SUSTAINED p-DIVISIBLE GROUPS AND A FOLIATION ON MODULI SPACES OF ABELIAN VARIETIES

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Abstract. We explain a concept of *sustained p-divisible groups*, discovered in collaboration with Frans Oort and motivated by the Hecke orbit problem. This concept leads to a scheme-theoretic definition of *central leaves* in moduli spaces of abelian varieties in characteristic $p > 0$. We also formulate a notion of *strongly Tate-linear* formal subschemes of the sustained deformation space $\mathbf{Def}^{\rm sus}(Y_0)$ of a p-divisible group Y_0, and a local rigidity question on whether every reduced and irreducible closed formal subscheme of $\mathbf{Def}^{\rm sus}(Y_0)$ stable under a *strongly non-trivial* action of a p-adic Lie group is strongly Tate-linear.

1. Introduction

This article is a survey of the notion of *sustained p-divisible groups.* This notion enunciates a fundamental feature of the family of p-divisible groups over a central leaf in a PEL moduli space in characteristic p, which is obvious from the original definition of central leaves. Local structure of central leaves are unveiled when one examines the space of sustained deformations of polarized p-divisible groups. A pleasant feature of this theory is that p-divisible groups appear serendipitously in a number of contexts, including stabilized Hom schemes and stabilized Isom schemes of p-divisible groups, sustained deformations of p-divisible groups, and formal completions of central leaves in PEL moduli spaces.

Proofs are omitted as a rule. The only exception is a sketch of the smoothness of the sustained deformation functors. We refer to chapter 5 of [11] for more information and complete proofs.

1.1. The notion of sustained p-divisible groups is motivated by the search of a scheme-theoretic definition of central leaves on a PEL modular variety $\mathscr{M}$ over $\overline{\mathbb{F}_p}$. The original definition of central leaves relies on the notion of *geometrically fiberwise constant* (gfc) p-divisible groups". The latter is a "point-wise" notion—it was unclear what a gfc p-divisible group over an Artinian local ring should be.

The sought-after answer is formulated below as a property the universal polarized p-divisible groups over central leaves which is stronger than gfc.

(Schematic definition of central leaves in PEL modular varieties)
Let $z_0 = [(A_0, \lambda_0, \mu_0)] \in \mathscr{M}(\overline{\mathbb{F}_p})$ be an $\overline{\mathbb{F}_p}$-point of $\mathscr{M}$, corresponding to an abelian variety A_0 over $\overline{\mathbb{F}_p}$ with prescribed endomorphisms $\lambda_0 : \mathscr{O}_E \to \mathrm{End}(A_0)$ and a polarization μ_0 compatible with λ_0, where $\mathscr{O}_E$ is a maximal order of a central simple algebra of finite dimension over $\mathbb{Q}$.

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The central leaf in $\mathscr{M}$ passing through z_0 is the largest locally closed subscheme $\mathcal{C}(z_0)$ of $\mathscr{M}$ such that the restriction to $\mathcal{C}(z_0)$ of the universal p-divisible group $(\boldsymbol{A}, \boldsymbol{\lambda}, \boldsymbol{\mu})[p^\infty]$ with PE structure over $\mathscr{M}$ has the following property.

For every $n \geq 0$, there exists a morphism $T_n \to \mathcal{C}(z_0)$ which is faithfully flat and of finite presentation, and an isomorphism

$$(A_0, \lambda_0, \mu_0)[p^n] \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} T_n \xrightarrow{\sim} (\boldsymbol{A}, \boldsymbol{\lambda}, \boldsymbol{\mu})[p^n] \times_{\mathscr{M}} T_n.$$

of polarized p-divisible groups over T_n.

Here $(\boldsymbol{A}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is the universal abelian scheme with PE structure over $\mathscr{M}$, and $\lambda_0[p^\infty] : \mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(A_0[p^\infty])$ and $\mu_0[p^\infty] : A_0[p^\infty] \to A_0[p^\infty]^t$ are the endomorphism structure and polarization on the p-divisible group $A_0[p^\infty]$ induced by λ_0 and μ_0 respectively. The notion of sustained p-divisible groups is defined in such a way that the above property of $(\boldsymbol{A}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is shortened to

$$(\boldsymbol{A}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \textit{ is strongly } \overline{\mathbb{F}_p}\textit{-sustained modeled on } (A_0[p^\infty], \mu_0[p^\infty], \lambda_0[p^\infty]).$$

The rest of this long introductory section offers a more leisurely tour. We recall in 1.2 and 1.3 Oort's idea on the foliation structures of a PEL modular variety [25] in positive characteristic p. I will take a shortcut and discuss only the notion of *strongly sustained p-divisible group.* The notion of *sustained p-divisible groups* is suppressed, to achieve a sharper focus. For the same reason, we consider only strongly sustained p-divisible groups and strongly sustained polarized p-divisible group. We will see in 4.5 that the definition of central leaves in Siegel modular varieties $\mathcal{A}_{g,d}$ in terms of sustained polarized p-divisible groups coincides with the original definition, which is reviewed in 1.2.

It is possible to extend our discussion to the case of p-divisible groups with prescribed endomorphism and polarization structure, and also the more general case of p-divisible groups with prescribed Tate-cycles. These generalizations are left to ambitious readers.

1.2. A foliation of $\mathcal{A}_g$ in characteristic $p > 0$.

The notion of *foliation* on the moduli space $\mathcal{A}_{g,d}$ of g-dimensional principally polarized abelian varieties in characteristic $p > 0$, due to Frans Oort, was announced in the conference *Moduli of Abelian Varieties, Texel '99*; see [25] for the published version. For any self-dual Newton polygon ξ of height $2g$, two foliations for the Newton stratum $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ in $\mathcal{A}_g$ associated to ξ are defined in [25], called the *central foliation* and the *isogeny foliation* of $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ respectively. Below is a list of their salient features.

(a) (foliation property) For every algebraically closed field $k \supseteq \mathbb{F}_p$ and every geometric point $x_0 \in \mathcal{W}_\xi(\mathcal{A}_{g,d})(k)$, there is a unique central leaf $\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d}}(x_0)$ passing through x_0 and a finite number of isogeny leaves $\mathcal{I}(x_0)_1, \ldots, \mathcal{I}(x_0)_{m(x_0)}$ passing through x_0. The central leaf $\mathcal{C}(x_0)$ and the isogeny leaves $\mathcal{I}(x_0)_j$ are reduced *closed* subschemes of the Newton stratum $\mathcal{W}_\xi(\mathcal{A}_{g,1}) \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$, and each isogeny leaf $\mathcal{I}(x_0)_j$ is irreducible by definition.

(b) Every central leaf $\mathcal{C}(x_0)$ as in (a) is smooth over k and all irreducible components of $\mathcal{C}(x_0)$ have the same dimension.

(c) Any two central leaves $\mathcal{C}(x_1), \mathcal{C}(x_2)$ in $\mathcal{W}_\xi(\mathcal{A}_{g,1})$ in the same Newton polygon stratum $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ have the same dimension. The common dimension of central leaves in $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ depends only on the Newton polygon ξ and not on the polarization degree d.

(d) (product structure) For every algebraically closed field $k \supseteq \mathbb{F}_p$, every positive integer $d \geq 1$, every symmetric Newton polygon ξ of height $2g$, every irreducible component $\mathcal{W}'$ of $\mathcal{W}_\xi(\mathcal{A}_{g,d})$, there exist a finite surjective k-morphism

$$f : \tilde{\mathcal{C}} \times_{\mathrm{Spec}(k)} \tilde{\mathcal{I}} \to \mathcal{W}'$$

such that $f(\{z\} \times \tilde{\mathcal{I}})$ is an isogeny leaf in $\mathcal{A}_{g,d}$ and $f(\tilde{\mathcal{C}} \times \{t\})$ is an irreducible component of a central leaf in $\mathcal{A}_{g,d}$, for every algebraically closed extension field K of k, every $z \in \tilde{\mathcal{C}}(K)$ and every $t \in \tilde{\mathcal{I}}(K)$.

Remark. (i) The Newton stratum $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ consists of all points of $\mathcal{A}_{g,1}$ whose Newton polygon is equal to ξ; it is denoted by $\mathcal{W}_\xi(\mathcal{A}_{g,d})^0$ in [25] and called the "open Newton polygon in $\mathcal{A}_{g,1}$ indexed by ξ". It is known that every Newton stratum $\mathcal{W}_\xi(\mathcal{A}_{g,1})$ in $\mathcal{A}_{g,1}$ is irreducible if ξ is not the supersingular Newton polygon, and every central leaf in $\mathcal{A}_{g,d}$ not contained in the supersingular Newton stratum is irreducible. See [7].

(ii) The definition of central leaves is given in 1.3.2, using the notion of *geometrically fiberwise constant p-divisible groups*. See also 2.10 for an "upgraded" definition of $\mathcal{C}(x_0)$ in terms of sustained polarized p-divisible groups.

(iii) The relation "being in the same isogeny leaf", denoted by $\sim_{\text{isleaf}}$, is *not* transitive, because each isogeny leaves is irreducible by definition. Let $\sim_{\text{isleaf}}$ be the equivalence relation generated by the relation $\sim_{\text{isleaf}}$, so that two points $x, y \in \mathcal{A}_{g,d,n}$ are equivalent under $\sim_{\text{isleaf}}$ if and only if there exist elements $x_0, x_1, \ldots, x_m$ in $\mathcal{A}_{g,d,n}$ with $x_0 = x$, $x_m = y$, and $x_i \sim_{\text{isleaf}} x_{i+1}$ for $i = 0, 1, \ldots, m-1$. Then the equivalence class in $\mathcal{A}_{g,d,n}$ for the relation $\sim_{\text{isleaf}}$ which contains a given geometric point $x_0 = (A_0, \mu_0, \zeta_0) \in \mathcal{A}_{g,d,n}(k)$ consists of all points $y = (A_y, \mu_y, \zeta_y) \in \mathcal{A}_{g,d,n}$ such that there exists an isogeny correspondence

$$(A_0, \mu_0) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \xleftarrow{\alpha} (A_1, \mu_1) \xrightarrow{\beta} (A_y, \mu_y) \times_{\mathrm{Spec}(\kappa(y_1))} \mathrm{Spec}(K)$$

over a field K containing both k and $\kappa(y)$, which respects the polarizations μ_0, μ_y and the kernels of the isogenies α, β are both of local-local type; see [25, §4]. This notion can be explained in terms of Rapoport–Zink spaces; see [8, 4.7.4].

Our attention in this article is focused on the central foliation. The isogeny leaves will not appear in the rest of this article.

1.3. Central leaves in moduli spaces of abelian varieties.

1.3.1. *Geometrically fiberwise constant p-divisible groups.*

We fix a prime number p in this paper. Let k be an algebraically closed field of characteristic p and let S be a scheme over k. Recall that a p-divisible group X over S, or respectively a polarized p-divisible group $(X, \lambda : X \to X^t)$ over S, is said to be *geometrically fiberwise constant* (abbreviated as *gfc*) relative to k if for any two (not necessarily closed) points $s_1, s_2 \in S$, there exist

- an algebraically closed field L containing k,
- k-morphisms $\iota_1 : \mathrm{Spec}(L) \to s_1$, $\iota_2 : \mathrm{Spec}(L) \to s_2$, and
- an L-isomorphism

$$\psi : X_{s_1} \times_{(s_1,\iota_1)} \mathrm{Spec}(L) \xrightarrow{\sim} X_{s_2} \times_{(s_2,\iota_2)} \mathrm{Spec}(L),$$

or respectively an L-isomorphism

$$\psi : (X_{s_1}, \lambda_{s_1}) \times_{(s_1,\iota_1)} \mathrm{Spec}(L) \xrightarrow{\sim} (X_{s_2}, \lambda_{s_2}) \times_{(s_2,\iota_2)} \mathrm{Spec}(L).$$

1.3.2. *Central leaves, the original definition.*

Let $d \geq 1$ be a positive integer. Let $n \geq 3$ be a positive integer prime to $p \cdot d$. Denote by $\mathcal{A}_{g,d,n}$ the moduli scheme classifying all g-dimensional polarized abelian schemes $(A \to S, \lambda : A \to A^t)$ over $\overline{\mathbb{F}_p}$ of polarization degree d, together with a symplectic level-n structure ζ.

For any geometric point $x_0 = [(A_0, \lambda_0, \zeta_0)] \in \mathcal{A}_{g,d,n}(\overline{\mathbb{F}_p})$ of the moduli space $\mathcal{A}_{g,d,n}$ over $\overline{\mathbb{F}_p}$, the *central leaf* $\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0)$ in $\mathcal{A}_{g,d,n}$ passing through x_0 is defined in [25] as the largest reduced subscheme of $\mathcal{A}_{g,d,n} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$ such that the principally polarized p-divisible group attached to the restriction to $\mathcal{C}(x_0)$ of the universal principally polarized abelian scheme is geometrically fiberwise constant.

Equivalently, as proved in [25], $\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0)$ is the reduced subscheme of $\mathcal{A}_{g,d,n} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$ whose k-points are given by

$$\mathcal{C}(x_0)(k) = \left\{ y \in \mathcal{A}_{g,d,n}(k) \,\middle|\, (A_y[p^\infty], \lambda_y[p^\infty]) \cong (A_0[p^\infty], \lambda_0[p^\infty]) \right\}.$$

The central leaf $\mathcal{C}(x_0)$ is a closed subscheme of the reduced locally closed subscheme $\mathcal{W}_\xi\big(\mathcal{C}_{\mathcal{A}_{g,d,n}} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} Spec(k)\big)$.

Remark. (a) Clearly every central leaf $\mathcal{C}(x_0)$ in $\mathcal{A}_{g,d,n}$ is stable under all prime-to-p Hecke correspondences on $\mathcal{A}_{g,d,n}$.

(b) We know from the Serre-Tate theorem that the deformation theory for any two closed points of a central leaf are isomorphic, so every central leaf is "homogeneous" in this weak sense. It follows quickly from this property that every central leaf $\mathcal{C}(x_0)$ is a *smooth* locally closed subscheme of the moduli scheme $\mathcal{A}_{g,d,n} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$.

(c) In view of the weak homogeneity property of central leaves, it seems quite appropriate to consider central leaves as "Shimura varieties in characteristic p". In contrast, generally the reduction modulo p of a Shimura variety is not homogenous in any sense. This is already the case in the case of modular curves: the deformation theory at points corresponding to superpersingular elliptic curves are quite different from the deformation theory at points corresponding to ordinary elliptic curves.

1.4. Motivation and definition of sustained p-divisible groups.
The above definition of central leaves suffers from an obvious deficiency of the definition of $\mathcal{C}(x_0)$ in the previous paragraph because of the "point-wise" nature of the notion of gfc: it does *not* tell us how to characterise the set $\mathcal{C}(x_0)(S)$ of all S-points of a central leaf $\mathcal{C}(x_0)$, for non-reduced k-schemes S. This defect is rectified through the notion of *strongly sustained polarized p-divisible groups*, first discovered in December 2012 when F. Oort visited the author in Taipei.

1.4.1. In a nutshell, given a scheme S ove a field $\kappa \supseteq \mathbb{F}_p$ and a polarized p-divisible group (Y_0, ν_0) over κ, a p-divisible group (X, μ) over S, is said to be *strongly κ-sustained modeled on Y_0* if

> *for every natural number n, the pair $(X[p^n], \mu[p^n])$ is locally in the flat topology of S isomorphic to $(Y_0[p^n], \nu_0[p^n])$.*

More precisely,

> *for every n, there exists a faithfully flat morphism $T_n \to S$ and an isomorphism from $(Y_0[p^n], \nu_0[p^n]) \times_{\mathrm{Spec}(\kappa)} T_n$ to $(X[p^n], \mu[p^n]) \times_S T_n$ over T_n.*

Here $X[p^n] := \mathrm{Ker}([p^n]_X)$ is the kernel of the endomorphism "multiplication by p^n" of X, and $\mu[p^n] : X[p^n] \to X^t[p^n]$ is the homomorphism induced by the polarization $\mu : X \to X^t$. Strongly sustained p-divisible groups are defined in a similar way. See §2 for details.

1.4.2. The definition of sustained p-divisible groups is partly based on the following properties of central leaves. Let $\mathcal{C}(x_0)$ be a central leaf in $\mathcal{A}_{g,d,n}$ attached to an $\overline{\mathbb{F}_p}$-point $x_0 = [(A_0, \mu_0, \zeta_0)] \in \mathcal{A}_{g,d,n}(k)$ as defined in 1.3.2. Let $\big(\boldsymbol{A}_{\mathcal{C}(x_0)}, \boldsymbol{\mu}_{\mathcal{C}(x_0)}\big) \to \mathcal{C}(x_0)$ be the restriction to $\mathcal{C}(x_0)$ of the universal polarized abelian scheme.

(a) *The p-divisible group $\boldsymbol{A}_{\mathcal{C}(x_0)}[p^\infty]$ attached to $\boldsymbol{A}_{\mathcal{C}(x_0)}$ admits a slope filtration, i.e. there exists a filtration $0 = \boldsymbol{X}_0 \subsetneqq \boldsymbol{X}_1 \subsetneqq \cdots \subsetneqq \boldsymbol{X}_m = \boldsymbol{A}_{\mathcal{C}(x_0)}[p^\infty]$, where $\boldsymbol{X}_i$ is a p-divisible group over $\mathcal{C}$ and the p-divisible group $\boldsymbol{X}_{i+1}/\boldsymbol{X}_i$ is isoclinic with slope s_i for $i = 1, \ldots, m$, and the slopes satisfy the inequalities $s_1 > s_2 > \cdots > s_m$.*

(b) *The polarized p-divisible group $\big(\boldsymbol{A}[p^\infty], \boldsymbol{\mu}[p^\infty]\big)_{\mathcal{C}(x_0)}$ over $\mathcal{C}(x_0)$ is strongly $\overline{\mathbb{F}_p}$-sustained modeled on $(A_0[p^\infty], \mu_0[p^\infty])$.*

The above properties (a), (b) were observed in 2004 and 2012 respectively; neither is obvious from the original definition of central leaves. For instance since the universal p-divisible group over $\mathcal{C}(x_0)$ has constant Newton polygon and $\mathcal{C}(x_0)$ is smooth, results in [30] and [27] implies that there exists an isogeny from a p-divisible group $\boldsymbol{Y}$ over $\mathcal{C}(x_0)$ to $\boldsymbol{A}_{\mathcal{C}(x_0)}[p^\infty]$ such that $\boldsymbol{Y}$ admits a slope filtration. But in general a p-divisible group with constant Newton polygon over an $\overline{\mathbb{F}_p}$-scheme does not admit a slope filtration.

We will see in 2.5 that (a) is an easy consequence of (b). On the other hand one of the main results in [17] shows that (b) implies that for every geometric point $\bar{x}$ of $\mathcal{C}(x_0)$, the polarized p-divisible groups $(A_{\bar{x}}[p^\infty], \mu_{\bar{x}}[p^\infty])$ and $(A_0[p^\infty], \mu_0[p^\infty]) \times_{\mathrm{Spec}(\overline{\mathbb{F}}_p)} \bar{x}$ are isomorphic over $\bar{x}$. So the property (b) encapsulates all prominent features of central leaves.

1.4.3. With the notion of sustained p-divisible groups at hand, it is natural to define the scheme-theoretic central leaf $\mathcal{C}(x_0)_{\mathrm{sus}}$ to be the largest locally closed *subscheme* of $\mathcal{A}_{g,d,n}$ such that the polarized p-divisible group attached to the polarized abelian scheme $(\boldsymbol{A}, \boldsymbol{\mu})_{\mathcal{C}(x_0)_{\mathrm{sus}}}$ is strongly $\overline{\mathbb{F}}_p$-sustained modeled on $(A_0[p^\infty], \lambda_0[p^\infty])$. This updated definition is adopted in 2.10, where we use the notion $\mathcal{C}(x_0)$ instead of $\mathcal{C}(x_0)_{\mathrm{sus}}$. The original definition of central leaves will not be used after this section.

Property (b) implies that the topological space underlying $\mathcal{C}(x_0)_{\mathrm{sus}}$ coincides with the central leaf $\mathcal{C}(x_0)$ defined in 1.3.2. We will see in 4.5 that

(c) *The scheme-theoretically defined central leaf $\mathcal{C}(x_0)_{\mathrm{sus}}$ is smooth over k, in particular it is reduced.*

Therefore the updated definition 2.10 of central leaves is fully backward compatible with the original definition 1.3.2.

1.4.4. An example. Let Y, Z be two isoclinic p-divisible groups over a perfect field $\kappa \supseteq \mathbb{F}_p$ such that $\mathrm{slope}(Y) < \mathrm{slope}(Z)$. Let $(R, \mathfrak{m})$ be an Artinian local κ-algebra, and let $\epsilon : R \to \kappa$ be a surjective κ-linear ring homomorphism. We give an alternative description of strongly κ-sustained p-divisible groups modeled on $Y \times Z$.

A p-divisible group X over R is strongly κ-sustained modeled on $Y \times Z$ if and only if the following properties hold.

(i) *X is isomorphic to an extension of Y by Z over R, i.e. there exists a faithfully flat homomorphism*

$$\beta : X \to Y \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$$

over R and an R-isomorphism

$$\alpha : Z \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R) \xrightarrow{\sim} \mathrm{Ker}(\beta).$$

(ii) *The extension class of Y by Z corresponding to X is p-divisible. In other words for every positive integer n, there exists an p-divisible group $\tilde{X}$ over R and a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & Z \times \mathrm{Spec}(R) \arrow[r] \arrow[d, "{[p^n]_Z}"'] & \tilde{X} \arrow[r] \arrow[d, "\simeq"] & Y \times \mathrm{Spec}(R) \arrow[r] \arrow[d, "1_Y"] & 0 \\
0 \arrow[r] & Z \times \mathrm{Spec}(R) \arrow[r, "\alpha"] & X \arrow[r, "\beta"] & Y \times \mathrm{Spec}(R) \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Remark. (a) Suppose that Y is not etale and Z is not multiplicative, then there exists a deformation of $Y \times Z$ which is not an extension of Y by Z if $R \supsetneqq \kappa$. Moreover there exists deformations of Y by Z which are not p-divisible.

(b) The subspace of the characteristic-p deformation space $\mathbf{Def}(Y \times Z)$ of $Y \times Z$ over which the universal deformation is an extension of Y by Z has a natural structure as a smooth formal group $\mathbf{Def}^{\mathrm{ext}}(Y \times Z)$ over κ, via the Baer sum of extensions. The statements (i) and (ii) says that the subspace $\mathbf{Def}^{\mathrm{sus}}(Y \times Z)$ of the deformation space of $Y \times Z$ corresponding to sustained deformations is the maximal p-divisible subgroup of $\mathbf{Def}^{\mathrm{ext}}(Y \times Z)$.

The formal scheme $\mathbf{Def}^{\mathrm{sus}}(Y \times Z)$ with reduced structure is the "gfc-defined central leaf" $\mathcal{C}^{\mathrm{gfc}}\big(\mathbf{Def}(Y \times Z)\big)$ in the deformation space $\mathbf{Def}(Y \times Z)$, i.e. the largest reduced formal subscheme of $\mathbf{Def}(Y \times Z)$ over which the universal p-divisible group is geometrically fiberwise constant. The phenomenon that $\mathcal{C}^{\mathrm{gfc}}\big(\mathbf{Def}(Y \times Z)\big)$ is naturally isomorphic to the maximal p-divisible subgroup of the smooth formal group $\mathbf{Def}^{\mathrm{ext}}(Y \times Z)$ was first observed in 2004.

(c) In the case when Y is a one-dimensional p-divisible group of height 3 and Z is the Serre dual of Y, we have $\dim(\mathbf{Def}(Y \times Z) = 9$, $\dim(\mathbf{Def}^{\mathrm{ext}}(Y \times Z)) = 4$, and $\mathbf{Def}^{\mathrm{sus}}(Y \times Z)$ has a natural structure as a 3-dimensional isoclinic p-divisible group with slope $1/3$ and height 9. The smooth formal group $\dim(\mathbf{Def}^{\mathrm{ext}}(Y \times Z)) = 4$ and has a one-dimensional unipotent smooth formal subgroup, whose intersection with $\mathbf{Def}^{\mathrm{sus}}(Y \times Z)$ is a non-trivial finite subgroup scheme of $\mathbf{Def}^{\mathrm{sus}}(Y \times Z)$ over κ.

1.5. Local structure of leaves—what sustained p-divisible groups are good for. We explained in 1.4 that the notion of strongly sustained p-divisible groups retains the essence of geometrically fiberwise constant p-divisible groups and provides further insight on properties of central leaves.

At the same time, this scheme-theoretic notion enables one to analyse the local structure of central leaves by deformation theory: for every p-divisible group X_0 (respectively every polarized p-divisible group (X_0, μ_0) over $\overline{\mathbb{F}_p}$, we have a local deformation space $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ (respectively $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ which classify all strongly $\overline{\mathbb{F}_p}$-sustained deformation of Y_0 (respectively (Y_0, μ_0)) over Artinian local $\overline{\mathbb{F}_p}$-algebras with residue field $\overline{\mathbb{F}_p}$. These deformation spaces are formally smooth over $\overline{\mathbb{F}_p}$. If $x_0 = [(A_0, \lambda_0, \zeta_0)]$ is an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$, the formal completion $\mathcal{C}(x_0)^{/x_0}$ of the central leaf $\mathcal{C}(x_0)$ in $\mathcal{A}_{g,1,n}$ is naturally isomorphic to the sustained deformation space $\mathbf{Def}^{\mathrm{sus}}(A_0[p^\infty], \lambda_0[p^\infty])$.

1.5.1. The notion of strongly sustained p-divisible groups allows us to take full advantage of deformation theory, which yields the following structural information on the formal completions $\mathcal{C}(x_0)^{/x_0}$ of a central leaf $\mathcal{C}(x_0)$ over $\overline{\mathbb{F}_p}$ as above.

(i) Every central leaf $\mathcal{C}(x_0)$ passing through an $\overline{\mathbb{F}_p}$-point x_0 of $\mathcal{A}_{g,d,n}$ is smooth over $\overline{\mathbb{F}_p}$, for every positive integer d.

(ii) The formal completion $\mathcal{C}(x_0)^{/x_0}$ of $\mathcal{C}(x_0)^{/x_0}$ is "built-up" from p-divisible formal groups over $\overline{\mathbb{F}_p}$ through a family of fibrations. We say that $\mathcal{C}(x_0)^{/x_0}$ has a *Tate-linear structure*, and regard such structure as a generalization of the classical Serre–Tate local coordinates for deformations of ordinary abelian varieties.

(iii) Suppose that $\mathscr{M}$ is a PEL modular subvariety of $\mathcal{A}_{g,1,n}$, x_0 is an $\overline{\mathbb{F}}_p$-point of $\mathscr{M}$, and $\mathcal{C}_{\mathscr{M}}(x_0)$ is the central leaf in $\mathscr{M}$ passing through x_0, and $\mathcal{C}_{\mathcal{A}_{g,1,n}}(x_0)$ is the central leaf in $\mathcal{A}_{g,1,n}$ passing through x_0. The general phenomenon is that $\mathcal{C}_{\mathscr{M}}(x_0)^{/x_0}$ should be a *Tate-linear formal subvariety* in a suitable sense. The idea of sustained p-divisible groups led us to a precise definition 6.2 of "strongly Tate-linear formal subvarieties" which describes this phenomenon.

Remark. The notion of Tate-linear formal subvarieties is related to the local rigidity property of the formal completions $\mathcal{C}(x_0)^{/x_0}$ as follows: every reduced irreducible formal subvariety of $\mathcal{C}(x_0)^{/x_0}$ which is stable under a *strongly nontrivial* action of a p-adic subgroup of $\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty])$ is expected to be a Tate-linear formal subvariety of $\mathcal{C}(x_0)^{/x_0}$. See 6.3.

1.5.2. We illustrate the local structure of a central leaf $\mathcal{C}(x_0)$ in a two cases, where $x_0 = [(A_0, \lambda_0, \zeta_0)]$ is an $\overline{\mathbb{F}}_p$-point of $\mathcal{A}_{g,1,n}$.

(a) Suppose that $A_{x_0}[p^\infty]$ is isomorphic to a product $Y_1 \times Y_2$ of isoclinic p-divisible groups of slopes s_1, s_2 respectively, $s_1 < s_2 = 1 - s_1$. The principal polarization λ_{x_0} on A_{x_0} induces a symmetric isomorphism $\delta : Y_2 \xrightarrow{\sim} Y_1^t$ from Y_2 to the Serre dual Y_1^t of Y_1.

> *The formal completion $\mathcal{C}(x_0)^{/x_0}$ of $\mathcal{C}(x_0)$ at x_0 has a natural structure as an isoclinic p-divisible formal group of height $g(g+1)/2$ and slope $s_2 - s_1$.*

The covariant Dieudonné module of $\mathcal{C}(x_0)^{/x_0}$ can be described explicitly: it is the largest $W(\overline{\mathbb{F}}_p)$-submodule of $\mathrm{Hom}^{\mathrm{sym}}_{W(\overline{\mathbb{F}}_p)}(\mathbb{D}_*(Y_1), \mathbb{D}_*(Y_1^t))$ which is stable under the semi-linear operators F, V on $\mathrm{Hom}^{\mathrm{sym}}_{W(\overline{\mathbb{F}}_p)}(\mathbb{D}_*(Y_1), \mathbb{D}_*(Y_1^t))\left[\frac{1}{p}\right]$ defined in the remark after 3.2. Here $\mathbb{D}_*(Y_i)$ is the covariant Dieudonneé module of Y_i for $i = 1, 2$, and $\mathrm{Hom}^{\mathrm{sym}}_{W(\overline{\mathbb{F}}_p)}(\mathbb{D}_*(Y_1), \mathbb{D}_*(Y_1^t))$ consists of all self-dual elements of $\mathrm{Hom}_{W(\overline{\mathbb{F}}_p)}(\mathbb{D}_*(Y_1), \mathbb{D}_*(Y_1^t))$.

(b) Suppose that $A_{x_0}[p^\infty]$ is isomorphic to a product of three isoclinic p-divisible groups Y_1, Y_0, Y_3, with slopes $s_1, s_0 = 1/2, s_3 = 1 - s_1$, $0 \le s_1 < s_2 < s_3 \le 1$, and heights $h_1, h_0, h_2 = h_1$ respectively. Then there exist

- a natural faithfully flat $\overline{\mathbb{F}}_p$-morphism $\pi : \mathcal{C}(x_0)^{/x_0} \to X$ from the smooth formal scheme $\mathcal{C}(x_0)$ to a p-divisible formal group $\mathbf{X}$ over $\overline{\mathbb{F}}_p$,
- a p-divisible formal group $\mathbf{Z}$ over $\overline{\mathbb{F}}_p$,
- a free action of $\mathbf{Z} \times \mathbf{X}$ on $\mathcal{C}(x_0)^{/x_0}$ over $\mathbf{X}$

such that

- the p-divisible formal group $\mathbf{Z}$ over $\overline{\mathbb{F}}_p$ is isoclinic of slope $1 - 2s_1$ and height $h_1(h_1+1)/2$,
- the p-divisible formal group $\mathbf{X}$ is isoclinic of slope $\frac{1}{2} - s_1$ and height $h_1 h_2$, and

- the morphism π together with the action of $\mathbf{Z}$ on $\mathcal{C}(x_0)^\times$ makes $\mathcal{C}(x_0)^{/x_0}$ a $\mathbf{Z}$-torsor over $\mathbf{X}$.

1.6. The rest of this article is organized as follows. In §2 we explain the notion of strongly sustained p-divisible group and some of its basic properties. In §3 we explain the stabilized Hom and Isom schemes attached to polarized p-divisible groups, which play important roles in the theory. In §4 we explain, in broad strokes, how to prove that central leaves in moduli spaces $\mathcal{A}_{g,d,n}$ are smooth. In §5 we explain the local structure of a central leaf in the case when the abelian variety involved has at most three distinct slopes. The corresponding sustained deformation space has a strong local rigidity property; see 5.8. A question on global rigidity is formulated in 5.10. In §6 we explain the notion of *Tate-linear formal subschemes* of the sustained deformation space of a p-divisible group. This notion is motivated by the local rigidity result 5.8 and the proof of 4.3 on the smoothness of sustained deformation spaces, and is needed for the statement of the local rigidity question 6.3 for sustained deformation spaces.

2. Strongly sustained p-divisible groups

2.1. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field, let Y_0 be a p-divisible group over κ, and let S be a κ-scheme.

(1) A p-divisible group X over S is *strongly κ-sustained modeled on* Y_0 if for every natural number n, the Isom scheme

$$\mathbf{Isom}_S\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^n]\big),$$

which represents the functor

$$T \;\mapsto\; \mathrm{Isom}_T\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} T,\, X[p^n] \times_S T\big) \qquad \forall\, S\text{-scheme } T$$

on the category of all S-schemes, is faithfully flat over S.

(2) Let $\mu_0 : Y_0 \to Y_0^t$ be a polarization on Y_0. A polarized p-divisible group $(X, \nu : X \to X^t)$ over S is *strongly κ-sustained modeled on* (Y_0, μ_0) if for every natural number n, the Isom scheme

$$\mathbf{Isom}_S\big((Y_0[p^n], \mu_0[p^n]) \times_{\mathrm{Spec}(\kappa)} S,\, (X[p^n], \nu[p^n])\big),$$

is faithfully flat over S.

2.2. Remark. (a) The above definition articulates the basic idea that locally for the fppf topology, the $[p^n]$-kernel $X[p^n]$ of a strongly κ-sustained p-divisible group X modeled on Y_0 is isomorphic to the $[p^n]$-kernel $Y[p^n]$ of the "constant" p-divisible group Y_0.

(b) Being "constant" is fundamentally a relative concept, which explains the appearance of the base field κ in 2.1. Note that the defintion of geometrically fiberwise constant p-divisible groups recalled in 1.3.1 also depends on the algebraically closed base field k.

(c) On the face of it, the idea of studying families p-divisible groups whose truncations are locally constant in the flat topology is analogous to the familiar notion of isotrivial families of algebraic varieties. However the phenomenon for sustained/gfc p-divisible groups is quite different from that of, say, isotrivial families of smooth projective curves. For instance every geometrically fiberwise constant family of smooth projective curves over a complete discrete valuation ring with algebraically closed residue field is constant, but this statement is false for p-divisible groups. This reflects the fact that we have a coarse moduli space of smooth projective curves with a given genus, but not for polarized p-divisible groups with a given height.

2.3. Remark. (a) It will be interesting to find a good generalization of defintion 2.1, in which the base scheme does not have to be the spectrum of a field $\kappa \supseteq \mathbb{F}_p$, and develop a satisfactory notion of a *family* of sustained p-divisible groups that is applicable to the family central leaves in a Newton polygon stratum of a Siegel modular variety $\mathcal{A}_{g,1,n}$.

(b) There is a related (and slightly weaker) notion of a *κ-sustained p-divisible group* over a κ-scheme S, which does not require the existence of a κ-model. More information can be found in [11]; see also [9].

2.4. Remark. The requirement that each truncated Barsotti–Tate group $X[p^n]$ is locally constant in the flat topology in definition 2.1 (i) can be strengthened to equivalent conditions (1a) or (1b) below, and to (1c) if X is isoclinic.

Suppose that $X \to S$ is a strongly κ-sustained p-divisible group modeled on a p-divisible group Y_0 over κ as in 2.1.

(1a) For each $n \in \mathbb{N}$, there exists a *finite locally free* morphism $T_n \to S$ and a T_n-isomorphism

$$Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} T_n \xrightarrow{\sim} X[p^n] \times_S T_n.$$

(1b) There exists a faithfully flat quasi-compact morphism $T \to S$ and a T-isomorphism

$$Y_0 \times_{\mathrm{Spec}(\kappa)} T \xrightarrow{\sim} X \times_S T.$$

(1c) If X is isoclinic, then for each $n \in \mathbb{N}$, there exists a *finite etale* morphism $T_n \to S$ and a T_n-isomorphism

$$Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} T_n \xrightarrow{\sim} X[p^n] \times_S T_n.$$

As the readers likely will expect, the obvious analogs of statements (i)–(iii) for a strongly κ-sustained *polarized* p-divisible group $(X \to S, \nu : X \to X^t)$ hold.

2.5. Proposition (The slope filtration on sustained p-divisible groups)**.** *Let S be a scheme over a field $\kappa \supseteq \mathbb{F}_p$. Let $X \to S$ be a strongly κ-sustained p-divisible group modeled on a p-divisible group Y_0 over κ. Let*

$$(0) = \mathrm{Fil}_0(Y_0) \subsetneqq \mathrm{Fil}_1(Y_0) \subsetneqq \cdots \subsetneqq \mathrm{Fil}_m(Y_0) = Y_0$$

be the slope filtration of Y_0 by p-divisible subgroups $\mathrm{Fil}_i(Y_0)$ over κ, $i = 0, 1, \ldots, m$, such that $\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$ is isoclinic for $i = 1, \ldots, m$, and the slopes s_i of

$\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$ *satisfy*

$$1 \geq s_1 > s_2 > \cdots > s_m \geq 0.$$

There exists a unique slope filtration

$$(0) = \mathrm{Fil}_0 X \subsetneqq \mathrm{Fil}_1 X \subsetneqq \cdots \subsetneqq \mathrm{Fil}_m X = X$$

of X by p-divisible subgroups $\mathrm{Fil}_i(X)$ *such that*

(a) *For each $i = 1, \ldots, m$, the p-divisible group* $\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$ *is isoclinic of slope s_i.*
(b) *The p-divisible group* $\mathrm{Fil}_i(X)$ *over S is strongly κ-sustained modeled on* $\mathrm{Fil}_i(Y_0)$ *for $i = 1, \ldots, m$.*
(c) *For any $i = 1, \ldots, m$, the isoclinic p-divisible group* $\mathrm{Fil}_i(X)/\mathrm{Fil}_{i-1}(X)$ *is strongly κ-sustained modeled on* $\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$.

2.6. Remark. (i) The existence of the slope filtration on a p-divisible group over a field was proved in [30, Cor. 13].

(ii) The proof 2.5 is an exercise of flat descent, transferring the slope filtration for the model Y_0 to X via the finite locally free covers

$$\mathbf{Isom}_S^{\mathrm{st}}\big((Y_0[p^n], \mu_0[p^n]) \times_{\mathrm{Spec}(\kappa)} S,\, (X[p^n], \lambda[p^n])\big)$$

of S constructed in 3.6.

(iii) In general a p-divisible group Y over a base scheme S in characteristic p with constant Newton polygons may not admit a slope filtration. All one can say is that if the base scheme S is noetherian and normal, then Y is isogenous to a *completely slope divisible* p-divisible group Z over S; see [27, Thm. 2.1]. We refer to [27, (1.2)] for the definition of completely slope divisible p-divisible groups, and to [11] for relations between completely slope divisible and sustained p-divisible groups.

(iv) Assume that the base scheme S in 2.5 is the spectrum of an Artinian local κ-algebra R with residue field κ. Then $X_0 := X \times_S \mathrm{Spec}(\kappa)$ is a model of the strongly κ-sustained p-divisible group $X \to S$ in 2.5, and $\mathrm{Fil}_i(X)/\mathrm{Fil}_{i-1}(X)$ is canonically isomorphic to $\big(\mathrm{Fil}_i(X_0)/\mathrm{Fil}_{i-1}(X_0)\big) \times_{\mathrm{Spec}(\kappa)} S$ for $i = 1, \ldots, m$.

2.7. Remark. (i) Sometimes it is convenient to reindex the slope filtration by the slopes themselves. In the context of 2.5, define a decreasing filtration $\mathrm{Fil}^{\bullet}_{\mathrm{can}} X$ on X by

$$\mathrm{Fil}^t_{\mathrm{can}} X := \mathrm{Fil}_{m(t)} X, \quad \text{where} \quad m(t) = \begin{cases} 0 & \text{if } \ t > s_1 \\ i & \text{if } \ s_{i+1} < t \leq s_i,\ i \in \{1, \ldots, m-1\} \\ m & \text{if } \ t \leq s_m \end{cases}$$

for every $t \in \mathbb{R}$.

(ii) Suppose that $X \to S$ and $Y \to S$ are strongly κ-sustained p-divisible groups, and $\phi : X \to Y$ is an S-homomorphism of p-divisible groups. Then

$$\phi(\mathrm{Fil}^t_{\mathrm{can}} X) \subseteq \mathrm{Fil}^t_{\mathrm{can}} Y \quad \forall\, t \in \mathbb{R}.$$

2.8. Proposition (Backward compatibility with gfc)**.** *Let S be a* reduced *scheme over a field $\kappa \supseteq \mathbb{F}_p$. Let $X \to S$ be a p-divisible group over S and let Y_0 be a p-divisible group over κ.*

(a) *If X_s is strongly κ-sustained modeled on Y_0 for every $s \in S$, then $X \to S$ is strongly κ-sustained modeled on Y_0.*

(b) *Let ν be a polarization on X and let μ_0 be a polarization on Y_0. If (X_s, λ_s) is strongly κ-sustained modeled on (Y_0, μ_0) for every $s \in S$, then (X, ν) is strongly κ-sustained modeled on (Y_0, μ_0).*

2.9. Proposition. *Let n be a positive integer such that $\gcd(n, p) = 1$, $n \geq 3$. Let $d > 0$ be a positive integer. Denote by $\mathcal{A}_{g,d,n}$ the fine moduli scheme over $\overline{\mathbb{F}_p}$ which classifies all polarized abelian schemes $(A \to S, \lambda : A \to A^t)$ of relative dimension g with $\deg(\lambda) = d^2$, plus a symplectic level-n structure ζ, where S is an $\overline{\mathbb{F}_p}$-scheme. Let $(\boldsymbol{A}, \boldsymbol{\nu})$ be the universal polarized abelian scheme over $\mathcal{A}_{g,d,n}$. Let $x_0 = [(A_0, \nu_0, \zeta_0)]$ be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$.*

There exists a unique locally closed subscheme $\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0)$ of $\mathcal{A}_{g,d,n}$ with the following property:

> *For every polarized abelian scheme with level-n-structure (B, μ, ψ) over an $\overline{\mathbb{F}_p}$-scheme T, the modular morphism $T \to \mathcal{A}_{g,d,n}$ factors through the inclusion $\mathcal{C}(x_0) \hookrightarrow \mathcal{A}_{g,d,n}$ if and only if the polarized p-divisible group $(B[p^\infty], \mu[p^\infty])$ over T is strongly κ-sustained modeled on $(A_0[p^\infty], \nu_0[p^\infty])$*

2.10. Definition (Updated definition of central leaves)**.** The locally closed subscheme

$$\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0) \ \subseteq \ \mathcal{A}_{g,d,n}$$

in 2.9 is called the *central leaf* in $\mathcal{A}_{g,d,n}$ passing through x_0.

Remark. As remarked at the end of 1.4, a priori it appears that the scheme-theoretic definition 2.10 might be different from the original definition of central leaves recalled in 1.3.2, i.e. there may exist a central leaf in $\mathcal{A}_{g,d,n}$ in the sense of the updated definition 2.10, temporarily denoted by $\mathcal{C}(x_0)_{\text{sus}}$, such that the set underlying $\mathcal{C}(x_0)_{\text{sus}}$ is equal to the central leaf $\mathcal{C}(x_0)$ defined in 2.10, but $\mathcal{C}(x_0)_{\text{sus}}$ is not reduced. However 4.5 implies that every central leaf $\mathcal{C}(x_0)_{\text{sus}}$ in $\mathcal{A}_{g,d,n}$ is smooth. So the new definition 2.10 is indeed fully compatible with the original definition 1.3.2.

3. Stabilized Hom schemes for p-divisible groups

3.1. Definition (Stabilized Hom schemes for p-divisible groups)**.** Let Y, Z be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. For each $n \in \mathbb{N}$, we have a commutative group scheme

$$H_n := \mathbf{Hom}(Y[p^n],\, Z[p^n])$$

of finite type over κ, which satisfies the university property that

$$\mathbf{Hom}(Y[p^n], Z[p^n])(S) = \mathrm{Hom}_S(Y[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, Z[p^n] \times_{\mathrm{Spec}(\kappa)} S)$$

for every κ-scheme S. In addition we have

- restriction homomorphisms $r_{n,n+i}\colon H_{n+i} \to H_n$, and
- closed embeddings $\iota_{n+i,n}\colon H_n \hookrightarrow H_{n+i}$ given by (a) the epimorphism $Y_{n+i} \twoheadrightarrow Y_n$ induced by $[p^i]_{Y[p^{n+i}]}$, and (b) the inclusion $Z[p^n] \hookrightarrow Z[p^{n+i}]$.

Note that

$$\iota_{n+i,n} \circ r_{n,n+i} = [p^i]_{H_{n+i}} \quad \forall\, n, i \in \mathbb{N}.$$

Define closed subgroup scheme $\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n$ of H_n over κ by

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n := \mathrm{Im}(r_{i,n+i}\colon H_{n+i} \to H_n) \qquad \text{for} i \gg 0$$

It is clear from the above definition that the arrows

$$\begin{tikzcd} H_{n+i} \arrow[r, bend left, "r_{n,n+i}"] & H_n \arrow[l, bend left, "\iota_{n+i,n}"] \end{tikzcd}$$

induce arrows

$$\begin{tikzcd} \mathbf{Hom}^{\mathrm{st}}(Y,Z)_{n+i} \arrow[r, shift left, "\pi_{n,n+i}"] & \mathbf{Hom}^{\mathrm{st}}(Y,Z)_n \arrow[l, shift left, "j_{n+i,n}"] \end{tikzcd} ,$$

and

$$j_{n+i,n} \circ \pi_{n,n+i} = [p^i]_{\mathbf{Hom}^{\mathrm{st}}(Y,Z)_{n+i}} \qquad \forall\, n, i \in \mathbb{N}.$$

3.2. Theorem. *Let Y, Z be p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$ as in* 3.1.

(a) *For each $n \in \mathbb{N}$, the commutative group scheme $\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n$ is* finite *over κ.*

(b) *The family*

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z) := \big(\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n, j_{n+i,n}, \pi_{n,n+i}\big)_{n\in\mathbb{N}}$$

of commutative groups schemes $\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n$ together with the homomorphisms $j_{n+i,n}, \pi_{n,n+i}$ is a p-divisible group over κ.

(c) *Suppose that the field κ is perfect, and let $\mathbb{D}_*(Y), \mathbb{D}_*(Z)$ be the covariant Dieudonné modules of Y and Z respectively. The covariant Dieudonné module of $\mathbf{Hom}^{\mathrm{st}}(Y,Z)$ is the largest $W(\kappa)$-submodule of $\mathrm{Hom}_{W(\kappa)}\big(\mathbb{D}_*(Y), \mathbb{D}_*(Z)\big) =: \boldsymbol{H}$ which is stable under the semi-linear operators F and V.*

(d) *Suppose that Y, Z are* isoclinic *over κ, with slopes s_Y and s_Z respectively.*

- *If $s_Y > s_Z$, then $\mathbf{Hom}^{\mathrm{st}}(Y,Z) = (0)$.*
- *If $s_Y \leq s_Z$, then $\mathbf{Hom}^{\mathrm{st}}(Y,Z)$ is isoclinic of slope $s_Z - s_Y$ and height* $\mathrm{ht}(Z)\cdot \mathrm{ht}(Y)$.

Remark. (i) In 3.2 (c), the semi-linear operators F, V on $\boldsymbol{H}_{\mathbb{Q}}$ are defined as follows: for every $h \in \boldsymbol{H}$, we have

$$F(h)(y) = F(h(Vy)) \in \boldsymbol{H}$$

and

$$V(h)(y) = V(h(V^{-1}y)) = p^{-1}\cdot V(h(F(y))) \in \boldsymbol{H}_{\mathbb{Q}}$$

for all $y \in \mathbb{D}_*(Y)$. So $F(\boldsymbol{H}) \subseteq \boldsymbol{H}$, while $V(\boldsymbol{H}) \subseteq \boldsymbol{H}_{\mathbb{Q}}$.

(ii) In covariant Dieudonné theory, the operator V on the Dieudonné module $\mathbb{D}_*(X)$ of a p-divisible group over a perfect field $\kappa \supseteq \mathbb{F}_p$ corresponds to the geometric Frobenius operator on X.

In the situation of 3.2 (d), the assumptions on Y and Z means that asymptotically, $V^n_{\mathbb{D}_*(Y)}$ is roughly $p^{n\cdot s_Y}$ times an isomorphism and $V^n_{\mathbb{D}_*(Z)}$ is roughly $p^{n\cdot s_Z}$ times an isomorphism for $n \gg 0$. Therefore the recipe in 3.2 (c) for the Dieudonné module of $\mathbf{Hom}^{\rm st}(Y,Z)$ implies 3.2 (d).

3.3. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field and let Y be a p-divisible group over κ.

(i) Define a projective system $\mathbf{End}^{\rm st}(Y)$ of finite ring schemes over κ by

$$\mathbf{End}^{\rm st}(Y) := \big(\mathbf{End}^{\rm st}(Y)_n\big)_{n\geq 1}, \quad \mathbf{End}^{\rm st}(Y)_n := \mathbf{Hom}^{\rm st}(Y,Y)_n.$$

The group of units

$$\mathbf{Aut}^{\rm st}(Y) := \big(\mathbf{Aut}^{\rm st}(Y)_n\big)_{n\geq 1}, \quad \mathbf{Aut}^{\rm st}(Y)_n := \big(\mathbf{End}^{\rm st}(Y)_n\big)^{\times},$$

in $\mathbf{End}^{\rm st}(Y)$ is a projective system of finite group schemes over κ.

(ii) Let μ be a polarization of Y. For each $n \geq 1$, denote by $\mathbf{Aut}^{\rm pst}(Y,\mu)_n$ the closed subgroup scheme of $\mathbf{Aut}(Y,\mu)_n$ consisting of automorphisms of $Y[p^n]$ which respect the homomorphism

$$\mu[p^n] = \mu|_{Y[p^n]} : Y[p^n] \to Y^t[p^n] = Y[p^n]^D,$$

where $Y[p^n]^D$ is the Cartier dual of $Y[p^n]$. Define a closed subgroup scheme $\mathbf{Aut}^{\rm st}(Y,\mu)_n$ of $\mathbf{Aut}^{\rm st}(Y)_n$ by

$$\mathbf{Aut}^{\rm st}(Y,\mu)_n := \mathrm{Im}\big(\mathbf{Aut}^{\rm pst}(Y,\mu)_{n+i} \longrightarrow \mathbf{Aut}^{\rm pst}(Y,\mu)_n\big), \quad i \gg 0.$$

Denote by $\mathbf{Aut}^{\rm st}(Y,\mu)$ the projective system of finite group schemes

$$\mathbf{Aut}^{\rm st}(Y,\mu) := \Big(\mathbf{Aut}^{\rm st}(Y,\mu)_n\Big)_{n\geq 1}.$$

3.4. Definition. Let (Y,μ) be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$.

(i) Pick an isogeny $\nu : Y^t \to Y$ such that $\nu\circ\mu = [p^m]_Y$ and $\mu\circ\nu = [p^m]_{Y^t}$ for a natural number $m \in \mathbb{N}$. Define a *quasi-isogeny* ι_μ on the p-divisible group $\mathbf{End}^{\rm st}(Y)$ by

$$\iota_\mu(h) = p^{-m}\cdot \nu\circ h^t\circ\mu \qquad h \in \mathbf{End}^{\rm st}(Y).$$

This quasi-isogeny depends only on μ and is independent of the choice of ν and m, and satisfies

$$\iota_\mu^2 = \mathrm{id}.$$

We call ι_μ the *Rosati involution* on $\mathbf{End}^{\rm st}(Y)$

(ii) Denote by $\mathbf{End}^{\rm st}(Y)^{\iota_\mu=-1}$ the largest p-divisible subgroup of $\mathbf{End}^{\rm st}(Y)$ on which the Rosati involution ι_μ operates as $-\mathrm{id}$.

3.5. Lemma. *Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$.*

(i) *Let $X \to S$ be a strongly κ-sustained p-divisible group modeled on Y_0. For each $n \in \mathbb{N}$, there exists a positive integer i_0 such that the schematic image*

$$\mathrm{Im}\Big(\mathbf{Isom}_S\big(Y_0[p^{n+i}]\times_{\mathrm{Spec}(\kappa)} S,\, X[p^{n+i}]\big) \longrightarrow \mathbf{Isom}_S\big(Y_0[p^n]\times_{\mathrm{Spec}(\kappa)} S,\, X[p^n]\big)\Big)$$

under the restriction homomorphism is independent of i for all $i \geq i_0$.

(ii) *The stabilized image*

$$\mathbf{Isom}^{\mathrm{st}}_S(Y_0, X)_n := \mathrm{Im}\Big(\mathbf{Isom}_S\big(Y_0[p^{n+i}] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^{n+i}]\big) \longrightarrow \mathbf{Isom}_S\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^n]\big)\Big), \quad i \gg 0$$

has a natural structure as a right torsor for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$. *Moreover the natural projections maps*

$$\mathbf{Isom}^{\mathrm{st}}_S(Y_0, X)_{n+i} \longrightarrow \mathbf{Isom}^{\mathrm{st}}_S(Y_0, X)_n$$

are faithfully flat and compatible with the projection maps

$$\mathbf{Aut}^{\mathrm{st}}(Y_0)_{n+i} \times_{\mathrm{Spec}(\kappa)} S \longrightarrow \mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S.$$

(iii) *Suppose that μ_0 is a polarization on Y_0 and (X, ν) is a strongly κ-sustained polarized p-divisible group modeled on (Y_0, μ_0). The obvious analogs of* (i) *and* (ii) *hold, and we have a projective family*

$$\Big(\mathbf{Isom}^{\mathrm{st}}_S((Y_0, \mu_0), (X, \nu))_n\Big)_{n \geq 1}$$

of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_n \times_{\mathrm{Spec}(\kappa)} S$, *compatible with the projections*

$$\mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_{n+i} \longrightarrow \mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_n.$$

3.6. Lemma. *Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$, and let μ_0 be a polarization on Y_0. Let S be a scheme over κ.*

(i) *Let $\big(T_n\big)_{n \geq 1}$ be a compatible family of right torsors for* $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$. *For each $n \geq 1$, let*

$$X_n := T_n \wedge^{\mathbf{Aut}^{\mathrm{st}}(Y_0)_n} Y_0[p^n]$$

be the contraction product of T_n with $Y_0[p^n]$ with respect to the natural action of $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ *on $Y_0[p^n]$. Then X_n is a BT_n group over S, and the family $\big(X_n\big)_{n \geq 1}$ together with the natural maps $X_n \hookrightarrow X_{n+1}$ and $X_{n+1} \twoheadrightarrow X_n$ is a p-divisible group over S.*

(ii) *The constructions in* 3.5 (i)–(ii) *and* 3.6 (i) *define an equivalence between*

- (a) *the category of strongly κ-sustained p-divisible groups over S modeled on Y_0, and*
- (b) *the category of projective systems of right torsors $\big(T_n\big)_{n \geq 1}$ for the group schemes* $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$ *over S which are compatible with the projective system of group schemes* $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)\big)_{n \geq 1}$, *in the sense that the projection map $T_{n+1} \to T_n$ is equivariant with respect to the projection* $\mathbf{Aut}^{\mathrm{st}}(Y_0)_{n+1} \to \mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ *for every n.*

(iii) *Let μ_0 be a polarization on Y_0. The obvious analog of* (i) *holds for strongly κ-sustained polarized p-divisible groups over S modeled on (Y_0, μ_0). This construction and* 3.5 (iii) *define an equivalence of categories between category of strongly*

κ-sustained polarized p-divisible groups over S modeled on (Y_0, μ_0) and the category of projective systems of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_n \times_{\mathrm{Spec}(\kappa)} S$ which are compatible with the projective system of group schemes $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)\big)_{n\geq 1}$.

4. Smoothness of sustained deformations

Let $\kappa \supseteq \mathbb{F}_p$ be a field. Let $\mathfrak{Art}_\kappa$ be the category whose objects are triples

$$(R,\, i : \kappa \to R,\, \epsilon : R \twoheadrightarrow \kappa),$$

where (R, i) is an Artinian local κ-algebra, and ϵ is a κ-linear surjective ring homomorphism whose kernel is the maximal ideal of R. A morphism in $\mathfrak{Art}_\kappa$ from (R_1, i_1, ϵ_1) to (R_2, i_2, ϵ_2) is a κ-linear ring homomorphism $h : R_1 \to R_2$ such that $\epsilon_2 \circ h_2 = \epsilon_1$.

4.1. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field and let Y_0 be a p-divisible group over κ.

(i) The functor $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of sustained deformations of Y_0 is the functor from $\mathfrak{Art}_\kappa$ to the category of sets which sends every object (R, i, ϵ) to the set of isomorphism classes of pairs

$$\big(X \to \mathrm{Spec}(R), \psi : Y_0 \xrightarrow{\sim} X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)\big),$$

where $X \to \mathrm{Spec}(R)$ is a strongly κ-sustained p-divisible group modeled on Y_0 and ψ is a κ-isomorphism.

Two pairs $(X_1 \to \mathrm{Spec}(R),\, \psi_1)$, $(X_2 \to \mathrm{Spec}(R),\, \psi_2)$ are isomorphic if there exists an isomorphism $\alpha : X_1 \to X_2$ of p-divisible groups over R such that $\big(\alpha \times_S \mathrm{Spec}(\kappa)\big) \circ \psi_1 = \psi_2$.

(ii) Let μ_0 be a polarization on Y_0. The functor $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ of sustained deformations of Y_0 is the functor from $\mathfrak{Art}_\kappa$ to the category of sets which sends every object (R, i, ϵ) to the set of isomorphism classes of triples

$$\big(X \to \mathrm{Spec}(R), \nu : X \to X^t, \psi : (Y_0, \mu_0) \xrightarrow{\sim} (X, \nu) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)\big),$$

where (X, ν) is a strongly κ-sustained p-divisible group over R modeled on (Y_0, μ_0), and ψ is an isomorphism from Y_0 to $X \times_S \mathrm{Spec}(\kappa)$ such that $\psi^*(\nu \times_S \mathrm{Spec}(\kappa)) = \mu_0$.

Two triples $(X_1 \to \mathrm{Spec}(R), \nu_1, \psi_1)$, $(X_2 \to \mathrm{Spec}(R), \nu_2, \psi_2)$ are isomorphic if there exists an isomorphism $\alpha : X_1 \to X_2$ over R such that $\alpha^*(\nu_2) = \nu_1$ and

$$\big(\alpha \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)\big) \circ \psi_1 = \psi_2.$$

Remark. Theorem 4.3 shows that the sustained deformation functors $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ are representable. Each is isomorphic to the formal spectrum of a formal power series over κ in a finite number of variables, and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ is a closed formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$.

4.2. Lemma. *Let $x_0 = [(A_0, \mu_0, \psi_0)]$ be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$ and let $\mathcal{C}(x_0)$ be the central leaf in $\mathcal{A}_{g,d,n}$ passing through x_0 as in 2.10. Denote by $\mathcal{C}(x_0)^{/x_0}$ the formal completion of $\mathcal{C}(x_0)$ at x_0. The natural morphism*

$$\mathcal{C}(x_0)^{/x_0} \longrightarrow \mathbf{Def}^{\mathrm{sus}}(A_0[p^\infty], \mu_0[p^\infty])$$

is an isomorphism.

4.3. Theorem. *Let $\kappa \supseteq \mathbb{F}_p$ be a field. Let Y_0 be a p-divisible group over κ and let μ_0 be a polarization of Y_0.*

(i) *The deformation functors $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ are formally smooth over κ.*

(ii) *The dimension of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is given by*
$$\dim\big(\mathbf{Def}^{\mathrm{sus}}(Y_0)\big) = \dim\big(\mathbf{End}^{\mathrm{st}}(Y_0)\big),$$
the dimension of the p-divisible group $\mathbf{End}^{\mathrm{st}}(Y_0)$.

(iii) *We have*
$$\dim\big(\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)\big) = \dim\big(\mathbf{End}^{\mathrm{st}}(Y_0)^{\iota_{\mu_0} = -\mathrm{id}}\big),$$
where ι_{μ_0} is the Rosati involution on the p-divisible group $\mathbf{End}^{\mathrm{st}}(Y_0)$ defined in 3.4.

4.4. We sketch a proof of the smoothness of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and the statement 4.3 (ii). The proofs of the smoothness of the deformation functor $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ and 4.3 (iii) are similar.

Let $h : (R', i', \epsilon') \to (R, i, \epsilon)$ be a morphism in such that $h : R' \to R$ is a surjection and $J := \mathrm{Ker}(h)$ is killed by the maximal ideal $\mathfrak{m}'$ of R'. In other words R' is a small extension of R. Let $S = \mathrm{Spec}(R)$, let $S' = \mathrm{Spec}(R')$, and let $S_0 := \mathrm{Spec}(\kappa)$. We need to show that, given a strongly κ-sustained p-divisible group X over R modeled on Y_0 plus a rigidification $\psi : Y_0 \xrightarrow{\sim} X \times_S S_0$, there exists a lifting of the pair (X, ψ) in $\mathbf{Def}^{\mathrm{sus}}(Y_0)(R)$ to a pair (X', ψ') in $\mathbf{Def}^{\mathrm{sus}}(Y_0)(R')$.

Let $(T_n)_{n\geq 1}$ be the projective family of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$ associated to the strongly κ-sustained p-divisible group over S as in 3.5, together with compatible trivializations
$$\phi_n : \mathbf{Aut}^{\mathrm{st}}(Y_0)_n \xrightarrow{\sim} T_n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$$
associated to ψ. According to 3.6 (ii), it suffices to show that the compatible family $\big(T_n, \phi_n\big)_{n\geq 1}$ of rigidified right torsors for $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S\big)_{n\geq 1}$ lifts to a compatible family $\big(T'_n, \phi'_n\big)_{n\geq 1}$ of rigidified right torsors for $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'\big)_{n\geq 1}$.

For each $n \geq 1$, we have a perfect complex $\ell_{T_n/S}$ of $\mathscr{O}_S$-modules of amplitude $\subseteq [-1, 0]$, called the co-Lie complex of T_n/S; see [13, Ch. 7, §2.4]. By [13, Ch. 7, Thm. 2.4.4], there is an obstruction element
$$o(T_n, S \hookrightarrow S') \in H^2(S, \ell^\vee_{T_n/S} \otimes^{\mathbb{L}}_R J)) \cong H^2(S_0, \ell^\vee_{T_n \times_S S_0/S_0} \otimes_\kappa J)$$
whose vanishing is the necessary and sufficient condition for the existence of a right torsor for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'$ which extends T_n. Since $\ell^\vee_{T_n \times_S S_0/S_0}$ is perfect of amplitude $\subseteq [0, 1]$ and S_0 is affine,
$$H^2(S_0, \ell^\vee_{T_n \times_S S_0/S_0} \otimes_\kappa J) = (0).$$
Therefore T_n can be extended to a right torsor for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'$, for every n.

The theorem [13, Ch. 7, Thm. 2.4.4] also tells us that the set of all liftings to S' of T_n has a natural structure as a torsor for

$$H^1(S_0, \ell^\vee_{T_n\times_S S_0/S_0}\otimes_\kappa J) = \nu_{T_n\times_S S_0/S_0}\otimes_\kappa J,$$

where

$$\nu_{T_n\times_S S_0/S_0} := H^1(\ell^\vee_{T_n\times_S S_0/S_0}) = H^1(\ell^\vee_{\mathbf{Aut}^{\mathrm{st}}(Y_0)_n/S_0}).$$

The last equality holds because the torsor $T_n\times_S S_0$ over $S_0 = \mathrm{Spec}(\kappa)$ is trivial.
Claim. The natural map

$$\nu_{T_{n+1}\times_S S_0/S_0}\otimes_\kappa J \longrightarrow \nu_{T_n\times_S S_0/S_0}\otimes_\kappa J$$

is an isomorphism for every $n\ge 1$.

Clearly the claim implies the existence of a projective system $\big(T'_n\big)_{n\ge1}$ of right torsors for $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\times_{\mathrm{Spec}(\kappa)} S'\big)_{n\ge1}$ which extends $\big(T_n\big)_{n\ge1}$. This finishes the proof of the smoothness of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$. The above claim also shows that the tangent space of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is naturally isomorphic to the κ-vector space $\nu_{T_n\times_S S_0/S_0}$, for any $n\ge 1$.

The key fact for the claim is the existence of a projective system of decreasing "slope filtration"

$$\Big(\mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n,\ t\in[0,1]\Big)_{n\ge1}$$

on the projective system $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\big)_{n\ge1}$, where each $\mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ is a normal subgroup scheme of $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ for each $t\in[0,1]$ and every $n\ge1$. This filtration has the following properties.

(a) The subquotient

$$\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n := \mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\big/\mathrm{Fil}^{>t}_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$$

is a commutative finite group scheme for every $t>0$ and ever $n\ge1$.

(b) $\nu_{\mathrm{gr}^0_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n/S_0} = (0)$ for every $n\ge1$.

(c) For every $t\in(0,1]$, the projective system

$$\Big(\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\Big)_{n\ge1}$$

"is" a p-divisible group, in the sense that there exist homomorphisms

$$\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\to \mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_{n+1}$$

which together with the projections make the family of commutative group schemes $\big(\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\big)_{n\ge1}$ a p-divisible group over κ.

Recall the fact that for any p-divisible group Z over κ, the natural map

$$\nu_{Z[p^{n+1}]/\kappa}\to\nu_{Z[p^n]/\kappa}$$

is an isomorphism for every $n\ge1$; see [15, Prop. 2.2.1]. The Claim follows from this fact and dévissage, using the above filtration on $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\big)_{n\ge1}$ and the exactness properties of co-Lie complexes of group schemes.

Note that the above argument also shows that

$$\dim(\mathbf{Def}^{\mathrm{sus}}(Y_0)) = \dim(\nu_{\mathbf{End}^{\mathrm{st}}(Y_0)_n}) = \dim(\mathbf{End}^{\mathrm{st}}(Y_0)) \quad \forall\, n \geq 1.$$

The second equality is a general property of p-divisible groups over fields of characteristic $p > 0$. We have finished the sketch of the proofs the smoothness of the deformation functor $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and the statement (ii). □

4.5. Corollary. *Let g, d, n be positive integers, with $n \geq 3$ and $\gcd(n, p) = 1$. For every $x_0 \in \mathcal{A}_{g,d,n}(\overline{\mathbb{F}_p})$, the central leaf $\mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0)$ as defined in 2.10 is smooth over $\overline{\mathbb{F}_p}$. Moreover the dimension of a central leaf in $\mathcal{A}_{g,d,n}$ depends only on the Newton polygon ξ of the central leaf, and is independent of the polarization degree d.*

Proof. This is an immediate consequence of 4.2 and 4.3. □

4.6. Remark. The moduli scheme $\mathcal{A}_{g,d,n}$ is known to be a local complete intersection for all $d \geq 1$. On the other hand it tends to exhibit many "unpleasant" phenomena when d is divisible by a high power of p. For instance $\mathcal{A}_{g,d,n}$ may be non-reduced at every point, there may exist Newton strata in $\mathcal{A}_{g,d,n}$ which have irreducible components of different dimensions, and the dimension of the supersingular locus in $\mathcal{A}_{g,d,n}$ can be substantially bigger than the dimension of the supersingular locus of $\mathcal{A}_{g,1,n}$, which is $\lfloor g^2/4 \rfloor$. Thus the fact that all central leaves in $\mathcal{A}_{g,d,n}$ are smooth might be a surprise.

5. Local properties of central leaves with at most 3 slopes

5.1. In this section we illustrate two general phenomena of sustained deformation spaces $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$, where Y_0 is a p-divisible group over an algebraically closed field $k \supseteq \mathbb{F}_p$ and μ_0 is a polarization on Y_0:

(a) The formal schemes $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ are "built up" from p-divisible formal groups through a family of fibrations whose fibers are p-divisible formal groups. Formal schemes with such properties are said to be *Tate-linear*.

(b) (Local rigidity of Tate-linear formal schemes) Suppose that a p-adic Lie group G operates on a formal scheme $\mathbf{D}$ with a Tate-linear structure, and the action is *non-trivial* in a strong sense. Then every irreducible reduced closed formal subscheme of $\mathbf{D}$ which is stable under the action of an open subgroup of G is a *Tate-linear formal subscheme* of $\mathbf{D}$.

For simplicity we will only illustrate phenomenon (a) for $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and phenomenon (b) for $\mathbf{D} = \mathbf{Def}^{\mathrm{sus}}(Y_0)$, both in the case when the p-divisible group Y_0 has at most three slopes. The formal schemes $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{D}$ we meet in these examples are either p-divisible formal groups over κ, or bi-extensions of p-divisible formal groups over κ. The precise definition of Tate-linear formal subschemes in either case is given in 5.6.

Note that the case when Y_0 is isoclinic is trivial, because $\mathbf{Def}^{\mathrm{sus}}(Y_0) = \mathrm{Spf}(k)$ if Y_0 is isoclinic.

5.2. Lemma. *Let Z_1, Z_2 be two isoclinic p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$, with slopes $s_1 > s_1$. Let $Y_0 := Z_1 \times Z_2$. Then $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is naturally isomorphic to the p-divisible formal group $\mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$.*

Note that $\mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$ is isoclinic of slope $s_1 - s_2$ and its height is $\mathrm{ht}(Z_1) \cdot \mathrm{ht}(Z_2)$, according to 3.2 (d).

5.3. Remark. (i) The proof of 5.2 is an easy consequence of 3.2 plus some diagram chasing with suitable Kummer sequences.

(ii) Usually a p-divisible group $Y \to S$ is considered as the *inductive* limit of its truncations $Y[p^n]$ in the category of sheaves on the flat site S_{fl} of S. If we adhere to this point of view, then the sheaf $\mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$ is *not* the "internal Hom" from Z_2 to Z_1 in the category of sheaves on S_{fl}. Rather it is essentially the "internal Ext^1" from Z_2 to Z_1.

5.4. Lemma. *Let Y_1, Y_2, Y_3 be isoclinic p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$, with slopes $s_1 > s_2 > s_3$. Let $Y_0 = Y_1 \times Y_2 \times Y_3$. Let $Z_{21} := \mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$, let $Z_{32} := \mathbf{Hom}^{\mathrm{st}}(Z_3, Z_2)$, and let $Z_{31} := \mathbf{Hom}^{\mathrm{st}}(Z_3, Z_1)$. The formal scheme $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ has a natural structure as a bi-extension of the p-divisible formal groups (Z_{21}, Z_{32}) by the p-divisible formal group Z_{31}.*

We refer to [19] for the definition and basic properties of bi-extensioins of formal groups.

5.5. Remark. Suppose that Y is a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$ with three slopes. Let

$$(0) \underset{\neq}{\subsetneq} Y_1 \underset{\neq}{\subsetneq} Y_2 \underset{\neq}{\subsetneq} Y$$

be the slope filtration of Y, so that

$$Z_1 := Y_1, \quad Z_2 := Y_2/Y_1, \quad Z_3 := Y/Y_2$$

are isoclinic p-divisible groups, of slopes

$$s_1 > s_2 > s_3$$

respectively. Proposition 2.5 defines maps

$$\pi_1 : \mathbf{Def}^{\mathrm{sus}}(Y) \to \mathbf{Def}^{\mathrm{sus}}(Y/Y_1) \quad \text{and} \quad \pi_1 : \mathbf{Def}^{\mathrm{sus}}(Y) \to \mathbf{Def}^{\mathrm{sus}}(Y_2).$$

The maps π_1 and π_2 define a morphism

$$\pi : \mathbf{Def}^{\mathrm{sus}}(Y) \longrightarrow \mathbf{Def}^{\mathrm{sus}}(Y/Y_1) \times_{\mathrm{Spec}(\kappa)} \mathbf{Def}^{\mathrm{sus}}(Y_2).$$

So far things look quite similar to 5.4. But unlike the situation in 5.4, here the map π does not make $\mathbf{Def}^{\mathrm{sus}}(Y)$ a bi-extension of $\big(\mathbf{Def}^{\mathrm{sus}}(Y/Y_1), \mathbf{Def}^{\mathrm{sus}}(Y_2)\big)$. The troubles are two fold.

(i) To begin with, $\mathbf{Def}^{\mathrm{sus}}(Y/Y_1)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_2)$ are torsors, not groups.
(ii) A more serious problem is that in general, the maps π_1, π_2 and π are not (formally) smooth.

The reason for (ii) is that the natural epimorphisms

$$\mathbf{End}^{\mathrm{st}}(Y) \longrightarrow \mathbf{End}^{\mathrm{st}}(Y/Y_1)$$

and

$$\mathbf{End}^{\mathrm{st}}(Y) \longrightarrow \mathbf{End}^{\mathrm{st}}(Y_2)$$

of p-divisible groups over κ may not be smooth, i.e. their kernels may not be p-divisible groups.

The moral here is that one needs to be careful when formulating the Tate-linear structure on the deformation spaces $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ when the p-divisible group Y_0 over κ is not isomorphic to a product of isoclinic p-divisible groups.

5.6. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field. Let Z, Z_1, Z_2, Z_3 be p-divisible formal groups over κ, and let B be a bi-extension of (Z_1, Z_2) by Z_3.

(a) A closed formal subscheme V of Z is *Tate-linear* if V is a p-divisible subgroup of Z.

(b) A closed formal subscheme V of B is *Tate-linear* if there exists a p-divisible subgroup Z_3' of Z_3 and a p-divisible subgroup U of $Z_1 \times Z_2$, such that V is stable under the action of Z_3' and the projection map $B \to Z_1 \times Z_2$ induces an isomorphism $V/Z_3 \xrightarrow{\sim} U$.

5.7. Definition. (a) An action of a p-adic Lie group on a p-divisible formal group Y over a field $\kappa \supseteq \mathbb{F}_p$ is said to be *strongly non-trivial* if no Jordan–Hölder component of the representation of $\mathrm{Lie}(G)$ on $\mathbb{D}_*\big(Y \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa^{\mathrm{alg}})\big)_{\mathbb{Q}}$ is the trivial representation of $\mathrm{Lie}(G)$. Here κ^{alg} denotes an algebraic closure of κ.

(b) Let Y_1, Y_2, Z be p-divisible formal groups over κ. An action of a p-adic Lie group G on a bi-extension of (Y_1, Y_2) by Z which respects the bi-extension structure is said to be *strongly non-trivial* if the induced actions of G on Y_1, Y_2 and Z are all strongly non-trivial.

5.8. Theorem (Local rigidity for p-divisible formal groups and their bi-extensions). *Let $\kappa \supseteq \mathbb{F}_p$ be an algebraically closed field and let G be a p-adic Lie group. Let Y_1, Y_2, Z be p-divisible formal groups over κ.*

(1) *If $V \subseteq Z$ is an irreducible closed formal subvariety of Z stable under a strongly non-trivial action of G on Z, then V is a formal subgroups of Z.*

(2) *Let B be a bi-extension of $Y_1 \times Y_2$ by Z. Suppose that $V \subset B$ is a reduced irreducible closed formal subscheme of B stable under a strongly non-trivial action of G on B.*
   - (a) *The formal subscheme V of the bi-extension B is Tate-linear.*
   - (b) *Furthermore if Y_1, Y_2 do not have any slope in common, then V is a sub-biextension of B.*

5.9. Remark. (a) See [4] for a proof of (a) and [11] for a proof of (b).

(b) It is expected that the method for (b) is can be extended to give a general local rigidity result for the sustained deformation space $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of any p-divisible group Y_0 over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$. See 6.3 for the precise statement.

(c) Let x_0 be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$ such that $A_{x_0}[p^\infty]$ is isomorphic to a product of at most three isoclinic p-divisible groups. Let V be a reduced closed subscheme of a central leaf $\mathcal{C} = \mathcal{C}(x_0)$ in $\mathcal{A}_{g,d,n}$ which is stable under all prime-to-p Hecke correspondences on $\mathcal{A}_{g,d,n}$. Then theorem 5.8 applies to the closed formal subscheme $V^{/z_0} \subseteq \mathcal{C}^{/z}$, for every $\overline{\mathbb{F}_p}$-point z_0 of V, with G being a compact open subgroup of the group of all $\mathbb{Q}_p$-points of the Frobenius torus associated to the closed point z_0. One concludes that the formal completion $V^{/z_0}$ of V at any closed point $z_0 \in V$ is a Tate-linear formal subscheme of the Tate-linear formal scheme $\mathcal{C}^{/z_0}$.

5.10. Global rigidity questions. Let $\mathcal{C} \subseteq \mathcal{A}_{g,1,n}$ be a leaf in $\mathcal{A}_{g,1,n}$ over $\overline{\mathbb{F}_p}$. Let $Z \subseteq \mathcal{C}$ be an irreducible closed subscheme of $\mathcal{C}$. Let $z \in Z(\overline{\mathbb{F}_p})$ be a closed point of Z. Suppose that $Z^{/z} \subseteq$ is stable under a strongly non-trivial action of a p-adic Lie group G which respects the Tate-linear structure on $\mathcal{C}^{/z}$. Assume that the abelian variety A_z has at most three distinct slopes, and $A_z[p^\infty]$ is isomorphic to a product of isoclinic p-divisible groups. We formulate an expectation and a question below.

5.10.1. Expectation. The formal subscheme $Z \subseteq \mathcal{C}$ is Tate-linear at every closed point of Z.

5.10.2. Question. Is Z (an irreducible component of) the reduction of a Shimura subvariety of the Siegel modular variety $\mathcal{A}_{g,1,n}$ over $\overline{\mathbb{Q}}$?

Remark. (a) In a few cases when Z is contained in the reduction of a "small" Shimura subvariety of the Siegel modular variety, one can show that the answer to 5.10.2 is affirmative.

(b) Most people seem to believe that the answer to 5.10.2 is "yes", but there is little evidence other than our inability to produce a counter-example. The case when $g = 2$ is already a challenge.

6. Strongly Tate-linear formal subschemes of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$

6.1. Definition. Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$. Let

$$\left(\mathbf{Aut}^{\mathrm{st}}(Y)_n\right)_{n\geq 1} =: \left(\Gamma_n\right)_{n\geq 1}$$

be the projective family of stabilized Aut group schemes of Y_0, and let

$$\left(\mathrm{Fil}^t\Gamma_n,\, t \in [0,1]\right)_{n\geq 1} := \left(\mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n,\ t \in [0,1]\right)_{n\geq 1}$$

be the slope filtration on $\left(\Gamma_n\right)_{n\geq 1}$. A projective family $\left(H_n\right)_{n\geq 1}$ of subgroup schemes $H_n \subseteq \Gamma_n$ is said to be *stable* if the vertical arrows in the diagram

$$\begin{array}{ccccc}
\mathrm{Fil}^{>t}H_{n+1} & \longrightarrow & \mathrm{Fil}^{\geq t}H_{n+1} & \longrightarrow & \mathrm{Fil}^{\geq t}H_{n+1}/\mathrm{Fil}^{>t}H_{n+1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fil}^{>t}H_n & \longrightarrow & \mathrm{Fil}^{\geq t}H_n & \longrightarrow & \mathrm{Fil}^{\geq t}H_n/\mathrm{Fil}^{>t}H_n
\end{array}$$

are faithfully flat for all $n \geq 1$ and all $t \in [0,1]$, where

$$\mathrm{Fil}^{>t}H_n := H_n \cap \mathrm{Fil}^{>t}\Gamma_n, \quad \mathrm{Fil}^{\geq t}H_n := H_n \cap \mathrm{Fil}^{\geq t}\Gamma_n,$$

and the projective system

$$\left(\mathrm{Fil}^{\geq t} H_n / \mathrm{Fil}^{>t} H_n\right)_{n\geq 1}$$

comes from a p-divisible group over κ, for every $t \in (0,1]$.

6.1.1. Definition. Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$, and let $\tilde{H} = \big(H_n\big)_{n\geq 1}$ be a stable family of subgroup schemes of the stabilized Aut group schemes $\big(\Gamma_n\big)_{n\geq 1}$ of Y_0 as in 6.1. Let

$$\tilde{T} := (T_n, \psi_n)_{n\geq 1}$$

be a compatible projective family of right Γ_n-torsors, in the sense that T_n is a right Γ_n-torsor and

$$\psi_n : T_{n+1} \wedge^{H_{n+1}} H_n \xrightarrow{\sim} T_n$$

is a isomorphism of right H_n-torsors for each $n \geq 1$. Here $T_{n+1} \wedge^{H_{n+1}} H_n$ denote the contraction product of T_{n+1} with H_n via the faithfully flat homomorphism $H_{n+1} \to H_n$, which has a natural structure as a right H_n-torsor.

The deformation functor $\mathbf{Def}(\tilde{T})$ of $\tilde{T}$ is the functor from $\mathfrak{Art}_\kappa$ to the category of sets, which sends every object (R, j, ϵ) in $\mathfrak{Art}_\kappa$ to the set of all isomorphism classes of projective families

$$\left(\boldsymbol{T}_n, \boldsymbol{\psi}_n, \zeta_n\right)_{n\geq 1},$$

where

- $\boldsymbol{T}_n$ is a right torsor for $H_{n,R} := H_n \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$,
- $\boldsymbol{\psi}_n : \boldsymbol{T}_{n+1} \wedge^{H_{n+1,R}} H_{n,R} \xrightarrow{\sim} T_n$ is an isomorphism of right $H_{n,R}$-torsors,
- $\zeta_n : T_n \xrightarrow{\sim} \boldsymbol{T}_n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$ is an isomorphism of right H_n-torsors,
- $\boldsymbol{\psi}_n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$ is naturally identified with ψ_n via the isomorphisms ζ_{n+1} and ζ_n

for every $n \geq 1$.

The argument for 4.3 also shows the following lemma 6.1.2.

6.1.2. Lemma. *Let Y_0 be a p-divisible group over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$, and let $\tilde{H} = \big(H_n\big)_{n\geq 1}$ be a stable family of subgroup schemes of the stabilized Aut group schemes $\big(\Gamma_n\big)_{n\geq 1}$ of Y_0 as in 6.1. Let*

$$T := (T_n, \psi_n)_{n\geq 1}$$

be a compatible projective family of right Γ_n-torsors over κ and let

$$\left(\alpha_n : T_n \wedge^{H_n} \Gamma_n \xrightarrow{\sim} \mathbf{Aut}^{\mathrm{st}}(Y_0)_n\right)_{n\geq 1}$$

be a compatible family of isomorphisms of right Γ_n-torsors. The deformation functor $\mathbf{Def}(\tilde{H})$ is formally smooth over κ, and the compatible family of isomorphisms $(\alpha_n)_{n\geq 1}$ defines a closed embedding

$$\mathbf{Def}(\tilde{H}) \hookrightarrow \mathbf{Def}^{\mathrm{sus}}(Y_0)\,.$$

6.2. Definition. Let Y_0 be a p-divisible group over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$. A smooth closed formal subscheme Z of $\mathbf{Def}^{\rm sus}(Y_0)$ is said to be a *strongly Tate-linear* formal subscheme of $\mathbf{Def}^{\rm sus}(Y_0)$ if there exist

- a stable projective family $\tilde{H} = \big(H_n\big)_{n\geq 1}$ of subgroup schemes of the stabilized Aut group schemes $\big(\Gamma_n\big)_{n\geq 1}$ of Y_0 as in 6.1,
- a compatible family of right H_n-torsors
$$\tilde{T} = \big(T_n\big)_{n\geq 1}$$
over κ, and
- a compatible family of isomorphisms
$$\Big(\alpha_n : T_n \wedge^{H_n} \Gamma_n \xrightarrow{\sim} \mathbf{Aut}^{\rm st}(Y_0)_n\Big)_{n\geq 1}$$
of right Γ_n-torsors over κ

such that Z is equal to the image of the close embedding
$$\mathbf{Def}(\tilde{T}) \hookrightarrow \mathbf{Def}^{\rm sus}(Y_0)$$
associated to $\big(\alpha_n\big)_{n\geq 1}$ as in 6.1.2.

6.2.1. Remark. (i) Formal completions of central leaves in the reduction modulo p of a Shimura subvariety in a Siegel modular variety are strongly Tate-linear. In particular, every central leaf $\mathcal{C}$ in a PEL type modular variety $\mathscr{M}$ over an algebraically closed field κ and every κ-point $z_0 \in \mathcal{C}$, the formal completion $\mathcal{C}^{/z_0}$ of $\mathcal{C}$ at z_0 is a strongly Tate-linear formal subscheme of $\mathbf{Def}^{\rm sus}(A_{z_0}[p^\infty])$, where A_{z_0} is the abelian variety with PEL structure corresponding to the point z_0 in the modular variety $\mathscr{M}$.

(ii) There are good reasons to view central leaves in the reduction of a Shimura subvariety of a Siegel modular variety as characteristic-p analogues of Shimura varieties. We explain this for the case of Siegel modular varieties.

(a) For any two closed points x_1, x_2 of a central leaf $\mathcal{C}$ in $\mathcal{A}_{g,d,n}$, there exists an isomorphism $\beta : (A_{x_1}, \mu_{x_1})[p^\infty] \xrightarrow{\sim} (A_{x_2}, \mu_{x_2})[p^\infty]$ and a β-equivariant isomorphism $\mathcal{C}^{/x_1} \xrightarrow{\sim} \mathcal{C}^{/x_2}$.

(b) Every central leaf in $\mathcal{A}_{g,d,n}$ is stable under the set of all $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$-Hecke correspondences. Moreover every $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$-Hecke orbit in a central leaf $\mathcal{C}$ in $\mathcal{A}_{g,d,n}$ is dense in $\mathcal{C}$ for the Zariski topology.

If one adopts this perspective, then a strongly Tate-linear formal subscheme Z of $\mathbf{Def}^{\rm sus}(Y_0)$ as in 6.2 can be regarded as a local version of "characteristic-p Shimura variety", and the stable projective family of subgroup schemes $\tilde{H}$ attached to Z as in 6.2 serves as an analog of the reductive $\mathbb{Q}$-group in the Shimura input datum of a Shimura variety.

6.2.2. Remark. Suppose that Z is a strongly Tate-linear formal subscheme of $\mathbf{Def}^{\rm sus}(Y_0)$ of the form $Z = \mathrm{Im}\big(\mathbf{Def}(\tilde{T}) \hookrightarrow \mathbf{Def}^{\rm sus}(Y_0)\big)$ for a compatible family of right H_n-torsors $\tilde{T} = \big(T_n\big)_{n\geq 1}$ as in 6.2. Let $\mathbf{Y}$ be the universal p-divisible group over $\mathbf{Def}^{\mathrm{sus}}(Y_0)$. Then we have a compatible family of isomorphisms

$$\phi_n : \mathbf{T}_n \wedge^{(H_n \times_{\mathrm{Spec}(\kappa)} Z)} (\Gamma_n \times_{\mathrm{Spec}(\kappa)} Z) \xrightarrow{\sim} \mathbf{Isom}^{\mathrm{st}}(Y_0, \mathbf{Y})_n \times_{\mathbf{Def}^{\mathrm{sus}}(Y_0)} Z$$

of right torsors for $\Gamma_n \times_{\mathrm{Spec}(\kappa)} Z$, where $\big(\mathbf{T}_n\big)_{n\geq 1}$ is the universal family of right H_n-torsors over $Z = \mathbf{Def}(\tilde{T})$. In the parlance of differential geometry, the stable family $\big(H\big)_{n\geq 1}$ of subgroup schemes of Γ_n is uniquely determined by Z as the smallest stable family of subgroup schemes of Γ_n such that the compatible family of right Γ_n-torsors

$$\Big(\mathbf{Isom}^{\mathrm{st}}(Y_0, \mathbf{Y})_n \times_{\mathbf{Def}^{\mathrm{sus}}(Y_0)} Z\Big)_{n\geq 1}$$

admits a "reduction of structural group" to the subgroup schemes H_n in a way that is compatible with the transition maps.

6.3. A local rigidity question for sustained deformation spaces.

NOTATION.

- Let Y_0 be a p-divisible group over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$.
- Let $\mathbf{End}^{\mathrm{st}}(Y_0)$ be the p-divisible group formed by the stabilized End group schemes of Y_0, and let $\mathrm{Fil}^{>0}\mathbf{End}^{\mathrm{st}}(Y_0)$ be the largest p-divisible formal subgroup of $\mathbf{End}^{\mathrm{st}}(Y_0)$.
- Let Z be a reduced and irreducible closed formal subscheme of the formal scheme $\mathbf{Def}^{\mathrm{sus}}(Y_0)$.
- Let G be a closed subgroup of the compact p-adic group $\mathrm{Aut}(Y_0)$.

Expectation. Suppose that Z is stable under the natural action of G on the sustained deformation space $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of Y_0, and the natural action of G on the p-divisible formal group $\mathrm{Fil}^{>0}\mathbf{End}^{\mathrm{st}}(Y_0)$ over κ is strongly non-trivial in the sense of 5.7 (a). Then Z is a strongly Tate-linear formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$.

Remark. (i) Theorem 5.8 says that the statement 6.3 holds when Y_0 is a product of at most three isoclinic p-divisible groups over κ. So there is considerable evidence supporting this expectation.

(ii) As remarked in 5.9 (b), it seems likely that the method for proving 5.8 (b) will also deliver the more general statement 6.3.

(iii) In applications to the Hecke orbit problem, we are given a reduced closed subscheme V of a central leaf $\mathcal{C}$ over $\overline{\mathbb{F}_p}$ in a Shimura subvariety $\mathcal{S}$ of a Siegel modular variety, such that V is stable under all prime-to-p Hecke correspondences on $\mathcal{S}$, and we want to show that $\mathcal{C}$ is equal to $\mathcal{S}$. Let z_0 be an $\overline{\mathbb{F}_p}$-point of the smooth locus of V, corresponding to a polarized abelian variety A_0 over $\overline{\mathbb{F}_p}$ with extra symmetries. Let $V^{/z_0}$ (respectively $\mathcal{C}^{/z_0}$) be the formal completion at z_0 of V (respectively $\mathcal{C}$). We have inclusions $V^{/z_0} \subseteq \mathcal{C}^{/z_0} \subseteq \mathbf{Def}^{\mathrm{sus}}(A_{z_0}[p^\infty])$. The fact that V is stable under all prime-to-p Hecke correspondences on $\mathcal{S}$ implies that there exists a compact open subgroup of the group G of p-adic points of the Frobenius torus attached to z_0, such that the formal subscheme $V^{/z_0}$ of $\mathbf{Def}(A_{z_0}[p^\infty])$ is stable under the natural action of G on ($\mathcal{C}^{/z_0}$ and) $\mathbf{Def}(A_{z_0}[p^\infty])$. So the rigidity statement 6.3 implies that $V^{/z_0}$ is a strongly Tate-linear formal subscheme of $\mathcal{C}^{/z_0}$

and $\mathbf{Def}(A_{z_0}[p^\infty])$. This conclusion is still some distance away from the desired conclusion, but it is a structural constraint on Hecke-stable subvarieties of $\mathcal{C}$.

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