $h : A_0 \to A$ and an isomorphism $f : B_0 \otimes_{A_0,h} A \xrightarrow{\sim} B$, such that $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ is flat at the image in $\mathrm{Spec}(B_0)$ of the point $\mathfrak{q} \in \mathrm{Spec}(B)$. The quoted fact follows from the special case when A is Noetherian; see e.g. [14] §22, Thm. 53 and Remark 2 after Thm. 53.

3.12. Lemma. *Let $\kappa \supset \mathbb{F}_p$ be a field. Let E be an extension field of κ. Let S be a scheme over E. Let $X \to S$ be a p-divisible group.*

(1) *If $X \to S$ is κ-sustained, then it is E-sustained.*
(2) *If $X \to S$ is strongly κ-sustained modeled on a p-divisible group X_0 over κ, then it is strongly E-sustained modeled on $X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(E)$.*
(3) *Suppose that $X \to S$ is strongly E-sustained modeled on a p-divisible group X_0 over E and X_0 is κ-sustained. Then $X \to S$ is κ-sustained.*

Proof. Let to the reader as an exercise. □

3.13. Remark. It is tempting to try to generalize the notion of sustained p-divisible group, so that the base field κ in the definition of sustained p-divisible groups is replaced by a general base scheme over $\mathbb{F}_p$, including the spectrum of an artinian local ring over $\mathbb{F}_p$. If one succeeds in finding such a generalization with good properties, one will have a good notion of *a family of leaves* of p-divisible group. A naive first attmpt is to replace the base field κ in 3.1 and 1.3 by a general scheme T in characteristic p, and replace the p-divisible group X_0 over κ in 3.1 by a p-divisible group X_1 over T. However we have not been successful in establishing enough properties for these more general definitions.

The following Lemma 3.14 is an immediate corollary of 2.22 and the definition of strongly sustained p-divisible groups.

3.14. Lemma. *Let κ be a field of characteristic p. Let $R = \kappa \oplus \mathfrak{m}$ be an Artinian local κ-algebra. Let X, Y be p-divisible groups over κ. Let E be an extension of X_R by Y_R whose closed fiber is $X \times Y$. Let $[E]$ be the element of $\mathcal{E}xt_{\mathrm{def}}(X,Y)(R)$ corresponding to the deformation$E$ of $X \times Y$. Let h be the element of $\mathscr{H}\!om'(X,Y)(R)$ corresponding to $[E]$ under the isomorphism*

$$\delta : \mathscr{H}\!om'(X,Y)^{\wedge} \xrightarrow{\sim} \mathcal{E}xt_{\mathrm{def}}(X,Y)$$

in 2.20. *The p-divisible group E over R is strongly κ-sustained modeled on $X \times Y$ if and only if the homomorphism $h : \mathrm{Spec}(R) \to \mathscr{H}\!om'(X,Y)$ factors through the maximal p-divisible subgroup $\mathscr{H}\!om'_{\mathrm{div}}(X,Y)$ of $\mathscr{H}\!om'(X,Y)$.*

4. Sustained p-divisible groups over fields

In the first part of this subsection will explain a stability result 4.3 for homomorphisms between $[p^n]$-torsion subgroups of two p-divisible groups for $n \gg 0$ over an algebraically closed field, and the related finiteness properties for p-divisible groups. Then we will use these results to explain the concept of sustained p-divisible groups when the base scheme is the spectrum of a field.

The essence of 4.3 is the finiteness results in Manin's famous thesis [13], and the statement of 4.3 is implicit in [13]. The proof of 4.3 below paraphrases part of the content of a letter from Thomas Zink to C.-L. Chai dated May 1st, 1999.

???Perhaps remove this sentence????? Let ξ be a Newton polygon with slopes in $[0,1]$, fixed throughout 3.

4.1. Lemma. *Let Y_0 be a p-divisible group over a field $\kappa \supset \mathbb{F}_p$. There exists a function $g_{Y_0} : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ satisfying the following properties.*

(i) *$g_{Y_0}(n) \geq n$ for all $n \in \mathbb{N}_{>0}$.*

(ii) *For every $n \in \mathbb{N}_{>0}$, every algebraically closed extension field k of κ, every endomorphism of $Y[p^n]_k$ which lifts to an endomorphism of $Y\big[p^{g_{Y_0}(n)}\big]_k$ can be extended to an endomorphism of Y_k. Here $Y[p^n]_k$ is short for $Y[p^n] \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\,k$; similarly for $Y\big[p^{g_{Y_0}(n)}\big]_k$ and Y_k.*

Proof. For each $n \in \mathbb{N}_{>0}$, consider the stabilized endomorphism group scheme $G_n(Y,Y)$ in the notation of 2.8. It has been shown in 2.9 XXXXXcheck thisXXX that $G_n(Y,Y)$ is the image of the natural restriction map $r_{n,N} : \mathscr{E}\mathcal{ND}(Y\big[p^N\big]) \to \mathscr{E}\mathcal{ND}(Y[p^n])$ for all N sufficiently large, and that the natural homomorphisms induced by the restriction maps $r_{n,n+i} : \mathscr{E}\mathcal{ND}(Y[p^{n+i}]) \to \mathscr{E}\mathcal{ND}(Y[p^n])$ $\pi_{n,n+i} : G_{n+i}(X,X) \to G_n(X,X)$ are fully faithful for all $i \in \mathbb{N}$.

For every positive integer n, define a map $g_{_{Y_0}} : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ by

$$g_{_{Y_0}}(n) := \min\left\{N \in \mathbb{N} \,|\, N \geq n,\ G_n(Y,Y) = \mathrm{Im}\left(r_{n,N} : \mathscr{E}\mathcal{ND}(Y\big[p^N\big]) \to \mathscr{E}\mathcal{ND}(Y[p^n])\right\}.$$

Let k be a extension field of κ. From the definition of $g_{_{Y_0}}$ we know that

$$G_n(X,X)(k) = \mathrm{Im}\left(r_{n,g_{_{Y_0}}(n)} : \mathrm{End}(Y\big[p^{g_{Y_0}(n)}\big]_k) \to \mathrm{End}(Y[p^n]_k)\right) \quad \forall\, n \in \mathbb{N}_{>0}.$$

The fact that $\pi_{n,n+1} : G_{n+1}(X,X) \to G_n(X,X)$ is faithfully flat for all n implies that for every algebraically closed extension field k of κ, the homomorphism $\pi_{n,n+1} : G_{n+1}(X,X)(k) \to G_n(X,X)$ is surjective for all $i \in \mathbb{N}$. Hence every element of $G_n(X,X)(k)$ extends to an element of $\varprojlim_{i\in\mathbb{N}} G_{n+i}(X,X)(k)$. The last projective limit is equal to (or canonically identified with) $\mathrm{End}(Y_k)$. $\square$

4.1.1. In the rest of 3 Y_0 denotes a a *minimal* p-divisible group over $\overline{\mathbb{F}_p}$ with Newton polygon ξ in the sense of [22].

Recall that the condition that Y_0 is minimal is equivalent to the condition that $\mathrm{End}(Y_0)$ is a maximal order of $\mathrm{End}(Y_0) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is a fact that any two minimal p-divisible groups over the an algebraically closed field $k \supset$ with the same Newton polygon ξ are isomorphic.

We remark that the only reason we picked the minimal p-divisible group Y_0 is that it is a canonical choice of a p-divisible group with Newton polygon ξ. Any other choice will also work for the proofs below. In particular will not need other properties of minimal p-divisible groups for the proof of Prop. 4.3.

The statement 4.2 below is a fundamental finiteness result for p-divisible group.

4.2. Proposition. *For every Newton polygon ξ and every prime number p, there exists a positive integer $c = c(p,\xi) > 0$ such that for every p-disible group Y over an algebraically closed field of characteristic p whose Newton polygon is ξ, there exits an isogeny $\alpha : Y \to Y_0$ with degree at most p^c.*

Proof. This finiteness property is a consequence of Theorems 3.4 and 3.5 of [13], stated below.

(A) *There exists a positive integer $D = D(\xi,p)$ such that over any algebraically closed field k of characteristic p, the number of isomorphism classes of* special *p-divisible groups over k with Newton polygon ξ is equal to D.*

Recall that a p-divisible group Y over an algebraically closed field $k \supset \mathbb{F}_p$ is *special* if Y can be decomposed as a product $Y \cong Y_1 \times \cdots \times Y_m$ of isotypic compoents $Y_1,\ldots,Y_m$ with distinct slopes $\lambda_1,\ldots,\lambda_m$ such that for each isoclinic component Y_i, $i = 1,\ldots,m$, we have

$$\mathrm{Ker}\big(\mathrm{Fr}^{s_i}_{Y_i/k}\big) = [p^{r_i}]_{Y_i},$$

where $\lambda_i = \frac{r_i}{s_i}$, $r_i, s_i \in \mathbb{N}, \gcd(r_i, s_i) = 1$, and $\mathrm{Fr}^{s_i}_{Y_i/k}$ is the s_i-th iterate of the relative Frobenius of Y_i. It is known that every special p-divisible group can be defined over $\overline{\mathbb{F}_p}$. Since homomorphisms between two p-divisible groups over algebraically closed fields of characteristic $p > 0$ stay unchanged under extension of algebraically closed base fields, the set of isomorphism classes of special p-divisible groups with Newton polygon ξ over an algebraically closed field k is independent of the base field k.

(B) *There exists a positive integer $c(p, \xi)$ such that for any algebraically closed field $k \supset \mathbb{F}_p$ and any p-divisible group Y over k, there exists a special p-divisible group Y' over k and an isogeny $Y \to Y'$ of degree at most $p^{c(p,\xi)}$.*

See also [24, Prop. 2.18] for the proof of another formulation of these two finiteness statements. □

4.3. Proposition. *For each Newton polygon ξ and each prime number p there exists a function $f = f_{p,\xi} : \mathbb{N} \to \mathbb{N}$, which has the following properties.*

(i) *$f(n) \geq n$ for all $n \in \mathbb{N}$.*

(ii) *For any $n \in \mathbb{N}$, any two p-divisible group Y_1, Y_2 over an algebraically closed field $k \supset \mathbb{F}_p$ with Newton polygon ξ and any homomorphism $\alpha_n : Y_1[p^n] \to Y_2[p^n]$ between the truncated BT-groups $Y_1[p^n]$ and $Y_2[p^n]$, if α_n can be extended to a homomorphism from $Y_1[p^{f(n)}] \to Y_2[p^{f(n)}]$ the α_n can be extended to a homomorphism from Y_1 to Y_2.*

PROOF. We will prove the following slightly more precise version of 4.3.

> *For any algebraically closed field $k \supset \mathbb{F}_p$, any two p-divisible group Y_1, Y_2 over k with Newton polygon ξ and for any homorphism $\alpha_n : Y_1[p^n] \to Y_2[p^n]$, if α_n admits an extend to an endomorphism of $\alpha' : Y_1[p^{g(n+2c)}] \to Y_2[p^{g(n+2c)}]$, then α_n can be extended to an homomorphism from Y_1 to Y_2. Here $g = g_{Y_0}$ as in* 4.1 and $c = c(p, \xi)$ as in 4.2.

From 4.2 we know that there exist isogenies

$$u_1 : Y_1 \to (Y_0)_k, \quad v_1 : (Y_0)_k \to Y_1, \quad u_2 : Y_2 \to (Y_0)_k, \quad v_2 : (Y_0)_k \to Y_2$$

such that $v_i \circ u_i = [p^c]_{Y_i}$ and $u_i \circ v_i = [p^c]_{(Y_0)_k}$ for $i = 1, 2$. Consider the endomorphism β' of $Y_0[p^{n+2c}]$ induced by the endomorphism

$$\left(u_2|_{Y_1[p^{g(n+2c)}]}\right) \circ \alpha' \circ \left(v_1|_{Y_1[p^{g(n+2c)}]}\right)$$

of $Y_0[p^{g(n+2c)}]_k$. From the basic property of the function g_{Y_0} we know that there exists an endomorphism $\tilde{\delta}$ of $(Y_0)_k$ whose restriction to $Y_0[p^{n+2c}]_k$ is equal to β'. Since the restriction to $Y[p^{2c}]$ of $v_2 \circ \tilde{\delta} \circ u_1$ is 0, there exists a unique homomorphism $\tilde{\alpha} : Y_1 \to Y_2$ such that

$$v_2 \circ \tilde{\delta} \circ u_1 = p^{2c} \cdot \tilde{\alpha}.$$

Let $\tilde{\alpha}_{n+2c} : Y_1[p^{n+2c}] \to Y_2[p^{n+2c}]$ be the homomorphisms from $Y_1[p^{n+2c}]$ to $Y_2[p^{n+2c}]$ induced by $\tilde{\alpha}$, and let α'_{n+2c} be the homomorphism from $Y_1[p^{n+2c}]$ to $Y_2[p^{n+2c}]$ induced by α'. The last displayed equality implies that $p^{2c} \cdot \alpha'_{n+2c} = p^{2c} \cdot \tilde{\alpha}_{n+2c}$. Therefore the α_n is equal to the homomorphism from $Y_1[p^n]$ to $Y_2[p^n]$ induced by $\tilde{\alpha}$. □

4.4. Corollary. *For every positive integers n and any Newton polygon ξ, there exists a positive integer $M = M(p, \xi)$ which depends only on p and ξ, such that if Y_1, Y_2 are two p-divisible groups over an algebraically closed field $k \supset \mathbb{F}_p$ with the same Newton polygon ξ such that $Y_1[p^M] \cong Y_2[p^M]$, then $Y_1 \cong Y_2$.*

Proof. It suffices to take $M(p, \xi) = f_{p,\xi}(1)$. $\square$

4.5. Remark. How the constant $c(p, \xi)$ in 4.2 and the constant $M(p, \xi)$ in 4.4 depends on the prime number p is an important open question. More generally the behavior of $f_{p,\xi}(n)$ as a function in three variables p, ξ, n is almost completely unknown. For instance it will be useful and interesting if one can find explicit upper bounds for $c(p, \xi)$, $M(p, \xi)$ and $f_{p,\xi}(n)$.

4.6. Corollary. *Let $\kappa \supset \mathbb{F}_p$ be a field, let S be a κ-scheme. Let $X \to S$ be a κ-sustained p-divisible group. The following two equivalent statements hold.*

(i) *Let k be an algebraically closed extension field of κ. For any two κ-morphisms $\tau_1, \tau_2 : \operatorname{Spec} k \to S$, the two p-divisible groups $\tau_1^* X$ and $\tau_2^* X$ over k are isomorphic.*

(ii) *$X \to S$ is* geometrically fiberwise constant *in the following sense: for any two points s_1, s_2 of S, there exists an algebraically closed extension field k of κ and κ-morphisms $\tau_i : s_i \to \operatorname{Spec}(k)$, $i = 1, 2$, such that the p-divisible groups $X_{s_1} \times_{s_1, \tau_1} \operatorname{Spec}(k)$ and $X_{s_1} \times_{s_2, \tau_2} \operatorname{Spec}(k)$ over the same base field k are isomorphic.*

Proof. Both (i) and (ii) are immediate consequences of 4.3 and the definition of κ-sustained p-divisible groups. The equivalence of (i) and (ii) follows from the fact that for any extension $L \subset M$ of algebraically closed fields and any two p-divisible groups Y, Z over L, the canonical map $\operatorname{Hom}_L(Y, Z) \to \operatorname{Hom}_M(Y_M, Z_M)$ is a bijection. $\square$

4.7. Corollary. *Let $X \to S$ be a strongly κ-sustained p-divisible group modeled on a p-divisible group Y_0 over κ, and let Y_1 be a p-divisible group over κ. Then $X \to S$ is strongly κ-sustained modeled on Y_1 if and only if there exists an extension field L of κ such that $Y_0 \times_{\operatorname{Spec} \kappa} \operatorname{Spec} L \cong Y_1 \times_{\operatorname{Spec} \kappa} \operatorname{Spec} L$.*

Proof. It follows immediate from the definition of sustained p-divisible groups that $X \to S$ is strongly κ-sustained modeled on Y_1 if and only if for every $n \in \mathbb{N}_{>0}$, the $\mathscr{I}\mathit{som}$ scheme $\mathscr{I}\mathit{som}_\kappa(Y_1[p^n], Y_0[p^n]$ is faithfully flat over κ. The last condition is equivalent to:

> *for every $n \in \mathbb{N}_{>0}$ there exists an algebraically closed extension field L_n of κ and an isomorphism between $Y_1[p^n]_{L_n}$ and $Y_0[p^n]_{L_n}$.*

From ??? to ??? K is an extension field of the base field κ.

4.8. Proposition. *Let K be an extension field of κ and let K^{alg} be an algebraic closure of K. Let X_0 be a p-divisible group over κ. A p-divisible group X over K is strongly κ-sustained modeled on X_0 if and only if $X \times_{\operatorname{Spec} K} \operatorname{Spec}(K^{\mathrm{alg}})$ is isomorphic to $X_0 \times_{\operatorname{Spec} \kappa} \operatorname{Spec}(K^{\mathrm{alg}})$.*

Proof. The "if" part follows from 3.4 (2); the "only if" part is a corollary of 4.4.

4.9. Corollary. *Let K be an extension field of κ and let K^{alg} be an algebraic closure of K. A p-divisible group X over K is strongly κ-sustained if and only if there exists a p-divisible group X_0 over κ such that $X \times_{\operatorname{Spec} K} \operatorname{Spec}(K^{\mathrm{alg}})$ is isomorphic to $X_0 \times_{\operatorname{Spec} \kappa} \operatorname{Spec}(K^{\mathrm{alg}})$.*

4.10. Remark. Let E/F be an extension of fields. For an algebraic-geometric object Z over E, the casual expression "Z has a model over F" and the equivalent expression "Z can be defined over F" has two possible meanings:

> (**stronger**) There exists an algebraic geometric object W over F of the same kind as Z such that there is an E-isomorphism between Z base change of W to E.
>
> (**weaker**) There exists an algebraic geometric object W over F of the same kind as Z such that there is an E^{alg}-isomorphism between the base change of Z to E^{alg} and the base change of W to E^{alg}.

Corollary 4.9 says that the a p-divisible group X over an extension field K of κ is strongly κ-sustained if and only if X can be defined over κ in the weaker sense above.

4.11. Lemma. *Suppose that K is a separable algebraic extension of κ. Let K^{alg} be an algebraic closure of K. A p-divisible group X over K is κ-sustained if and only if*

$$X \times_{\operatorname{Spec} K,\tau_1} \operatorname{Spec}(K^{\mathrm{alg}}) \xrightarrow{\sim} X \times_{\operatorname{Spec} K,\tau_1} \operatorname{Spec}(K^{\mathrm{alg}})$$

for all κ-linear ring homomorphisms $\tau_1, \tau_2 : K \to K^{\mathrm{alg}}$.

Proof. The "only if" part is a special case of Corollary 4.6.

For the "if" part, According to defintion 1.3, we need to show that for every positive integer n, the structural morphism

$$\pi_n \colon \mathscr{I}\mathcal{SOM}_{K\otimes_\kappa K}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]) \longrightarrow \operatorname{Spec}(K \otimes_\kappa K)$$

is faithfully flat. Because the morphism π_n is of finite presentation, there exists a finite subextension E/κ of K/κ and a commutative finite group scheme Y_n over E such that $Y_n \times_{\operatorname{Spec}(E)} \operatorname{Spec}(K) \cong X[p^n]$. The assumption that

$$X \times_{\operatorname{Spec} K,\tau_1} \operatorname{Spec}(K^{\mathrm{alg}}) \cong X \times_{\operatorname{Spec} K,\tau_1} \operatorname{Spec}(K^{\mathrm{alg}})$$

for all κ-linear ring homomorphisms $\tau_1, \tau_2 : K \to K^{\mathrm{alg}}$ implies that the structural morphism

$$\pi_{Y_n} : \mathscr{I}\mathcal{SOM}_{E\otimes_\kappa E}(\mathrm{pr}_1^* Y_n, \mathrm{pr}_2^* Y_n) \longrightarrow \operatorname{Spec}(E \otimes_\kappa E)$$

of $\mathscr{I}\mathcal{SOM}_{E\otimes_\kappa E}(\mathrm{pr}_1 Y_n, \mathrm{pr}_2 Y_n)$ is surjective, because its base change to $K^{\mathrm{alg}} \otimes_\kappa K^{\mathrm{alg}}$ is. On the other hand π_{Y_n} is flat because $E \otimes_\kappa E$ is a finite étale extension of a field. We conclude that π_{Y_n} is faithfully flat, which implies that its base change

$$\mathscr{I}\mathcal{SOM}_{K\otimes_\kappa K}(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]) \longrightarrow \operatorname{Spec}(K \otimes_\kappa K)$$

to $K \otimes_\kappa K$ is faithfully flat. $\square$

4.12. Proposition. *Let K be an extension field of κ. Let X be a κ-susted p-divisible group over K. There exists a finite extension field E of κ and a κ-linear embedding of $\tau : E \to K^{\mathrm{alg}}$ such that the base change X_L of X to the composit field $L := K\cdot\tau(E)$ admits an E-model.*

Proof. We may and do assume that K is algebraically closed. Proposition 4.12 is a consequence of the main results of [13, §3]. Theorem 3.4 of [13] tells us that there exists a *special* p-divisible group Y_0 over a finite field κ' and an isogeny $u_0 : (Y_0)_K \to X$ such that for every isogeny $u : Y \to X$ with Y special, there exists a unique isomorphism $\beta : Y \to Y_0$ such that $u_0 \circ \beta = u$. Replacing κ by the compositum of κ with the finite field κ', we may and do assume that $\kappa' \subseteq \kappa$. Let p^c be the degree of the isogeny u_0.

Since X is κ-sustained, we know that for every $\mathbb{F}$-linear automorphism τ of K, the twist ${}^\tau X$ of X by τ is isomorphic to X. On the other hand, there is an isogeny ${}^\tau u : (Y_0)_K \to {}^\tau X$ with the same university property as u_0 above. So there exists an automorphism $\alpha_\tau \in \mathrm{Aut}((Y_0)_K)$ and an isomorphism $\beta_\tau : X \to {}^\tau X$ such that $\beta_\tau \circ u_0 = {}^\tau u_0 \circ \alpha_\tau$. Consequently

$$\alpha_\tau(\mathrm{Ker}(u_0)) = \mathrm{Ker}({}^\tau u_0) \quad \forall \tau \in \mathrm{Aut}(K/\kappa).$$

On the other hand we know that the image of $\mathrm{Aut}((Y_0)_K)$ in $\mathrm{Aut}((Y_0)[p^c]_K$ is a finite group. so $\mathrm{Ker}(u_0)$ must be (the base field extension of) a E-rational subgroup scheme of $Y_0[p^c]_E$ for some finite extension field E of κ. Therefore X admits an E-model. □

4.13. Lemma. *Let X be an κ-sustained p-divisible group over K. Suppose that X is isoclinic and K/k is algebraic and purely inseparable, then there exists a p-divisible group X_0 over κ and an isomorphism $(X_0)_K \xrightarrow{\sim} X$. In particular X is strongly κ-sustained.*

4.14. Example. In this example $\kappa = \mathbb{F}_{p^2}$. Recall that $\overline{\mathbb{F}_p}$ is an algebraic closure of κ. We will construct a κ-sustained p-divisible group over finite extension field of $\mathbb{F}_p$ which does not have a κ-model.

Let Y_1 be the p-divisible group attached to a supersingular elliptic curve over κ. Let $X_1 := Y_1 \times_{\mathrm{Spec}\,\kappa} Y_1$. The following facts are well-known.

(1) The p-torsion subroup $X_1[p]$ of X_1 is isomorphic to α_p^2.
(2) Over the base field κ, the scheme of subgroups of $X_1[p]$ isomorphic to α_p is naturally isomorphic to $\mathbb{P}^1$ over κ.
(3) Let s be a closed point of $\mathbb{P}^1_\kappa$, let $\kappa(s)$ be the residue field of s and let X_s be the corresponding α_p-quotient of X_1 over $\kappa(s)$. Suppose that there exists p-divisible group Z over a subextension F/κ of $\overline{\mathbb{F}_p}/\kappa$ such that $X_s \times_{\mathrm{Spec}\,\kappa(s)} \mathrm{Spec}\,\overline{\mathbb{F}_p} \cong Z \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,\overline{\mathbb{F}_p}$, then $E \subseteq F$. In short: $\kappa(s)$ is the smallest field of definition of X_s (when we work over the base field $\kappa = \mathbb{F}_{p^2}$).
(4) For two closed points s_1, s_2 of $\mathbb{P}^1_\kappa$, the base extension to $\overline{\mathbb{F}_p}$ of their corresponding α_p-quotients X_{s_1} and X_{s_2} are isomorhic if and only if s_1 and s_2 lie in the same $\mathrm{PGL}_2(\mathbb{F}_{p^2})$-quotient. Here PGL_2 operates on $\mathbb{P}^1$ via linear fractional transformations as usual.

    The reason that $\mathrm{PGL}_2(\mathbb{F}_{p^2})$ appears here is that $\mathrm{End}(Y_1)$ operates on $\mathrm{Lie}(Y_1)$ via a surjective ring homomorphism $h : \mathrm{End}(Y_1) \twoheadrightarrow \mathbb{F}_{p^2}$, hence $\mathrm{Aut}(X_1)$ operates on $\mathrm{Lie}(X_1)$ via the surjective group homomorphism $\mathrm{Aut}(X_1) \to \mathrm{GL}_2(\mathbb{F}_{p^2}$ attached to h.

For any closed point s of $\mathbb{P}^1_\kappa$, the corresponding α_p-quotient $X_s \to \operatorname{Spec}\kappa(s)$ is κ-sustained if and only if for every element $\tau \in \operatorname{Gal}(\kappa(s)/\kappa)$, the τ-twist ${}^\tau X_s$ of X_s is isomorphic to X_s after base changed to $\overline{\mathbb{F}_p}$. In view of the fact (3) above, the condition for X_s to be κ-sustained is that $\tau \cdot s \in \mathrm{PGL}_2(\mathbb{F}_{p^2}) \cdot s$ for all $\tau \in \operatorname{Gal}(\kappa(s)/\kappa$. Since $\kappa(s)$ is generated by the Frobenius element $\mathrm{Fr}_{\kappa(s)/\kappa}$, the last condition is equivalent to ${}^{\mathrm{Fr}_{\kappa(s)/\kappa}} \cdot s \in \mathrm{PGL}_2(\mathbb{F}_{p^2}) \cdot s$.

Here is a specific example. Let $a \in \mathbb{A}^1(\overline{\mathbb{F}_p}) \subset \mathbb{P}^1(\overline{\mathbb{F}_p})$ be a solution of the equation

$$t^{p^2} - t - 1 = 0.$$

The α_p-quotient X_a of X_1 attached to the closed point a of $\mathbb{P}^1_\kappa$ is a κ-sustained p-divisible group over $\mathbb{F}_{p^2}$. Clearly $a \notin \mathbb{A}^1(\kappa)$, which implies that X_a does not have a κ-model according to (3).

5. Slope filtrations and stabilized Isom groups

In this subsection we show that on every κ-sustained p-divisible group there exists a natural slope filtration; see 1.6. The notion of stabilized $\mathscr{I}\!\mathit{som}$ group schemes introduced is also useful for proving a differential criterion 5.9 for a p-divisible group to be sustained.

5.1. Definition. Let S be a scheme over $\mathbb{F}_p$. A p-divisible group $X \to S$ is *isoclinic* of slope $\lambda \in [0,1] \cap \mathbb{Q}$ if there exists positive integers m_0, c such that

$$X[p^{\lfloor \lambda n \rfloor - c}] \subset \operatorname{Ker}\left(F^{(p^n)}_{X/S}\right) \subset X[p^{\lceil \lambda n \rceil + c}]$$

for all $n \geq m_0$.

5.1.1. Remark. There is a related weaker notion, which may be termed *weakly isoclinic p-divisible groups*, namely all fibers are isoclinic of the same slope. We will not use this notion, because every sustained p-divisible group admits a natural slope filtration whose associated graded pieces are isoclinic in the sense of 5.1.

5.2. Definition. Let S be a scheme over $\mathbb{F}_p$. A *slope filtration* on a p-divisible group $X \to S$ is a increasing filtration of X by p-divisible subgroups such that the associated graded pieces are isoclinic p-divisible groups whose slopes are strictly decreasing. More precisely, over each connected component U of S, suppose that the filtration on X is

$$X = X_0 \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_m = 0,$$

where each X_i is a p-divisible subgroup of X. The requirement is that X_i/X_{i+1} is a strongly isoclinic p-divisible group for $i = 0, 1, 2, \ldots, m-1$, and

$$\mathrm{slope}(X_0/X_1) < \mathrm{slope}(X_1/X_2) < \cdots < \mathrm{slope}(X_{m-1}).$$

It is proved in [27, Cor. 13] that every p-divisible group over a field $K \supset \mathbb{F}_p$ admits a slope filtration.

5.3. Lemma. *Let $X \to S$ and $Y \to S$ be two isoclinic p-divisible groups over a base scheme S over $\mathbb{F}_p$, with slopes λ_X and λ_Y respectively. If $\lambda_X > \lambda_Y$, then* $\operatorname{Hom}_S(X, Y) = (0)$.

Proof. This is a consequence of the more precise statement 5.4

5.4. Lemma. *Let $X \to S$ and $Y \to S$ be two isoclinic p-divisible groups over a base scheme S over $\mathbb{F}_p$, with slopes λ and μ respectively. Let c, d be natural numbers such that*

$$X[p^{\lfloor m/\lambda \rfloor - c}] \subseteq \mathrm{Ker}(\mathrm{Fr}^m_{X/S}) \subseteq X[p^{\lceil m/\lambda \rceil + c}]$$

$$Y[p^{\lfloor m/\lambda \rfloor - d}] \subseteq \mathrm{Ker}(\mathrm{Fr}^m_{Y/S}) \subseteq Y[p^{\lceil m/\mu \rceil + d}]$$

for every positive integer m. Assume that that $\lambda > \mu$. Let n be a positive integer. Let $h_n : X[p^n] \to Y^{\lceil}p^n]$ be an S-homomorphism. Suppose that there exists

- *a positive integer N such that*
$$N \cdot (1 - \tfrac{\mu}{\lambda}) \geq n + d + \mu(\tfrac{c}{\lambda} + 1),$$
- *a faithfully flat morphism $f : T \to S$, and*
- *a T-homomorphism $h_N : X[p^N] \times_S T \to Y[p^N]$*

such that $f^* h_n = h_N$. Then $h_n = 0$.

Proof. It suffices to show that $f^*h_n = 0$, so we may and do assume that $T = S$ and $f = \mathrm{id}_S$. The relation $X[p^{\lfloor m/\lambda \rfloor - c}] \subseteq \mathrm{Ker}(\mathrm{Fr}^m_{X/S}) \subseteq X[p^{\lceil m/\lambda \rceil + c}]$ implies that

$$\mathrm{Ker}\left(\mathrm{Fr}^{\lfloor (r-c-1)/\lambda \rfloor}_{X/S}\right) \subseteq X[p^r] \subseteq \mathrm{Ker}\left(\mathrm{Fr}^{\lceil (r+c)/\lambda \rceil}_{X/S}\right)$$

for all positive integers r. From $X[p^N] \subset \mathrm{Fr}^{\lfloor (N+c)/\lambda \rfloor}_{X/S}$, we see that

$$h_N(X[p^N]) \subseteq \mathrm{Ker}\left(\mathrm{Fr}^{\lceil (N+c)/\lambda \rceil}_{Y/S}\right) \subseteq Y[p^{\lceil (N+c)/\lambda \rceil \cdot \mu + d}).$$

The assumption that $N \cdot (1 - \frac{\mu}{\lambda}) \geq n + d + \mu(\frac{c}{\lambda} + 1$ implies that

$$h_N(X[p^N]) \subseteq Y[p^{\lceil (N+c)/\lambda \rceil \cdot \mu + d}) \subseteq Y[p^{N-n}],$$

therefore the composition

$$X[p^N] \xrightarrow{[p^{N-n}]_X} X[p^n] \xrightarrow{h_n} Y[p^n],$$

is equal to 0 because it is equal to the composition

$$X[p^N] \xrightarrow{h_N} Y[p^N] \xrightarrow{[p^{N-n}]_Y} Y[p^n].$$

Therefore $h_n = 0$ because $[p^{N-n}]_X : X[p^N] \to X[p^n]$ is faithfully flat. $\square$

5.5. Corollary. *Let S be an $\mathbb{F}_p$-scheme. A p-divisible group $X \to S$ admits at most one slope filtration.*

Proof. Let $X = X_0 \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_m = 0$ and $X = X'_0 \supsetneqq X'_1 \supsetneqq X'_2 \supsetneqq \cdots \supsetneqq X'_m = 0$ be two slope filtrations of X. Note that the different slopes $\lambda_1, \ldots, \lambda_m$ and their multiplicities for either in filtration are the same, for they are determined by the fiber X_s of any point $s \in S$.

Apply 5.3 to the homomorphism $X_{m-1} \hookrightarrow X \to X/X'_1$, we see that $X_{m-1} \subseteq X'_1$. Apply 5.3 to the homomorphism $X_{m-1} \hookrightarrow X'_1 \to X'_1/X'_2$, we see that $X_{m-1} \subseteq X'_2$. Apply the same argument successively, we see that $X_{m-1} \subseteq X'_{m-1}$. By symmetry we also have $X'_{m-1} \subseteq X_{m-1}$, so $X_{m-1} = X'_{m-}$. Now we have two slope filtrations on the p-divisible group X/X_{m-1}. An induction on the length of the slope filtrations finishes the proof. $\square$

5.5.1. Remark. The statement of 5.3 becomes false for general (possibly non-Noetherian) base schemes S in characteristic p if the assumption that $\lambda_X > \lambda_Y$ is replaced by the assumption that $\lambda_X < \lambda_Y$. Here is an example. Let S be the spectrum of the quotient ring

$$R = \mathbb{F}_p[x_1, x_2, \ldots]/(x_1^p, x_2^p - x_1, x_3^p - x_2, \ldots)$$

of the polynomial ring over $\mathbb{F}_p$ with variables x_i, $i \in \mathbb{N}$, by the ideal I generated by the elements $x_i^p - x_{i-1}$ with $i \in \mathbb{N}$, where $x_{-1} = 0$ by convention. Let X be the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ over S, and let Y be $\mathbb{G}_m[p^\infty]$ over S. Let

$$h_n : (p^{-n}\mathbb{Z})/\mathbb{Z})_S \to \mathbb{G}_m[p^n]_S$$

be the S-homomorphism which sends the generator $p^{-n} \bmod \mathbb{Z}$ of $X[p^n]$ to the element of $\mu_{p^n}(R)$ corresponding to the ring homomorphism

$$\alpha_n : \mathbb{F}_p[T]/\left((1+T)^{p^n} - 1\right) \longrightarrow R, \quad T \mapsto 1 + x_n \bmod I.$$

The family $\{h_n | n \in \mathbb{N}_{>0}\}$ of S-homomorphisms defines a non-zero homomrphism from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\mathbb{G}_m[p^\infty]$ over S.

5.6. Definition. Let S be a κ-scheme. Let $X \to S$ and $Y \to S$ be κ-sustained p-divisible groups over S.

(1) For any positive integer n, define a sheaf $\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X,Y)_n$ on S_{fppf} by

$$\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X,Y)_n := \mathrm{Image}\left(\mathscr{I}\mathit{som}_S(X[p^N], Y[p^N]) \to \mathscr{I}\mathit{som}_S(X[p^n], Y[p^n])\right)$$

for $N \gg n$. Here $N \gg n$ means that $N \geq n + C$ for a constant C depending only on X, Y.

The meaning of the sheaf-theoretic image in the above displayed formula is as follows. For any κ-scheme T and any κ-morphism $T \to S$, $\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X,Y)_n(Y)$ consists of all isomorphisms $h : X[p^n]\times_S T \to Y[p^n]\times_S T$ such that there exists an fppf morphism $f : T_1 \to T$ and a homomorphism $h_1 : X[p^N]\times_S T_1 \to Y[p^N]\times_S T_1$ over T_1 with the property that h_1 is equal to the pull-back of h by the fppf covering morphism $f : T_1 \to T$.

(2) In the case when $X = X_0 \times_{\mathrm{Spec}(\kappa)} S$, we abuse the notation and write $\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X_0,Y)_n$ for $\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X_0 \times_{\mathrm{Spec}(\kappa)} S, Y)_n$. When $Y = X$ we denote the sheaf $\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X,X)_n$ by $\mathscr{A}\mathit{ut}^{\mathrm{st}}_S(X)$.

5.6.1. Remark. (a) For any $n \geq 1$, the sheaf $\mathscr{I}\mathit{som}^{\mathrm{st}}_S(X,Y)_n(Y)$ is non-trivial if and only if for every (or equivalently, for some) $s \in S$, the p-divisible group $Y \to S$ is strongly $\kappa(s)$-sustained modeled on Y_0; in other words $X_s \times \bar{s}$ is isomorphic to $Y_s \times_s \bar{s}$ for every (or equivalently, for some) $s \in S$, where $\bar{s}$ is a geometric point above s. This is a consequence of 4.1.

(b) The fact that there exists a constant C such that the images of the group schemes $\mathscr{I}\mathit{som}_S(X[p^N], Y[p^N])$ and $\mathscr{I}\mathit{som}_S(X[p^M], Y[p^M])$ in

$$\mathscr{I}\mathit{som}_S(X[p^n], Y[p^n])$$

are equal for all $M, N \geq n+C$ follows from (a) and the stabilization results in 2.7and 2.8.

(c) It follows from 2.7 that $\mathscr{ISOM}_\kappa^{\mathrm{st}}(X_0)_n$ is representable by a finite group scheme over κ.

5.7. Lemma. *Let $\kappa \supset \mathbb{F}_p$ be a field, and let S be a κ-scheme. Let $X \to S$ and $Y \to S$ be κ-sustained p-divisible groups.*

(1) *Both $\mathscr{ISOM}_S^{\mathrm{st}}(X)_n$ and $\mathscr{ISOM}_S^{\mathrm{st}}(Y)_n$ are representable by a finite locally free group schemes over S.*

(2) *For every positive integer n, the S_{fppf}-sheaf $\mathscr{ISOM}_S^{\mathrm{st}}(X, Y)_n$ is either trivial or is representable by a finite locally free scheme over S. The natural right action of $\mathscr{ISOM}_S^{\mathrm{st}}(X)_n$ and the natural left action of $\mathscr{ISOM}_S^{\mathrm{st}}(Y)_n$ on $\mathscr{ISOM}_n^{\mathrm{st}}(X, Y)_n$, which commute with each other, give $\mathscr{ISOM}_n^{\mathrm{st}}(X, Y)_n$ compatible structures of a right $\mathscr{ISOM}_S^{\mathrm{st}}(X)_n$-torsor as well as a left torsor under $\mathscr{ISOM}_S^{\mathrm{st}}(Y)_n$.*

Proof. Let s be a point of S, and let $E := \kappa(s)$ be the residue at s, so $X_E := X \times_{\mathrm{Spec}(\kappa)} \to S \times_{\mathrm{Spec}(\kappa)} =: S_E$ is strongly κ-sustained modeled on X_{0E}. Over $\mathscr{I}_n := \mathscr{ISOM}_S(X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S, X[p^n])$, the sheaf $\mathscr{AUT}^{\mathrm{st}}(X)_n \times_S \mathscr{I}_n$ is representable by the $\mathscr{I}_n$-scheme $\mathscr{AUT}^{\mathrm{st}}(X_0) \times_{\mathrm{Spec}(\kappa)} \mathscr{I}_n$. By descent the sheaf $\mathscr{AUT}^{\mathrm{st}}(X)_n$ is representable by a scheme which is finite and locally free over S. Similarly for $\mathscr{AUT}^{\mathrm{st}}(Y)_n$. We have proved (1).

The sheaf $\mathscr{ISOM}_S^{\mathrm{st}}(X, Y)_n$ is trivial if $X_s \times_s \bar{s}$ is not isomorphic to $X_s \times_s \bar{s}$ over $\bar{s}$. Suppose that $X_s \times_s \bar{s}$ is isomorphic to $X_s \times_s \bar{s}$ over $\bar{s}$. Let

$$T_n := \mathscr{ISOM}_S(X_s[p^n] \times_s S, X[p^n]) \times_S \mathscr{ISOM}_S(X_s[p^n] \times_s S, X[p^n]).$$

Clearly T_n is faithfully flat and of finite presentation over S. As in the proof of of (1), for each positive integer n the sheaf $\mathscr{ISOM}_S^{\mathrm{st}}(X, Y)_n \times_S T_n$ is representable by a finite locally free scheme over T_n. By descent $\mathscr{ISOM}_S^{\mathrm{st}}(X, Y)_n$ is representable by a finite locally free sheaf over S.

5.8. Lemma. *Let $(R, \mathfrak{m})$ be a Noetherian local ring which contains a subfield $\kappa \supset \mathbb{F}_p$. Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of R, and let $K \subset R$ be a coefficient subfield of $\hat{R}$, which exists by Cohen's structure theorem for complete Noetherian local rings. Let X be a p-divisible group over R and let $X_0 = X \times_{\mathrm{Spec} R} \mathrm{Spec}(K)$ be the closed fiber of the p-divisible group X. The following statements are equivalent.*

(1) *X is κ-sustained.*

(2) *X is strongly K-sustained modeled on X_0 and the p-divisible group X_0 is κ-sustained.*

(3) $\hat{X} = X \times_{\mathrm{Spec} R} \mathrm{Spec}(\hat{R}) \to \mathrm{Spec}(\hat{R})$ *is κ-sustained.*

(4) $\hat{X} = X \times_{\mathrm{Spec} R} \mathrm{Spec}(\hat{R}) \to \mathrm{Spec}(\hat{R})$ *is strongly K-sustained modeld on X_0 and X_0 is κ-sustained.*

(5) $X_i := X \times_{\mathrm{Spec} R} \mathrm{Spec}(R/\mathfrak{m}^{i+1}) \to \mathrm{Spec}(R/\mathfrak{m}^{i+1}) =: S_i$ *is κ-sustained for every $i \in \mathbb{N}$.*

(6) *X_i is strongly K-sustained modeled on X_0 and X_0 is κ-sustained.*

Proof. The equivalences (1) $\iff$ (2), (3) $\iff$ (4) and (5) $\iff$ (6) follow from 3.12. Because $R \to \hat{R}$ is faithfully flat, we get from 3.4 (2). that (1) $\iff$ (3). The implication (4) $\implies$ (6) is immediate from 3.4 (1).

Assume that the statement (6) holds. The assumption implies that for every positive integer n and every natural number i, the stabilized $\mathscr{I}\!som$ sheaf

$$T_{n,S_i} := \mathscr{I}\!som^{\rm st}_{S_i}\left(X_0[p^n] \times_{\mathrm{Spec}(K)} S_i, X[p^n] \times_S S_i\right)$$

is representable by a faithfully flat finite locally free scheme over S_i. Clearly the natural morphism

$$T_{n,S_{i+1}} \times_{S_{i+1}} S_i \longrightarrow T_{n,S_i}$$

is an isomorphism for every $i \in \mathbb{N}$ and every $n \in \mathbb{N}_{>0}$. For a fixed n, the inductive system of the T_{n,S_i}'s is a formal scheme $\hat{T}_n$ over $\mathrm{Spf}(\hat{R}) =: \hat{S}$ which is finite over $\hat{S}$. By Grothendieck's algebraization theorem, there exists a finite locally free scheme T_n over R and an isomorphism α_n from $T_n \times_{\mathrm{Spec}(R)} \hat{S}$ to $\hat{T}_n$; this pair (T_n, α_n) is unique up to unique isomorphism. Moreover the tautological isomorphism over $\hat{T}_n$ from the formal scheme $X_0[p^n] \times_{\mathrm{Spec}(K)} \hat{T}_n$ to $X[p^n] \times_S \hat{T}_n$ algebraizes to an unique T_n-isomorphism from $X_0[p^n] \times_{\mathrm{Spec}(K)} T_n$ to $X[p^n] \times_S T_n$. So the $\mathscr{I}\!som$ scheme $\mathscr{I}\!som_S(X_0[p^n]_S, X[p^n])$ has a section over the finite locally free cover $T_n \to S$, for every $n > 0$. It follows that the p-divisible group $X \to S$ is strongly K-sustained modeled on X_0. We have proved that (6) $\implies$ (4). $\square$

5.9. Corollary. *Let S be a locally noetherian scheme over a field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a p-divisible group. Let X_0 be a p-divisible group over κ.*

(1) *If X is strongly κ-sustained modeled on X_0, then for every point $s \in S$ and every $i \in \mathbb{N}$, $X \times_S \mathrm{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^{i+1})$ is strongly κ-sustained modeled on X_0. Conversely if $X \times_S \mathrm{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^{i+1})$ is strongly κ-sustained modeled on X_0 for every $i \in \mathbb{N}$ and every closed point $s \in S$, then X is strongly κ-sustained modeled on X_0.*

(2) *If X is κ-sustained, then for every point $s \in S$ and every $i \in \mathbb{N}$, $X \times_S \mathrm{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^{i+1})$ is κ-sustained. Conversely if S is Noetherian connected and $X \times_S \mathrm{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^{i+1})$ is κ-sustained for every $i \in \mathbb{N}$ and every closed point $s \in S$. Then X is κ-sustained.*

Proof. The statement (1) is immediate from 5.8. The statement (2) follows from 5.8 and 3.10. $\square$

5.10. Corollary. *Let R be a commutative Noetherian local ring which contains a field $\kappa \supset \mathbb{F}_p$. Let I be an ideal of R such that $1 + I \subset R^\times$; equivalently every maximal ideal of R contains I. Let $\hat{R}$ be the I-adic completion of R. Let X be a p-divisible group over R.*

(1) *Let X_0 be a p-divisible group over κ. The following statements are equivalent.*
   (1a) *The p-divisible group X over R is strongly κ-sustained modeled on X_0.*
   (1b) *The p-divisible group $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\hat{R})$ over $\hat{R}$ is strongly κ-sustained modeled on X_0.*

(1c) *The p-divisible group $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/I^{i+1})$ over R/I^{i+1} is strongly κ-sustained modeled on X_0 for every $i \in \mathbb{N}$.*

(2) *The following statements are equivalent.*

(2a) *The p-divisible group X over R is κ-sustained.*

(2b) *The p-divisible group $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\hat{R})$ over $\hat{R}$ is κ-sustained.*

(2c) *The p-divisible group $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/I^{i+1})$ over R/I^{i+1} is κ-sustained for every $i \in \mathbb{N}$.*

PROOF. The statement (1) is a corollary of 5.9. The implications (2a) $\Longrightarrow$ (2b) $\Longrightarrow$ (2c) are clear. The implication (2c) $\Longrightarrow$ (2a) is an application of Grothendieck's existence theorem for formal schemes similar to the proof of 5.8. Details are left to the reader as an exercise. □

5.11. Lemma. *Let S be a κ-scheme and let $X \to S$ be a strongly κ-sustained* isoclinic *p-divisible group.*

(1) *For each positive integer n, the S_{fppf}-sheaf $\mathscr{I}\mathit{som}_S^{\mathrm{st}}(X_0, X)$ is representable by an scheme finite étale over S, which is a right torsor for the finite étale group scheme $\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0, X)_n \times_{\mathrm{Spec}(\kappa)} S$ over S.*

(2) *For any positive integers $m > n$, the natural map*

$$\mathscr{I}\mathit{som}_S^{\mathrm{st}}(X_0, X)_m \to \mathscr{I}\mathit{som}_S^{\mathrm{st}}(X_0, X)_m$$

is a faithfully flat morphism from an $(\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_m \times_{\mathrm{Spec}(\kappa)} S)$-torsor to torsor for $(\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_n \times_{\mathrm{Spec}(\kappa)} S)$, which is compatible with the epimorphism

$$\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_m \twoheadrightarrow \mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_n.$$

(3) *The natural map*

$$\mathscr{I}\mathit{som}_S^{\mathrm{st}}(X_0, X)_n \wedge^{\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_n \times_{\mathrm{Spec}(\kappa)} S} X_0[p^n]_S \longrightarrow X[p^n]$$

is an isomorphism for every positive integer n, where the left-hand side of the above displayed formula is the contraction product of the right torsor $\mathscr{I}\mathit{som}_S^{\mathrm{st}}(X_0, X)_n$ for $\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_n \times_{\mathrm{Spec}(\kappa)} S$ with the scheme $X_0[p^n]_S$ with left action by $\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_n \times_{\mathrm{Spec}(\kappa)} S$. In other words the BT_n-group $X[p^n]$ is isomorphic to the twist of the "constant" BT_n-group $X_0[p^n]_S$ by the étale torsor $\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0)_n \times_{\mathrm{Spec}(\kappa)} S$.

XXXX Basically has been proved. Perhaps add details later XXXX

5.12. Lemma. *Let E be an purely inseparable algebraic extension field of κ.*

(1) *Let Z be an* isoclinic *p-divisible group over E. If Z is κ-sustained, then there exists a p-divisible group Z_0 over κ, uniquely determined by Z, such that Z is isomorphic to the base extension of Z_0 to E.*

(2) *Let S be scheme over κ. Let $X \to S$ be a κ-sustained* isoclinic *p-divisible group over S. If $X \to S$ is strongly E-sustained, then it is strongly κ-sustained.*

PROOF. By 5.11, we know that for every positive integer n, the faithfully flat morphism

$$\mathscr{I}_n := \mathscr{I}\mathit{som}_{E \otimes_\kappa E}\left(\mathrm{pr}_1^* Z[p^n], \mathrm{pr}_2^* Z[p^n]\right) \longrightarrow \mathrm{Spec}(E \otimes_\kappa E)$$

is finite étale. The pull-back of $\mathscr{I}_n \to \mathrm{Spec}(E \otimes_\kappa E)$ by the diagonal morphism $\Delta_E : \mathrm{Spec}(E) \to \mathrm{Spec}(E \otimes_\kappa E)$ has natural section σ_n which corresponds to the identity map on $Z[p^n]$. Because the morphism Δ_E is radicial, σ_n extends uniquely to a section $\tilde{\sigma}_n$ of the finite étale cover $\mathscr{I}_n \to \mathrm{Spec}(E \otimes_\kappa E)$. Over $\mathrm{Spec}(E \otimes_\kappa E \otimes_\kappa E)$, there is a unique section of

$$\mathscr{I}\!\mathit{SOM}_{E\otimes_\kappa E\otimes_\kappa E}\left(\mathrm{pr}_1^* Z[p^n], \mathrm{pr}_3^* Z[p^n]\right) \longrightarrow \mathrm{Spec}(E \otimes_\kappa E \otimes_\kappa E)$$

whose pull-back by the diagonal morphism $\Delta_{E,3} : \mathrm{Spec}(E) \to \mathrm{Spec}(E \otimes_\kappa E \otimes_\kappa E)$ is corresponds to the identity map for $Z[p^n]$. Therefore σ_n satisfies the 1-cocycle condition for descent. We have proved (1).

Let Y be a p-divisible group over E which is an E-model of the strongly sustained p-divisible group $X \to S$. We know from the assumptions of (2) that for every positive integer n,

$$\mathscr{I}\!\mathit{SOM}_{S\times_{\mathrm{Spec}(\kappa)}S}\left(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]\right) \longrightarrow S \times_{\mathrm{Spec}(\kappa)} S$$

and

$$\mathscr{I}\!\mathit{SOM}_S(Y[p^n] \times_{\mathrm{Spec}(E)} S, X[p^n]) \longrightarrow S$$

are faithfully flat. Let $q_1, q_2 : \mathrm{Spec}(E \otimes_\kappa E) \to \mathrm{Spec}(E)$ be the two projections. Pulling back the last displayed map by the morphism $\mathrm{pr}_i : S \times_{\mathrm{Spec}(E)} S \to S$, we see that

$$\mathscr{I}\!\mathit{SOM}_{S\times_{\mathrm{Spec}(E)}S}\left(q_i^* Y[p^n] \times_{\mathrm{Spec}(E\otimes_\kappa E)} (S \times_{\mathrm{Spec}(E)} S), \mathrm{pr}_i^* X[p^n]\right) \longrightarrow S \times_{\mathrm{Spec}(\kappa)} S$$

is faithfully flat for $i = 1, 2$. The faithfully flatness of the first and last of the above three dispalyed morphisms implies that there exists a faithfully flat morphism $f : T \to S \times_{\mathrm{Spec}(\kappa)} S$ of finite presentation and an over T between

$$q_i^* Y[p^n] \times_{\mathrm{Spec}(E\otimes_\kappa E)} (S \times_{\mathrm{Spec}(E)} S) \times_{S\times_{\mathrm{Spec}(E)}S} T$$

for $i = 1, 2$. Therefore

$$\mathscr{I}\!\mathit{SOM}_{\mathrm{Spec}(E\otimes_\kappa E)}\left(q_1^* Y[p^n], q_2^* Y[p^n]\right) \longrightarrow \mathrm{Spec}(E \otimes_\kappa E)$$

is faithfully flat and $Y \to \mathrm{Spec}(E)$ is κ-sustained. By (1) the p-divisible group Y descends to a p-divisible group Y_0 over κ and $X \to S$ is strongly κ-sustained modeled on Y_0. □

5.13. Lemma. *Let X be a p-divisible group over a field $\kappa \supset \mathbb{F}_p$. Let*

$$X = X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_{m-1} \supsetneqq X_m = 0$$

be the slope filtration of X. For each positive integer n, and each natural number i with $0 \le i \le m-1$, the subgroup scheme $X_i[p^n] \subset X[p^n]$ is stable under the action of the finite group scheme $\mathscr{A}\!\mathit{UT}^{\mathrm{st}}(X)_n$.

Proof. This is an immediate consequence of 5.4.

5.14. Lemma. *Let S be a κ-scheme. Let $X \to S$ be a strongly κ-sustained p-divisible group modeled on a p-divisible group Y over κ. There exists a slope filtration on the p-divisible group $X \to S$ whose associated graded pieces are strongly κ-sustained.*

Proof. For each positive integer $n > 0$, over the faithfully flat finite group scheme $\mathscr{I}_n := \mathscr{ISOM}^{\mathrm{st}}(Y,X)_n \to S$, we have a tautological isomorphism

$$\alpha_n : Y[p^n] \times_{\mathrm{Spec}(\kappa)} \mathscr{I}_n \xrightarrow{\sim} X \times_S \mathscr{I}_n.$$

The slope filtration

$$Y = Y_0 \supsetneqq Y_1 \supsetneqq \cdots \supsetneqq Y_m = 0$$

on Y induces a filtration on $Y[p^n]$, which is transfered by α_n to a filtration $\mathrm{Fil}_{X[p^n],\mathscr{I}_n}$ on $X[p^n] \times_S \mathscr{I}_n \to \mathscr{I}_n$. By 5.13, the pull-backs of $\mathrm{Fil}_{X[p^n],\mathscr{I}_n}$ by the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{I}_n \times_S \mathscr{I}_n \to \mathscr{I}_n$ give the same filtration on $X[p^n] \times_S (\mathscr{I}_n \times_S \mathscr{I}_n)$. Therefore $\mathrm{Fil}_{X[p^n],\mathscr{I}_n}$ descends to a filtration $\mathrm{Fil}_{X[p^n]}$ on $X[p^n] \to S$. As n varies these filtrations on $X[p^n] \to S$ define a fitration

$$X = X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_m = 0$$

of the p-divisible group $X \to S$ by p-divisible subgroups, whose associated graded pieces are p-divisible groups. Moreover by descent from the faithfully flat morphisms $\mathscr{I}_n \to S$, the iterated Frobenii on the associated graded pieces X_i/X_{i-1} satisfies the same divisibility conditions by powers of p as Y_i/Y_{i-1}, for each $i = 0, 1, \ldots, m$. Therefore X_i/X_{i-1} is isoclinic and κ-sustained for $i = 0, 1, \ldots, m$.

5.15. Corollary. *Let X_0, Y_0 be isoclinic p-divisible groups X_0, Y_0 over a perfect field $\kappa \supset \mathbb{F}_p$ with $\mathrm{slope}(X_0) < \mathrm{slope}(Y_0)$. Let $R = \kappa \oplus \mathfrak{m}$ be an Artinian local κ-algebra. with residue field κ. Let E be a p-divisible group over R such that $E \otimes_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$ is isomorphic to $X_0 \times Y_0$. Then E is strongly κ-sustained modeled on $X_0 \times Y_0$ if and only if there exists a morphism $h : \mathrm{Spec}(R) \to \mathscr{HOM}'_{\mathrm{div}}(X_0, Y_0)$ such that E is isomorphic to the extension $E_{\delta\circ h}$ of $X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$ by $Y_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$.*

Proof. The "if" part is immediate from 3.14. Conversely suppose that E is strongly κ-sustained modeled on $X_0 \times Y_0$. By 5.14 E is an extension of $X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$ by $Y_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$, therefore there exists a a morphism $h : \mathrm{Spec}(R) \to \mathscr{HOM}'_{\mathrm{div}}(X_0, Y_0)$, uniquely determined by E, such that E_h is isomorphic to E. Lemma 3.14 tells us that h factors through $\mathscr{HOM}'_{\mathrm{div}}(X_0, Y_0)$. $\square$

5.16. Proposition. *Let S be a κ-scheme. Every κ-sustained p-divisible group $X \to S$ admits a natural slope filtration whose associated graded pieces are κ-sustained.*

Proof. Let s be a point of S. Let $E := \kappa(s)$, let $S_E := S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(E)$ and let $X_E := X \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(E)$. The p-divisible group $X_E \to S_E$ is strongly E-sustained modeled on X_s. By 5.14, X_E admits a slope filtration. The pull-backs of this slope filtration to $S_E \times_S S_E$ via the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : S_E \times_S S_E \to S_E$ are equal by 5.5, therefore the slope filtration on X_E descends to a slope filtration on X. Each graded piece of the slope filtration on X is κ-sustained because its fiber product over S with S_E is κ-sustained. $\square$

6. Pointwise criterion for sustained p-divisible groups

The main result of this subsection is 6.7: for a p-divisible group $X \to S$ over a *reduced* base κ-scheme to be strongly κ-sustained modeled on a given p-divisible group X_0, it is necessary and sufficient that every fiber X_s of $X \to S$ is strongly κ-sustained modeled on X_0. It follows that when the base field $\kappa \supset \mathbb{F}_p$ is an algebraically closed, the latter condition is equivalent to the notion of *geometrically fiberwise constant* p-divisible groups in [21].

6.1. Completely slope divisible p-divisible groups. Let S be a κ-scheme. Recall from [27] and [23] that a p-divisible group $X \to S$ is *completely slope divisible* if there exist natural numbers $m, r_0, r_1, \ldots, r_{m-1}, s_0, s_1, \ldots, s_{m-1}$ and a decreasing filtration

$$X = X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_{m-1} \supsetneqq X_m = 0$$

of X by p-divisible subgroups such that X_i/X_{i+1} is a p-divisible group for $i = 0, \ldots, m-1$, satisfying the following properties:

(i) $s_i > 0$ for all $i = 0, 1, \ldots, m-1$ and

$$0 \leq \frac{r_0}{s_0} < \frac{r_1}{s_1} < \cdots < \frac{r_{m-1}}{s_{m-1}} \leq 1$$

(ii) For each $i = 0, 1, \ldots, m-1$, the restriction to X_i of the s_i-th iterate $\mathrm{Fr}^{s_i}_{X_i/S}$ of the relative Frobenius is divisible by p^{r_i}.

(iii) The isogeny $p^{-r_i}\mathrm{Fr}^{s_i}_{X_i/S} : X_i \to X_i^{(p^{s_i})}$ induces an isomorphism

$$p^{-r_i}\mathrm{Fr}^{s_i}_{(X_i/X_{i+1})/S} : X_i/X_{i+1} \xrightarrow{\sim} (X_i/X_{i+1})^{(p^{s_i})}, \quad \forall\, i = 0, 1, \ldots, m-1.$$

In particular X_i/X_{i+1} is an isoclinic p-divisible group with slope $\frac{r_i}{s_i}$ for $i = 0, 1, \ldots, m-1$.

Recall that a scheme over $\mathbb{F}_p$ is *perfect* if the absolute Frobenius $\mathrm{Fr}_S : S \to S$ is an isomorphism. A commutative ring R over $\mathbb{F}_p$ is perfect if the absolute Frobenius map

$$\mathrm{Fr}_R : R \to R, \qquad x \mapsto x^p \ \forall\, x \in R$$

defines a ring automorphism of R. Clearly every perfect commutative ring over $\mathbb{F}_p$ is reduced. For any commutative ring R over $\mathbb{F}_p$, the *perfection* R^{perf} of R is the inductive limit

$$R^{\mathrm{perf}} := \varinjlim \left(R \xrightarrow{\mathrm{Fr}_R} R \xrightarrow{\mathrm{Fr}_R} R \xrightarrow{\mathrm{Fr}_R} \cdots \xrightarrow{\mathrm{Fr}_R} R \xrightarrow{\mathrm{Fr}_R} \cdots \right)$$

of rings. Every homomorphism $R \to R'$ from a commutative ring R over $\mathbb{F}_p$ to a perfect commutative ring R' over $\mathbb{F}_p$ factors through a unique ring homomorphism $R^{\mathrm{perf}} \to R'$.

6.2. The following fact are known; see [23, 1.3–1.5, 1.10].

(a) If S is a connected perfect scheme over $\mathbb{F}_p$, then every completely slope divisible p-divisible group over S is isomorphic to a direct sum of isoclinic completely slope divisible groups over S.

(b) If X is a p-divisible group X over an algebraically closed field $k \supset \mathbb{F}_p$ such that $p^{-r}\mathrm{Fr}^s_X : X \to X^{(p^s)}$ is an isomorphism, then X is isomorphic to the base field extension of a p-divisible group over $\mathbb{F}_q$.

(c) For every completely p-divisible group X over an algebraically closed field $k \supset \mathbb{F}_p$, there exists a finite subfield $\mathbb{F}_q \subset k$ and isoclinic completely slope divisible p-divisible groups Y_i over $\mathbb{F}_q$ such that $X \cong \oplus_i Y_i \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} k$.

(d) Let S be a scheme over $\mathbb{F}_p$ and let $X \to S$ be an *isoclinic* completely slope divisible p-divisible group such that $p^{-r}\mathrm{Fr}^s_{X/S} : X \xrightarrow{\sim} X^{(p^s)}$ is an isomorphism. Suppse that S contains a finite field κ with p^s elements and S is connected. There exists a p-divisible group X_0 over κ such that for every positive integer n, there exists a finite étale morphism $T_n \to S$ such that $X[p^n] \times_S T$ is isomorphic to $X_0 \times_{\operatorname{Spec}\kappa} T$. In particular X is strongly κ-sustained modeled on X_0.

Note that (c) is an immediate consequence of (a) and (b) above. The statement (d) has been formulated slightly differently from [23, 1.10] for our purpose. Part of the reason for the change of formulation is that "completely slope divisible" is an absolute notion , while "κ-sustained" is a relative notion which depends critically on the base field κ.

6.3. Lemma. *Suppose that $\kappa \supset \mathbb{F}_p$ is an algebracially closed field and S is a perfect scheme over κ. Let $X \to S$ be a completely slope divisible p-divisible group, and let $Y \to S$ be a p-divisible group over S and let $\xi : X \to Y$ be an S-isogeny. Assume that exists a p-divisible group Y_0 over κ such that Y_s is strongly κ-sustained modeled on Y_0 for every $s \in S$. Then Y is isomorphic to $Y_0 \times_{\operatorname{Spec}(\kappa)} S$. In particular Y is strongly κ-sustained modeled on Y_0.*

Proof. The conclusion of 6.4 is Zariski local for S, so we may and do assume that S is connected. According to 6.2 (a)–(c), there exists a p-divisible group X_0 over κ and an isomorphism $\alpha\colon X_0 \times_{\operatorname{Spec}(\kappa)} S \xrightarrow{\sim} X$. We identify X with $X_0 \times_{\operatorname{Spec}(\kappa)} S$ via α and regard ξ as an isogeny from $X_0 \times_{\operatorname{Spec}(\kappa)} S$ to Y in the rest of the proof.

According to 4.7, the assumption means that $Y_s \times_t \bar{s} \cong Y_0 \times_{\operatorname{Spec}(\kappa)} \bar{s}$ for every $s \in S$, where $\bar{s}$ stands for a geometric point above s. We will show that the kernel $\operatorname{Ker}(\xi)$ of ξ is equal to $G \times_{\operatorname{Spec}(\kappa)} S$ for a finite subgroup scheme G of X_0, which is uniquely determined by ξ. For every $s \in S$, the fiber Y_s is isomorphic to $Y_0 \times_{\operatorname{Spec}(\kappa)} \bar{s}$, therefore the fundamental finiteness results of [13] recalled as facts (A), (B) in the proof of 4.2 implies that $\operatorname{Ker}(\xi_s) : X_0 \times_{\operatorname{Spec}(\kappa)} s \to Y_s$ is equal to $G_s \times_{\operatorname{Spec}(\kappa)} s$ for a uniquely determined finite subgroup scheme G_s of X_0. Clearly for any two poinots s_1, s_2 of S such that s_1 specializes to s_2, we have $G_{s_1} = G_{s_2}$. Because S is connected, $G_{s_1} = G_{s_2}$ for all $s_1, s_2 \in S$; denote by G_0 this subgroup scheme of X_0.

We have shown that the two closed finite locally free subgroup schemes $\operatorname{Ker}(\xi)$ and $G_0 \times_{\operatorname{Spec}(\kappa)} S$ of $X_0 \times_{\operatorname{Spec}(\kappa)} S$ have the same fiber at every point of S. Since R is reduced, for any regular formal function f on $X_0 \times_{\operatorname{Spec}(\kappa)} S$, f vanishes on

$\mathrm{Ker}(\xi)$ (respectively $G_0 \times_{\mathrm{Spec}(\kappa)} S$) if and only if f vanishes on every fiber of $\mathrm{Ker}(\xi)$ (respectively $G_0 \times_{\mathrm{Spec}(\kappa)} S$). We conclude that $\mathrm{Ker}(\xi) = G_0 \times_{\mathrm{Spec}(\kappa)} S$, therefore Y is isomorphic to $Y_0 \times_{\mathrm{Spec}(\kappa)} S$. □

The technical lemma below on pseudo-torsors is an important ingredient of the proof of the fiberwise criterion 6.7 for strongly sustained p-divisible groups over a reduced base scheme.

6.4. Lemma. *Let $G \to S$ be a flat group scheme of finite presentation over a scheme S. Let $P \to S$ be a separated scheme of finite presentation over S, and let $\mu : P \times_S G \to P$ be an action of G on P over S. Assume that the G-action μ makes P a right pseudo-G-torsor. In other words, for every scheme T and every morphism $T \to S$, $P(T)$ is either empty or is a right principal homogeneous space for $G(T)$; equivalently $(\mathrm{id}_I, \mu)\colon G \times_S P \xrightarrow{\sim} P \times_S P$ is an isomorphism of S-schemes.*

(a) *The quotient P/G of the S_{fppf}-sheaf P by the S_{fppf}-sheaf G is representable by an S-scheme T.*
(b) *The morphism $P \to T$ gives P a structure as a right G-torsor for the site S_{fppf}.*
(c) *The structural morphism $\pi : T \to S$ factors as a composition $T \xrightarrow{j} T' \xrightarrow{\pi'} S$ of morphisms of schemes, where π' is a finite morphism of finite presentation and j is a quasi-compact open immersion.*
(d) *The morphism $\pi' : T \to S$ is a universal monomorphism and induces a continuous injection from $|T|$ to $|S|$, where $|T|$ and $|S|$ are the topological spaces undertying T and S respectively.*
(e) *For every $t \in T$, π' induces an isomorphism between the residue field of $\pi'(s)$ and the residue field of t. In particular π' is unramified.*
(f) *Let $t \in T$ be a point of T such that $\mathscr{O}_{S,\pi'(t)}$ is reduced.*
   - *If π' is étale at t, then π' induces an isomorphism of schemes between an open neighborhood of t and an open neighborhood of $\pi'(t)$.*
   - *If π' is not étale at t, then $\pi'(\mathrm{Spec}(\mathscr{O}_{T,t}))$ is contained in a closed subset of $\mathrm{Spec}(\mathscr{O}_{S,\pi'(t)})$ which is not equal to $\mathrm{Spec}(\mathscr{O}_{S,\pi'(t)})$. In particular there exists a point $s \in S$ whose closure in S contains $\pi'(t)$ such that $s \neq \pi'(t_1)$ for any $t_1 \in T$ whose closure in T contains t.*

(g) *Suppose that S is reduced and $\pi' : |T| \to |S|$ is a bijection such that every specialization in S comes from a specialization in T. Then π' is an isomorphism. In other words $P \to S$ is faithfully flat, making P a G-torsor over S.*
(h) *Suppose that S is reduced and locally Noetherian, and $\pi'(T)$ is Zariski dense in S. Then there exists a dense open subset U of S such that π' induces an isomorphism of schemes from ${\pi'}^{-1}(U)$ to U. In particular $P \times_S U \to U$ is a torsor for $G \times_S U$.*

Proof. The equivalence relation on P defined by the right action of G is flat because and G is flat over S. According to [25, `http://stacks.math.columbia.edu/tag/04S5`Tag04S5], the sheaf P/G is representable by an algebraic space T over S. The statement (b) is a consequence of the definition of the P/G as a sheaf on S_{fppf}. The statement (c), except that T and T' are schemes, follows from Zariski's main theorem [25, `http://stacks.math.columbia.edu/tag/05W7`Tag 05W7]. But then T' is a scheme because T' is finite over the scheme S, and the open subspace $T \subset T'$ is also a scheme, so T and T' are schemes. We have proved the statementes (a)–(c).

It is clear from the definition of T that $\pi' : T \to S$ is a universal monomorphism, in the sense that for every scheme Z the map $T(Z) \to S(Z)$ induced by π' is an injection. The rest of the statement (d) and the statement (e) are easy consequences of this fact.

The statement (f) follows from (e), the assumption that $\mathscr{O}_{S,\pi'(t)}$ is reduced and the local structure of unramified morphisms in EGA $\mathrm{IV}_{\mathrm{IV}}$ 18.4.6. The statement (g) is a corollary of the statement (f).

To prove the statement (h), suffices to show that it holds when $X \to S$ is replaced by $X \times_S V$ for every affine open subscheme $V \subset S$. In other words we may assume that S is affine, hence Noetherian. The assumptions in (h) implies that the image $\pi'(|T|)$ of π' is a Zariski dense subset of $|S|$. On the other hand by Chevalley's theorem we know that $\pi'(|T|)$ is a constructible subset of $|S|$ because π' is of finite presentation and S is Noetherian. It is an elementary fact that a dense constructible subset of a noetherian space contains a dense open subset. Because π' is a universal monomorphism, it follows that there exists a dense open subscheme $U \subset S$ such that π' induces an isomorphism from $T \times_S U \to U$. $\square$

6.5. Proposition. *Let S be a connected* reduced *κ-scheme. Let $X \to S$ be a completely slope divisible p-divisible group with a slope filtration*

$$X = X_0 \supsetneq X_1 \supsetneq \cdots \supset X_m \supsetneq X_{m+1} = 0$$

for some natural number m, and let r_i, s_i, $i = 0, 1, m$, be natural numbers satisfying conditions 6.1 (i)–(iii)*. Assume that κ contains a finite subfield with p^s elements, where s is a positive common multiple of $s_1, \ldots (s_m$. Then $X \to S$ is strongly κ-sustained.*

Proof. Let $q = p^2$. Let s be a point of S and let $\bar{s}$ be a geometric point above s. By 6.2 (b), there exists a p-divisible group Y_0 over κ such that $Y_0 \times_{\mathrm{Spec}(\kappa)} \bar{s}$ is isomorphic to $X_s \times_s \bar{s}$.

For every positive integer n, $\mathscr{I}(Y_0, X)_n := \mathscr{I}\mathit{som}_S(Y_0[p^n]_S, X[p^n]) \to S$ is a right pseudo-torsor for $\mathscr{A}\mathit{ut}(Y_0[p^n])_S$. Let $T_n := \mathscr{I}(Y_0, X)_n / \mathscr{A}\mathit{ut}(Y_0[p^n])_S$ as in 6.4.

Let S^{perf} be the perfection of S. By 6.2 (a), $T_n \times_S S^{\mathrm{perf}} \to S^{\mathrm{perf}}$ is an isomorphism, therefore $T_n \to S$ is a homeomorphism. From Lemma 6.4 (f)–(g), we see that $T_n \to S$ is an isomorphism of schemes and $\mathscr{I}(Y_0, X)_n \to S$ is faithfully flat, for every positive integer n. Therefore the completely slope divisible p-divisible group $X \to S$ is strongly κ-sustained modeled on Y_0. $\square$

6.6. Example. We will construct a completely slope-divisible p-divisible group $X \to S$, where S is the spectrum of a weakly perfect ring $R = \overline{\mathbb{F}_p}[t]/(t^{p^2})$, such that

$X \to S$ is not $\overline{\mathbb{F}_p}$-sustained. (Because the scheme S has a $\overline{\mathbb{F}_p}$-rational closed point, $\overline{\mathbb{F}_p}$-sustained is equivalent to strongly $\overline{\mathbb{F}_p}$-sustained for p-divisible groups over S.)

We say that a commutative ring R is *weakly perfect* if the absolute Frobenius homomorphism for R is a surjective. A scheme is said to be weakly perfect if and only if every stalk of its structural sheaf is weakly perfect. Clearly a scheme is perfect if and only if it is reduced and weakly perfect.

This example shows that the statement of 6.5 becomes *false* if assumption that the base scheme S is perfect is weakened to *weakly perfect*. The same example also also shows that if "perfect" is replaced by "weakly perfect" in 6.2 (a), then the resulting statement becomes false. "weakly perfect"

Sketch. Let Y, Z be isoclinic simple completely slope divisible groups over $\overline{\mathbb{F}_p}$, with slopes $\frac{1}{3}$ and $\frac{1}{2}$ respectively. Consider the family $H_n = H_n(Y,Z) := \mathscr{H}\mathit{om}(Y[p^n], Z[p^n])$ of $\mathscr{H}\mathit{om}$ schemes over $\overline{\mathbb{F}_p}$. Let $G_n = G_n(Y,Z) \subset H_n(Y,Z)$ be the stabilized $\mathscr{H}\mathit{om}$ schemes over $\overline{\mathbb{F}_p}$. For each $n \in \mathbb{N}_{>0}$, we have a coboundary homomorphism

$$\delta_n : H_n(Y,Z) \longrightarrow \mathscr{E}\mathit{xt}(Y,Z)[p^n]$$

of presheaves on the category of all affine $\overline{\mathbb{F}_p}$-schemes. The family of coboundary homomorphisms δ_n defines a homomorphism

$$\delta : \varinjlim_n H_n(X,Y) \to \mathscr{E}\mathit{xt}(Y,Z)$$

of presheaves.

The maximal unipotent subgroup $\mathscr{H}\mathit{om}'(Y,Z)^\wedge_{\mathrm{sub,unip}}$ of the smooth formal group

$$\mathscr{H}\mathit{om}'(Y,Z)^\wedge = \varinjlim_{n\to\infty} H_n(Y,Z)^\wedge$$

is a non-trivial smooth unipotent formal group over $\overline{\mathbb{F}_p}$; its intersection with the maximal p-divisible subgroup $\mathscr{H}\mathit{om}'_{\mathrm{div}}(Y,Z)^\wedge$ of $\mathscr{H}\mathit{om}'(Y,Z)$ is is a finite unipotent group scheme U over $\overline{\mathbb{F}_p}$. The quotient of $\mathscr{H}\mathit{om}'(Y,Z)^\wedge$ by $\mathscr{H}\mathit{om}'_{\mathrm{div}}(Y,Z)^\wedge$ is a smooth unipotent formal group $\mathscr{H}\mathit{om}'(Y,Z)^\wedge_{\mathrm{quot,unip}}$ over $\overline{\mathbb{F}_p}$. The kernel of the natural homorphism $\mathscr{H}\mathit{om}'(Y,Z)^\wedge_{\mathrm{sub,unip}} \to \mathscr{H}\mathit{om}'(Y,Z)^\wedge_{\mathrm{quot,unip}}$ is the finite group scheme U.

There exists an R-valued point $c \in \mathscr{H}\mathit{om}'(Y,Z)^\wedge(R)$, which comes from an element $c_{n_0} \in H_n(Y,Z)(R)$ for some n such that the image of c in the quotient $\mathscr{H}\mathit{om}'(Y,Z)^\wedge_{\mathrm{quot,unip}}(R)$ is non-trivial. In other words there exists $m_0 > 0$ such that the image of c_{n_0} in

$$\left(\mathrm{coker}([p^{m_0+n_0}] : H_{m_0+n_0} \to H_{n_0})\right)(R)$$

is non-zero. More explicitly, there does not exist a faithfully flat commutative R-algebra R_1 of finite presentation such that the image of c_{n_0} in $H_{m_0+n_0}(R_1)$ under the composition

$$H_{n_0}(R) \hookrightarrow H_{m_0+n_0}(R) \hookrightarrow H_{m_0+n_0}(R_1)$$

is equal to $p^{m_0+n_0}$ times an element of $H_{m_0+n_0}(R_1)$.

XXXXX fill in more details here on the existence of such a c_{n_0} XXXXXXX

Let X be the extension of Y_R by Z_R given by the class $\delta(c)$. The resulting p-divisible group X over R is *not* strongly κ-sustained modeled on $Y \times_{\mathrm{Spec}(\kappa)} Z$ because c_{n_0} is not divisible by $p^{m_0+n_0}$ in the sense above. On the other hand X is by definition the (middle term of the) push-out of the short exact sequence

$$0 \longrightarrow Y[p^{n_0}]_R \longrightarrow Y_R \xrightarrow{[p^{n_0}]_Y} Y_R \longrightarrow 0$$

by the composition

$$Y[p^{n_0}]_R \xrightarrow{c_{n_0}} Z[p^{n_0}]_R \to Z_R.$$

We claim that for all $M \gg 0$, the relative Frobenius $\mathrm{Fr}^{3M}_{X/R} : X \to X^{(p^{3M})}$ satisfies the divisiblity condition that $X[p^M] \subset \mathrm{Ker}\left(\mathrm{Fr}^{3M}_{X/R}\right)$, so X is completely slope divisible. This claim follows from the following statements below.

- The $3M$-th iterate of the relative Frobenius homomorphism $\mathrm{Fr}^{3M}_{Z/R}$ sends $Z[p^{3M+n_0}]_R$ into $Z[p^{n_0}]_R^{(p^{3M})}$.
- The pull-back of $c_{n_0} : Y[p^{n_0}]_R \to Z[p^{n_0}]_R$ under the absolute Frobenius map $\mathrm{Fr}^{3M}_S : S \to S$ for S is 0 if M is sufficiently large.
- For every positive integer n, the BT_n group $X[p^n]$ is the (middle term of the) push out of the short exact sequence

$$0 \longrightarrow Y[p^{n_0}]_R \longrightarrow Y[p^{n_0+n}]_R \xrightarrow{[p^{n_0}]_Y} Y[p^n]_R \longrightarrow 0$$

  by $c_{n_0} : Y[p^{n_0}]_R \xrightarrow{c_{n_0}} Z[p^{n_0}]_R$.

XXXX fill in more details here about checking the divisibility condition XXXX

At this point we have produced a completely slope divisible p-divisible group X over a weakly perfect ring R, such that X is not κ-sustained. Of course the last sentence implies that X is not isomorphic to a direct sum of isoclinic completely slope divisible p-divisible groups over R. Finally we remark that there exists a finitely generated $\overline{\mathbb{F}_p}$-subalgebra R' of R and a p-divisible group X' over R' such that X is isomorphic to $X' \times_{\mathrm{Spec}(R')} \mathrm{Spec}(R)$. It suffices to take R_1 to be any finitely generated $\overline{\mathbb{F}_p}$-subalgebra R' of R such that $c_{n_0} \in H_{n_0}(R)$ comes from an element of $H_{n_0}(R')$.

6.7. Theorem. *Let S be a* reduced *κ-scheme, where $\kappa \supset \mathbb{F}_p$ is a field. Let $X \to S$ be a p-divisible group, and let X_0 be a p-divisible group over κ. Suppose that for every point $s \in S$, the geometric fiber $X_s \times_s \bar{s}$ is isomorphic to $X_0 \times_{\mathrm{Spec}(\kappa)} \bar{s}$; equivalenly X_s is strongly κ-sustained modeled on X_0 for every $s \in S$. Then $X \to S$ is strongly κ-sustained modeled on X_0.*

PROOF. For each positive integer n, consider the pseudo-torsor

$$\mathscr{I}(X_0, X)_n := \mathscr{I}\!\mathit{som}_S(Y_0[p^n]_S, X[p^n]) \to S$$

for the group scheme $\mathscr{A}\!\mathit{ut}(X_0[p^n])_S$ and the structural morphism

$$\pi_n : T_n := \mathscr{I}(Y_0, X)_n / \mathscr{A}\!\mathit{ut}(X_0[p^n])_S \to S$$

of the quotient of $\mathscr{I}(X_0, X)_n$ by $\mathscr{A}\!\mathit{ut}(X_0[p^n])_S$.

Claim. *For any two points $s, s' \in S$ such that s is contained in the closure of s' in S, there exist points $t, t' \in T_n$ such that $\pi_n(t) = s$, $\pi_n(t') = s'$ and t is contained in the closure t'.*

This claim implies that $\mathscr{I}(X_0, X)_n \to S$ is faithfully flat, according to 6.4 (f)–(g), and we will be done.

There exists a valuation ring R_1 in the residue field $\kappa(s')$ of s' centered at s whose fraction field is equal to $\kappa(s')$. Let L be a compositum field of $\kappa(s')$ and κ^{alg}, i.e. an extension field of both $\kappa(s')$ and κ^{alg} which is generated by these two fields. Let R_2 be the normalization of R_1 in L, and let R_3 be the perfection of R_2. Let $f : \mathrm{Spec}(R_3) \to S$ be the compostion of the natural morphisms $\mathrm{Spec}(R_3) \to \mathrm{Spec}(R_2)$, $\mathrm{Spec}(R_2) \to \mathrm{Spec}(R_1)$ and $\mathrm{Spec}(R_1) \to S$. It is clear that f sends the generic point s'_3 of $\mathrm{Spec}(R_3)$ to s' and the closed point s_3 of $\mathrm{Spec}(R_3)$ to s.

By [23, Thm. 2.1], over the normal scheme $\mathrm{Spec}(R_2)$ there exists a p-divisible group $Z \to \mathrm{Spec}(R_2)$ which completely slope divisible, and an isogeny $\xi : Z \to X \times_S \mathrm{Spec}(R_3)$. By Lemma 6.3, the p-divisible group $X \times_S \mathrm{Spec}(R_3)$ is strongly κ^{alg}-sustained modeld on $X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa^{\mathrm{alg}})$. The last statement implies that there exists a morphism

$$g : \mathrm{Spec}(R_3) \to T_n \times_S \mathrm{Spec}(R_3)$$

such that the composition of g with the monomorphism

$$T_n \times_S \mathrm{Spec}(R_3) \to \mathrm{Spec}(R_3)$$

is equal to the identity morphism on $\mathrm{Spec}(R_3)$. The last statement implies that the claim holds with $t' = \mathrm{pr}_1(g(s'_3))$ and $t = \mathrm{pr}_1(g(s_3))$, where $\mathrm{pr}_1 : T_n \times_S \mathrm{Spec}(R_3) \to T_n$ is the projection to the first factor of the fiber product $T_n \times_S \mathrm{Spec}(R_3)$. We have proved the claim and Theorem 6.7. □

6.8. Corollary. *Let S be a* reduced *scheme over the base field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a p-divisible group, and let X_0 be a p-divisible group over κ. Assume that Newton polygon of every fiber of $X \to S$ is equal to the Newton polygon of X_0, and there is a dense open subset $U \subset S$ such that $X \times_S U \to U$ is strongly κ-sustained modeled on X_0. Then $X \to S$ is strongly κ-sustained modeled on X_0.*

PROOF. Let Z be the reduced closed subscheme of S such that $|Z| = |S| \smallsetminus |U|$. By 6.5, it suffice to show that for every point $z \in Z$, the fiber X_z of X at z is strongly κ-sustained modeld on X_0.

Since U is schematically dense in S, the restriction map

$$\mathscr{O}_{S,z} \to \Gamma\left(\mathrm{Spec}(\mathscr{O}_{S,z}) \smallsetminus \mathrm{Spec}(\mathscr{O}_{Z,z}), \mathscr{O}_S\right)$$

is injective, so $\mathrm{Spec}(\mathscr{O}_{S,z}) \smallsetminus \mathrm{Spec}(\mathscr{O}_{Z,z})$ is non-empty. Hence there exists a point $s' \in \mathrm{Spec}(\mathscr{O}_{S,z}) \smallsetminus \mathrm{Spec}(\mathscr{O}_{Z,z}) \subset U$ such that s belongs to the closure of s'.

As in the proof of 6.7,

- let R_1 be a valuation ring R_1 in the residue field $\kappa(s')$ of s' centered at z whose fraction field is equal to $\kappa(s')$,
- let L be a compositum field of $\kappa(s')$ and κ^{alg}, i.e. an extension field of both $\kappa(s')$ and κ^{alg} which is generated by these two fields,
- let R_2 be the normalization of R_1 in L,

- let R_3 be the perfection of R_2, and
- let $\xi : Z \to X \times_S \operatorname{Spec}(R_2)$ be an isogeny from a completely slope divisible p-divisible group Z over the normal κ^{alg}-algebra R_2 to the p-divisible group $X \times_S \operatorname{Spec}(R_2)$.

By 6.2 (d), there exists a completely slope divisible p-divisible group Z_0 over κ^{alg} and an isomorphism $\beta : Z_0 \times_{\operatorname{Spec}(\kappa^{\mathrm{alg}})} \operatorname{Spec}(R_3) \xrightarrow{\sim} Z$. The proof of 6.3 shows that the generic fiber of the kernel of the isogeny

$$(\xi \times_{\operatorname{Spec}(R_2)} \operatorname{Spec}(R_3)) \circ \beta : Z_0 \times_{\operatorname{Spec}(\kappa^{\mathrm{alg}})} \operatorname{Spec}(R_3) \longrightarrow X \times_S \operatorname{Spec}(R_3)$$

is equal to $G_0 \times_{\operatorname{Spec}(\kappa^{\mathrm{alg}})} \operatorname{Spec}(L^{\mathrm{perf}})$ for a finite subgroup scheme $G_0 \subset Z_0$ over κ^{alg}. Hence

$$\operatorname{Ker}\big((\xi \times_{\operatorname{Spec}(R_2)} \operatorname{Spec}(R_3)) \circ \beta\big) = G_0 \times_{\operatorname{Spec}(\kappa^{\mathrm{alg}})} \operatorname{Spec}(R_3).$$

It follows that the fiber X_z of $X \to S$ at a point $z \in Z$ is strongly κ-sustained modeled on X_0, for every $z \in Z$. We have shown that all fibers of $X \to S$ are strongly κ-sustained modeled on X_0, hence $X \to S$ is strongly κ-sustained modeled on X_0 by 6.7. $\square$

6.8.1. Lemma. *Let S be a* reduced noetherian *scheme over the base field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a p-divisible group, and let X_0 be a p-divisible group over κ. Suppose that there is a Zariski dense subset $B \subset |S|$ such that X_b is strongly κ-sustained modeled on X_0 for every element $b \in B$. Then there exists a Zariski dense open subscheme $U \subset S$ such that the p-divisible group $X \times_S U \to U$ is strongly κ-sustained modeled on X_0.*

Proof. It is a general fact that the subset of S consisting of all points of S whose fiber has the same Newton polygon as X_0 is a constructible subset of S; see [12, Thm. 2.3.1]. Passing to a dense open subset of S, we may and do assume that all fibers of $X \to S$ have the same Newton polygon ξ.

Let $M = M(\xi, p)$ as in 4.4. Consider the pseudo-torsor

$$\mathscr{I}(X_0, X)_M := \mathscr{I\!som}_S\left(Y_0[p^M]_S, X[p^M]_S\right) \to S$$

for the group scheme $\mathscr{A\!ut}(X_0[p^M])_S$ and the structural morphism

$$\pi_M : T_M := \mathscr{I}(Y_0, X)_M / \mathscr{A\!ut}(X_0[p^M])_S \to S$$

of the quotient of $\mathscr{I}(X_0, X)_M$ by $\mathscr{A\!ut}(X_0[p^M])_S$. By 6.4 (h), there exists a dense open subset $U \subset S$ such that π_M induces an isomorphism from $T_M \times_S U$ to U. In other words $\mathscr{I\!som}_U(Y_0[p^M]_U, X[p^M]_U)$ is faithfully flat over U.

By 4.4, the property that $\mathscr{I\!som}_U(Y_0[p^M]_U, X[p^M]_U) \to U$ is faithfully flat implies that for every $u \in U$, the p-divisible group X_u is strongly κ-sustained. We conclude by 6.7 that the p-divisible group $X \times_S U \to U$ is strongly κ-sustained modeled on X_0.

6.8.2. Lemma. *Let S be a Noetherian scheme over a field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a p-divisible group over S, and let X_0 be a p-divisible group over κ. Let ξ be the Newton polygon of X_0.*

(a) *The subset of S consisting of all points $s \in S$ such that the fiber X_s of $X \to S$ at s whose Newton polygon is equal to ξ is a locally closed subset $|S_\xi|$ of $|S|$.*
(b) *The subset of S consisting of all points $s \in S$ such that the fiber X_s of $X \to S$ at s is strongly κ-sustained modeld on X_0 is a closed subset $|S_{X_0}|$ of $|S_\xi|$.*
(c) *Denote by S_{X_0} the locally closed reduced subscheme of S whose underlying set is $|S_{X_0}|$. The p-divisible group $X \times_S S_{X_0} \to S_{X_0}$ is strongly κ-sustained modeled on X_0.*

PROOF. The statement (a) is proved in [12, 2.3.1]. Let $M = M(\xi, p)$ as in 4.4. By 4.4, $|S_{X_0}|$ is the set-theoretic image of $\mathscr{I}\!\mathit{som}_S \left(Y_0[p^M]_S, X[p^M]_S\right)$ in S. This implies the statement (b) because $\mathscr{I}\!\mathit{som}_S \left(Y_0[p^M]_S, X[p^M]_S\right)$ is of finite presentation over S. The statement (c) follows from 6.8.2 and 6.8. □

7. Deformation of sustained p-divisible groups

In this section we will study the sustained deformations artinian local rings, of given p-divisible group X_0 over a field. This functor of sustained deformations is We define some stabilized Isom-schemes attached to sustained p-divisible groups.

7.1. Definition. Let S be a scheme over a field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ and $Y \to S$ be κ-sustained p-divisible groups. Let X_0 and Y_0 be p-divisible groups over κ. In (4)–(7) below we assume that $X \to S$ and $Y \to S$ are strongly κ-sustained modeled on X_0 and Y_0 respectively.

(1) Define $\mathscr{H}\!\mathit{om}'_S(X, Y)$ by
$$\mathscr{H}\!\mathit{om}'_S(X,Y) := \varinjlim_n \mathscr{H}\!\mathit{om}_S(X[p^n], Y[p^n]),$$
the inductive system of $\mathscr{H}\!\mathit{om}_S(X[p^n], Y[p^n])$'s, such that for each n the transition map
$$\mathscr{H}\!\mathit{om}_S(X[p^n], Y[p^n]) \longrightarrow \mathscr{H}\!\mathit{om}_S(X[p^{n+1}], Y[p^{n+1}])$$
is the composition of monomorphisms
$$\mathscr{H}\!\mathit{om}_S(X[p^n], Y[p^n]) \hookrightarrow \mathscr{H}\!\mathit{om}_S(X[p^{n+1}], Y[p^n]) \hookrightarrow \mathscr{H}\!\mathit{om}_S(X[p^{n+1}], Y[p^{n+1}]),$$
where the first arrow is induced by $X[p^{n+1}] \twoheadrightarrow X[p^n]$ and the second arrow is induced by $Y[p^n] \hookrightarrow Y[p^{n+1}]$.

(2) It follows from 2.7 and the definition of strongly sustained p-divisible groups that there exists a natural number n_0 such that the image
$$\mathrm{Im}\left(\mathscr{H}\!\mathit{om}_S(X[p^{n+N}], Y[p^{n+N}]) \to \mathscr{H}\!\mathit{om}_S(X[p^n], Y[p^n])\right)$$
of the restriction homomorphism between two Hom sheaves for either the fppf or the fpqc topology, is equal to the sheaf image
$$\mathrm{Im}\left(\mathscr{H}\!\mathit{om}_S(X[p^{n+N+1}], Y[p^{n+N+1}) \to \mathscr{H}\!\mathit{om}_S(X[p^n], Y[p^n])\right),$$
for all $N \geq n_0$. For each positive integer n, define
$$\mathscr{H}\!\mathit{om}^{\mathrm{st}}_S(X,Y)_n := \mathrm{Im}\left(\mathrm{Hom}_S(X[p^N], Y[p^N]) \to \mathrm{Hom}_S(X[p^n], Y[p^n])\right)$$

for $N \gg 0$ as a sheaf on S_{fppf}, and define $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ to be the inductive system

$$\mathscr{HOM}'_{\mathrm{div}}(X,Y) := \varinjlim_{n} \mathscr{HOM}_S^{\mathrm{st}}(X,Y)_n$$

of $\mathscr{HOM}^{\mathrm{st}}(X,Y)_n \subset \mathrm{Hom}_S(X[p^n], Y[p^n])$'s.

(3) For each $n \in \mathbb{N}_{>0}$, let $\mathscr{AUT}^{\mathrm{st}}(X_0)_{n,\mathrm{conn}}$ be the neutrual component of the stabilized Aut group $\mathscr{AUT}^{\mathrm{st}}(X_0)_n$ of $X_0[n]$, and let $\mathscr{AUT}^{\mathrm{st}}(X_0)_{n,\mathrm{et}}$ be the maximal étale quotient of $\mathscr{AUT}^{\mathrm{st}}(X_0)_n$. Define $\mathscr{AUT}^{\mathrm{st}}(Y_0)_{n,\mathrm{conn}}$ and $\mathscr{AUT}^{\mathrm{st}}(Y_0)_{n,\mathrm{et}}$ similarly.

(4) For each $n \in \mathbb{N}_{>0}$, let $\mathscr{ISOM}^{\mathrm{st}}(X_0,X)_n$ be the open subscheme of the stabilized Hom-scheme $\mathscr{HOM}^{\mathrm{st}}(X_0,X)_n$ of isomorphisms, characterized by the following property: for any S-scheme T and any S-morphism $f : T \to \mathscr{HOM}^{\mathrm{st}}(X_0,X)_n$ of $\mathscr{HOM}^{\mathrm{st}}(X_0,X)_n$, this morphism f factors through the open subscheme $\mathscr{ISOM}^{\mathrm{st}}(X_0,X)_n$ of $\mathscr{HOM}^{\mathrm{st}}(X_0,X)_n$ if and only if the homomorphism $\alpha_f : X_0[p^n] \times_{\mathrm{Spec}(\kappa)} T \to X[p^n] \times_S T$ corresponding to the T-point f of $\mathscr{HOM}^{\mathrm{st}}(X_0,X)_n$ is an isomorphism.

(5) Similar to (4) above, for each $n \in \mathbb{N}_{>0}$ let $\mathscr{AUT}^{\mathrm{st}}(X,X)_n$ the open subscheme of the stabilized Hom-scheme $\mathscr{END}^{\mathrm{st}}(X,X)_n$ of automorphisms, characterized by the following property: for any S-scheme T and any S-morphism $f : T \to \mathscr{END}^{\mathrm{st}}(X)_n$ of $\mathscr{END}^{\mathrm{st}}(X)_n$, this morphism f factors through the open subscheme $\mathscr{AUT}^{\mathrm{st}}(X)_n$ of $\mathscr{END}^{\mathrm{st}}(X)_n$ if and only if the homomorphism $\alpha_f : X[p^n] \times_S T \to X[p^n] \times_S T$ corresponding to the T-point f of $\mathscr{END}^{\mathrm{st}}(X)_n$ is an automorphism.

(6) Let $\mathrm{gr}_\bullet X_0$ (respectively $\mathrm{gr}_\bullet X$, $\mathrm{gr}_\bullet Y_0$ and $\mathrm{gr}_\bullet Y$) be the product of the associated graded pieces $\mathrm{gr}_i(X_0)$ of the slope filtration of X_0 (respectively X, Y_0 and Y). For each $n \in \mathbb{N}_{>0}$, let

$$\mathscr{ISOM}^{\mathrm{st}}(\mathrm{gr}_\bullet X_0, \mathrm{gr}_\bullet X)_{n,\mathrm{et}} := \prod_i \mathscr{ISOM}^{\mathrm{st}}(\mathrm{gr}_i(X_0), \mathrm{gr}_i(X))_n\,,$$

a finite étale group scheme over S. Define $\mathscr{ISOM}^{\mathrm{st}}(Y_0,Y)_{n,\mathrm{et}}$ similarly.

(7) For each $n \in \mathbb{N}_{>0}$, let $\mathscr{ISOM}^{\mathrm{st}}(X_0,X)_{n,\mathrm{et}}$ be the image of $\mathscr{ISOM}^{\mathrm{st}}(X_0,X)_n$ in $\mathscr{ISOM}^{\mathrm{st}}(\mathrm{gr}_\bullet X_0, \mathrm{gr}_\bullet X)_{n,\mathrm{et}}$.

7.2. Lemma. *Let X_0 and Y_0 be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. Let S be a scheme over the κ, and let $X \to S$, $Y \to S$ be strongly κ-sustained p-divisible groups modeled on X_0 and Y_0 respectively.* 7.1.

(1) *For every $n \in \mathbb{N}$, the S-schemes $\mathscr{HOM}_S^{\mathrm{st}}(X,Y)_n$, $\mathscr{AUT}^{\mathrm{st}}(X_0)_n$, $\mathscr{AUT}^{\mathrm{st}}(Y_0)_n$, $\mathscr{AUT}^{\mathrm{st}}(X)_n$, $\mathscr{AUT}^{\mathrm{st}}(Y)_n$, $\mathscr{ISOM}^{\mathrm{st}}(X_0,X)_n$, $\mathscr{ISOM}^{\mathrm{st}}(Y_0,Y)_n$ are finite locally free over S.*

(2) *The scheme $\mathscr{ISOM}^{\mathrm{st}}(X_0,X)_n$ has a natural structure as a right torsor for $\mathscr{AUT}^{\mathrm{st}}(X_0)_n \times_{\mathrm{Spec}(\kappa)} S$, plus a compatible as a left $\mathscr{AUT}^{\mathrm{st}}(X)_n$-torsor structure Similarly the scheme $\mathscr{ISOM}^{\mathrm{st}}(Y_0,Y)_n$ has a natural structure as a left torsor for $\mathscr{AUT}^{\mathrm{st}}(Y_0) \times_{\mathrm{Spec}(\kappa)} S$, plus a compatible right $\mathscr{AUT}^{\mathrm{st}}(Y)_n$-torsor structure.*

(3) *The inductive system $\mathscr{HOM}'(X,Y)$ defines a smooth commutative formal group over S whose relative dimension is $\dim_S(Y) \cdot \dim_S(X^t)$.*

(4) *For each* $n \in \mathbb{N}_{>0}$*, the sheaf* $\mathscr{HOM}^{\rm st}_S(X,Y)_n$ *is representable by a finite locally free group scheme over* S*.*
(5) *The inductive system* $\mathscr{HOM}'_{\rm div}(X,Y)$ *is a* p*-divisible group over* S*, which is strongly* κ*-sustained modeled on* $\mathscr{HOM}'_{\rm div}(X_0,Y_0)$*.*
(6) *For each* n*, the natural maps*

$$\mathscr{ISOM}^{\rm st}(X_0,X)_n/\mathscr{AUT}^{\rm st}(X_0)_{n,{\rm conn}} \longrightarrow \mathscr{ISOM}^{\rm st}(X_0,X)_{n,{\rm et}}$$

and

$$\mathscr{ISOM}^{\rm st}(Y_0,Y)_n/\mathscr{AUT}^{\rm st}(Y_0)_{n,{\rm conn}} \longrightarrow \mathscr{ISOM}^{\rm st}(Y_0,Y)_{n,{\rm et}}$$

are isomorphisms.

Remark. In 7.2 (3), a *commutative formal group* over a scheme S is understood to be a covariant functor from the category ${\rm Nilp}_S$ of nilpotent commutative S-algebras to the category of abelian groups. By definition a *nilpotent commutative* S*-algebra* is a commutative algebra of the form $R\oplus N$ over an affine open subset ${\rm Spec}(R)$ of S, such that there exists a positive integer j with the property that $N^j=(0)$. The statement that $\mathscr{HOM}'(X,Y)$ is a smooth commutative formal group over S means that the natural map $\mathscr{HOM}'(X,Y)(R\oplus N_1)\to \mathscr{HOM}'(X,Y)(R\oplus N_2)$ is surjective for any surjection $R\oplus N_1 \twoheadrightarrow R\oplus N_2$ in ${\rm Nilp}_S$.

Proof. We will give a proof of the first part of 7.2 (3) on the smoothness of $\mathscr{HOM}'(X,Y)$. The proofs of 7.2 (1), (2), (4), (5) are left as exercises. The second part of of 7.2 on the relative dimension of $\mathscr{HOM}'(X,Y)$ is easier than the first part, and is left as an exercise as well.

We may and do assume that $S={\rm Spec}(R)$ is affine. It suffices to show that the natural map $\mathscr{HOM}'(X,Y)(R\oplus N_1)\to \mathscr{HOM}'(X,Y)(R\oplus N_2)$ is surjective for every surjection $\beta: R_1=R\oplus N_1\twoheadrightarrow R\oplus N_2=R_2$ of commutative nilpotent S-algebras such that $J:={\rm Ker}(\beta)$ satisfies $J^2=0$.

Let ${\rm Inf}^1\mathscr{HOM}'(X,Y)$ be the first infinitesimal neighborhood of $\mathscr{HOM}'(X,Y)$. Recall from 2.18 that $\mathscr{HOM}'(X_0,Y_0)$ is a commutative smooth formal group over κ. Combined with the assumption that X and Y are strongly κ-sustained model on X_0 and Y_0 respectively, it follows that ${\rm Inf}^1\mathscr{HOM}'(X,Y)$ there exists a projective R-module of finite rank M which represents ${\rm Inf}^1\mathscr{HOM}'(X,Y)$, in the following sense: there exist a natural isomorphism

$$\zeta_{R'\oplus J'}: {\rm Inf}^1\mathscr{HOM}'(X,Y)\xrightarrow{\sim} M\otimes_R J'$$

for every commutative R-algebra R' and every nilpotent R'-algebra $R'\oplus J'$ with ${J'}^2=(0)$, which is functorial in $R'\oplus J'$.

Let $\underline{M\otimes_R J}$ be the group scheme over $S={\rm Spec}(R)$ whose points over any commutative R-algebra T is $M\otimes_R J\otimes_R T$. Suppose we are given a surjection $\beta: R_1\twoheadrightarrow R_2$ as above, and an element $h\in \mathscr{HOM}'(X,Y)(R\oplus N_2)$. From the assumption that X and Y are strongly κ-sustained, we see that the sheaf of liftings of h to $(R_2\oplus N_2)\otimes_{R_1} R'$, where R' runs through all commutative R-algebras which are faithfully flat of finite presentation and quasi-finite over R, defines a fppf torsor Q for the group $\underline{M\otimes_R J}$. This torsor has a section over S because S is affine. We have shown that there exists an element in $\mathscr{H}\!om'(X,Y)(R_1 \oplus N_1)$ whose image in $\mathscr{H}\!om'(X,Y)(R_2 \oplus N_2)$ is h. We have proved 7.2 (3). $\square$

7.3. Definition. Let $\kappa \supset \mathbb{F}_p$ be a perfect field of characteristic p. Let X_0 be a p-divisible group over κ. Let

$$X_0 \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_m = 0$$

be the slope filtration of X_0, such that X_i/X_{i+1} is isoclinic for $i = 0, 1, \ldots, m-1$, and $\mathrm{slope}(X_0/X_1) < \mathrm{slope}(X_1/X_2) < \cdots < \mathrm{slope}(X_{m-1}/X_m) < \mathrm{slope}(X_{m-1})$. We define two subfunctors $\mathcal{D}ef_{\mathrm{MExt}}(X_0)$ and $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ of the deformation functor $\mathcal{D}ef(X_0) : \mathfrak{Art}^+_\kappa \to \mathfrak{Sets}$. It will be shown that both subfunctors are formally smooth over κ.

(a) The functor $\mathcal{D}ef_{\mathrm{MExt}}(X_0)$ sends an object $(R, j : \mathsf{k} \to R, \epsilon : R \to \mathsf{k})$ of $\mathfrak{Art}^+_k$ to the set of isomorphism classes of pairs

$$\left(\mathscr{X}_0 \supsetneqq \mathscr{X}_1 \supsetneqq \cdots \supsetneqq \mathscr{X}_m = 0,\ \xi : X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathsf{k}) \xrightarrow{\sim} \mathscr{X}_0 \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\mathsf{k})\right),$$

where each $\mathscr{X}_i$ is a p-divisible group over R for $i = 0, 1, \ldots, m$ such that $\mathscr{X}_i/\mathscr{X}_{i+1}$ is a p-divisible group over i for $i = 0, 1, \ldots, m-1$, and ξ is an isomorphism of p-divisible groups which induces an isomorphism from $X_i \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathsf{k})$ to $\mathscr{X}_i \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\mathsf{k})$ for $i = 0, 1, \ldots, m$.

(b) The functor $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ is the subfunctor of $\mathcal{D}ef_{\mathrm{MExt}}(X_0)$ which sends an object $(R, j : \mathsf{k} \to R, \epsilon : R \to \mathsf{k})$ of $\mathfrak{Art}^+_k$ to the set of isomorphism classes of pairs

$$\left(\mathscr{X}_0 \supsetneqq \mathscr{X}_1 \supsetneqq \cdots \supsetneqq \mathscr{X}_m = 0,\ \xi : X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathsf{k}) \xrightarrow{\sim} \mathscr{X}_0 \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\mathsf{k})\right),$$

where $\mathscr{X}_0$ is strongly k-sustained modeled on $X_0 \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\mathsf{k})$ and $\mathscr{X}_0 \supsetneqq \mathscr{X}_1 \supsetneqq \cdots \supsetneqq \mathscr{X}_m = 0$ is the slope filtration on $\mathscr{X}_0$.

Remark. The first functor $\mathcal{D}ef_{\mathrm{MExt}}(X_0)$ is the locus in the local deformation space $\mathcal{D}ef(X_0)$ where the slope filtration on the closed fiber X_0 extends to a filtration of the deformation, so that the deformed p-divisible group is a successive extension of the isoclinic graded pieces of the slope filtration. Note that graded pieces of the slope filtration are rigid: they are isoclinic, hence don't deform. The second functor $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ is the locus in $\mathcal{D}ef(X_0)$ where the deformation is κ-sustained.

It is easy to see that the natural/obvious arrow from $\mathcal{D}ef_{\mathrm{MExt}}(X_0)$ to $\mathcal{D}ef(X_0)$ induces an injection for every object $(R, j : \mathsf{k} \to R, \epsilon : R \to \mathsf{k})$ of $\mathfrak{Art}^+_k$, making $\mathcal{D}ef_{\mathrm{MExt}}(X_0)$ a subfunctor of $\mathcal{D}ef(X_0)$. This is an immediate consequence of the rigidity of homomorphisms between p-divisible groups: two homomorphism of p-divisible groups over an artinian local ring R with the same source and target whose restrictions the closed fiber are equal over R.

7.4. Theorem. *Let $\kappa \supset \mathbb{F}_p$ be a perfect field and let X_0 be a p-divisible group over κ with slope filtration $X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_{m-1} \supsetneqq X_m = 0$. For $i = 0, 1, \ldots, m-1$, let $d_i := \dim(X_i/X_{i+1})$, $c_i := \dim((X_i/X_{i+1})^t)$, $h_i := c_i + d_i$ and $\lambda_i := \frac{d_i}{h_i}$ be respectively the dimension, codimension, slope and height of the isoclinice p-divisible group X_i/X_{i+1}.*

(1) *The functors $\mathcal{D}ef_{\mathrm{MExt}}(X_a/X_b)$ and $\mathcal{D}ef(X_a/X_b)_{\mathrm{sus}}$ on $\mathfrak{Art}_\kappa^+$ are both pro-representable, Each of them is isomorphic to the formal spectrum of formal power series rings over κ in a finite number of variables.*

(2) *Assume that the slope filtration of X_0 splits, so that $X_0 \cong \prod_{i=0}^{m-1}(X_i/X_{i+1})$ in (2)–(4) below. The natural maps*

$$\pi^{X_a/X_c}_{X_a/X_b} : \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_c) \longrightarrow \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_b)$$

$$\pi^{X_a/X_c}_{X_b/X_c} : \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_c) \longrightarrow \mathcal{D}ef_{\mathrm{MExt}}(X_b/X_c)$$

and

$$\pi^{X_a/X_c}_{X_a/X_b,\,X_b/X_c} : \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_c) \longrightarrow \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_b) \times_{\mathrm{Spec}(\kappa)} \mathcal{D}ef_{\mathrm{MExt}}(X_b/X_c)$$

are formally smooth for all $0 \le a < b < c \le m$.

(3) *The natural maps*

$$\pi^{X_a/X_c}_{X_a/X_b} : \mathcal{D}ef(X_a/X_c)_{\mathrm{sus}} \longrightarrow \mathcal{D}ef(X_a/X_b)_{\mathrm{sus}}$$

$$\pi^{X_a/X_c}_{X_b/X_c} : \mathcal{D}ef(X_a/X_c)_{\mathrm{sus}} \longrightarrow \mathcal{D}ef(X_b/X_c)_{\mathrm{sus}}$$

and

$$\pi^{X_a/X_c}_{X_a/X_b,\,X_b/X_c} : \mathcal{D}ef(X_a/X_c)_{\mathrm{sus}} \longrightarrow \mathcal{D}ef(X_a/X_b)_{\mathrm{sus}} \times_{\mathrm{Spec}(\kappa)} \mathcal{D}ef(X_b/X_c)_{\mathrm{sus}}$$

are formally smooth for all natural numbers a, b, c with $0 \le a < b < c \le m$.

(4) *The natural maps*

$$\pi^{X_a/X_d}_{X_a/X_c,\,X_b/X_d} : \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_d) \longrightarrow \mathcal{D}ef_{\mathrm{MExt}}(X_a/X_c) \times_{\mathcal{D}ef_{\mathrm{MExt}}(X_b/X_c)} \mathcal{D}ef_{\mathrm{MExt}}(X_b/X_d)$$

and

$$\pi^{X_a/X_d}_{X_a/X_c,\,X_b/X_d} : \mathcal{D}ef(X_a/X_d)_{\mathrm{sus}} \longrightarrow \mathcal{D}ef(X_a/X_c)_{\mathrm{sus}} \times_{\mathcal{D}ef(X_b/X_c)_{\mathrm{sus}}} \mathcal{D}ef(X_b/X_d)_{\mathrm{sus}}$$

are formally smooth for all $a, b, c, d \in \mathbb{N}$ such that $0 \le a < b < c < d \le m$.

(5) *The equalities*

– $\dim(\mathcal{D}ef_{\mathrm{MExt}}(X_a/X_b)) = \sum_{a \le i < j < b} d_j\, c_i$ *and*

– $\dim(\mathcal{D}ef(X_a/X_b)_{\mathrm{sus}}) = \sum_{a \le i < j < b} (d_j c_i - d_i c_j) = \sum_{a \le i < j < b} (\lambda_j - \lambda_i) h_i h_j$

hold for all $0 \le a < b \le m$.

7.5. Multiple extensions. We explain the formal structure of the category of successive extensions with fixed graded pieces in an abelian category.

7.5.1. Definition. Let $\mathfrak{C}$ be an abelian category, and let $Z_0, \ldots, Z_{m-1}$ be objects in $\mathfrak{C}$. Let $\mathfrak{MExt}(Z_0, Z_1, \ldots, Z_{m-1})$ be the groupoid category whose objects are tuples of the form

$$\Big(X_0 \hookleftarrow X_1 \hookleftarrow \cdots \hookleftarrow X_{m-1} \hookleftarrow X_m = 0;\ \big(\psi_i : X_i/X_{i+1} \xrightarrow{\sim} Z_i\big)_{0 \le i \le m-1}\Big)$$

where

- $X_0 \hookleftarrow X_1 \hookleftarrow \cdots \hookleftarrow X_{m-1}$ is a decreasing chain of objects in $\mathfrak{C}$, and
- $\psi_0, \psi_1, \ldots, \psi_{m-1}$ are isomorphisms in $\mathfrak{C}$.

A morphism from $\Big(X_0 \hookleftarrow X_1 \hookleftarrow \cdots \hookleftarrow X_{m-1} \hookleftarrow X_m = 0;\ \big(\psi_i : X_i/X_{i+1} \xrightarrow{\sim} Z_i\big)\Big)$ to $\Big(X_0' \hookleftarrow X_1' \hookleftarrow \cdots \hookleftarrow Y_{m-1}' \hookleftarrow Y_m' = 0;\ \big(\psi_i' : Y_i'/X_{i+1}' \xrightarrow{\sim} Z_i\big)\Big)$ is a family of isomorphisms $h_i : X_i \xrightarrow{\sim} X_i'$ for $i = 0, 1, \ldots, m$ such that

- $h_i \circ j_{i+1} = j_i' \circ h_{i+1}$ for $i = 0, 1, \ldots, m-1$, where $j_{i+1} : X_{i+1} \hookrightarrow X_i$ and $j_i' : X_{i+1}' \hookrightarrow X_i'$ denote the inclusions in the two chains, and
- $\psi_i' \circ \bar{h}_i = \psi_i$ for $i = 0, 1, \ldots, m-1$, where $\bar{h}_i : X_i/X_{i+1} \xrightarrow{\sim} X_i'/X_{i+1}'$ is the isomorphism induced by (h_i, h_{i+1}).

7.5.2. Let $\mathfrak{C}$ be an abelian category and let $Z_0, \ldots, Z_m$ be objects in $\mathfrak{C}$. Let $0 \leq b \leq m-1$ be a natural number between 0 and $m-1$.

$$\Phi_{[0,b-1],[b,m-1]} : \mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_0, \ldots, \ldots, Z_{m-1}) \longrightarrow \mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_0, \ldots, Z_{b-1}) \times \mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_b, \ldots, Z_{m-1})$$

be the functor which sends an object

$$\Big(X_0 \hookleftarrow X_1 \hookleftarrow \cdots \hookleftarrow X_{m-1} \hookleftarrow X_m = 0;\ \big(\psi_i : X_i/X_{i+1} \xrightarrow{\sim} Z_i\big)_{0 \leq i \leq m-1}\Big)$$

in $\mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_0, \ldots, Z_{m-1})$ to the object

$$\Big(\big(X_0/X_b \hookleftarrow X_1/X_b \hookleftarrow \cdots \hookleftarrow X_{b-1}/X_b \hookleftarrow 0;\ \big(\bar{\psi}_i : (X_i/X_b)/(X_{i+1}/X_b) \xrightarrow{\sim} X_i/X_{i+1} \xrightarrow{\psi_i} Z_i\big)_{0 \leq i \leq b-1}\big),$$
$$\big(X_b \hookleftarrow \cdots \hookleftarrow X_{m-1} \hookrightarrow 0; \big(\psi_j : X_j/X_{j+1} \xrightarrow{\sim} Z_j\big)_{b \leq j \leq m-1}\big)\Big)$$

in the product category $\mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_0, \ldots, Z_{b-1}) \times \mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_b, \ldots, Z_{m-1})$.

The standard Baer construction shows that for any given objects $(E_1, E_2) \in \mathrm{Ob}\big(\mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_0, \ldots, Z_{b-1}) \times \mathfrak{M}\mathfrak{E}\mathfrak{x}\mathfrak{t}(Z_b, \ldots, Z_{m-1})\big)$ the set of isomorphism classes of the category $\Phi^{-1}_{[0,b-1],[b,m-1]}((E_1, E_2))$ has a natural structure as an abelian group. In fact the Baer construction does something more: it gives a functorial composition law which is functorially associative, strictly commutative, and the usual compatibility relations of the functorial associativity and strict commutativity are satisfied, so that $\Phi^{-1}_{[0,b-1],[b,m-1]}((E_1, E_2))$ acquires a natural structure as a *strictly commutative Picard category* in the sense of 1.4.2 of SGA4 exposé 18.

7.5.3. Definition. A *strictly commutative Picard category* is a non-empty category P such that all morphisms are isomorphisms, together with the following structures

- a functor $+ : P \times P \to P$,
- a natural transformation of functors (strict commutativity isomorphisms)

  $$\tau : + \circ (\text{flip}) \to +\,,$$

  written as

  $$\tau_{x,y} : x + y \to y + x$$

- a natural transformation of functors (associativity isomorphisms)

  $$\alpha : + \circ (+ \times\ \mathrm{id}_P) \implies + \circ (\mathrm{id}_P\ \times\ +)\ :\ P \times P \times P \longrightarrow P,$$

  from $P \times P \times P$ to P, from the composition

  $$P \times P \times P \xrightarrow{+\times \mathrm{id}_P} P \times P \xrightarrow{+} P$$

to the composition

$$P \times P \times P \xrightarrow{\mathrm{id}_P \times +} P \times P \xrightarrow{+} P\,,$$

written as

$$\alpha_{x,y,z} : (x+y)+z \to x+(y+z)$$

which satisfy the following conditions.

(1) For every $x \in P$, the functor $P \to P$ which sends y to $x+y$ is an equivalence.

(2) $\tau_{x,x} = \mathrm{id}_{x+x}$.

(3) (Pentagon axiom for strict associativity) For any four objects x, y, z, w in P, the diagram

$$\begin{tikzcd}
x+(y+(z+w)) & (x+y)+(z+w) \arrow[l, "\alpha_{x,y,z+w}"'] & ((x+y)+z)+w \arrow[l, "\alpha_{x+y,z,w}"'] \arrow[d, "\alpha_{x,y,z}+1_w"] \\
x+((y+z)+w) \arrow[u, "1_X+\alpha_{y,z,w}"] & & (x+(y+z))+w \arrow[ll, "\alpha_{x,y+z,w}"]
\end{tikzcd}$$

commutes.

(4) (Hexagon axiom for compatibility of associativity and commutativity) For any three objects x, y, z in P, the diaggram

$$\begin{tikzcd}
(x+y)+z \arrow[r, "\alpha_{x,y,z}"] \arrow[d, "\tau_{x,y}+1_z"'] & x+(y+z) \arrow[r, "1_x+\tau_{y,z}"] & x+(z+y) \\
(y+x)+z \arrow[r, "\alpha_{y,x,z}"'] & y+(x+z) \arrow[r, "\tau_{y,x+z}"'] & (x+z)+y \arrow[u, "\alpha_{x,z,y}"']
\end{tikzcd}$$

commutes.

7.5.4. Let $Z_0, Z_1, \ldots, Z_{m-1}$ be objects in an abelian category $\mathfrak{C}$.

1. For every triple i, j, k of natural numbers such that $0 \le i < j < k \le m$, we have a natural projection functor

$$\Phi^{[i,k)}_{[i,j),[j,k)} : \mathfrak{MExt}(Z_i, \ldots, Z_{k-1}) \longrightarrow \mathfrak{MExt}(Z_i, \ldots, Z_{j-1}) \times \mathfrak{MExt}(Z_j, \ldots, Z_{k-1})$$

According to 7.5.2, fibers of the functor $\Phi^{[i,k)}_{[i,j),[j,k)}$ have a functorial comosition law via the standard Baer construction, making each fiber a strictly commutative Picard category. This enriches the standard fact that the set of isomorphism class of each fiber is a commutative group.

Suppose now that we have a sequence of four natural numbers a, b, c, d such that $0 \le a < b < c < d \le m$.

2. We have projections

- $\Phi^{[a,d)}_{[a,c)} : \mathfrak{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathfrak{MExt}(Z_a, \ldots, Z_{c-1})$
- $\Phi^{[a,d)}_{[b,d)} : \mathfrak{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathfrak{MExt}(Z_b, \ldots, Z_{d-1})$
- $\Phi^{[a,d)}_{[b,c)} : \mathfrak{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathfrak{MExt}(Z_b, \ldots, Z_{c-1})$

- $\Phi^{[a,c)}_{[b,c)} : \mathfrak{MExt}(Z_a, \ldots, Z_{c-1}) \longrightarrow \mathfrak{MExt}(Z_b, \ldots, Z_{c-1})$
- $\Phi^{[b,d)}_{[b,c)} : \mathfrak{MExt}(Z_b, \ldots, Z_{d-1}) \longrightarrow \mathfrak{MExt}(Z_b, \ldots, Z_{c-1})$

Clearly we have

$$\Phi^{[a,c)}_{[b,c)} \circ \Phi^{[a,d)}_{[a,c)} = \Phi^{[a,d)}_{[b,c)} = \Phi^{[b,d)}_{[b,c)} \circ \Phi^{[a,d)}_{[b,d)}.$$

3. The last displayed equality defines a natural transformation of functors

$$\Phi^{[a,d)}_{[a,c),[b,d)} : \mathfrak{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathfrak{MExt}(Z_a, \ldots, Z_{c-1}) \times_{\mathfrak{MExt}(Z_b,\ldots,Z_{c-1})} \mathfrak{MExt}(Z_b, \ldots, Z_{d-1}),$$

whose target $\mathfrak{MExt}(Z_a, \ldots, Z_{c-1}) \times_{\mathfrak{MExt}(Z_b,\ldots,Z_{c-1})} \mathfrak{MExt}(Z_b, \ldots, Z_{d-1})$ is the 2-fiber product of the categories $\mathfrak{MExt}(Z_a, \ldots, Z_{c-1})$ and $\mathfrak{MExt}(Z_b, \ldots, Z_{d-1})$ over the category $\mathfrak{MExt}(Z_b, \ldots, Z_{c-1})$.

According to **1** above, there is a functorial composition law on fibers of the functor $\Phi^{[a,d)}_{[b,d)}$; denote it by $+_1$. Similarly there is a composition law on fibers of the functor $\Phi^{[a,d)}_{[a,c)}$, denoted by $+_2$. We will write them as $+_1^{[a,c),[bd)}$ and $+_2^{[a,c),[bd)}$ respectively when more precision is needed.

4. For every pair of natural numbers i, j with $0 \le i < j \le m$, denote by

$$\mathrm{MExt}(Z_i, Z_{i+1}, \ldots, Z_{j-1})$$

the set of isomorphism classes of objects in $\mathfrak{MExt}(Z_i, Z_{i+1}, \ldots, Z_{j-1})$. The functor $\Phi^{[a,d)}_{[a,c),[b,d)}$ in **3** induces a map

$$\pi^{[a,d)}_{[a,c),[b,d)} : \mathrm{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathrm{MExt}(Z_a, \ldots, Z_{c-1}) \times_{\mathrm{MExt}(Z_b,\ldots,Z_{c-1})} \mathrm{MExt}(Z_b, \ldots, Z_{d-1}),$$

The functorial compositions $+_1^{[a,c),[bd)}$ and $+_2^{[a,c),[bd)}$ yields two relative group laws

$$+_1 : \mathrm{MExt}(Z_a, \ldots, Z_{d-1}) \times_{\mathrm{MExt}(Z_b,\ldots,Z_{d-1})} \mathrm{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathrm{MExt}(Z_a, \ldots, Z_{d-1})$$

and

$$+_2 : \mathrm{MExt}(Z_a, \ldots, Z_{d-1}) \times_{\mathrm{MExt}(Z_a,\ldots,Z_{c-1})} \mathrm{MExt}(Z_a, \ldots, Z_{d-1}) \longrightarrow \mathrm{MExt}(Z_a, \ldots, Z_{d-1})$$

8. Some examples

8.1. *An $\mathbb{F}_{p^2}$-sustained p-divisible group without an $\mathbb{F}_{p^2}$-model.*

We recall a well-known construction of supersingular p-divisible groups of height 4. Let Y_1 be the p-divisible group over $\mathbb{F}_p$ whose covariant Diedonné module is

$$M_1 := \mathrm{Car}_p(\mathbb{F}_p)/\mathrm{Car}_p(\mathbb{F}_p)(F - V).$$

Let $e \in M_1$ be the image of 1 in M_1, so that M_1 is a free $W(\mathbb{F}_p)$-module with basis $e_1, f_1 := Fe_1 = Ve_1$. The p-divisible group Y_1 contains a unique subgroup α. isomorphic to α_p. The quotient homomorphism $Y_1 \to Y_1/\alpha$ corresponds to the inclusion $Y_1 \hookrightarrow Y_1 + p^{-1}f_1$, consider as a $\mathrm{Car}_p(\mathbb{F}_p)$-linear map of $\mathrm{Car}_p(\mathbb{F}_p)$-modules. Let $X_1 := Y_1 \times_{\mathrm{Spec}(\mathbb{F}_p)} Y_1$, which contains $\alpha \times_{\mathrm{Spec}(\mathbb{F}_p)} \alpha$.

For every field $k \supset \mathbb{F}_p$, we have a natural isomorphism

$$k \xrightarrow{\sim} \mathrm{End}_k(\alpha),$$

which induces a bijection from $\mathbb{P}^1(k)$ to the set of all k-epimorphisms

$$(\alpha \times_{\mathrm{Spec}(\mathbb{F}_p)} \alpha) \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(k) \twoheadrightarrow \alpha \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(k).$$

For every element $t \in \mathbb{P}^1(k)$, let α_t be the kernel of the epimorphism $h_t : \alpha\times\alpha \to \alpha$, and define a supersingular p-divisible group X_t over k by

$$X_t := X_1/\alpha_t\,.$$

It is well-known that the following statements hold for every perfect field k containing $\mathbb{F}_{p^2}$.

- The p-divisible group X_t has α-number 2 if and only if t belongs to $\mathbb{P}^1(\mathbb{F}_{p^2})$.
- For any two points $t_1, t_2 \in \mathbb{P}^1(k)$, the p-divisible groups X_{t_1} and X_{t_2} are isomorphic over k if and only if t_1 and t_2 belong to the same $\mathrm{GL}_2(\mathbb{F}_{p^2})$-orbit.
- Every p-divisible group over k is isomorphic to X_t for some $t \in \mathbb{P}^1(k)$.

There exist plenty of examples of elements $s \in k = \mathbb{A}^1(k) \subset \mathbb{P}^1(k)$ satisfying the following conditions:

(i) $s \notin \mathbb{F}_{p^2}$

(ii) There exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{p^2})$ such that $s^p = \frac{as+b}{cs+d}$.

Condition (ii) implies that $\sigma^* X_s$ is isomorphic to X_s for every element $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^2}(s)/\mathbb{F}_p)$, so the p-divisible group X_s is $\mathbb{F}_p$-sustained, and of course also $\mathbb{F}_{p^2}$-sustained. Condition (i) implies that X_s does not have a model over $\mathbb{F}_{p^2}$, therefore not strongly $\mathbb{F}_{p^2}$-sustained.

8.2. *Formal completion of a central stream with slopes $\frac{1}{3}$ and $\frac{2}{3}$*

8.2.1. Denote by $G_{1,2}$ the one-dimensional p-divisible group over $\mathbb{F}_p$ of slope $1/3$ whose covariant Dieudonné module M is a free $W(\mathbb{F}_p)$ module with basis e_1, e_2, e_3 such that

$$Ve_1 = e_2,\ Ve_2 = e_3,\ Ve_3 = pe_2; \quad pe_1 = Fe_2,\ pe_2 = Fe_3,\ e_3 = Fe_2.$$

Denote by $G_{2,1}$ the two-dimensional p-divisible group over $\mathbb{F}_p$ whose covariant Dieudonné module N is a free $W(\mathbb{F}_p)$-module with basis f_1, f_2, f_3 such that

$$Ff_1 = f_2,\ Ff_2 = f_3,\ Ff_3 = pf_1; \quad pf_1 = Vf_2,\ pf_2 = Vf_3,\ f_3 = Vf_1.$$

Let $e_1^\vee, e_2^\vee, e_3^\vee$ be the $W(\mathbb{F}_p)$-basis of $\mathrm{Hom}_{W(\mathbb{F}_p)}(M, W(\mathbb{F}_p))$ dual to the $W(\mathbb{F}_p)$-basis e_1, e_2, e_3 of M. Let

$$H := \mathrm{Hom}_{W(\mathbb{F}_p)}(M, N) \in h_{ij} := f_i \otimes e_j^\vee \quad \forall i,j \in \{1,2,3\}.$$

The natural action of F on H and V on $H[1/p]$ on the basis element h_{ij} is given explicitly as follows.

$$\begin{array}{cccccc}
Vh_{11} = h_{32} & Vh_{12} = h_{33} & Vh_{13} = p^{-1}h_{31} & ph_{11} = Fh_{32} & ph_{12} = Fh_{33} & h_{13} = Fh_{31} \\
Vh_{21} = ph_{12} & Vh_{22} = ph_{13} & Vh_{23} = h_{11} & h_{21} = Fh_{12} & h_{22} = Fh_{13} & ph_{23} = Fh_{11} \\
Vh_{31} = ph_{22} & Vh_{32} = ph_{23} & Vh_{33} = h_{21} & h_{31} = Fh_{22} & h_{32} = Fh_{23} & ph_{33} = Fh_{21}
\end{array}$$

Let $h'_{13} := ph_{13}$, and let $\tilde{h}_{31} := p^{-1}h_{31}$. Let

- $H_1 :=$ the $W(\mathbb{F}_p)$-span of h'_{13} and all h_{ij}'s with $(i,j) \neq (1,3)$, and
- $H_2 :=$ the $W(\mathbb{F}_p)$-span of h'_{31} and all h_{ij}'s with $(i,j) \neq (3,1)$.

Clearly H_1 is the largest $W(\mathbb{F}_p)$-submodule $H_{F,V}$ of H stable under the actions of F and V, and H_2 is the smallest $W(\mathbb{F}_p)$-submodule $H^{F,V}$ of $H[1/p]$ containing H which is stable under the actions of F and V. Therefore H_1 is the covariant Dieudonné module of $\mathcal{D}ef(G_{2,1}\times G_{1,2})_{\rm sus} = \mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}$, the maximal p-divisible subgroup of the smooth formal group $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})$ over $\mathbb{F}_p$, while H_2 is the covariant Dieudonné module of the $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})^{\rm div}$, the maximal p-divisible quotient of $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})$.

It is a remarkable coincidence that H_1 and H_3 are both direct sum of three Dieudonné submodules isomorphic to M: we have

$$H_1 = \big(Wh_{22}+Wh'_{13}+Wh_{31}\big)+\big(Wh_{23}+Wh_{11}+Wh_{32}\big)+\big(Wh_{12}+Wh_{33}+Wh_{21}\big)$$

and

$$H_2 = \big(Wh_{13}+W\tilde{h}_{31}+Wh_{22}\big)+\big(Wh_{23}+Wh_{11}+Wh_{32}\big)+\big(Wh_{12}+Wh_{33}+Wh_{21}\big),$$

and each of the summand grouped under a pair of parentheses is a Dieudonné submodule. The intersection $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}\cap\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm unip}$ of the maximal p-divisible subgroup and the maximal smooth unipotent subgroup of $\mathcal{D}ef(G_{1,2},G_{2,1})$ is equal to the kernal of the natural isogeny

$$\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}\longrightarrow\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})^{\rm div};$$

this isogeny corresponds to the inclusion map $H_1\hookrightarrow H_2$ of covariant Dieudonné modules. We conclude that

$$\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}\cap\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm unip}\cong\alpha_{p^2}:={\rm Ker}({\rm Fr}^2:\mathbb{G}_a\to\mathbb{G}_a).$$

Note also that there is a unique W-submodule of H_2 which contains H_1 which is stable under both F and V and not equal to H_1 or H_2, namely $H_3:=H_1+W\cdot\tilde{h}_{31}$. The inclusion $H_1\hookrightarrow H_3$ corresponds to an isogeny of p-divisible groups, whose kernel is the unique subgroup scheme of order p contained in the subgroup scheme. $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}\cap\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm unip}$. Notice also that the subgroup scheme $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}\cap\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm unip}$ of $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}$ is contained in the one-dimensional p-divisible formal subgroup of $\mathcal{E}xt_{\rm def}(G_{1,2},G_{2,1})_{\rm div}$ whose covariant Dieudonné module is $Wh_{22}+Wh'_{1,3}+Wh_{31}$, because the inclusion map induces a map from the pair $Wh_{22}+Wh'_{1,3}+Wh_{31}\hookrightarrow Wh_{22}+Wh_{1,3}+W\tilde{h}_{31}$ to the pair $H_1\hookrightarrow H_2$.

8.2.2. We need to set up notation related to duality and polarization before computing the maximal closed formal subscheme $\mathcal{D}ef(G_{1,2}\times G_{2,1},\nu\times\nu^t)_{\rm sus}$ of the deformation space $\mathcal{D}ef(G_{1,2}\times G_{2,1},\alpha\times\nu^t)\subset\mathcal{D}ef(G_{1,2}\times G_{2,1})$ of a polarization $\nu\times\nu^t$ of $G_{1,2}\times G_{2,1}$.

Recall that a polarization on a p-divisible group X over a base scheme S is an S-isogenay μ from X to its Serre dual X^t such that $\mu^t=\mu$, after using the canonical isomorphism $X\xrightarrow{\sim}(X^t)^t$ to identify the Serre dual of X^t with X. The Serre-dual of $G_{1,2}\times G_{2,1}$ is $G^t_{2,1}\times G^t_{1,2}$. It is well-known that $G_{1,2}$ and $G_{2,1}$ are dual to each other. Therefore every polarization μ on $G_{12}\times G_{2,1}$ is necessarily of the form $\mu=\nu\times\nu^t$, where $\nu:G_{1,2}\to G^t_{2,1}$ is an isogeny, and $\nu^t:G_{2,1}\to G^t_{1,2}$ is the Serre-dual of ν.

Recall also when the base scheme S is the spectrum of a perfect field k, the covariant Dieudonné module $\mathrm{D}_*(X^t)$ of a p-divisible group X over k naturally identified with the $W(k)$-linear dual of $\mathrm{D}_*(X)$, and the action of the semi-linear operators F and V on $\mathrm{D}_*(X)^\vee := \mathrm{Hom}_{W(k)}(\mathrm{D}_*(X), W(k))$ is given by

$$(Fh)(x) = \sigma(h(V(x))), \quad (Vh)(x) = \sigma^{-1}(h(F(x)) \quad \forall x \in \mathrm{D}_*(X)$$

for every element $h \in \mathrm{D}_*(X)^\vee$, where σ is the ring automorphism of $W(k)$ which induces the absolute Frobenius $a \mapsto a^p$ on the residue field k of $W(k)$.

We go back to $G_{1,2}$ and $G_{2,1}$, with $\mathbb{F}_p$ as the base field, and write W for $W(\mathbb{F}_p)$. Let $N^\vee = Wf_1^\vee + Wf_2^\vee + Wf_3^\vee$ be the W-linear dual of N, where $f_1^\vee, f_2^\vee, f_3^\vee$ are the dual basis of f_1, f_2, f_3. We have

$$Vf_2^\vee = f_1^\vee, Vf_3^\vee = f_2^\vee, Vf_1^\vee = pf_3^\vee, \quad Ff_1^\vee = pf_2^\vee, Ff_2^\vee = pf_3^\vee, Ff_3^\vee = f_1^\vee.$$

(1) The following W-linear isomorphism

$$\beta_1 : M \to N^\vee \qquad \beta_1 : e_1 \mapsto f_3^\vee,\ e_2 \mapsto f_2^\vee,\ e_3 \mapsto f_1^\vee$$

is an isomorphism of Dieudonné modules.

Similarly we have $M^\vee = We_1^\vee + We_2^\vee + We_3^\vee$,

$$Ve_3^\vee = e_1^\vee, Ve_2^\vee = pe_3^\vee, Ve_1^\vee = pe_2^\vee, Fe_3^\vee = e_2^\vee, Fe_2^\vee = e_1^\vee, Fe_1^\vee = pe_3^\vee.$$

and the dual

$$\beta_1^\vee : N \to M^\vee \qquad \beta_1^\vee : f_1 \mapsto e_3^\vee,\ f_2 \mapsto e_2^\vee,\ f_3 \mapsto e_1^\vee$$

of β_1 is an isomorphism of Dieudonné modules. Note that the matrix representation of β_1 and $\beta_1^\vee$ are both $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. The bases used for the matrix representations are (e_1, e_2, e_3) for M, $(f_1^\vee, f_2^\vee, f_3^\vee)$ for $N^\vee$, (f_1, f_2, f_3) for N and $(e_1^\vee, e_2^\vee, e_3^\vee)$ for $M^\vee$ respectively.

It is clear from the above discussion that isomorphism

$$\beta_1 \times \beta_1^\vee : M \oplus N \longrightarrow N^\vee \oplus M^\vee$$

of Dieudonné modules corresponds to a principal polarization

$$\mu_1 = \nu_1 \times \nu_1^t : G_{1,2} \times G_{2,1} \longrightarrow G_{2,1}^t \times G_{1,2}^t\,.$$

(2) To get a poloarization of degree p^2 on $G_{1,2} \times G_{2,1}$, take the homomorphisms

$$\beta_2 : M \to N^\vee \qquad e_1 \mapsto f_2^\vee,\ e_2 \mapsto f_1^\vee,\ e_3 \mapsto pf_3^\vee$$

of Dieudonné modules, and its dual

$$\beta_2^\vee : N \to M^\vee \qquad f_1 \mapsto e_2^\vee,\ f_2 \mapsto e_1^\vee,\ f_3 \mapsto pe_3^\vee.$$

Their external product

$$\beta_2 \times \beta_2^t : M \oplus N \longrightarrow N^\vee \oplus M^\vee$$

corresponds to a polarization

$$\mu_2 = \nu_2 \times \nu_2^t : G_{1,2} \times G_{2,1} \longrightarrow G_{2,1}^t \times G_{1,2}^t$$

such that $\mathrm{Ker}(\mu_2) \cong \alpha_p \times \alpha_p$. The matrix representations of β_2 and $\beta_2^\vee$ are both equal to $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & p \end{bmatrix}$.

(3) Define a homomorphism $\beta_3 : M \to N^\vee$ of Dieudonné modules by

$$\beta_2 : M \to N^\vee \qquad e_1 \mapsto f_1^\vee,\ e_2 \mapsto pf_3^\vee,\ e_3 \mapsto pf_2^\vee.$$

The dual of β_3 is the homomorphism

$$\beta_3^\vee : N \to M^\vee \qquad f_1 \mapsto e_1^\vee,\ f_2 \mapsto e_3^\vee,\ f_3 \mapsto pe_2^\vee.$$

The matrix representations of β_2 and $\beta_2^\vee$ are both equal to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{bmatrix}$.

Let $\nu_3 : G_{1,2} \to G_{2,1}^t$ and $\nu_3^t : G_{2,1} \to G_{1,2}^t$ be the isogenies corresponding to β_3 and $\beta_3^\vee$ respectively. The kernel of the polarization

$$\mu_3 := \nu_3 \times \nu_3^t : G_{1,2} \times G_{2,1} \to G_{2,1}^t \times G_{1,2}^t$$

is isomorphic to $\alpha_{p^2} \times \alpha_{p^2}$.

8.2.3. *How to compute the sustained locus in $\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu \times \nu^t)$*

xxxxxxxx refer/add justification

Let $\mu = \nu \times \nu^t : G_{1,2} \times G_{2,1} \to G_{2,1}^t \times G_{1,2}^t$ be a polarization of $G_{1,2} \times G_{2,1}$. We have two homomorphisms of p-divisible groups from $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\text{p-div}}$ to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{1,2}^t)_{\text{p-div}}$:

(i) the composition of the canonical isomorphism

$$\mathrm{can} : \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \xrightarrow{\sim} \mathcal{E}xt_{\mathrm{def}}(G_{2,1}^t, G_{1,2}^t)_{\mathrm{div}}$$

with the homomorphism

$$\mathcal{E}xt_{\mathrm{def}}(\nu, G_{1,2}^t)_{\mathrm{div}} : \mathcal{E}xt_{\mathrm{def}}(G_{2,1}^t, G_{1,2}^t)_{\mathrm{div}} \to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{1,2}^t)_{\mathrm{div}}$$

given by functorialty of the functor $\mathcal{E}xt_{\mathrm{def}}(?, G_{1,2}^t)_{\mathrm{div}}$ isogeny $\nu : G_{1,2} \to G_{2,1}^t$.

(ii) the functorial homomorphism

$$\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, \nu^t)_{\mathrm{div}} : \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1}) \to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{1,2}^t)$$

given by the functor $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, ?)_{\mathrm{div}}$ and the isogeny $\nu^t : G_{2,1} \to G_{1,2}^t$.

The sustained locus $\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu \times \nu^t)_{\mathrm{sus}}$ of the local deformation space $\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu \times \nu^t)$ of the polarized p-divisible group $(G_{1,2} \times G_{2,1}, \nu \times \nu^t)$ is the equalizer of the above two homomorphisms:

$$\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu \times \nu^t)_{\mathrm{sus}} = \mathrm{Ker}\big[\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, \nu^t)_{\mathrm{div}} - \mathcal{E}xt_{\mathrm{def}}(\nu, G_{1,2}^t)_{\text{p-div}} \circ \mathrm{can}\big]$$

We hav seen that the Dieudonné module $\mathrm{Hom}_W(M, N)_{F,V}$ of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}}$ is the the W-submodule of $\mathrm{Hom}_W(M, N)$ spanned by $pf_1 \otimes e_3^\vee$ and $f_i \otimes e_j^\vee$ with $(i, j) \neq (1, 3)$. In view of the isomorphism $\beta_1^\vee : N \xrightarrow{\sim} M^\vee$ of Dieudonné modules which sends f_1 to $e_3^\vee$, we conclude that the Dieudonné module $\mathrm{Hom}_W(M, M^\vee)_{F,V}$ of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{1,2})_{\mathrm{div}}$ is the W-submodule of $\mathrm{Hom}_W(M, M^\vee)$ spanned by $pe_3^\vee \otimes e_3^\vee$ and $f_i \otimes e_j^\vee$ with $(i, j) \neq (3, 3)$.

Let $\beta : M \to N^\vee$ and $\beta^\vee : N \to M^\vee$ be the homomorphisms between Dieudonné modules corresponding to ν and $\nu^\vee$ respectively. Clearly the homomorphism from $\mathrm{Hom}_W(M,N)_{\mathrm{div}}$ to $\mathrm{Hom}_W(M,M^\vee)_{\mathrm{div}}$ corresponding to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2},\nu^t)_{\mathrm{div}}$ is the restriction to $\mathrm{Hom}_W(M,N)_{\mathrm{div}}$ of the homomorphism

$$\mathrm{Hom}_W(M,N) \to \mathrm{Hom}_W(M,M^\vee), \qquad h \mapsto \beta^\vee \circ h \quad \forall h \in \mathrm{Hom}_W(M,N),$$

while the homomorphism from $\mathrm{Hom}_W(M,N)_{F,V}$ to $\mathrm{Hom}_W(M,M^\vee)_{F,V}$ corresponding to $\mathcal{E}xt_{\mathrm{def}}(\nu, G^t_{1,2})_{\mathrm{div}} \circ \mathrm{can}$ is the restriction to $\mathrm{Hom}_W(M,N)_{\mathrm{div}}$ of the homomorphism

$$\mathrm{Hom}_W(M,N) \to \mathrm{Hom}_W(M,M^\vee), \qquad h \mapsto h^\vee \circ \beta \quad \forall h \in \mathrm{Hom}_W(M,N).$$

To summarize the discussion: Let $\nu \times \nu^t : G_{1,2} \times G_{2,1} \to G^t_{2,1} \times G^t_{1,2}$ be a polarization of $G_{1,2} \times G_{2,1}$. Let $\beta : M \to N$ and $\beta : N^\vee \to M^\vee$ be the homomorphism between Dieudonné modules corresponding to ν and ν^t respectively. Let

$$\xi(\nu) : \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G^t_{1,2})_{\mathrm{div}}$$

be the homomorphism between p-divisible group corresponding to the homomorphism

$$\Xi(\beta) : \mathrm{Hom}_W(M,N)_{F,V} \to \mathrm{Hom}_W(M,M^\vee)_{F,V} \qquad h \mapsto \beta^\vee \circ h - h^\vee \circ \beta$$

for all $h \in \mathrm{Hom}_W(M,N)_{F,V}$. Then the sustained locus $\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu \times \nu^t)_{\mathrm{sus}}$ in the deformation space of the polarized p-divisible group $(G_{1,2} \times G_{2,1}, \nu \times \nu^t)$ is a subgroup of the p-divisible group $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}}$ is equal to the kernel of $\xi(\nu)$. Note that when $\beta : M \to N^\vee$ is an isomorphism, the equality $\beta^\vee \circ h - h^\vee \circ \beta$ is equivalent to the more familiar form $h = (\beta^\vee)^{-1} \circ h^\vee \circ \beta$, and

$$h \mapsto (\beta^\vee)^{-1} \circ h^\vee \circ \beta$$

is an involution on the Dieudonné module $\mathrm{Hom}_W(M,N)_{F,V}$ of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}}$.

8.2.4. *The sustained locus in the principally polarized case*

Consider the principal polarization $\nu_1 \times \nu_1^t$ on $G_{1,2} \times G_{2,1}$ in 8.2.2. The sustained locus in the deformation space $\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu_1 \times \nu_1^t)$ is equal to the kernel of

$$(\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, \nu_1^t)_{\mathrm{div}} - \mathcal{E}xt_{\mathrm{def}}(\nu_1, G^t_{1,2})_{\mathrm{div}} \circ \mathrm{can}) : \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \longrightarrow \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{1,2})_{\mathrm{div}}$$

We use the base $(e_j), (e_i^\vee), (f_j), (f_j^\vee)$, $1 \le i,j \le 3$, for M, $M^\vee$, N, $N^\vee$, and the resulting coordinates for $\mathrm{Hom}_W(M,N)$ and $\mathrm{Hom}_W(M,M^\vee)$ Thus a typical element of the Dieudonné module $\mathrm{Hom}_W(M,N)_{F,V} = \mathrm{D}_*(\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}})$ is represented by a matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{with} \quad a,b,d,e,f,g,h,i \in W,\ c \in pW.$$

Similarly a typical element of $\mathrm{Hom}_W(M,N)_{F,V} = \mathrm{D}_*(\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}})$ is represented by a matrix

$$\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} \quad \text{with} \quad r,s,t,u,v,w,x,y \in W,\ z \in pW.$$

The homomorphism

$$\mathcal{E}xt_{\mathrm{def}}(G_{1,2},\nu_1^t)_{\mathrm{div}} : \mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}} \longrightarrow \mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{1,2})_{\mathrm{div}}$$

is represented by the map

$$\mathrm{Hom}_W(M,N)_{F,V} \longrightarrow \mathrm{Hom}_W(M,M^\vee)_{F,V}$$

given by the following explicit formula

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

Similarly the homomorphism

$$\mathcal{E}xt_{\mathrm{def}}(\nu_1, G_{1,2}^t)_{\mathrm{div}} \circ \mathrm{can} : \mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}} \longrightarrow \mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{1,2})_{\mathrm{div}}$$

corresponds to the map from $\mathrm{Hom}_W(M,N)_{F,V}$ to $\mathrm{Hom}_W(M,M^\vee)_{F,V}$ given by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g & d & a \\ h & e & b \\ i & f & c \end{pmatrix}$$

The difference of the above maps is the homomorphism

$$\Xi(\beta_1) : \mathrm{Hom}_W(M,N)_{F,V} \longrightarrow \mathrm{Hom}_W(M,M^\vee)_{F,V}$$

given by

$$\Xi(\beta_1) : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} 0 & h-d & i-a \\ d-h & 0 & f-b \\ a-i & b-f & 0 \end{pmatrix}$$

for all $a,b,d,e,f,g,h,i \in W$, $c \in pW$.

The kernel of $\Xi(\beta_1)$ is a direct summand of $\mathrm{Hom}_W(M,N)_{F,V}$, equal to

$$\bigl[W\cdot(h_{12}+h_{23}) + W\cdot(h_{11}+h_{33}) + W\cdot(h_{21}+h_{32})\bigr] + \bigl[W\cdot h_{22} + W\cdot h'_{13} + W\cdot h_{31}\bigr].$$

Moreover each of the two W-submodules of rank three grouped together under square brackets in the above displayed formula is stable under F and V, and isomorphic to M as Dieudonné modules. The image of $\Xi(\beta_1)$ is a W-direct summand of $\mathrm{Hom}_W(M,M^\vee)$, equal to

$$W\cdot(e_2^\vee\otimes e_3^\vee - e_3^\vee\otimes e_2^\vee) + W\cdot(e_3^\vee\otimes e_1^\vee - e_1^\vee\otimes e_3^\vee) + W\cdot(e_1^\vee\otimes e_2^\vee - e_2^\vee\otimes e_1^\vee),$$

where $e_i^\vee\otimes e_j^\vee$ denotes the element of $\mathrm{Hom}_W(M,M^\vee)$ which sends e_l to $\delta_{jl}e_i^\vee$ for all $i,j,l=1,2,3$. A simple caluculation shows that the action of V on this Dieudonné submodule is given by:

- $V\cdot(e_2^\vee\otimes e_3^\vee - e_3^\vee\otimes e_2^\vee) = e_3^\vee\otimes e_1^\vee - e_1^\vee\otimes e_3^\vee$,
- $V\cdot(e_3^\vee\otimes e_1^\vee - e_1^\vee\otimes e_3^\vee) = e_1^\vee\otimes e_2^\vee - e_2^\vee\otimes e_1^\vee$,
- $V\cdot(e_1^\vee\otimes e_2^\vee - e_2^\vee\otimes e_1^\vee) = p(e_2^\vee\otimes e_3^\vee - e_3^\vee\otimes e_2^\vee)$.

We conclude that the sustained locus of the deformation space $\mathcal{D}ef(G_{1,2}\times G_{2,1},\nu_1\times\nu_1^t)$ is equal to the p-divisible subgroup of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}}$, whose covariant Dieudonné module is the direct summand $\mathrm{Ker}(\Xi(\beta_1))$ of $\mathrm{Hom}_W(M,N)_{F,V}$.

Is the subgroup scheme $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \cap \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ contained in the sustained locus of $\mathcal{D}ef(G_{1,2}\times G_{2,1}, \nu_1\times\nu_1^t)$? The answer is "yes": this subgroup scheme, isomorphic to α_{p^2}, is contained in the one-dimensional p-divisible subgroup of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}}$ whose Dieudonné module is the submodule $Wh_{22}+Wh'_{13}+Wh_{31}$ of $\mathrm{Hom}_W(M,N)_{F,V}$. The last submodule is contained in $\mathrm{Ker}(\Xi(\beta_1))$ as we just saw, which means that the one-dimensional p-divisible group corresponding to it is contained in $\mathcal{D}ef(G_{1,2}\times G_{2,1}, \nu_1\times\nu_1^t)_{\mathrm{sus}}$.

8.2.5. *The sustained locus for a polarization of degree p^2*

Consider now the polarization $\nu_2\times\nu_2^t$ on $G_{1,2}\times G_{2,1}$ in 8.2.2. This time the isogeny $\nu_2 : G_{1,2}\to G_{2,1}^t$ corresponds to the map $\beta_2 : M\to N^\vee$ with matrix representation $\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&p\end{pmatrix}$. The homomorphism from $\mathrm{Hom}_W(M,N)_{F,V}$ to $\mathrm{Hom}_W(M,M^\vee)_{F,V}$ corresponding to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2},\nu_2^t)_{\mathrm{div}}$ is given by

$$\begin{pmatrix} a&b&c\\ d&e&f\\ g&h&i\end{pmatrix} \mapsto \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&p\end{pmatrix}\cdot\begin{pmatrix} a&b&c\\ d&e&f\\ g&h&i\end{pmatrix} = \begin{pmatrix} d&e&f\\ a&b&c\\ pg&ph&pi\end{pmatrix}$$

The homomorphism from $\mathrm{Hom}_W(M,N)_{F,V}$ to $\mathrm{Hom}_W(M,M^\vee)_{F,V}$ which corresponds to $\mathcal{E}xt_{\mathrm{def}}(\nu_2, G_{1,2}^t)\circ\mathrm{can}$ is given by

$$\begin{pmatrix} a&b&c\\ d&e&f\\ g&h&i\end{pmatrix} \mapsto \begin{pmatrix} a&d&g\\ b&e&h\\ c&f&i\end{pmatrix}\cdot\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&p\end{pmatrix} = \begin{pmatrix} d&a&pg\\ e&b&ph\\ f&c&pi\end{pmatrix}$$

The difference of these two maps is the linear map $\Xi(\beta_2)$ from $\mathrm{Hom}_W(M,N)_{F,V}$ to $\mathrm{Hom}_W(M,M^\vee)_{F,V}$ given by

$$\Xi(\beta_2) : \begin{pmatrix} a&b&c\\ d&e&f\\ g&h&i\end{pmatrix} \mapsto \begin{pmatrix} 0&e-a&f-pg\\ a-e&0&c-ph\\ pg-f&ph-c&0\end{pmatrix}$$

We have

$$\mathrm{Ker}(\Xi(\beta_2)) = \big(Wh_{12}+Wh_{33}+Wh_{21}\big)\ +\ \big(W(h_{11}+h_{22})+W(h'_{13}+h_{32})+W(h_{31}+ph_{23})\big)$$

Moreover

- $Vh_{12}=h_{33}$, $Vh_{33}=h_{21}$, $Vh_{21}=ph_{12}$,
- $V(h_{11}+h_{22}) = h'_{13}+h_{32}$, $V(h'_{13}+h_{32}) = h_{31}+ph_{23}$, $V(h_{31}+ph_{23}) = p(h_{11}+h_{22})$.

The following statements are easily checked.

(1) $\mathrm{Ker}(\Xi(\beta_2))_{\mathbb{Q}}\cap H_2 = \mathrm{Ker}(\Xi(\beta_2)) + W\cdot p^{-1}(h_{31}+ph_{23})$
(2) The homomorphism $\Xi(\beta_2)$ extends to a W-linear homomorphism from $H_2 = \mathrm{Hom}_W(M,N)+W\tilde{h}_{31}$ to $\mathrm{Hom}_W(M,M^\vee)_{F,V}$, because $\tilde{h}_{31}$ is sent to $e_3^\vee\otimes e_3\in\mathrm{Hom}_W(M,M^\vee)_{F,V}$.
(3) The image of the monomorphism

$$H_2/(\mathrm{Ker}(\Xi(\beta_2))_{\mathbb{Q}}\cap H_2)\longrightarrow \mathrm{Hom}_W(M,M^\vee)_{F,V}$$

induced by $\Xi(\beta_2)_{\mathbb{Q}}$ is free direct summand of $\mathrm{Hom}_W(M, M^\vee)_{F,V}$ as a W-module.

We know that the kernel of the homomorphism

$$\xi(\nu_2) : \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \longrightarrow \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G^t_{1,2})_{\mathrm{div}}$$

of p-divisible groups corresponding to the homomorphism

$$\Xi(\beta_2) : \mathrm{Hom}_W(M, N) \longrightarrow \mathrm{Hom}_W(M, M^\vee)$$

of Dieudonné modules is equal to the sustained locus of the deformation space $\mathcal{D}ef(G_{1,2} \times G_{2,1}, \nu_2 \times \nu_2^t))$. Let $\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$ be the maximal p-divisible subgroup of $\mathrm{Ker}(\xi(\nu_2))$; and the quotient $\mathrm{Ker}(\xi(\nu_2))/\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$ is a commutative finite group scheme over $\mathbb{F}_p$. The statement (1) means that the intersection

$$\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}} \cap \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}} =: L_1$$

is the unique non-trivial subgroup scheme of order p of

$$L_2 := \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \cap \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}} \cong \nu_{p^2}.$$

The statement (2) means that the finite group scheme $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \cap \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ is contained in $\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$. The statement (3) implies that the natural map

$$L_2/L_1 \longrightarrow \mathrm{Ker}(\xi(\nu_2))/\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$$

is an isomorphism. In particular

$$\mathrm{Ker}(\xi(\nu_2)) = L_2 + \mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}\,.$$

8.2.6. *The sustained locus for a polarization of degree* p^4

The homomorphism from $\mathrm{Hom}_W(M, N)_{F,V}$ to $\mathrm{Hom}_W(M, M^\vee)_{F,V}$ corresponding to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, \nu_3^t)_{\mathrm{div}}$ is given by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ pg & ph & pi \\ pd & pe & pf \end{pmatrix}$$

The homomorphism from $\mathrm{Hom}_W(M, N)_{F,V}$ to $\mathrm{Hom}_W(M, M^\vee)_{F,V}$ which corresponds to $\mathcal{E}xt_{\mathrm{def}}(\nu_3, G^t_{1,2}) \circ \mathrm{can}$ is given by

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{pmatrix} = \begin{pmatrix} a & pg & pd \\ b & ph & pe \\ c & pi & pf \end{pmatrix}$$

The difference of these two maps is the linear map $\Xi(\beta_2)$ from $\mathrm{Hom}_W(M, N)_{F,V}$ to $\mathrm{Hom}_W(M, M^\vee)_{F,V}$ given by

$$\Xi(\beta_3) : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} 0 & b - pg & c - pd \\ pg - b & 0 & pi - pe \\ pd - c & pe - pi & 0 \end{pmatrix}$$

We have

$$\mathrm{Ker}(\Xi(\beta_3)) = \big(Wh_{23} + Wh_{11} + Wh_{32}\big) + \big(W(h_{22} + h_{23}) + W(h_{21} + h'_{13}) + W(ph_{12} + h_{31})\big)$$

and

- $Vh_{23} = h_{11}$, $Vh_{11} = h_{32}$, $Vh_{32} = ph_{23}$,

- $V(h_{22}+h_{23}) = (h_{21}+h'_{13})$, $V(h_{21}+h'_{13}) = ph_{12}+h_{31}$, $V(ph_{12}+h_{31}) = p(h_{22}+h_{23})$.

So both $Wh_{23}+Wh_{11}+Wh_{32}$ and $W(h_{22}+h_{23})+W(h_{21}+h'_{13})+W(ph_{12}+h_{31})$ are isomorphic to M as Dieudonné modules. Moreover an easy computation shows that

(1) $\mathrm{Ker}(\Xi(\beta_3))_{\mathbb{Q}} \cap H_2 = \mathrm{Ker}(\Xi(\beta_3) + W\cdot p^{-1}(ph_{12}+h_{31})$.

(2) The image of $H_1 = \mathrm{Hom}_W(M,N)_{F,V}$ under $\Xi(\beta_3)$ is the submodule

$$W\cdot(e_1^\vee\otimes e_2^\vee - e_2^\vee\otimes e_1^\vee) + pW\cdot(e_1^\vee\otimes e_3^\vee - e_3^\vee\otimes e_1^\vee) + pW\cdot(e_2^\vee\otimes e_3^\vee - e_3^\vee\otimes e_2^\vee)$$

of $\mathrm{Hom}_W(M,M^\vee)_{F,V}$.

(3) The image of H_2 under $\Xi(\beta_3)_{\mathbb{Q}}$ is contained in $\mathrm{Hom}_W(M,M^\vee)_{F,V}$. (In fact $\Xi(\beta_3)_{\mathbb{Q}}(H_2) = \Xi(\beta_3)(H_1)$.)

Let $\xi(\nu_3)$ be the homomorphism from $\mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}}$ to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{1,2}^t)_{\mathrm{div}}$ corresponding to $\Xi(\beta_3)$. The sustained locus of $\mathcal{D}ef(G_{1,2}\times G_{2,1},\nu_3\times\nu_3^t)$ is equal to $\mathrm{Ker}(\xi(\nu_3))$. The statements (1)–(3) above have the following consequences.

(a) It follows from (1) that the intersection of the maximal p-divisible subgroup $\mathrm{Ker}(\xi(\nu_3))_{\mathrm{div}}$ with L_2 is equal to L_1.

(b) It follows from (2) and (3) that $\mathrm{Ker}(\xi(\nu_3))/\mathrm{Ker}(\xi(\nu_3))_{\mathrm{div}}$ is a commutative finite group scheme over $\mathbb{F}_p$ of order p^2, and the natural map from L_2/L_1 to $\mathrm{Ker}(\xi(\nu_3))/\mathrm{Ker}(\xi(\nu_3))_{\mathrm{div}}$ is a monomorphism whose cokernel is isomorphic to α_p.

A further computation with Dieudonné modules shows that the quotient group scheme $\mathrm{Ker}(\xi(\nu_3))/\mathrm{Ker}(\xi(\nu_3))_{\mathrm{div}}$ is isomorphic to α_{p^2}. Thus the sustained locus $\mathcal{D}ef(G_{1,2}\times G_{2,1},\mu_3)_{\mathrm{sus}}$, being naturally isomorphic to $\mathrm{Ker}(\xi(\nu_3))$, has a natural structure as a formal group. We have a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\xi(\nu_3))_{\mathrm{div}} \longrightarrow \mathrm{Ker}(\xi(\nu_3)) \longrightarrow \alpha_{p^2} \longrightarrow 0.$$

In particular $\mathcal{D}ef(G_{1,2}\times G_{2,1},\mu_3)_{\mathrm{sus}}$ is not reduced, just like $\mathcal{D}ef(G_{1,2}\times G_{2,1},\mu_2)_{\mathrm{sus}}$.

8.2.7. *Arbitrary polarizations on $G_{1,2}\times G_{2,1}$*

Every polarization μ on $G_{1,2}\times G_{2,1}$ has the form $\mu = u\,p^n\mu_i$ for a uniquely determined triple (i,n,u), with $i\in\{1,2,3\}$, $n\in\mathbb{N}$ and $u\in{\mathbb{Z}_p}^\times$. Moreover $\xi(u\,p^n\mu_i) = u\,p^n\xi(\mu_i)$ for all (i,n,u). So

$$\mathcal{D}ef(G_{1,2}\times G_{2,1},u\,p^n\mu_i)_{\mathrm{sus}} = \mathrm{Ker}\big(u\,p^n\xi(\nu_i)) = \mathrm{Ker}\big(p^n\xi(\nu_i))\big)$$

for all (i,n,u). The maximal p-divisible subgroup of $\mathrm{Ker}\big(p^n\xi(\nu_i))\big)$ is equal to $\mathrm{Ker}\big(\xi(\nu_i))\big)_{\mathrm{div}}$. Moreover we have a natural isomorphism

$$\mathrm{Ker}\big(p^n\xi(\nu_i))\big)/\mathrm{Ker}\big(\xi(\nu_i))\big)_{\mathrm{div}} \xleftarrow{\sim} Y_i[p^n],$$

where Y_i is the one-dimensional p-divisible group over $\mathbb{F}_p$ with slope $1/3$ and height 3 whose Dieudonné module is

$$\big(\Xi(\beta_i)(\mathrm{Hom}_W(M,N)_{F,V})\big)[1/p] \cap \mathrm{Hom}_W(M,M^\vee)_{F,V}\,.$$

(Of cousre Y_i is non-canonically isomorphic to $G_{1,2}$.)

8.2.8. *How to "read off" the kernel of a homomorphism between p-divisible groups from their Dieudonné modules.*

In this subsection we first described the p-disibible group structure on the the sustained locus $\mathcal{D}ef(G_{1,2}\times G_{2,1})_{\mathrm{sus}}$ of the deformation space of $G_{1,2}\times G_{2,1}$, by giving its covariant Dieudonné module. This p-divisible group is $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}}$, and its Dieudonné module, denoted by $\mathrm{Hom}_W(M,N)_{F,V}$, is the maximal W-submodule of $\mathrm{Hom}_W(M,N)$ stable under the semi-linear operators F and V on $\mathrm{Hom}_W(M,N)[1/p]$. For any polarization $\mu = u\,p^n(\nu_i\times\nu_i^t) : G_{1,2}\times G_{2,1}\longrightarrow G_{2,1}^t\times G_{1,2}^t$, we identified the the sustained locus $\mathcal{D}ef(G_{1,2}\times G_{2,1},\mu)_{\mathrm{sus}}$ as the kernel of a homomorphism

$$p^n\xi(\nu_i) : \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \longrightarrow \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{1,2}^t)_{\mathrm{div}}$$

of p-divisible groups, and we gave formulas for the homomorphisms

$$p^n\Xi(\beta_i) : \mathrm{Hom}_W(M,N)_{F,V} \longrightarrow \mathrm{Hom}_W(M,M^\vee)_{F,V}$$

of Dieudonné modules corresponding to $p^n\xi(\nu_i)$. Then we stated what these formulas tells us about the kernels $\mathrm{Ker}(p^n\xi(\nu_i))$'s. Here we explain the general procedure of "reading off the kernel" of a homomorphism between p-divisible formal groups from the homomorphism between their covariant Dieudonné modules.

Suppose we have a homomorphism $\psi : Y\to Z$ of p-divisible groups over a perfect field $k\supset\mathbb{F}_p$.

Let $\Psi = \mathrm{D}_*(\psi) : P\to Q$ be the homomorphism induced by ψ, from $P=\mathrm{D}_*(Y)$ to $Q := \mathrm{D}_*(Z)$. Recall that P and Q are both free modules of finite rank over $W{:=}\,W(k)$, with semi-linear actions by F and V; the operator V is topologically nilpotent on both P and Q. The W-linear map is compatible with the operators F, V on P and Q.

Let $U := \mathrm{Ker}(\psi)$, and let U_{div} be the maximal p-divisible subgroup of U. Let $\bar\psi$ is the homomorphism from the p-divisible group Y/U_{div} to the p-divisible group Z induced by ψ. Let Z_1 be the image of $\bar\psi$, a p-divisible subgroup of Z. The homomorphism $\bar\phi$ factors through Z_1, and we have $\bar\psi = j\circ\varphi$, where $\varphi : Y/U_1\to Z_1$ is the isogeny inducec by $\bar\phi$ and $j : Z_1\hookrightarrow Z$ is the inclusion.

We have a commutative diagram

$$(*)\qquad \begin{tikzcd}
0 \arrow[r] & U_{\mathrm{div}} \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d, "="'] & Y/U_{\mathrm{div}} \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & U \arrow[r] & Y \arrow[r, "\psi"] & Z_1 \arrow[r] & 0
\end{tikzcd}$$

where

(1) The top horizontal row of the above commutative diagram $(*)$ corresponds to the short exact sequence

$$0\longrightarrow \mathrm{Ker}(\Psi)\longrightarrow P\longrightarrow P/\mathrm{Ker}(\Psi)\longrightarrow 0$$

of Dieudonné modules. In particular we have a natural isomorphisms

$$\mathrm{D}_*(U_{\mathrm{div}}) \xrightarrow{\sim} \mathrm{Ker}(\Psi),\quad \mathrm{D}_*(Y/U_{\mathrm{div}}) \xrightarrow{\sim} P/\mathrm{Ker}(\Psi)\cong \Psi(P)\,.$$

(2) The above coomutative diagram also gives an natural isomorphism

$$\mathrm{Ker}(\bar{\psi}) = \mathrm{Ker}(\phi) \xrightarrow{\sim} U/U_{\mathrm{div}} .$$

(3) The Dieudonné module $Q_1 = \mathrm{D}_*(Z_1)$ of the p-divisible subgroup $Z_1 \subset Z$ the a W-direct summand of $Q = \mathrm{D}_*(Z)$ given by

$$Q_1 = Q \cap \Psi(P)_{\mathbb{Q}},$$

so that $Q_1/\Psi(P) = (Q/\Psi(P))_{\mathrm{tor}}$, the set of all $W(k)$-torsion elements in $Q/\Psi(P)$.

(4) The isogeny $\phi : Y/U_{\mathrm{div}} \to Z_1$ of p-divisible groups corresponds to the isogeny

$$\Phi_1 : P/\mathrm{Ker}(\Psi) \longrightarrow Q_1$$

of Dieudonné modules induced by Ψ.

In particular, the left at the left of the short exact sequence

$$U_{\mathrm{div}} \rightarrowtail U \twoheadrightarrow U/U_{\mathrm{div}}$$

is given by $\mathrm{D}_*(U_{\mathrm{div}}) = \mathrm{Ker}(\Psi)$, while the term at the right, being isomorphic to the kernel of the isogeny $\phi : Y/U_{\mathrm{div}} \to Z_1$, is described by (4).

8.3. *A family of p-divisible groups over $\mathbb{P}^1$ whose restriction to the formal completion at one point of $\mathbb{P}^1$ is the p-divisible group over $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$*

8.3.1. The basic idea underlying the construction used in 8.1 is the following: suppose that we have a p-divisible group Y over a field $K \supset \mathbb{F}_p$. Suppose that Y contains a finite subgroup scheme G isomorphic to $\alpha_p \times \alpha_p$ over the base field K. Subgroups of G which are isomorphic to α_p are parametrized by $\mathbb{P}^1_{/K}$. Use this $\mathbb{P}^1$ as the base scheme, we can divide the constant p-divisible group $Y \times_{\mathrm{Spec}(K)} \mathbb{P}^1 \to \mathbb{P}^1$ by the tautological family of subgroups $\boldsymbol{\alpha} \to \mathbb{P}^1$ of $G \times_{\mathrm{Spec}(K)} \mathbb{P}^1$, and obtain a p-divisible group $X := Y \times_{\mathrm{Spec}(K)} \mathbb{P}^1/\boldsymbol{\alpha} \longrightarrow \mathbb{P}^1$ over $\mathbb{P}^1$. In favorable situations, the isomorphism type of a fiber X_t in this family either determines or "almost determines" the parameter t, so that the base scheme $\mathbb{P}^1$ serves as some kind of "moduli space" for p-disibivle groups in this family.

8.3.2. Consider the case when $Y = G_{1,2} \times G_{2,1}$ over $\mathbb{F}_p$ in the notation of 8.2. Both $G_{1,2}$ and $G_{2,1}$ contains a unique subgroup scheme isomorphic to α_p. Since the base field is the prime field $\mathbb{F}_p$, the Frobenius and the Verschiebung for $G_{1,2}$ are endomorphisms of $G_{1,2}$; the same holds for $G_{2,1}$. The subgroup scheme $G_1 := \mathrm{Fr}_{G_{1,2}}$ is the unique subgroup scheme of $G_{1,2}$ isomorphic to α_p, while the subgroup scheme $G_2 := \mathrm{Ver}_{G_{2,1}}$ is the unique subgroup scheme of $G_{2,1}$ isomorphic to α_p. Let $G := G_1 \times G_2$.

Let $\mathcal{P}$ be the scheme over $\mathbb{F}_p$ such that for every commutative $\mathbb{F}_p$-algebra R, $\mathcal{P}(R)$ is the set of all isomorphism classes of epimorphisms $h : (G_1 \times G_2) \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R) \twoheadrightarrow \alpha_{\mathbf{L}}$, where $\mathbf{L} = \mathrm{Spec}_R(\oplus_{n\in\mathbb{N}}\mathcal{L}^{\otimes -n})$ is the line bundle attached to an invertible R-module $\mathcal{L}$, and $\alpha_{\mathbf{L}} = \mathrm{Ker}\left(\mathrm{Fr}_R^{(p)} : \mathbf{L} \to \mathbf{L}^{(p)}\right)$. This scheme $\mathcal{P}$ is non-canonically isomorphic to $\mathbb{P}^1$ over $\mathbb{F}_p$. It has two special $\mathbb{F}_p$-points:

- the point ∞, represented by an epimorphism $h_\infty : G \twoheadrightarrow G_1$ whose kernel is the second factor G_2 of G, and the restriction of h_∞ to the second factor G_1 gives is the identity map on G_1.
- the point 0, represented by the epimorphism $h_0 : G \to G_2$ whose kernel is the first factor G_1 of G and the restriction of h_0 to the second factor G_2 is id_{G_2}.

Let $\mathcal{U} = \mathcal{P} \smallsetminus \{\infty\}$, a scheme over $\mathbb{F}_p$ isomorphic to $\mathbb{A}^1$. Its points over a an $\mathbb{F}_p$-algebra R corresponds to epimorphisms of the form

$$u : (G_1 \times G_2) \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R) \longrightarrow G_2, \quad h_\beta : (x_1, x_2) \mapsto x_2 - \beta_h(x_1)$$

for all functorial points $x_1 \in G_{1,2}$ and $x_2 \in G_{2,1}$, where

$$\beta_u : G_1 \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R) \longrightarrow G_2 \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R)$$

is an R-homomorphism from $G_1 \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R)$ to $G_2 \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R)$. Note that the kernel of a homomorphism u as above is equale to the graph of the homomorphism $(1, \beta_u) : G_1 \to G_1 \times G_2$.

We have a standard commutative group structure of $(\mathcal{U}, 0)$, isomorphic to $\mathbb{G}_a$ with $0 \in \mathcal{U}$ as the zero element. It is easy to see that this group structure on $\mathcal{U}$ is compatible with the group structure for the parameter β's: we have

$$\beta_{u_1} + \beta_{u_2} = \beta_{u_1 + u_2}$$

for any two points u_1, u_2 of U over the same base ring R. Here the sum $u_1 + u_2$ is taken under the standard group law for $(\mathcal{U}, 0)$, while $\beta_{u_1} + \beta_{u_2}$ is the sum of two homomorphisms from G_1 to G_2.

Over $\mathcal{U}$ we have a tautological epimorphism

$$\mathbf{u} : (G_1 \times G_2) \times \mathcal{U} \to G_2 \times \mathcal{U}$$

Define a p-divisible group $\mathcal{X} \to \mathcal{U}$ by

$$\mathcal{X} := (G_{1,2} \times G_{2,1}) \times \mathcal{U} / \mathrm{Ker}(\mathbf{u}),$$

and let $\mathcal{Y} := (G_{1,2} \times G_{2,1}) \times \mathcal{U}$ and let

$$\pi : \mathcal{Y} \to \mathcal{X}$$

be the isogeny whose kernel is $\mathrm{Ker}(\mathbf{u})$. Note that the fiber X_0 of the family $\mathcal{X}$ above the point $0 \in \mathcal{U}$ is the p-divisible group $G_{1,2} \times G_{2,1}/G_1 = (G_{1,2}/G_1) \times G_{2,1}$ over $\mathbb{F}_p$. The endomorphism $\mathrm{Ver}_{G_{1,2}} : G_{1,2} \to G_{1,2}$ induces an isomorphism $G_{1,2}/G_1 \xrightarrow{\sim} G_{1,2}$.

It follows immediately from the definition of $\mathcal{X}$ that the composition of the first projection $\mathrm{pr}_1 : \mathcal{Y} \to G_{1,2} \times \mathcal{U}$ with the quotient map

$$G_{1,2} \times \mathcal{U} \to (G_{1,2} \times \mathcal{U})/(G_1 \times \mathcal{U})$$

factors through the quotient map $\mathcal{Y} \to \mathcal{Y}/\mathrm{Ker}(\mathbf{u})$ and induces an epimorphism $\mathcal{X} \to (G_{1,2}/G_1) \times \mathcal{U}$, whose kernel is the composition of the inclusion $G_{2,1} \times \mathcal{U} \hookrightarrow \mathcal{Y}$ with the quotient map $\mathcal{Y} \to \mathcal{X}$. We conclude that $\mathcal{X}$ sits in the middle of a short exact sequence

$$E_{\mathcal{X}} = \Big(0 \longrightarrow G_{2,1} \times \mathcal{U} \longrightarrow \mathcal{X} \longrightarrow (G_{1,2}/G_1) \times \mathcal{U} \longrightarrow 0 \Big)$$

over $\mathcal{U}$. We will often identify the quotient $G_{1,2}/G_1$ with $G_{1,2}$ via the isomorphism $G_{1,2}/G_1 \xrightarrow{\sim} G_{1,2}$ induced by $\mathrm{Ver}_{G_{1,2}}$, if confusion is unlikely to arize.

Let $q_1, q_2 : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ be the first and second projection from $\mathcal{U}\times\mathcal{U}$ to $\mathcal{U}$, and denote by $\varsigma : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ the group law on the commutative algebraic group $\mathcal{U}$ isomorphic to $\mathbb{G}_a$. Let $q_1^*E_{\mathcal{X}}$, $q_2^*E_{\mathcal{X}}$ and $\varsigma^*E_{\mathcal{X}}$ be the pull-back of the extension $E_{\mathcal{X}}$ by q_1, q_2 and ς respectively.

8.3.3. Lemma. *Notation as in the previous two paragraphs. There is a natural isomorphism of extensions of $(G_{1,2}/G_1)\times\mathcal{U}$ by $G_{2,1}\times\mathcal{U}$, from the Baer sum of the extensions $q_1^*E_{\mathcal{X}}$ and $q_2^*E_{\mathcal{X}}$, to the extension $\varsigma^*E_{\mathcal{X}}$.*

PROOF. It is easy to see that the Baer sum of the extensions $q_1^*E_{\mathcal{X}}$ and $q_2^*E_{\mathcal{X}}$ is naturally isomorphic to the extension $\Sigma_*E_{\mathcal{X}}$ over $\mathcal{U}\times\mathcal{U}$ of $G_{1,2}/G_1$ by $G_{2,1}$ constructed as follows.

(i) Let $\tilde{\mathcal{X}}$ be the quotient of the constant p-divisible group $(G_{1,2}\times G_{2,1}\times G_{2,1})\times(\mathcal{U}\times\mathcal{U})$ over $\mathcal{U}\times\mathcal{U}$ by the graph of the $\mathcal{U}\times\mathcal{U}$-homomorphism

$$(q_1^*\beta_{\mathbf{u}}, q_2^*\beta_{\mathbf{u}}) : G_1\times(\mathcal{U}\times\mathcal{U}) \longrightarrow (G_1\times G_1)\times(\mathcal{U}\times\mathcal{U}),$$

where $q_i^*\beta_{\mathbf{u}}$ is the base change to $\mathcal{U}\times\mathcal{U}$ of the universal/tautological homomorphism $\beta_{\mathbf{u}} : G_1\times\mathcal{U} \to G_2\times\mathcal{U}$ over $\mathcal{U}$, for $i=1,2$. The projection from $(G_{1,2}\times G_{2,1}\times G_{2,1})\times(\mathcal{U}\times\mathcal{U})$ to $G_{1,2}$, composed with the quotient map $G_{1,2}\to G_{1,2}/G_1$, induces an epimorphism $\tilde{\mathcal{X}}\to G_{1,2}/G_1$, which forms part of the data of an extension

$$E_{\tilde{X}} = \Big(0 \longrightarrow (G_{2,1}\times G_{2,1})_{\mathcal{U}\times\mathcal{U}} \longrightarrow \tilde{\mathcal{X}} \longrightarrow (G_{1,2}/G_1)_{\mathcal{U}\times\mathcal{U}} \longrightarrow 0\Big)\,.$$

(ii) Let $\Sigma_*E_{\mathcal{X}}$ be the push-forward of the extension $E_{\tilde{X}}$ by the group law $\Sigma : G_{2,1}\times G_{2,1}\to G_{2,1}$ of $G_{2,1}$.

On the other hand the base change to $\mathcal{U}\times\mathcal{U}$ of the homomorphism

$$\Psi : G_{1,2}\times G_{2,1}\times G_{2,1} \longrightarrow G_{1,2}\times G_{2,1}, \quad (x, y_1, y_2)\mapsto (x, y_1+y_2)$$

induces a homomorphism

$$\psi : \tilde{\mathcal{X}} \longrightarrow \mathcal{Y}_{\mathcal{U}\times\mathcal{U}}/\mathrm{Ker}(\varsigma^*\mathbf{u})$$

from the quotient of $\mathcal{Y}_{\mathcal{U}\times\mathcal{U}}$ by the graph of the $(q_1^*\beta_{\mathbf{u}}, q_2^*\beta_{\mathbf{u}})$ to the quotient of $\mathcal{Y}_{\mathcal{U}\times\mathcal{U}}$ by the kernel of $\varsigma^*\mathbf{u}$. The homomorphism ψ induces an isomorphism of extensions, from $\Sigma_*E_{\mathcal{X}}$ to $\varsigma^*E_{\mathcal{X}}$. $\square$

Let $\hat{\mathcal{U}} := \mathcal{U}^{/0}$ be the formal completeion of $\mathcal{U}$ at the point $0\in\mathcal{U}$, and let $E_{\mathcal{X}}|_{\hat{\mathcal{U}}}$ be the base change to $\hat{\mathcal{U}}$ of the extension $\mathcal{X}$ of $G_{1,2}$ by $G_{2,1}$. The extension $E_{\mathcal{X}}|_{\hat{\mathcal{U}}}$ over $\hat{\mathcal{U}}$ defines a morphism $\upsilon : \hat{\mathcal{U}}\to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$, and Lemma 8.3.3 says that υ is a homomorphism of formal groups. Since the formal group $\hat{\mathcal{U}}$ is isomorphic to $\widehat{\mathbb{G}}_a$, the homomorphism υ factors as the composition of a homomorphism $\upsilon_1 : \hat{\mathcal{U}}\to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ and the inclusion $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}} \hookrightarrow \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$ of the maximal unipotent subgroup $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ in $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$.

8.3.4. Lemma. *The homomorphism $\upsilon_1 : \hat{\mathcal{U}}\to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ above is an isomorphism.*

PROOF. It suffices to show that the derivative $d\,v$ of the homomorphism v_1 is injective. Since $\mathcal{U}$ is one-dimension, this is the same as showing that $d\,v$ is nontrival.

Let $S_1 = \mathrm{Spec}(\mathscr{O}_{\mathcal{U},0}/\mathfrak{m}_{\mathcal{U},0}^2)$ be the first order infinitesimal neighborhood of $0 \in \mathcal{U}$. The restriction $\mathcal{X}_{S_1}$ of $\mathcal{X}$ to $S_1 \hookrightarrow \mathcal{U}$ is the quotient of (the base extension to S_1 of) $Y = G_{1,2} \times G_{2,1}$ by the finite subgroup scheme $\mathrm{Ker}(\mathbf{u}|_{S_1})$ over S_1.

We want to show that the extension $\mathcal{X}_{S_1}$ of $G_{1,2}$ by $G_{2,1}$ over S_1 is not isomorphic to the base change to S_1 of the trivial extension X_0.

Suppose that there is an isomorphism $f : \mathcal{X}_{S_1} \xrightarrow{\sim} \mathcal{Y}_{S_1}$ of extensions over S_1. The composition $f \circ \pi|_{S_1}$ of f with the restriction to S_1 of the quotient map $\pi : \mathcal{Y} \to \mathcal{X}$ is an endomorphism of $\mathcal{Y}_{S_1} = (G_{1,2} \times G_{2,1}) \times S_1$. From the rigidity of homomorphism between p-divisible groups we know that $f \circ \pi|_{S_1}$ is equal to the base extension to S_1 of its closed fiber. Therefore its kernel, which is $\mathrm{Ker}(\mathbf{u}|_{S_1})$, is equal to the base extension to S_1 of G_1. This contradiction proves that dv_1 is an isomorphism. □

8.3.5. *Another proof of* 8.3.4.

Here we give another proof of 8.3.4, by showing that the derivative dv of the homomorphism $v_1 : \hat{\mathcal{U}} \to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ is surjective, using the crystalline Dieudonné theory. At the same time we will exhibit the tangent space of the formal group $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ explicitly as a one-dimensioinal subspace of the tangent space of $\mathcal{D}ef(G_{1,2} \times G_{2,1})$.

We will keep the notation in 8.3.4, and shorten $\mathfrak{m}_{\mathcal{U},0}$ to $\mathfrak{m}$. Let ϵ be a non-zero element of the one-dimension $\mathbb{F}_p$-vector space $\mathfrak{m}/\mathfrak{m}^2$. We have

$$S_1 = \mathrm{Spec}\big(\mathbb{F}_p \oplus (\mathfrak{m}/\mathfrak{m}^2)\big) = \mathrm{Spec}(\mathbb{F}_p[\epsilon]) = \mathrm{Spec}(\mathbb{F}_p \oplus \mathbb{F}_p\,\epsilon).$$

Let $\mathrm{D}_*(G_{1,2}) = We_1 \oplus We_2 \oplus We_3$ and $\mathrm{D}{*}(G_{2,1}) = Wf_1 \oplus Wf_2 \oplus Wf_3$ be the covariant Dieudonné modules of $G_{1,2}$ and $G_{2,1}$ in 8.2. The crystalline Diedonné module $\mathrm{D}_{\mathrm{crys}}(G_{1,2})(\mathbb{F}_p[\epsilon] \twoheadrightarrow \mathbb{F}_p)$ of $G_{1,2}$ evaluated at $\mathbb{F}_p[\epsilon] \twoheadrightarrow \mathbb{F}_p$ with trivial divided power structure is canonically isomorphic to $\big(\mathrm{D}_*(G_{1,2})/p\mathrm{D}_*(G_{1,2})\big) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\epsilon]$; similarly for $\mathrm{D}_{\mathrm{crys}}(G_{2,1})(\mathbb{F}_p[\epsilon] \twoheadrightarrow \mathbb{F}_p)$.

According to the crystalline deformation theory, the category of deformations of the p-divisible group $G_{1,2} \times G_{2,1}$ over S_1 is equivalent to the category of free rank-three $\mathbb{F}_p[\epsilon]$-submodules $\mathcal{F}$ of the free rank-6 $\mathbb{F}_p[\epsilon]$-module

$$\mathcal{D} := \big(\mathrm{D}_*(G_{1,2})/p\mathrm{D}_*(G_{1,2})\big) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\epsilon] \,\oplus\, \big(\mathrm{D}_*(G_{2,1})/p\mathrm{D}_*(G_{2,1})\big) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\epsilon]$$

such that $\mathcal{F} \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p$ is equal to the three-dimensional $\mathbb{F}_p$-vector subspace

$$\big(V\,\mathrm{D}_*(G_{1,2})/p\mathrm{D}_*(G_{1,2})\big) \oplus \big(V\,\mathrm{D}_*(G_{2,1})/p\mathrm{D}_*(G_{2,1})\big).$$

of the six-dimensional $\mathbb{F}_p$-vector space

$$\big(\mathrm{D}_*(G_{1,2})/p\mathrm{D}_*(G_{1,2})\big) \oplus \big(\mathrm{D}_*(G_{2,1})/p\mathrm{D}_*(G_{2,1})\big).$$

Similarly the category of deformations of $G_{1,2} \times G_{2,1}$ over S_1 such that the slope filtration

$$0 \to G_{2,1} \to G_{1,2} \times G_{2,1} \to G_{1,2} \to 0$$

also deforms is equivalent to the category of all free rank-three $\mathbb{F}_p[\epsilon]$-submodules $\mathcal{F}$ of the free rank-6 $\mathbb{F}_p[\epsilon]$-module satisfying the above condition, such that

$$\mathcal{F} \cap \big(\mathrm{D}_*(G_{2,1})/p\mathrm{D}_*(G_{2,1})\big) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\epsilon] = \big(V\,\mathrm{D}_*(G_{2,1})/p\mathrm{D}_*(G_{2,1})\big) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\epsilon]\,.$$

Using the chosen bases e_1, e_2, e_3 for the free $k[\epsilon]$-module $\big(\mathrm{D}_*(G_{1,2})/p\mathrm{D}_*(G_{1,2})\big) \otimes_{\mathbb{F}_p} k[\epsilon]$ and f_1, f_2, f_3 for $\big(\mathrm{D}_*(G_{2,1})/p\mathrm{D}_*(G_{2,1})\big) \otimes_{\mathbb{F}_p} k[\epsilon]$, it is not difficult to see that $k[\epsilon]$-points of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$ correspond to $k[\epsilon]$-submodules $\mathcal{F}$ of $\mathcal{D}$ of the form

$$\mathcal{F} = k[\epsilon]\cdot f_3 + k[\epsilon]\cdot(e_2 + a\epsilon f_1 + b\epsilon f_2) + k[\epsilon]\cdot(e_3 + c\epsilon f_1 + d\epsilon f_2)$$

with $a, b, c, d \in \mathbb{F}_p$. Clearly every direct summand $\mathcal{F}$ of $\mathcal{D}$ as above correspond to a unique quadruple a, b, c, d of elements in $\mathbb{F}_p$. This constitutes an explicit description of the tangent space of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$.

It is not easy to figure out directly which ones among the submodules $\mathcal{F}$ as above correspond to elements of the Lie algebra of $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$, but it is easier to determine all submodules $\mathcal{F}$ which correpsond to elements in the derivative $d\upsilon_1$ of the homomorphism $\upsilon_1 : \hat{\mathcal{U}} \to \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$. We want the p-divisible group $X_{\mathcal{F}}$ over $\mathbb{F}_p[\epsilon]$ corresponding to $\mathcal{F}$ to be connected to $Y_0 \times S_1$ by a pair of isogenies

$$Y_0 \times S_1 \xrightarrow{\gamma} X_{\mathcal{F}} \xrightarrow{\delta} Y_0 \times S_1$$

such that the kernel of the closed fiber γ_0 of γ is the subgroup scheme G_1 of $G_1 \times G_2 \subset G_{1,2} \times G_{2,1}$, the kernal of the closed fiber δ_0 of δ is G_2, and the composition $\delta \circ \gamma$ is an endomorphism of $Y_0 \times S_1$ whose kernel is $(G_1 \times G_2) \times S_1$. The above conditions translate into the following:

(i) The homomorphism Γ from $\mathcal{D}$ to $\mathcal{D}$ (corresponding to γ) given by

$$\Gamma : e_1 \mapsto e_2,\ e_2 \mapsto e_3,\ e_3 \mapsto 0,\ f_1 \mapsto f_1,\ f_2 \mapsto f_2,\ f_3 \mapsto f_3$$

satisfies

$$\Gamma\big(\mathbb{F}_p[\epsilon]e_2 + \mathbb{F}_p[\epsilon]e_3 + \mathbb{F}_p[\epsilon]f_3\big) \subseteq \mathcal{F}$$

(i) The homomorphism Δ from $\mathcal{D}$ to $\mathcal{D}$ (corresponding to δ) given by

$$\Delta : e_1 \mapsto e_1,\ e_2 \mapsto e_2,\ e_3 \mapsto e_3,\ f_1 \mapsto f_2,\ f_2 \mapsto f_3,\ f_3 \mapsto 0$$

satisfies

$$\Delta\big(\mathcal{F}\big) \subseteq \mathbb{F}_p[\epsilon]e_2 + \mathbb{F}_p[\epsilon]e_3 + \mathbb{F}_p[\epsilon]f_3$$

It is immediately seen that for an $\mathbb{F}_p[\epsilon]$-direct summand $\mathcal{F}$

$$\mathcal{F} = k[\epsilon]\cdot f_3 + k[\epsilon]\cdot(e_2 + a\epsilon f_1 + b\epsilon f_2) + k[\epsilon]\cdot(e_3 + c\epsilon f_1 + d\epsilon f_2)$$

of $\mathcal{D}$ with $a, b, c, d \in \mathbb{F}_p$, $\mathcal{F}$ satisfies condition (i) if and only if $c = d = 0$, while $\mathcal{F}$ satisfies condition (ii) if and only if $a = c = 0$. In particular the set of all $\mathbb{F}_p$-submodules $\mathcal{F}$ of $\mathcal{D}$ of the form

$$\mathcal{F} = k[\epsilon]\cdot f_3 + k[\epsilon]\cdot e_2 + k[\epsilon]\cdot(e_3 + d\epsilon f_2)$$

with $d \in \mathbb{F}_p$ correspond to tangent vectors in the image of the derivative of υ_1. In particular the image of $d\upsilon_1$ is non-zero. Since both the source and the target of $d\upsilon_1$ is one dimensional, the homomorphism υ_1 from $\hat{\mathcal{U}}$ to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ is an isomorphism.

8.4. *An example of a p-divisible group over $\mathbb{F}_p(t)$ which is $\mathbb{F}_p(t^{p^2})$-sustained but does not have an $\overline{\mathbb{F}_p}((t))$-model*

We will continue with the notation in 8.3 in this subsection. Choose and fix an isomorphism $\mathcal{U} \xrightarrow{\sim} \mathbb{A}^1 = \mathrm{Spec}(\mathbb{F}_p[x])$ such that the point "$0 \in \mathcal{U}$" goes to the $\mathbb{F}_p$-point of $\mathbb{A}^1$ with $x = 0$. Let t be inverse image of x under this isomorphism, so that $\mathcal{U} = \mathrm{Spec}(\mathbb{F}_p[t])$.

8.4.1. Lemma. *Let $\mathcal{X}_\eta$ be the generic fiber of the p-divisible group $\mathcal{X} \to \mathcal{U}$. This p-divisible group $\mathcal{X}_\eta$ over $\mathbb{F}_p(t)$ is $\mathbb{F}_p(t^{p^2})$-sustained.*

Proof. By descent it suffice to prove that the base change

$$\mathcal{X}_{\mathbb{F}_p((t))} := \mathcal{X}_\eta \times_{\mathrm{Spec}\,\mathbb{F}_p(t)} \mathrm{Spec}\big(\mathbb{F}_p((t))\big)$$

of $\mathcal{X}_\eta$ to the fraction field $\mathbb{F}_p((t))$ of $\mathbb{F}_p[[t]]$ is $\mathbb{F}_p((t^{p^2}))$-sustained. Here we have used the base change from the field $\mathbb{F}_p(t^{p^2})$ to its t^{p^2}-adic completion $\mathbb{F}_p((t^{p^2}))$, and $F(t) \otimes_{\mathbb{F}_p(t^{p^2})} \mathbb{F}_p((t^{p^2})) = \mathbb{F}_p((t))$.

We have seen in 8.3.4 that $\mathcal{X}_{\mathbb{F}_p((t))}$ can be identified as the generic fiber of the restriction to $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}$ of the universal p-divisible group $\mathbf{X}_{\mathcal{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})}$ over $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1}) \arrow[r, "\pi"] & \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})/\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \\
\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}} \arrow[u, hook] \arrow[r, "\pi_{\mathrm{u}}"] & \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}}/\big(\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}} \cap \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}}\big) \arrow[u, "\cong"']
\end{tikzcd}$$

The bottom horizontal arrow π_{u} is naturally identified with the map $\mathrm{Spf}\big(\mathbb{F}_p[[t]]\big) \to \mathrm{Spf}\big(\mathbb{F}_p[[t^p]]\big)$ corresponding to the inclusion $\mathbb{F}_p[[t^{p^2}]] \hookrightarrow \mathbb{F}_p[[t]]$. To simplify the notation, let

$$\mathbf{U}_1 := \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}},$$

$$L_2 := \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{unip}} \cap \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})_{\mathrm{div}} \cong \alpha_{p^2}$$

and let

$$\mathbf{U}_2 := \mathbf{U}_1/L_2\,.$$

We have a natural isomorphism

$$\zeta : \mathbf{U}_1 \times L_2 \xrightarrow{\sim} \mathbf{U}_1 \times_{\mathbf{U}_2} \mathbf{U}_1\,, \qquad \zeta : (u, a) \mapsto (u, u + a).$$

We want to show that $\mathbf{X}_{\mathbb{F}_p((t))}$ is $\kappa((t^{p^2}))$-sustained. Clearly it suffices to show that the Isom-scheme

$$\mathscr{I}\!\mathit{SOM}_{\mathbf{U}_1 \times_{\mathbf{U}_2} \mathbf{U}_1}\big(\mathrm{pr}_1^*\mathbf{X}_{\mathbf{U}_1}[p^n],\ \mathrm{pr}_2^*\mathbf{X}_{\mathbf{U}_1}[p^n]\big) \longrightarrow \mathbf{U}_1 \times_{\mathbf{U}_2} \mathbf{U}_1$$

is faithfully flat for very $n \in \mathbb{N}$. Both $\mathrm{pr}_1^*\mathbf{X}$ and $\mathrm{pr}_2^*\mathbf{X}$ are extensions of $G_{1,2}$ by $G_{2,1}$ $\mathbf{U}_1 \times_{\mathbf{U}_2} \mathbf{U}_1$. The isomorphism $\mathbf{U}_1 \times_{\mathbf{U}_2} \mathbf{U}_1 \xleftarrow{\sim} \mathbf{U}_1 \times L_2$ implies that we have a commutative diagram

$$
\begin{tikzcd}
\mathscr{ISOM}_{\mathbf{U}_1\times_{\mathbf{U}_2}\mathbf{U}_1}\big(\mathbf{X}_{\mathbf{U}_1}[p^n]\times_{\mathbf{U}_2}\mathbf{U}_1,\ \mathbf{U}_1\times_{\mathbf{U}_2}\mathbf{X}_{\mathbf{U}_1}[p^n]\big) \arrow[r, "f_n"] & \mathbf{U}_1\times_{\mathbf{U}_2}\mathbf{U}_1 \\
\mathscr{ISOM}_{\mathbf{U}_1\times L_2}\big((G_{1,2}\times G_{2,1})[p^n]\times(\mathbf{U}_1\times L_2),\ \mathbf{U}_1\times(\mathbf{X}_{\mathbf{U}_1}\overset{g}{\times}_{\mathbf{U}_1}L_2)[p^n]\big) \arrow[u, "\psi_n"] \arrow[r] & U\times L_2 \arrow[u, "\cong"', "\zeta"]
\end{tikzcd}
$$

for every $n\in\mathbb{N}$, where the left vertical arrow ψ comes from the functoriality of the Baer sum. The bottom horizontal arrow g_n is faithfully flat because L_2 is contained in $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}}$ as we saw in 8.2.1. Therefore g is faithfully flat. Thus the top horizontal arrow f_n admits a section after a faithfully flat base change, which implies that f_n is also faithfully flat. $\square$

Remark. (i) The p-divisible group $\mathcal{X}_\eta$ over $\mathbb{F}_p(t)$ does not have an $\mathbb{F}_p(t^{p^2})$-model nor an $\mathbb{F}_p(t^p)$-model. This is easy to verify with the argument used in 8.1.

(ii) We can apply the construction in 8.3 to the set-up in 8.1. In 8.1, we started with a one-dimensional p-divisible group Y_1 over $\mathbb{F}_p$ with slope $1/2$ and height 2. The p-divisible group $Y_3 := Y_1\times Y_1$ contains a subgroup scheme G_3 over $\mathbb{F}_p$ isomorphic to $\alpha_p\times\alpha_p$. Let $\mathcal{P}_3$ be the scheme over $\mathbb{F}_p$ isomorphic to $\mathbb{P}^1$ classifying α_p-type quotients of G_3, and let $\mathbf{u}_3$ be the tautological quotient map on $G_3\times\mathcal{P}_3$. Let $\mathcal{X}_3 := (Y_3\times\mathcal{P}_3)/(\mathrm{Ker}(\mathbf{u}_3)$; it is an isoclinic p-divisible group over $\mathbf{P}_3$ of slope $1/2$ and height 4. Let η_3 be the generic point of $\mathcal{P}_3$, and let $\mathcal{X}_{3\,\eta_3} := \mathcal{X}_3\times_{\mathcal{P}_3}\eta_3$. Thus the construction of the p-divisible group $\mathcal{X}_{3\,\eta_3}$ over $\eta_3\cong\mathrm{Spec}(\mathbb{F}_p(t))$is entirely analogous to $\mathcal{X}_\eta$. Similar to $\mathcal{X}_\eta$, there does not exist a p-divisible group over $\mathbb{F}_p(t^{p^2})$ whose base extension to $\mathbb{F}_p(t)$ is isomorphic to $\mathcal{X}_{3\,\eta_3}$ over $\mathbb{F}_p(t)$. However because $\mathcal{X}_\eta$ is isoclinic we know from 1.5.2 (= 4.13) that $\mathcal{X}_{3,\,\eta_3}$ is *not* $\mathbb{F}_p(t^{p^2})$-sustained.

8.5. *An example of a sustained polarized p-divisible groups such that the sustained locus in its deformation space is reduced but not geometrically reduced*

In this subsection we will combine the considerations in 8.4 and 8.2.5 to produce a field $K\supset\mathbb{F}_p$ and a subfield $\kappa\subset K$, and a polarized p-divisible group (X,μ) over K which is κ-sustained, such that the maximal κ-sustained locus T in the deformation space of (X,μ) is reduced, but *not* geometrically reduced over κ.

We will use the notation in 8.4 and 8.2.5. Recall that we have maps

$$\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1}) \xrightarrow{\ \pi\ } \mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})/\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}} \xleftarrow{\ \sim\ } \mathbf{U_2}=\mathrm{Spf}(\mathbb{F}_p[[t^{p^2}]])$$

We will abuse notation and pretend that $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})$ is a scheme over the affine scheme $\mathrm{Spec}(\mathbb{F}_p[[t^{p^2}]])$. This should not cause any problem because all formal schemes involved are affine. Let

$$T := \mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})\times_{\mathrm{Spec}(\mathbb{F}_p[[t^{p^2}]])}\mathrm{Spec}\big(\mathbb{F}_p((t^{p^2}))\big),$$

a $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}}$-torsor over $\mathbb{F}_p((t^{p^2}))$. Denote by $\mathbf{X}_T$ the restriction to T of the universal p-divisible group $\mathbf{X}_{\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})}$ over $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})$.

Let $\Sigma\subset\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})$ be the subgroup of $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})$ generated by $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{unip}}$ and the subgroup $\mathrm{Ker}(\xi(\nu_2))$ of $\mathscr{E}xt_{\mathrm{def}}(G_{1,2},G_{2,1})_{\mathrm{div}}$ in 8.2.5.

The polarization $\mu_2 = \nu_2 \times \nu_2^t$ on $G_{1,2} \times G_{2,1}$ extends uniquely to a polarization $\tilde{\mu}_2$ of degree p^2 on the universal p-divisible group $\mathbf{X}_\Sigma$ over Σ. Recall from 8.2.5 that $\mathrm{Ker}(\xi(\nu_2))$ is generated by its maximal p-divisible subgroup $\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$ and the subgroup $L_2 \subset \mathbf{U}_1$. Their intersection $L_2 \cap \mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$ is equal to the subgroup L_1 of L_2, and we have a natural isomorphism $L_2/L_1 \xrightarrow{\sim} \mathrm{Ker}(\xi(\nu_2))/\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$. Moreover we have a natural isomorphism

$$\mathrm{Spf}(\mathbb{F}_p[[t^{p^2}]]) = \mathbf{U}_1/L_2 \xrightarrow{\sim} \Sigma/\mathrm{Ker}(\xi(\nu_2)).$$

Note that

$$\Sigma/\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}} \xleftarrow{\sim} \mathbf{U}_1/L_1 = \mathrm{Spf}\big(\mathbb{F}_p[[t^p]]\big).$$

Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spf}\big(\mathbb{F}_p[[t]]\big) = \mathbf{U} \arrow[r, hook] \arrow[d] & \Sigma \arrow[r, hook] \arrow[d] & \mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1}) \arrow[d] \\
\mathrm{Spf}\big(\mathbb{F}_p[[t^{p^2}]]\big) = \mathbf{U}_2 \arrow[r, "\cong"] & \Sigma/\mathrm{Ker}(\xi(\nu_2)) & \mathbf{U}_2 \arrow[l, "\cong"']
\end{tikzcd}$$

Apply base change from $\mathbb{F}_p[[t^{p^2}]]$ to $\mathbb{F}_p((t^{p^2}))$, we get $\mathbb{F}_p((t^{p^2}))$-embeddings

$$\mathrm{Spec}\big(\mathbb{F}_p((t))\big) \hookrightarrow S \hookrightarrow T,$$

where

$$S := \Sigma \times_{\mathrm{Spec}(\mathbb{F}_p[[t^{p^2}]])} \mathrm{Spec}\big(\mathbb{F}_p((t^{p^2}))\big)\,.$$

Note that S has a natural structure as a $\mathrm{Ker}(\xi(\nu_2))_{\mathrm{div}}$-torsor over $\mathrm{Spec}\big(\mathbb{F}_p((t^p))\big)$, while we have seen that T is a torsor over the p-divisible group $\mathcal{E}xt_{\mathrm{def}}(G_{1,2}, G_{2,1})$.

Over T we have a p-divisible group $\mathbf{X}_T$. The polarization $\tilde{\mu}_2$ on $\mathbf{X}_\Sigma$ induces a polarization $\tilde{\mu}_{2,S}$ on $\mathbf{X}_T \times_T S$. Let $K = \mathbb{F}_p((t))$, $\kappa((t^{p^2}))$. Let $(X_K, \tilde{\mu}_{2,K})$ be the restriction to $\mathrm{Spec}(K)$ of $(\mathbf{X}_T, \tilde{\mu}_{2,T})$. The significance of the embeddings $\mathrm{Spec}\big(\mathbb{F}_p((t))\big) \hookrightarrow S \hookrightarrow T$ is as follows.

(1) T is the maximal K-sustained locus in the deformation space of X_K. Moreover X_K is κ-sustained.
(2) S is the maximal subscheme of T over which the polarization $\tilde{\mu}_{2,K}$ extends. The polarized p-divisible group $(X_S, \tilde{\mu}_{2,S})$ is κ-sustained, and S is the sustained locus in the deformation space of $(X_K, \tilde{\mu}_{2,K})$.
(3) The κ-scheme S is reduced but not geometrically reduced: $S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(K)$ is not reduced.

(xxxxxxx write more details xxxxxx?)

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