CHAPTER 7
SUSTAINED p-DIVISIBLE GROUPS: A FOLIATION RETRACED

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This chapter is a survey of the notion of *sustained p-divisible groups*, which provides a scheme-theoretic definition of the notion of *central leaves* introduced in [24]. Detailed proofs will appear in a planned monograph by the authors on Hecke orbits [8].

Here is a survey of results mentioned in this chapter:

- In 1.4 and 1.6 we give the definition (in various forms) of a sustained p-divisible group, and the resulting definition of a central leaf.
- In 3.2 we show that every sustained p-divisible group admits a unique slope filtration.
- In Theorem 4.3 we show that the old definition of *geometrically fiber constant* and the new definition of *sustained* agree over a reduced base scheme.
- In Theorem 5.1 we analyze properties and smoothness of central leaves in the unpolarized case.
- In Proposition 5.3 we analyze properties and smoothness of central leaves in the polarized case.

1. What is a sustained p-divisible group

1.1. General idea of a sustained p-divisible group. In a nutshell,

> *a p-divisible group $X \to S$ over a characteristic-p base scheme S is said to be* sustained *if its p^n-torsion subgroup scheme $X[p^n]$ is S-locally constant for the flat topology of S, for all $n \in \mathbb{N}$.*

See 1.4 for the precise definition. This concept enriches the notion of *geometrically fiberwise constant* family of p-divisible groups in [24] and paves the path for unveiling the exquisite local structure of central leaves in moduli spaces of abelian varieties and deformation spaces of p-divisible groups.

In music a sustained (*sostenuto* in Italian) tone is constant, but as the underlying harmony may change it does not feel constant at all. Here we have borrowed the adjective "sustained" to describe a family of "the same objects" whereas the whole family need not be constant.

1.2. Constancy and base field. Careful readers are likely to have zeroed in on an important point which was glossed over in the opening paragraph: being "constant" over a base scheme is fundamentally a *relative* concept. There is no satisfactory definition of a "constant family $Y \to S$ of algebraic varieties" over

*C.-L. Chai was supported by NSF grant DMS 1200271 and a Simons fellowship.

a general base scheme S as far as we know. For a scheme S over a field κ, a κ-constant family over S is a morphism $Y \to S$ which is S-isomorphic to $Y_0 \times_{\mathrm{Spec}(\kappa)} S$ for some κ-scheme Y_0. Similarly a base field κ needs to be fixed, before one can properly define and study sustained p-divisible groups relative to the chosen base field κ.

Not surprisingly, the meaning of "sustainedness" changes with the base field: if S is a scheme over a field $\kappa \supset \mathbb{F}_p$ and $X \to S$ is a p-divisible group over S, being sustained relative to a subfield δ of κ is a stronger condition than being sustained relative to κ itself. As an example, if the base field $\kappa \supset \mathbb{F}_p$ is perfect and K is an algebraic extension field of κ, then a p-divisible group over a field K is κ-sustained if and only if for any pair of κ-linear embeddings $\tau_1, \tau_2 : K \hookrightarrow \Omega$ to an algebraic closure Ω of K, the p-divisible groups $X \otimes_{K,\tau_1} \Omega$ and $X \otimes_{K,\tau_1} \Omega$ over Ω are isomorphic.

1.3. Definition. Let $\kappa \supset \mathbb{F}_p$ be a field, and let S be a reduced scheme over κ. A p-divisible group $X \to S$ is *geometrically fiberwise constant relative to* κ if for any algebraically closed field k which contains κ and any two points $s_1, s_2 \in S(k)$, there exists some k-isomorphism from s_1^*X to s_2^*X.

Remark. Let $k \supset \mathbb{F}_p$ be an algebraically closed based field, let $n \geq 3$ be an positive integer which is prime to p, and let $\mathscr{A}_{g,n}$ be the moduli space of g-dimensional principally polarized abelian varieties over k with symplectic level-n structure. A *central leaf* $\mathscr{C}$ over k in $\mathscr{A}_{g,n}$ is defined in [24] to be a reduced locally closed k-subscheme of $\mathscr{A}_{g,n}$ which is maximal among all reduced locally closed k-subschemes S of $\mathscr{A}_{g,n}$ such that the restriction to S of the universal p-divisible group is geometrically fiberwise constant relative to k. This "point-wise" definition was the best available at the time. The geometric picture that the family of central leaves and the family of isogeny leaves in a given Newton polygon stratum form two essentially transversal foliations provided new insight on the structure of the moduli spaces $\mathscr{A}_{g,n}$.

Admittedly the point-wise definition of central leaves is a bit awkward to work with. For instance it makes studying differential properties of central leaves a bit of a challenge. As another example, it can be shown, with some effort, that the restriction to a central leaf of the universal p-divisible group over $\mathscr{A}_{g,n}$ admits a slope filtration. This is an important property of central leaves; in contrast this statement does not hold for Newton polygon strata of $\mathscr{A}_{g,n}$. However the proof does not provide a transparent explanation of the existence of slope filtration on central leaves.

The definition below resolves the above plights in one stroke.

1.4. Definition. Let $\kappa \supset \mathbb{F}_p$ be a field, and let S be a κ-scheme. Let X_0 be a p-divisible group over κ.

(a1) A p-divisible group over $X \to S$ is *strongly κ-sustained modeled on* X_0 if the S-scheme

$$\mathscr{I}\!\mathit{som}_S\big(X_0[p^n]_S, X[p^n]\big) \to S$$

of isomorphisms between the truncated Barsotti-Tate groups $X_0[p^n]$ and $X[p^n]$ is faithfully flat over S for every $n \in \mathbb{N}$. Here $X_0[p^n]_S$ is short for $X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S$; we will use similar abbreviation for base change in the rest of this article if no confusion is likely to arise.

(a2) A p-divisible group $X \to S$ is *strongly κ-sustained* if there exists a p-divisible group X_0 over κ such that $X \to S$ is κ-sustained modeled on X_0. In this case we say that X_0 is a κ-model of X.

(b1) A p-divisible group over $X \to S$ is *κ-sustained* if the $S \times_{\mathrm{Spec}(\kappa)} S$-scheme

$$\mathscr{I}\!\mathit{SOM}_{S\times_{\mathrm{Spec}(\kappa)} S}\big(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]\big) \to S \times_{\mathrm{Spec}\,\kappa} S$$

is faithfully flat over $S \times_{\mathrm{Spec}\,\kappa} S$ for every $n \in \mathbb{N}$, where

$$\mathrm{pr}_1, \mathrm{pr}_2 : S \times_{\mathrm{Spec}(\kappa)} S \to S$$

are the two projections from $S \times_{\mathrm{Spec}(\kappa)} S$ to S.

(b2) Let $X \to S$ be a κ-sustained p-divisible group over S as in (b1). Let K be a field containing the base field κ. A κ-sustained p-divisible group X_1 over K is said to be a *K/κ-model* of $X \to S$ if the structural morphism

$$\mathscr{I}\!\mathit{SOM}_{S_K}\big(X_1[p^n] \times_{\mathrm{Spec}(K)} S_K, X[p^n] \times_S S_K\big) \longrightarrow S_K$$

of the above Isom scheme is faithfully flat, for every positive integer n, where $S_K := S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(K)$.

(c) The above definitions extend to polarized p-divisible groups, giving us two more notions:

- a polarized p-divisible group $(X \to S, \mu : X \to X^t)$ is *strongly κ-sustained modeled on a polarized p-divisible group* (X_0, μ_0) over κ if the S-scheme

$$\mathscr{I}\!\mathit{SOM}_S\big((X_0[p^n]_S, \mu_0[p^n]_S), (X[p^n], \mu[p^n])\big) \to S$$

is faithfully flat over S for every $n \in \mathbb{N}$.

- a polarized p-divisible group $(X \to S, \mu : X \to X^t)$ is *κ-sustained* if

$$\mathscr{I}\!\mathit{SOM}_{S\times_{\mathrm{Spec}\,\kappa} S}\big(\mathrm{pr}_1^*(X[p^n], \mu[p^n]), \mathrm{pr}_2^*(X[p^n], \mu[p^n])\big) \to S \times_{\mathrm{Spec}\,\kappa} S$$

is faithfully flat for every $n \in \mathbb{N}$.

(d) Just as in the unpolarized case, given an over-field K of κ, a *K/κ-model* of a κ-sustained polarized p-divisible group (X, μ) over a κ-scheme S is a κ-sustained polarized p-divisible group (X_3, μ_3) over K such that the Isom scheme

$$\mathscr{I}\!\mathit{SOM}_{S_K}\big((X_2[p^n], \mu_2[p^n]) \times_{\mathrm{Spec}(K)} S_K, (X[p^n], \mu[p^n]) \times_S S_K\big) \longrightarrow S_K$$

is faithfully flat over S_K, for every positive integer n.

1.5. Remarks on the definition. (a) Suppose that $X \to S$ is κ-sustained and $S(\kappa) \neq \emptyset$, then $X \to S$ is strongly κ-sustained with X_s as a model, for any κ-rational point s of S. This fact follows immediately from the definition. Similarly for any point $s \in S$, the fiber X_s above s of a κ-sustained p-divisible group X is a κ-sustained p-divisible group over the residue field $\kappa(s)$ of s, and X_s is a $\kappa(s)/\kappa$ model of $X \to S$.

(b) Clearly a p-divisible group X over a κ-scheme S is strongly κ-sustained if and only if it is κ-sustained and admits a κ/κ-model. We will abbreviate "κ/κ-model" to "κ-model" when no confusion is likely to arise. We would like to consider the notion "κ-sustained" as more fundamental, and "strongly κ-sustained" as the special case when there exists a κ-model for the κ-sustained p-divisible group in question.

(d) Being sustained is weaker than being *quasi-isotrivial*. Every ordinary family of p-divisible group $X \to S$ of constant relative dimension and height is strongly $\mathbb{F}_p$-sustained, but there are many examples of ordinary p-divisible groups over a κ-scheme S which are not κ-quasi-isotrivial. For instance p-divisible group

$$\mathcal{E}[p^\infty]|_{\mathcal{M}_n^{\mathrm{ord}}} \longrightarrow \mathcal{M}_n^{\mathrm{ord}}$$

attached to the restriction to the ordinary locus $\mathcal{M}_n^{\mathrm{ord}}$ of the universal elliptic curve

$$\mathcal{E} \longrightarrow \mathcal{M}_n$$

is not $\overline{\mathbb{F}_p}$ -quasi-isotrivial. Here $\mathcal{M}_n$ is the modular curve over $\overline{\mathbb{F}_p}$, with level-structure n, where $n \geq 3$ and $\gcd(n,p) = 1$. Moreover for every positive integer $m \geq 1$, the BT_n-group $\mathcal{E}[p^m] \to \mathcal{M}_n^{\mathrm{ord}}$ is not $\overline{\mathbb{F}_p}$ -quasi-isotrivial either.

To clarify what we mean: we say that a p-divisible group $X \to S$ is *quasi-κ-isotrivial* (respectively *locally κ-isotrivial*) if there exist

- a Zariski open cover $\big(U_i\big)_{i\in I}$ of S,
- etale surjective morphisms (respectively finite etale morphisms)

$$\big(f_i : T_i \to U_i\big)_{i\in I},$$

- a p-divisible groups X_0 over κ, and
- T_i-isomorphisms $X_0 \times_{\mathrm{Spec}(\kappa)} T_i \xrightarrow{\sim} X \times_S T_i$ of p-divisible groups for all $i \in I$.

The above terminology is based on the notion of "quasi-isotrivial" and "locally isotrivial" for torsors, as in SGA3 exposé IV, 6.5, p. 249. For instance for a group scheme $G \to S$, a G-torsor $\mathcal{P} \to S$ is quasi-isotrivial if there exists a Zariski open cover $\big(U_i\big)_{i\in I}$, etale surjective morphisms $\big(f_i : T_i \to U_i\big)_{i\in I}$ and sections $e_i : T_i \to \mathcal{P} \times_S T_i$ of the $G \times_S T_i$-torsor $\mathcal{P} \times_S T_i$ for all $i \in I$.

1.5.1. Elementary properties. Let S be a κ-scheme. Let $X \to S$ be a p-divisible group. The following properties are easily verified from the definition.

(1) If $X \to S$ is strongly κ-sustained over S modeled on a p-divisible group X_0 over κ, then $X \to S$ is κ-sustained.
(2) If $X \to S$ is a κ-sustained and $s_0 \in S(\kappa)$ is a κ-rational point of S, then $X \to S$ is strongly κ-sustained modeled on s_0^*X.
(3) Let $f : T \to S$ be a morphism of κ-schemes. If $X \to S$ is κ-sustained (respectively strongly κ-sustained modeled on a p-divisible group X_0 over κ), then $f^*X \to T$ is also κ-sustained (respectively strongly κ-sustained modeled on X_0).

(4) Suppose that $f : T \to S$ is a faithfully flat morphism and $f^*X \to T$ is κ-sustained (respectively strongly κ-sustained modeled on a p-divisible group) X_0 over κ, then $X \to S$ is also κ-sustained (respectively strongly κ-sustained modeled on X_0).

(5) Let U, V be open subschemes of S.
   - If both restrictions $X_U \to U$ and $X_V \to V$ are both strongly κ-sustained modeled on a p-divisible group X_0 over κ, then $X_{U\cup V} \to U \cup V$ is also strongly κ-sustained modeled on X_0.
   - If both $X_U \to U$ and $X_V \to V$ are κ-sustained and $U \cap V \neq \emptyset$, then the p-divisible group $X_{U\cup V} \to U \cup V$ is also κ-sustained.

It is instructive to examine the case when the base scheme S is the spectrum of a field K containing the base field κ. The following properties, although not immediately obvious from the definition, make it clear that κ-sustainedness is a relative concept and depends crucially on the base field κ.

(6A) Let X_0 be a p-divisible group over κ. A p-divisible group X over K is strongly κ-sustained modeled on X_0 if and only if one of the following equivalent conditions hold.
   - There exists an algebraically closed field L containing κ and a κ-linear embedding $\tau : K \to L$ such that $X \times_{\mathrm{Spec}(K),\tau} \mathrm{Spec}(L)$ is L-isomorphic to $X_0 \times_{Spec(\kappa)} \mathrm{Spec}(L)$.
   - The base extension $X \times_{\mathrm{Spec}(K),\tau} \mathrm{Spec}(L)$ of X from K to L is L-isomorphic to $X_0 \times_{Spec(\kappa)} \mathrm{Spec}(L)$ for every algebraically closed field L containing κ and every κ-linear embedding $\tau : K \to L$.

(6B) Suppose that X is a κ-sustained p-divisible group over K.
   (i) For any algebraically closed field L containing κ and any two κ-linear homomorphisms $\tau_1, \tau_2 : K \to L$, the p-divisible groups $\tau_1^* X$, $\tau_2^* X$ are isomorphic over L.
   (ii) There exists a finite extension field κ_1 of κ, a κ-linear embedding $\tau : \kappa_1 \hookrightarrow L$ of κ_1 into an algebraic closure L of K, and a p-divisible group X_1 over κ_1 such that the p-divisible group $X_L := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is strongly κ_1-sustained modeled on X_1.

(6C) Assume that the base field $\kappa \supset \mathbb{F}_p$ is *perfect*. A p-divisible group X over K is strongly κ-sustained if and only if the statement (6B) (i) hold.

We are now in a position to give a scheme-theoretic definition of central leaves in terms of sustained p-divisible groups.

1.6. Definition (a schematic definition of central leaves). Let $n \geq 3$ be a positive integer relatively prime to p. Let $\kappa \supset \mathbb{F}_p$ be a field.

(i) Let $x_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,n}(\kappa)$ be a κ-rational point of $\mathscr{A}_{g,n}$. The *central leaf* $\mathscr{C}(x_0)$ *in* $\mathscr{A}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z}[\mu_n,1/n])} \mathrm{Spec}(\kappa)$ *passing through* x_0 is the maximal member among the family of all locally closed subscheme of $\mathscr{A}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z}[\mu_n,1/n])} \mathrm{Spec}(\kappa)$ such that the principally polarized p-divisible group attached to the restriction to $\mathscr{C}(x_0)$ of the universal abelian scheme is strongly κ-sustained modeled on $(A_0[p^\infty], \lambda_0[p^\infty])$. (This definition depends on the fact that there exist a member $\mathscr{C}(x_0)$ in this family which contains every member of this family.)

(ii) A *central leaf over* κ is a maximal member $\mathscr{C}$ in the family of all locally closed subscheme of $\mathscr{A}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z}[\mu_n,1/n])} \mathrm{Spec}(\kappa)$ such that the principally polarized p-divisible group attached to the restriction to $\mathscr{C}$ of the universal abelian scheme is κ-sustained.

Of course there is a backward compatibility issue: we have to show that the above definitions coincide with the definition in [24] of central leaves when the field κ is algebraically closed; in particular $\mathscr{C}(x_0)$ is reduced (and smooth over κ).

1.7. The rest of this article is organized as follows.

- In §2 we define the stabilized Hom schemes $\mathscr{H}om^{\mathrm{st}}(X[p^n], Y[p^n])$ and the p-divisible group $\mathscr{H}om'_{\mathrm{div}}(X,Y) = \varinjlim_n \mathscr{H}om^{\mathrm{st}}(X[p^n], Y[p^n])$ associated to p-divisible groups X, Y over a field κ.
- In §3 we explain descent related to sustained p-divisible groups. In particular a strongly sustained p-divisible group X over a κ-scheme modeled on a p-divisible group X_0 over a field $\kappa \supset \mathbb{F}_p$ corresponds to projective system of right torsors for the stabilized Aut group schemes $\mathscr{A}ut^{\mathrm{st}}(X_0[p^n])$. Properties of the stabilized Aut group schemes quickly lead to the existence of a natural slope filtration on every sustained p-divisible group; see 3.2.
- The main theorem 4.3 of §4 shows that the definition of central leaves in 1.6 is compatible with the definition in [24] using the notion of geometrically fiberwise constant p-divisible groups.
- Deformation of sustained p-divisible groups is discussed in §5. Isoclinic sustained p-divisible groups are Galois twists of constant isoclinic p-divisible groups and do not deform, hence every sustained deformation of a p-divisible group X_0 is obtained by deforming its slope filtration. This leads to an assembly of maps between sustained deformations spaces of subquotients of X_0 with respect to its slope filtration. These maps shows that the space $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ of sustained deformations of X_0 can be "assembled" from the space of sustained deformations of p-divisible groups with two slopes. Such structural description of $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ can be regarded as a generalization of the Serre–Tate coordinates for ordinary p-divisible groups and abelian varieties.

  When X_0 has exactly two slopes whose slope filtration corresponds to a short exact sequence $0 \to Z \to X_0 \to Y \to 0$, where Y and Z are isoclinic p-divisible groups with $\mathrm{slope}(Y) < \mathrm{slope}(Z)$, the space $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ of sustained deformations of X_0 is a torsor for the p-divisible formal group $\mathscr{H}om'_{\mathrm{div}}(Y, Z)$; see 5.6. The full description of the structure of the space of sustained deformations of a p-divisible group in the general case is somewhat technical. A full documentation will appear in a planned monograph [8] on Hecke orbits.

2. Stabilized Hom schemes for truncations of p-divisible groups

2.1. In this section we discuss a fundamental stabilization phenomenon for the Hom schemes and Isom schemes between the truncated Barsotti–Tate groups attached to two p-divisible groups over the same base field. These stabilized Hom, Isom and Aut schemes enter the theory of sustained p-divisible groups in (at least) two ways.

(a) The category of strongly sustained p-divisible groups over a base κ-scheme S modeled on a p-divisible group X_0 over a field κ is equivalent to the category of projective families of torsors over S for the stabilized Aut schemes for $X_0[p^n]$, indexed by $n \in \mathbb{N}$.

(b) Given two p-divisible groups X, Y over a field $\kappa \supset \mathbb{F}_p$, the direct limit of stabilized Hom schemes from $X[p^n]$ to $Y[p^n]$ is a p-divisible group, denoted by $\mathscr{H}om'_{\rm div}(X,Y)$. Such p-divisible group appear as "building blocks" of formal completions of central leaves.

It is tempting to think of the bifunctor $(X,Y) \rightsquigarrow \mathscr{H}om'_{\rm div}(X,Y)$ as an "internal Hom" for the category of p-divisible groups. However the Hom functor between two p-divisible groups is the *projective* limit of the Hom functors between their truncations, not an inductive limit. For many purposes it is better to consider $(X,Y) \rightsquigarrow \mathscr{H}om'_{\rm div}(X,Y)$ as a disguised Ext functor. The relation between $\mathscr{H}om'_{\rm div}(X,Y)$ and the functor $\mathscr{E}xt_{\rm def}(X,Y)$ of "deforming $X \times Y$ by forming extensions of X by Y over Artinian κ-algebras" is given in 2.10–2.11.

2.2. An outline of the basic stabilization phenomenon. Let $\kappa \supset \mathbb{F}_p$ be a field. Let X, Y be p-divisible groups over κ. Let $\mathscr{H}om(X[p^n], Y[p^n])$ be the scheme of homomorphisms from $X[p^n]$ to $Y[p^n]$ over κ-schemes; see 2.3. It is a group scheme of finite type over κ.

(i) For each n, the image in $\mathscr{H}om(X[p^n], Y[p^n])$ of the restriction homomorphism

$$r_{n,n+i} : \mathscr{H}om(X[p^{n+i}], Y[p^{n+i}]) \longrightarrow \mathscr{H}om(X[p^n], Y[p^n])$$

stabilizes as $i \to \infty$, to a *finite* group scheme $\mathscr{H}om^{\rm st}(X[p^n], Y[p^n])$ over κ.

(ii) Besides the restriction maps $H_{n+i} \to H_n$ used in (i) above, there are naturally defined monomorphisms

$$\iota_{n+1,n} : \mathscr{H}om(X[p^n], Y[p^n]) \to \mathscr{H}om(X[p^{n+1}], Y[p^{n+1}]),$$

which induce monomorphisms

$$\mathscr{H}om^{\rm st}(X[p^n], Y[p^n]) \to \mathscr{H}om^{\rm st}(X[p^{n+1}], Y[p^{n+1}]).$$

The inductive system $\left(\mathscr{H}om^{\rm st}(X[p^n], Y[p^n])\right)_{n\in\mathbb{N}}$ is a p-divisible group, denoted by $\mathscr{H}om'_{\rm div}(X,Y)$.

The precise definitions and statements about the stabilization process can be found in 2.3–2.6; the key stabilization statement is lemma 2.4. The importance of this stabilization phenomenon is twofold.

- A strongly κ-sustained p-divisible group $X \to S$ over a κ-scheme S is essentially a *projective system* of torsor over S for the stabilized Aut groups $\mathscr{A}\mathit{ut}^{\mathrm{st}}(X_0[p^n])$.
- The sustained locus of the characteristic-p deformation space of a p-divisible group over a perfect field κ is built up from p-divisible groups isogenous to p-divisible groups of the form $\mathscr{H}\mathit{om}'_{\mathrm{div}}(Y,Z)$, where Y,Z are isoclinic p-divisible groups over κ. Note that the p-divisible group $\mathscr{H}\mathit{om}'_{\mathrm{div}}(Y,Z)$ is an *inductive system* of stabilized Hom schemes $\mathscr{H}\mathit{om}^{\mathrm{st}}(Y[p^n], Z[p^n]$.

2.3. Definition. Let $\kappa \supset \mathbb{F}_p$ be a field and let X, Y be p-divisible groups over κ.

(1) For every $n \in \mathbb{N}$ we have a commutative affine algebraic group

$$\mathscr{H}\mathit{om}(X[p^n], Y[p^n])$$

of finite type over κ, which represents the functor

$$S \mapsto \mathrm{Hom}_S(X[p^n] \times_{\mathrm{Spec}(\kappa)} S, Y[p^n] \times_{\mathrm{Spec}(\kappa)} S)$$

on the category of all κ-schemes S. We often shorten $\mathscr{H}\mathit{om}(X[p^n], Y[p^n])$ to $H_n(X,Y)$, and sometimes we will shorten it further to H_n if there is no danger of confusion.

(2) For any $n, i \in \mathbb{N}$, let $j_{X,n+i,n} : X[p^n] \to X[p^{n+i}]$ be the inclusion homomorphism, and let $\pi_{X,n,n+i} : X[p^{n+i}] \to X[p^n]$ be the faithfully flat homomorphism such that $[p^i]_{X[p^{n+i}]} = j_{X,n+i,n} \circ \pi_{X,n,n+i}$. Define $\pi_{Y,n,n+i}$ and $j_{Y,n+i,n}$ similarly.

(3) For all $n, i \in \mathbb{N}$, denote by

$$r_{n,n+i} : H_{n+i} \to H_n$$

the homomorphism such that

$$\alpha \circ j_{X,n+i,n} = j_{Y,n+i,n} \circ r_{n,n+i}(\alpha)$$

for any κ-scheme S and any homomorphism $\alpha : X[p^{n+i}]_S \to Y[p^{n+i}]_S$. In other words $r_{n,n+i}$ is defined by restricting homomorphisms from $X[p^{n+i}]$ to $Y[p^{n+i}]$ to the subgroup scheme $X[p^n] \subset X[p^{n+i}]$.

(4) For all $n, i \in \mathbb{N}$, denote by

$$\iota_{n+i,n} : H_n \to H_{n+i}$$

the homomorphism such that

$$\iota_{n+i,n}(\beta) = j_{Y,n+i,n} \circ \beta \circ \pi_{X,n,n+i}$$

for any κ-scheme S and any homomorphism $\beta : X[p^n]_S \to Y[p^n]_S$. This homomorphism $\iota_{n+i,n}$ is a closed embedding.

2.4. Lemma. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$.*

(1) *The homomorphisms $\left(r_{n,m}\right)_{m \geq n,\, m,n \in \mathbb{N}}$ form a* projective system *of commutative group schemes*

$$\left(H_n(X,Y),\, r_{n,m}\right)_{m,n \in \mathbb{N},\, m \geq n}$$

indexed by $\mathbb{N}$. *Similarly homomorphisms* $\big(\iota_{n,m}\big)_{m\le n,\, m,n\in\mathbb{N}}$ *form a* inductive system *of commutative group schemes*

$$\big(H_n(X,Y),\, \iota_{n,m}\big)_{m,n\in\mathbb{N},\, m\le n}$$

indexed by $\mathbb{N}$. *Moreover we have*

$$r_{n,n+i}\circ \iota_{n+i,n} = [p^i]_{H_n} \quad \text{and} \quad \iota_{n+i,n}\circ r_{n,n+i} = [p^i]_{H_{n+i}}$$

for all $n, i \in \mathbb{N}$.

(2) *For* $n, i, j \in \mathbb{N}$ *we have a commutative diagram*

$$\begin{array}{ccccccc}
H_i & \xrightarrow{\iota_{n+i,i}} & H_{n+i} & \xrightarrow{r_{i,n+i}} & H_n & \longrightarrow & 0 \\
\uparrow{\scriptstyle r_{i,i+j}} & & \uparrow{\scriptstyle r_{n+i,n+i+j}} & & \uparrow{\scriptstyle =} & & \\
H_{i+j} & \xrightarrow{\iota_{n+i+j,i+j}} & H_{n+i+j} & \xrightarrow{r_{i+j,n+i+j}} & H_n & \longrightarrow & 0
\end{array}$$

with exact rows. This diagram applied twice, the second time with $j = 0$, *induces monomorphisms*

$$H_{n+i+j}/\iota_{n+i+j}(H_{i+j}) \xrightarrow{\nu_{n;i,i+j}} H_{n+i}/\iota_{n+i,i}(H_i) \xrightarrow{\nu_{n;i}} H_n$$

whose composition is equal to $\nu_{n;i+j}$.

(3) *There exists a positive integer* n_0 *such that the monomorphism*

$$\nu_{n;1,2}\colon H_{n+2}/\iota_{n+2,n+1}(H_{n+1}) \rightarrowtail H_{n+1}/\iota_{n+1,n}(H_n)$$

is an isomorphism of commutative finite group schemes over κ *for every integer* $n \ge n_0$.

2.5. Definition. Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. Let n_0 be a positive integer such that 2.4 (3) holds, i.e. the monomorphism $\nu_{n;1,2}$ is an isomorphism for every $n \ge n_0$.

(1) Define commutative finite group schemes $G_n = G_n(X,Y)$ over κ, $n \in \mathbb{N}$, by

$$G_n := H_{n+n_0}/\iota_{n+n_0,n_0}(H_{n_0}).$$

(2) Define monomorphisms $\nu_n\colon G_n \rightarrowtail H_n$ by $\nu_n = \nu_{n;i_0}$ for all $n \in \mathbb{N}$, where $\nu_{n;i_0}$ is defined in 2.4 (2).

(3) For every $n, i \in \mathbb{N}$, denote by

$$j_{n+i,n}\colon G_n \longrightarrow G_{n+i}$$

the homomorphisms over κ induced by $\iota_{n+i+n_0}\colon H_{n+n_0} \longrightarrow H_{n+n_0+i}$.

(4) For every $n, i \in \mathbb{N}$, denote by

$$\pi_{n,n+i} : G_{n+i} \longrightarrow G_n$$

the homomorphism induced by $r_{n+n_0,n+n_0+i}\colon H_{n+n_0+i} \longrightarrow H_{n+n_0}$.

2.6. Definition. (1) Denote by $\mathscr{H}om'(X,Y)$ the *inductive* system of commutative group schemes $H_n(X,Y)$ of finite type over κ, with transition maps

$$\iota_{n+m,n}\colon H_n(X,Y) \to H_{n+m}(X,Y).$$

(The superscript $'$ in $\mathscr{H}\!om'(X,Y)$ is meant to indicated that the arrows in the projective system $\mathscr{H}\!om'(X,Y)$ of $\mathscr{H}\!om$-schemes $H_i(X,Y) = \mathscr{H}\!om(X[p^i], Y[p^i])$ are reversed, giving rise to an inductive system instead.)

(2) We will write $\mathscr{H}\!om^{\rm st}(X,Y)_n$ for the group scheme $G_n(X,Y)$ over the base field κ. We will call it the stabilized $\mathscr{H}\!om$ scheme at truncation level n.

(3) Denote by $\mathscr{H}\!om'_{\rm div}(X,Y)$ the p-divisible group

$$\left(\mathscr{H}\!om^{\rm st}(X,Y)_n,\ j_{n+m,n},\ \pi_{n,n+m} \right)_{n,m\in\mathbb{N}}.$$

In other words $\mathscr{H}\!om'_{\rm div}(X,Y)[p^n] = \mathscr{H}\!om^{\rm st}(X,Y)_n$ for all $n \in \mathbb{N}$.

(4) Let $\nu : \mathscr{H}\!om'_{\rm div}(X,Y) \rightarrowtail \mathscr{H}\!om'(X,Y)$ be the monomorphism defined by the compatible family of monomorphisms

$$\nu_n : \mathscr{H}\!om'_{\rm div}(X,Y)[p^n] \rightarrowtail G_n(X,Y) \to H_n(X,Y) = \mathscr{H}\!om'(X,Y)[p^n]$$

Remark. The limit of the inductive system $\mathscr{H}\!om'(X,Y)$ is canonically identified with $\mathscr{H}\!om(\mathbf{T}_p(X), Y)$, the sheafified $\mathscr{H}\!om$ (or the internal Hom) in the category of sheaves of abelian groups, from the *projective system*

$$\mathbf{T}_p(X) := \left(X[p^n], \pi_{X,n,n+m} : X[p^{m+n}] \to X[p^n] \right)$$

to the *inductive system*

$$\left(Y[p^n], j_{Y,n+m,n} : Y[p^n] \to Y[p^{n+m}] \right).$$

The group scheme $\mathscr{H}\!om(X[p^i], Y[p^i])$ is the kernel of the endomorphism "multiplication by p^n" of this inductive limit:

$$\mathscr{H}\!om(X[p^i], Y[p^i]) = \mathscr{H}\!om(\mathbf{T}_p(X), Y)[p^n].$$

2.7. Corollary. (a) *The formation of $\mathscr{H}\!om'(X,Y)$ and $\mathscr{H}\!om'_{\rm div}(X,Y)$ commute with extension of base fields: for every homomorphism of fields $\kappa \to \kappa'$, the natural maps*

$$\mathscr{H}\!om'(X,Y)_{\kappa'} \longrightarrow \mathscr{H}\!om'(X_{\kappa'}, Y_{\kappa'})$$

and

$$\mathscr{H}\!om'_{\rm div}(X,Y)_{\kappa'} \longrightarrow \mathscr{H}\!om'_{\rm div}(X_{\kappa'}, Y_{\kappa'})$$

are isomorphisms.

(b) *The monomorphism $\nu : \mathscr{H}\!om'_{\rm div}(X,Y) \rightarrowtail \mathscr{H}\!om'(X,Y)$ identifies the inductive system $\mathscr{H}\!om'_{\rm div}(X,Y)$ as the maximal p-divisible subgroup of $\mathscr{H}\!om'(X,Y)$, which satisfies the following universal property: for every p-divisible group Z over a κ-scheme S and every S-homomorphism*

$$f : Z \to \mathscr{H}\!om'(X,Y),$$

there exists a unique homomorphism $g : Z \to \mathscr{H}\!om'_{\rm div}(X,Y)_S$ over S such that $f = \nu \circ g$.

2.8. Proposition. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. Suppose that X, Y are both isoclinic and let λ_X and λ_Y be their slopes.*

(a) *If $\lambda_X < \lambda_Y$, then the p-divisible group $\mathscr{H}\!om'_{\rm div}(X,Y)$ is isoclinic of slope $\lambda_Y - \lambda_X$.*

(b) *If $\lambda_X > \lambda_Y$, then $\mathscr{H}\!om'_{\rm div}(X,Y) = 0$.*

(c) *If $\lambda_X = \lambda_Y$, then $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ is an étale p-divisible group.*

2.9. Corollary. *Let X, Y be p-divisible groups over a field κ. If every slope of X is strictly bigger than every slope of Y, then $\mathscr{HOM}'_{\mathrm{div}}(X,Y) = 0$.*

We end this section with an example which illustrates a remark in the last paragraph of 2.1, that $\mathscr{HOM}'_{\mathrm{div}}(X,Y)$ is better thought of part of the "divisible part" of the sheaf $\mathscr{EXT}(X,Y)$ of extensions of X by Y, in the setting of deformations of $X \times_{\mathrm{Spec}(\kappa)} Y$ via extensions.

2.10. Definition. Let $\kappa \supset \mathbb{F}_p$ be a perfect field of characteristic $p > 0$. Let X, Y be p-divisible groups over κ.

(a) Denote by $\mathfrak{Art}_\kappa^+$ the category of augmented commutative Artinian local algebras over κ. Objects of $\mathfrak{Art}_\kappa^+$ are pairs

$$(R,\ j\colon \mathsf{k} \to R,\ \epsilon : R \to \mathsf{k})$$

where

- R is a commutative Artinian local algebra over κ,
- k is a perfect field containing κ,
- j and ϵ are κ-linear ring homomorphisms such that $\epsilon \circ j = \mathrm{id}_{\mathsf{k}}$.
- $\epsilon : R/\mathfrak{m} \longrightarrow R$ is a homomorphisms of κ-algebras such that the composition $\mathrm{pr} \circ \epsilon$ of ϵ with the quotient map $\mathrm{pr} : R \to \mathsf{k}$ is equal to id_{k}.

Morphisms from $(R_1,\ j_1\colon \mathsf{k}_1 \to R_1, \epsilon_1 : R_1 \to \mathsf{k}_1)$ to $(R_2,\ j_2\colon \mathsf{k}_2 \to R_2, \epsilon_2\colon R_2 \to \mathsf{k}_2)$ are κ-linear homomorphisms from R_1 to R_2 which are compatible with the augmentations ϵ_1 and ϵ_2.

(b) Define $\mathcal{E}xt_{\mathrm{def}}(X,Y) : \mathfrak{Art}_\kappa^+ \longrightarrow \mathfrak{Sets}$ to be the functor from $\mathfrak{Art}_\kappa^+$ to the category of sets which sends any object $(R,\ j\colon \mathsf{k} \to R,\ \epsilon\colon R \to \mathsf{k})$ of $\mathfrak{Art}_\kappa^+$ to the set of isomorphism classes of

$$\Big(0 \to Y_R \to E \to X_R \to 0,\ \zeta : E_{\mathsf{k}} \xrightarrow{\sim} X_{\mathsf{k}} \times Y_{\mathsf{k}}\Big),$$

where $0 \to Y_R \to E \to X_R \to 0$ is a short exact sequence of p-divisible groups over R, and ζ is an isomorphism from the closed fiber of the extension E to the split extension $0 \to Y_{\mathsf{k}} \to X_{\mathsf{k}} \times Y_{\mathsf{k}} \to X_{\mathsf{k}} \to 0$ which induces the identity maps $\mathrm{id}_{X_{\mathsf{k}}}$ and $\mathrm{id}_{Y_{\mathsf{k}}}$ on both X_{k} and Y_{k}. The Baer sum construction gives $\mathcal{E}xt_{\mathrm{def}}(X,Y)(R)$ a natural structure as an abelian group, for every object (R,ϵ) in $\mathfrak{Art}_\kappa^+$, so we can promote $\mathcal{E}xt_{\mathrm{def}}(X,Y)$ to a functor from $\mathfrak{Art}_\kappa^+$ to the category of abelian groups.

The functor $\mathcal{E}xt_{\mathrm{def}}(X,Y)$ can be identified as a subfunctor of the deformation functor of the p-divisible group $X \times_{\mathrm{Spec}(\kappa)} Y$; it is the largest closed formal subscheme of $\mathcal{D}ef(X \times_{\mathrm{Spec}(\kappa)} Y)$ over which the deformation of $X \times_{\mathrm{Spec}(\kappa)} Y$ is an extension of X by Y.

2.11. Proposition. *Let X, Y be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$.*

(i) *There is a natural isomorphism commutative smooth formal groups*

$$\delta : \mathscr{HOM}'(X,Y) \xrightarrow{\sim} \mathcal{E}xt_{\mathrm{def}}(X,Y)$$

over κ of dimension $\dim(Y) \cdot \dim(X^t)$.

(ii) *The isomorphism* $\delta : \mathscr{H}\!om'(X,Y) \xrightarrow{\sim} \mathscr{E}xt_{\rm def}(X,Y)$ *in* (i) *induces an isomorphism*

$$\delta\colon \mathscr{H}\!om'_{\rm div}(X,Y) \xrightarrow{\sim} \mathscr{E}xt_{\rm def}(X,Y)_{\rm div},$$

where $\mathscr{E}xt_{\rm def}(X,Y)_{\rm div}$ *is the maximal* p*-divisible subgroup of* $\mathscr{E}xt_{\rm def}(X,Y)$*.*

3. Descent of sustained p-divisible groups, torsors for stabilized Isom schemes and the slope filtration

In this section we discuss some basic properties of sustained p-divisible groups related to descent; notable among them are the torsor interpretation 3.1 and the existence of slope filtration 3.2. We also compare sustained p-divisible groups with completely p-divisible groups, and unravel what it means to say that a p-divisible group over a field K is sustained over a subfield κ of K.

Proposition 3.1 below says that a strongly κ-sustained p-divisible group $X \to S$ modeled on a p-divisible group X_0 over κ gives rise to a compatible family of torsors over S for the stabilized Aut group $\mathscr{A}\!ut^{\rm st}(X_0[p^n])$, and the strongly sustained p-divisible group X can be recovered from this family of torsors. It is a consequence of the stabilization results in §2, the definition of strongly sustained p-divisible groups and descent.

3.1. Proposition. *Let* $\kappa \supset \mathbb{F}_p$ *be a field and let* X_0 *be a* p*-divisible group over* κ*. Let* $X \to S$ *be a strongly* κ*-sustained* p*-divisible group over* S *modeled on* X_0*. Let* $\mathscr{I}\!som_S(X_0[p^n], X[p^n])$ *be the* S*-scheme of isomorphisms from* $X_0[p^n]$ *to* $X[p^n]$ *over* S*-schemes.*

(1) *There exists a natural number* n_0 *such that the image of the restriction morphisms* $r_{n,n+i} : \mathscr{I}\!som_S(X_0[p^{n+i}], X[p^{n+i}]) \to \mathscr{I}\!som_S(X_0[p^n], X[p^n])$ *and* $r_{n,n+i} : \mathscr{I}\!som_S(X_0[p^{n+i+1}], X[p^{n+i+1}]) \to \mathscr{I}\!som_S(X_0[p^n], X[p^n])$ *are equal as fppf sheaves, for all* $i \geq n_0$*.*

(2) *The stabilized image in* $\mathscr{I}\!som_S(X_0[p^n], X[p^n])$ *is represented by a closed subscheme* $\mathscr{I}\!som_S^{\rm st}(X_0[p^n], X[p^n])$*, which is a right torsor for (the base change to* S *of) the finite group scheme* $\mathscr{A}\!ut^{\rm st}(X_0[p^n])$ *over* κ*. Denote the torsor* $\mathscr{I}\!som_S^{\rm st}(X_0[p^n], X[p^n])$ *by* T_n*, and let* $G_n := \mathscr{A}\!ut^{\rm st}(X_0[p^n])$ *for each* $n \in \mathbb{N}$*.*

(3) *The natural transition maps* $T_{n+1} \to T_n$ *are faithfully flat and compatible with the transition maps* $G_{n+1} \to G_n$*.*

(4) *There is a compatible family of natural isomorphisms*

$$\alpha_n : T_n \times^{G_n} X_0[p^n] \xrightarrow{\sim} X[p^n], \quad n \in \mathbb{N},$$

where $T_n \times^{G_n} X_0[p^n] = T_n \times_S X_0[p^n]/G_n$ *is the contraction product of the* G_n*-torsor with* $X_0[p^n]$*.*

3.1.1. Remark. The stabilization phenomenon in 3.1 above generalizes to sustained p-divisible groups, as follows. Suppose that $Y \to S$ is a κ-sustained p-divisible group $\to S$ over a κ-scheme S. Then

$$\mathrm{Image}\left(\mathscr{A}\!ut_S(X[p^{n+i}]) \longrightarrow \mathscr{A}\!ut_S(X[p^n])\right)$$

stabilizes for $i \gg 0$ to a finite locally free group scheme $\mathscr{AUT}_S^{\rm st}(X[p^n])$, and

$$\text{Image}\left\{\text{Isom}_{S\times_{\text{Spec}(\kappa)}S}\left(\text{pr}_1^*(X)[p^{n+i}], \text{pr}_2^*(X)[p^{n+i}]\right)\right.$$
$$\left.\longrightarrow \text{Isom}_{S\times_{\text{Spec}(\kappa)}S}\left(\text{pr}_1^*(X)[p^n], \text{pr}_2^*(X)[p^n]\right)\right\}$$

stabilizes for $i \gg 0$ to a bi-torsor $\mathscr{ISOM}^{\rm st}_{S\times_{\text{Spec}(\kappa)}S}\big(\text{pr}_1^*(X)[p^n], \text{pr}_2^*(X)[p^n]\big)$ for $\mathscr{AUT}_S^{\rm st}(X[p^n])$.

3.1.2. Corollary. *Let $\kappa \supset \mathbb{F}_p$ be a field. Let K be a purely inseparable algebraic extension field of κ. Suppose that X is an isoclinic p-divisible group over K which is κ-sustained. Then X descends to κ, i.e. there exists a p-divisible group X_0 over κ and an isomorphism $X \xrightarrow{\sim} X_0 \times_{\text{Spec}\kappa} \text{Spec}(K)$. In particular X is strongly κ-sustained.*

Corollary 3.1.2 follows from the special case of 3.1 when X is isoclinic and consequently the stabilized Aut groups $\mathscr{AUT}^{\rm st}(X_0[p^n])$ are finite étale, plus descent.

3.1.3. Corollary. *Let $\kappa \supset \mathbb{F}_p$ be a field. Let R be an artinian κ-algebra, and let $\epsilon : R \twoheadrightarrow \kappa$ be a surjective κ-linear ring homomorphism. Suppose that $Y \to \text{Spec}(R)$ is an isoclinic κ-sustained p-divisible group. Then Y is naturally isomorphic to the constant p-divisible group*

$$(Y \times_{(\text{Spec}(R),\epsilon^\cdot)} \text{Spec}(\kappa)) \times_{\text{Spec}(\kappa)} \text{Spec}(R).$$

Corollary 3.1.3 follows from a descent argument similar to 3.1.2, using the stabilized Isom schemes

$$\mathscr{ISOM}^{\rm st}_{S\times_{\text{Spec}(\kappa)}S}(\text{pr}_1^*X[p^n], \text{pr}_2^*X[p^n]$$

for a κ-sustained p-divisible group $X \to S$.

Proposition 3.2 below asserts that there exists a functorial slope filtration on every κ-sustained p-divisible group X over S. If X is κ-sustained with a κ-model X_0 is obtained by descending the "constant slope filtration", which exists over the projective system of faithfully covers $\mathscr{ISOM}_S^{\rm st}(X_0[p^n], X[p^n])$ of S, down to S.

3.2. Proposition (Existence and uniqueness of slope filtration)**.** *Let S be a scheme over a field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a κ-sustained p-divisible group over S.*

(1) *There exists a natural number $m \geq 1$, rational numbers $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} \leq 1$ and sustained p-divisible subgroups $X_i \to S$, $0 \leq i \leq m$, with*

$$X = X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_m = 0$$

such that $X_i/X_{i+1} \to S$ is an isoclinic sustained p-divisible group of slope λ_i, for each $i = 0, \ldots, m-1$. Moreover the numbers $m, \lambda_0, \ldots, \lambda_{m-1}$ and the p-divisible subgroups $X_i \to S$ of X in (1) *are uniquely determined by the p-divisible group $X \to S$.*

(2) *Let $Y \to S$ be another sustained p-divisible group over S. Let $\text{Fil}^\bullet_{\text{slope}}X$ and $\text{Fil}^\bullet_{\text{slope}}Y$ be the filtration on X and Y defined below. Then*

$$\alpha(\text{Fil}^\lambda_{\text{slope}}X) \subseteq \text{Fil}^\lambda_{\text{slope}}Y \quad \forall\, \lambda \in [0,1]$$

for every S-homomorphism $\alpha : X \to Y$.

Definition. The filtration $\mathrm{Fil}^{\bullet}_{\mathrm{slope}}X$ on a κ-sustained p-divisible group $X \to S$ is the decreasing filtration

$$X = X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_m = 0$$

as in 3.2 (1), re-indexed with jumps at $\lambda_0, \ldots, \lambda_{m-1}$; i.e.

$$\mathrm{Fil}^{\lambda}_{\mathrm{slope}}X/\mathrm{Fil}^{\lambda+}_{\mathrm{slope}}X \cong \begin{cases} X_i/X_{i+1} & \text{if } \lambda = \lambda_i \text{ for some } i = 0, \ldots, m-1 \\ 0 & \text{otherwise} \end{cases}$$

To be more explicit,

$$\mathrm{Fil}^{\lambda}_{\mathrm{slope}}X := \begin{cases} X_m = 0 & \text{if } \lambda > \lambda_{m-1} \\ X_i & \text{if } \lambda_i \geq \lambda > \lambda_{i-1},\ 1 \leq i \leq m-1 \\ X_0 = X & \text{if } \lambda \leq \lambda_0 \end{cases}$$

for any real number $\lambda \in [0, 1]$.

4. Pointwise criterion for sustained p-divisible groups

The main result of this section is theorem 4.3: for a p-divisible group $X \to S$ over a *reduced* base κ-scheme to be strongly κ-sustained modeled on a given p-divisible group X_0, it is necessary and sufficient that every fiber X_s of $X \to S$ is strongly κ-sustained modeled on X_0. It follows that when the base field $\kappa \supset \mathbb{F}_p$ is an algebraically closed, the latter condition is equivalent to the notion of *geometrically fiberwise constant* p-divisible groups in [24]. The proof of 4.3 is technical; see [8] for details. We will formulate the technical lemmas which give some indication of the intermediate steps, but omit all proofs.

4.1. Lemma. *Suppose that $\kappa \supset \mathbb{F}_p$ is an algebraically closed field and S is a perfect scheme over κ. Let $X \to S$ be a completely slope divisible p-divisible group, and let $Y \to S$ be a p-divisible group over S and let $\xi : X \to Y$ be an S-isogeny. Assume that there exists a p-divisible group Y_0 over κ such that Y_s is strongly κ-sustained modeled on Y_0 for every $s \in S$. Then Y is isomorphic to $Y_0 \times_{\mathrm{Spec}(\kappa)} S$. In particular Y is strongly κ-sustained modeled on Y_0.*

The technical lemma 4.2 below on pseudo-torsors is an important ingredient of the proof of the fiberwise criterion 4.3 for strongly sustained p-divisible groups over a reduced base scheme.

4.2. Lemma. *Let $G \to S$ be a flat group scheme of finite presentation over a scheme S. Let $P \to S$ be a separated scheme of finite presentation over S, and let $\mu : P \times_S G \to P$ be an action of G on P over S. Assume that the G-action μ makes P a right pseudo-G-torsor. In other words, for every scheme T and every morphism $T \to S$, $P(T)$ is either empty or is a right principal homogeneous space for $G(T)$; equivalently $(\mathrm{id}_I, \mu)\colon G \times_S P \xrightarrow{\sim} P \times_S P$ is an isomorphism of S-schemes.*

(a) *The quotient P/G of the S_{fppf}-sheaf P by the S_{fppf}-sheaf G is representable by an S-scheme T.*

(b) *The morphism $P \to T$ gives P a structure as a right G-torsor for the site S_{fppf}.*

(c) *The structural morphism* $\pi : T \to S$ *factors as a composition* $T \xrightarrow{j} T' \xrightarrow{\pi'} S$ *of morphisms of schemes, where* π' *is a finite morphism of finite presentation and* j *is a quasi-compact open immersion.*

(d) *The morphism* $\pi' : T \to S$ *is a universal monomorphism and induces a continuous injection from* $|T|$ *to* $|S|$*, where* $|T|$ *and* $|S|$ *are the topological spaces underlying* T *and* S *respectively.*

(e) *For every* $t \in T$*,* π' *induces an isomorphism between the residue field of* $\pi'(s)$ *and the residue field of* t*. In particular* π' *is unramified.*

(f) *Let* $t \in T$ *be a point of* T *such that* $\mathscr{O}_{S,\pi'(t)}$ *is reduced.*
   - *If* π' *is étale at* t*, then* π' *induces an isomorphism of schemes between an open neighborhood of* t *and an open neighborhood of* $\pi'(t)$*.*
   - *If* π' *is not étale at* t*, then* $\pi'(\mathrm{Spec}(\mathscr{O}_{T,t}))$ *is contained in a closed subset of* $\mathrm{Spec}(\mathscr{O}_{S,\pi'(t)})$ *which is not equal to* $\mathrm{Spec}(\mathscr{O}_{S,\pi'(t)})$*. In particular there exists a point* $s \in S$ *whose closure in* S *contains* $\pi'(t)$ *such that* $s \neq \pi'(t_1)$ *for any* $t_1 \in T$ *whose closure in* T *contains* t*.*

(g) *Suppose that* S *is reduced and* $\pi' : |T| \to |S|$ *is a bijection such that every specialization in* S *comes from a specialization in* T*. Then* π' *is an isomorphism. In other words* $P \to S$ *is faithfully flat, making* P *a* G*-torsor over* S*.*

(h) *Suppose that* S *is reduced and locally Noetherian, and* $\pi'(T)$ *is Zariski dense in* S*. Then there exists a dense open subset* U *of* S *such that* π' *induces an isomorphism of schemes from* ${\pi'}^{-1}(U)$ *to* U*. In particular* $P \times_S U \to U$ *is a torsor for* $G \times_S U$*.*

4.3. Theorem. *Let* S *be a* reduced κ*-scheme, where* $\kappa \supset \mathbb{F}_p$ *is a field. Let* $X \to S$ *be a* p*-divisible group, and let* X_0 *be a* p*-divisible group over* κ*. Suppose that for every point* $s \in S$*, the geometric fiber* $X_s \times_s \bar{s}$ *is isomorphic to* $X_0 \times_{\mathrm{Spec}(\kappa)} \bar{s}$*; equivalently* X_s *is strongly* κ*-sustained modeled on* X_0 *for every* $s \in S$*. Then* $X \to S$ *is strongly* κ*-sustained modeled on* X_0*.*

Remark. Theorem 4.3 says that a strongly geometrically fiberwise constant p-divisible group relative to κ over a reduced κ-scheme is strongly κ-sustained. So over a reduced κ-scheme, being strongly κ-sustained is equivalent to being strongly geometrically fiberwise constant relative to κ. The latter notion was introduced in [24] to define the central foliation on Siegel modular varieties.

4.4. Corollary. *Let* S *be a* reduced *scheme over a field* $\kappa \supset \mathbb{F}_p$*. Let* $X \to S$ *be a* p*-divisible group, and let* X_0 *be a* p*-divisible group over* κ*. Assume that Newton polygon of every fiber of* $X \to S$ *is equal to the Newton polygon of* X_0*, and there is a dense open subset* $U \subset S$ *such that* $X \times_S U \to U$ *is strongly* κ*-sustained modeled on* X_0*. Then* $X \to S$ *is strongly* κ*-sustained modeled on* X_0*.*

Remark. Corollary 4.4 is a generalization of a result in [24], where it is shown that every central leaf $\mathcal{C}$ in $\mathscr{A}_g$ is closed in the open Newton polygon stratum containing $\mathcal{C}$.

4.5. Lemma. *Let* S *be a* reduced noetherian *scheme over a field* $\kappa \supset \mathbb{F}_p$*. Let* $X \to S$ *be a* p*-divisible group, and let* X_0 *be a* p*-divisible group over* κ*. Suppose* *that there is a Zariski dense subset $B \subset |S|$ such that X_b is strongly κ-sustained modeled on X_0 for every element $b \in B$, where $|S|$ denotes the topological space underlying S. Then there exists a Zariski dense open subscheme $U \subset S$ such that the p-divisible group $X \times_S U \to U$ is strongly κ-sustained modeled on X_0.*

4.6. Lemma. *Let S be a Noetherian scheme over a field $\kappa \supset \mathbb{F}_p$. Let $X \to S$ be a p-divisible group over S, and let X_0 be a p-divisible group over κ. Let ξ be the Newton polygon of X_0.*

(a) *The subset of S consisting of all points $s \in S$ such that the fiber X_s of $X \to S$ at s whose Newton polygon is equal to ξ is a locally closed subset $|S_\xi|$ of $|S|$.*
(b) *The subset of S consisting of all points $s \in S$ such that the fiber X_s of $X \to S$ at s is strongly κ-sustained modeld on X_0 is a closed subset $|S_{X_0}|$ of $|S_\xi|$.*
(c) *Denote by S_{X_0} the locally closed reduced subscheme of S whose underlying set is $|S_{X_0}|$. The p-divisible group $X \times_S S_{X_0} \to S_{X_0}$ is strongly κ-sustained modeled on X_0.*

5. Deformation of sustained p-divisible groups and local structure of central leaves

In this section we consider deformations of a p-divisible group X_0 over a field $\kappa \supset \mathbb{F}_p$ which are κ-sustained, and the structure of the space of such sustained deformations $\mathcal{D}ef(X_0)_{\rm sus}$. The general phenomenon is that $\mathcal{D}ef(X_0)_{\rm sus}$ is "built-up" from p-divisible groups via a system of fibrations, with p-divisible groups as fibers. This "big-picture description" is literally true when X_0 is a product of isoclinic p-divisible groups. Otherwise the description needs to be taken with a grain of salt, because the fibers may be an extension of a connected finite group scheme by a p-divisible group. The p-divisible groups which appear as the "building blocks" of $\mathcal{D}ef(X_0)_{\rm sus}$ are of the form $\mathscr{H}om'_{\rm div}(Y,Z)$, where Y, Z are isoclinic p-divisible groups over κ with $\mathrm{slope}(Y) < \mathrm{slope}(Z)$. Their Dieudonné modules are given in 5.3.

5.1. Theorem. *Let $\kappa \supset \mathbb{F}_p$ be a perfect field. Let X_0 be a p-divisible group over κ. Let $\mathcal{D}ef(X_0)$ be the equi-characteristic-p deformation space of X_0, isomorphic to the formal spectrum of a power series ring over κ in $\dim(X_0)\cdot\dim(X_0^t)$ variables.*

(1) *There exists a closed formal subscheme $\mathcal{D}ef(X_0)_{\rm sus}$ of $\mathcal{D}ef(X_0)$, uniquely characterized by the following properties.*
   – *The p-divisible group*
   $$\mathscr{X}_{\rm univ} \times_{\mathcal{D}ef(X_0)} \mathcal{D}ef(X_0)_{\rm sus} \to \mathcal{D}ef(X_0)_{\rm sus}$$
   is strongly k-sustained modeled on X_0.
   – *If Z is a closed formal subscheme of $\mathcal{D}ef(X_0)$ such the p-divisible group*
   $$\mathscr{X}_{\rm univ} \times_{\mathcal{D}ef(X_0)} Z \longrightarrow Z$$
   is strongly k-sustained modeled on X_0, then Z is a subscheme of $\mathcal{D}ef(X_0)_{\rm sus}$.

(2) *The formal scheme $\mathcal{D}ef(X_0)_{\rm sus}$ is formally smooth over k.*

(3) *Suppose that X_0 is isogenous to a product*

$$Y_1 \times_{\mathrm{Spec}(\kappa)} Y_2 \times_{\mathrm{Spec}(\kappa)} \cdots \times_{\mathrm{Spec}(\kappa)} Y_r \,,$$

where Y_i is an isoclinic p-divisible groups over κ with slopes λ_i for each i, and $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_r \le 1$. Then

$$\dim(\mathcal{D}ef(X_0)_{\mathrm{sus}}) = \sum_{1 \le i < j \le r} (\lambda_j - \lambda_i) \cdot \mathrm{ht}(Y_i) \cdot \mathrm{ht}(Y_j)\,.$$

It is unknown whether the sustained deformation space of every 2-divisible group X_2 over a perfect field $\kappa \supset \mathbb{F}_2$; see **??**.

5.2. Proposition. *Let $\kappa \supset \mathbb{F}_p$ be a perfect field. Let X_0 be a g-dimensional p-divisible group over κ and let $\mu_0 : X_0 \xrightarrow{\sim} X_0^t$ be a polarization of X_0. Suppose that X_0 is isogenous to a product*

$$Y_1 \times_{\mathrm{Spec}(\kappa)} Y_2 \times_{\mathrm{Spec}(\kappa)} \cdots \times_{\mathrm{Spec}(\kappa)} Y_r \,,$$

where Y_i is an isoclinic p-divisible groups over κ with slopes λ_i for each i, and $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_r \le 1$, so that Y_i^t is isogenous to Y_{r+1-i} for all $i = 1, \ldots, r$. Let $\mathcal{D}ef(X_0, \mu_0)$ be the equi-characteristic-p deformation space of X_0, isomorphic to the formal spectrum of a power series ring in $g\cdot(g+1)/2$ variables over κ.

(1) *There exists a closed formal subscheme $\mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}}$ of $\mathcal{D}ef(X_0, \mu_0)$, uniquely characterized by the following properties.*
   - *The polarized p-divisible group*

   $$(\mathscr{X}_{\mathrm{univ}}, \mu_{\mathrm{univ}}) \times_{\mathcal{D}ef(X_0,\mu_0)} \mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}} \to \mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}}$$

   is strongly k-sustained modeled on (X_0, μ_0).
   - *If Z is a closed formal subscheme of $\mathcal{D}ef(X_0, \mu_0)$ such the p-divisible group*

   $$(\mathscr{X}_{\mathrm{univ}}, \mu_{\mathrm{univ}}) \times_{\mathcal{D}ef(X_0,\mu_0)} Z \longrightarrow Z$$

   is strongly k-sustained modeled on (X_0, μ_0), then Z is a subscheme of $\mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}}$.

(2) *The dimension of $\mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}}$ is*

$$\dim(\mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}}) = \frac{1}{2} \sum_{1 \le i < j \le r,\, i+j \ne r+1} (\lambda_j - \lambda_i) \cdot \mathrm{ht}(Y_i) \cdot \mathrm{ht}(Y_j) + \sum_{1 \le i \le \lfloor r/2 \rfloor} (1 - 2\lambda_i) \cdot \frac{\mathrm{ht}(Y_i) \cdot (\mathrm{ht}(Y_i) + 1)}{2}$$

(3) *The deformation space $\mathcal{D}ef(X_0, \mu_0)_{\mathrm{sus}}$ is formally smooth.*

The proof of theorem 5.1 uses the deformation theory of torsors in [13, Chap. VII §2] and the interpretation of strongly sustained p-divisible groups as torsors for the stabilized Aut group schemes.

Proposition 5.3 gives a formula of the Dieudonné module of $\mathscr{HOM}'_{\rm div}(X,Y)$ in terms of the Dieudonné modules of X and Y.

5.3. Proposition. *Let X,Y be p-divisible groups over a perfect field $\kappa\supset\mathbb{F}_p$. The covariant Dieudonné module $\mathrm{M}_*(\mathscr{HOM}'_{\rm div}(X,Y))$ of the p-divisible formal group $\mathscr{HOM}'_{\rm div}(X,Y)$ is naturally isomorphic to the largest $W(\kappa)$-submodule of the $W(k)$-module $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X),\mathrm{M}_*(Y))$ which is stable under the actions of both operators F and V on $\mathrm{Hom}_{W(\kappa)}(\mathrm{M}_*(X),\mathrm{M}_*(Y))[1/p]$.*

5.4. Slope filtration and sustained deformations

The existence of the slope filtration on a sustained p-divisible group immediately implies the following:

> *A κ-sustained deformations of a p-divisible group X_κ over κ is determined by the deformation of the slope filtration of X_κ.*

To elaborate, suppose that $\kappa\supset\mathbb{F}_p$ is field and $R\twoheadrightarrow\kappa$ is an augmented artinian κ-algebra. Let X_κ be a p-divisible group over κ and let X be a κ-sustained p-divisible group over R whose closed fiber is X_κ. Let

$$X=X_0\supsetneqq X_1\supsetneqq\cdots\supsetneqq X_m=0$$

be the slope filtration of X as in 3.2 (1). Let $Y_i=(X_i/X_{i+1})\times_{\mathrm{Spec}(R)}\mathrm{Spec}(\kappa)$ for $i=0,\dots,r-1$. Then X_i/X_{i+1} is an isoclinic strongly κ-sustained p-divisible group over the artinian local ring R modeled on Y_i for each $i=0,1,\dots,r-1$. Through the natural isomorphisms $\psi_i: X_i/X_{i+1}\xrightarrow{\sim}Y_i\times_{\mathrm{Spec}(\kappa)}\mathrm{Spec}(R)$ given by 3.1.3, we can regard X as a successive extension of the constant p-divisible groups $Y_i\times_{\mathrm{Spec}(\kappa)}\mathrm{Spec}(R)$, $i=0,1,\dots,r-1$. Thus every sustained deformation of X_κ arises from a deformation of its slope filtration.

5.5. Remark. At many places in this chapter we made the assumption that the base field κ is perfect, although this assumption may very well be superfluous in the majority of cases. Part of the reason is that perfection is often assumed in the literature on deformation theory and in Dieudonné theory, and we have not made an effort to eliminate this assumption as much as possible.

5.6. The first non-trivial case for the sustained deformation space $\mathcal{D}ef(X_0)_{\rm sus}$ occurs when the p-divisible group X_0 has exactly two slopes. This very special case reveals a new phenomenon, that the sustained locus in the local deformation space of a p-divisible group are "built up" from p-divisible formal groups through successive fibrations.

We will use the following notations.

- Let $k\supset\mathbb{F}_p$ be a perfect field. Let Y and Z be isoclinic p-divisible groups over k such that $\mathrm{slope}(Y)<\mathrm{slope}(Z)$. Let $X_0=Y\times_{\mathrm{Spec}(k)}Z$.
- Let $\mathrm{M}(Y)$, $\mathrm{M}(Z)$ be the covariant Dieudonné modules of Y and Z respectively. Both are finite free modules over $W(k)$, plus semi-linear actions F and V. (The covariant theory is normalized so that $\mathrm{M}(Z)$ is the Cartier module for Z, and same for $\mathrm{M}(Y)$ if $\mathrm{slope}(Y)>0$.)

  The operator V corresponds to the relative Frobenius homomorphisms $Y\to Y^{(p)}$ and $Z\to Z^{(p)}$ over k, while F corresponds to the Verschiebung homomorphisms $Y^{(p)}\to Y$ and $Z^{(p)}\to Z$ over k.

- On $\mathrm{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))[1/p]$ we have natural actions of F and V defined by
$$(F \cdot h)(x) := F(h(Vx)), \quad (V \cdot h)(x) := V(h(V^{-1}x))$$
for all $h \in \mathrm{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))[1/p]$ and all $x \in \mathrm{M}(X)$. Note that the fee $W(k)$-module
$$\mathrm{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))$$
is stable under F but not necessarily under V.

Recall that the isomorphism $\delta : \mathscr{H}om'(Y, Z) \xrightarrow{\sim} \mathcal{E}xt_{\mathrm{def}}(Y, Z)$ in 2.11 induces an isomorphism
$$\delta : \mathscr{H}om'_{\mathrm{div}}(Y, Z) \xrightarrow{\sim} \mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}.$$
Combining 5.3, 2.8, 2.11, we obtain the following structural description of the space $\mathcal{D}ef(X_0)_{\mathrm{sus}}$ of strongly sustained deformations of X_0.

5.6.1. Proposition. *Notation as in subsection* 5.6.

(1) *The group-valued functor $\mathcal{E}xt_{\mathrm{def}}(Y, Z)$ is representable by a commutative smooth formal group over k.*
(2) *The commutative smooth formal group $\mathcal{E}xt_{\mathrm{def}}(Y, Z)$ over k has a largest p-divisible subgroup over k, denoted by $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}$. Similarly the formal group $\mathcal{E}xt_{\mathrm{def}}(Y, Z)$ has a largest unipotent smooth formal subgroup over k, denoted by*
$$\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{unip}};$$
it is killed by p^N for some $N \in \mathbb{N}$.
(3) *The smooth formal group $\mathcal{E}xt_{\mathrm{def}}(Y, Z)$ over k is generated by the two smooth formal subgroups $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}$ and $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{unip}}$ over k. The intersection $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}} \cap \mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{unip}}$ of these two smooth formal subgroups is a commutative finite group scheme over k.*
(4) *The p-divisible subgroup $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{unip}}$ is equal to the k-sustained locus in $\mathcal{D}ef(X_0)$, in the following sense.*
   - *The restriction to $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}$ of the universal p-divisible group*
   $$\mathscr{X}_{\mathrm{univ}} \to \mathcal{D}ef(X_0)$$
   over $\mathcal{D}ef(X_0)$ is strictly κ-sustained modeled on X_0.
   - *If $V \subseteq \mathcal{D}ef(X_0)$ is a closed formal subscheme of $\mathcal{D}ef(X_0)$ such that the restriction to V of the universal p-divisible group over $\mathcal{D}ef(X_0)$ is strictly κ-sustained modeled on X_0, then $V \subseteq \mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}$.*
(5) *The p-divisible formal group $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}$ is isoclinic; its slope is equal to* $\mathrm{slope}(Z) - \mathrm{slope}(Y)$, *and its height is equal to* $\mathrm{height}(Y) \cdot \mathrm{height}(Z)$.
(6) *The covariant Dieudonné module of $\mathcal{E}xt_{\mathrm{def}}(Y, Z)_{\mathrm{div}}$ is the largest $W(k)$-submodule of* $\mathrm{Hom}_{W(k)}(\mathrm{M}(Y), \mathrm{M}(Z))$ *which is stable under the actions of both F and V.*

5.6.2. Remark. Let Y, Z be isoclinic p-divisible groups over a perfect field $\kappa \supset \mathbb{F}_p$ as in 5.6. Suppose that X_1 is an extension of Y by Z over κ but not necessarily a product. Then $\mathcal{D}ef(X_1)_{\rm sus}$ has a natural structure as a torsor for the commutative smooth formal group $\mathscr{HOM}'_{\rm div}(Y, Z)$. This torsor has a natural base point, corresponding to constant deformations.

5.6.3. Proposition. *Let Y_1, Y_2, Y_3 be three isoclinic p-divisible groups over a perfect field $\kappa \supset \mathbb{F}_p$, and* $\mathrm{slope}(Y_1) < \mathrm{slope}(Y_2) < \mathrm{slope}(Y_3)$. *Let*

$$\mathcal{D}ef(Y_1 \times Y_2 \times Y_3)_{\rm sus} =: \mathcal{D} \subset \mathcal{D}ef(Y_1 \times Y_2 \times Y_3)$$

be the largest closed formal subscheme of the characteristic-p deformation space $\mathcal{D}ef(Y_1 \times Y_2 \times Y_3)$ over which the deformation is strongly $\overline{\mathbb{F}_p}$-sustained. Then $\mathcal{D}$ has a natural structure as a biextension *of the p-divisible formal groups* $\big(\mathscr{HOM}'_{\rm div}(Y_1, Y_2), \mathscr{HOM}'_{\rm div}$ *by the p-divisible formal group.* $\mathscr{HOM}'_{\rm div}(Y_1, Y_3)$.

See [18] for the basics of biextensions. This biextension structure on the space $\mathcal{D}ef(Y_1 \times Y_2 \times Y_3)_{\rm sus}$ of sustained deformations arises as follows. Let $\mathscr{X} \to \mathcal{D}$ be the universal sustained p-divisible group over $\mathcal{D}ef(Y_1 \times Y_2 \times Y_3)_{\rm sus} := \mathcal{D}$, and let

$$\mathscr{X} = \mathscr{X}_0 \supsetneqq \mathscr{X}_1 \supsetneqq \mathscr{X}_2 \supsetneqq \mathscr{X}_3 = 0$$

be the slope filtration of $\mathcal{X}$. Then we have canonical isomorphisms

$$\mathscr{X}_0/\mathscr{X}_1 \xrightarrow{\sim} Y_1 \times \mathcal{D}, \quad \mathscr{X}_1/\mathscr{X}_2 \xrightarrow{\sim} Y_2 \times \mathcal{D}, \quad \mathscr{X}_2 \xrightarrow{\sim} Y_3 \times \mathcal{D}.$$

The sustained p-divisible group $\mathscr{X}_0/\mathscr{X}_2$ over $\mathcal{D}$ defines a morphism

$$\pi_1 : \mathcal{D} \to \mathcal{D}ef(Y_1 \times Y_2)_{\rm sus}\,,$$

and the sustained p-divisible group $\mathscr{X}$ over $\mathcal{D}$ defines a morphism

$$\pi_2 : \mathcal{D} \to \mathcal{D}ef(Y_2 \times Y_3)_{\rm sus}.$$

In addition, the Baer sum construction provides two commutative group laws

$$+_1 : \mathcal{D} \times_{\pi_2, \mathcal{D}ef(Y_2\times Y_3)_{\rm sus}, \pi_2} \mathcal{D} \longrightarrow \mathcal{D}$$

and

$$+_2 : \mathcal{D} \times_{\pi_2, \mathcal{D}ef(Y_1\times Y_2)_{\rm sus}, \pi_1} \mathcal{D} \longrightarrow \mathcal{D}.$$

It is easy to see by formal arguments that the group laws $+_1$ and $+_2$ are compatible in the sense of [18], and making $\mathcal{D} = \mathcal{D}ef(Y_1 \times Y_2 \times Y_3)_{\rm sus}$ a biextension of $\big(\mathscr{HOM}'_{\rm div}(Y_1, Y_2), \mathscr{HOM}'_{\rm div}(Y_2, Y_3)\big)$ by $\mathscr{HOM}'_{\rm div}(Y_1, Y_3)$.

5.7. Open Question. Suppose that $\kappa \supset \mathbb{F}_p$ is a nonperfect field and X, Y are p-divisible groups over κ. How to describe the display of the p-divisible group $\mathscr{HOM}'_{\rm div}(X, Y)$ in terms of the displays of X and Y?

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