Mumford's example of non-flat $\mathcal{P}ic^{\tau}$

Ching-Li Chai, March 6, 2008

In the 1962 March 31 letter to Mumford, Grothendieck said "I am quite interested in your letter. Concerning your example of a non-flat $\mathcal{P}ic^{\tau}$, I am convinced there are still stronger counter-examples ...". In Remarque 2.9 of [3] Grothendieck mentioned this example, and said that it is a deformation of an Igusa surface over an Artinian ring. Such an example is reconstructed from the above hint. Basic information on the de Rham cohomology of Igusa surfaces are collected in § 1; the example is explained in 2.4. *No* priority or innovation is claimed.

§1. Igusa surfaces

(1.1) An *Igusa surface* is a proper smooth surface over a field $k \supset \mathbb{F}_2$, constructed as follows. The "input data" consists of two one-dimensional abelian varieties E_1, E_2 over k, and a non-trivial 2-torsion point $a \in E_2[2](k)$. Notice that the existence of a non-trivial 2-torsion point implies that E_2 is ordinary.

Let σ be the automorphism of $E_1 \times E_2$ such that $\sigma\colon (x, y) \mapsto (-x, y + a)$, an involution with no fixed point. The Igusa surface attached to the above input data is $X := (E_1 \times E_2)/G$, where $G := \{1, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$; see [1].

The following morphisms are useful:

(1) the quotient map $\pi : E_1 \times E_2 \longrightarrow X$,

(2) the morphism $f\colon X \longrightarrow E_3 := E_2/\{1, a\}$, induced by $\mathrm{pr}_2 : E_1 \times E_2 \longrightarrow E_2$.

Notice that f gives X a structure as an E_1-torsor over E_3.

It is easy to compute the ℓ-adic Betti numbers of X for $\ell \neq p$ using the quotient morphism π, because σ operates on $H^1(E_1 \times E_2, \mathbb{Q}_\ell) = H^1(E_1, \mathbb{Q}_\ell) \oplus H^1(E_2, \mathbb{Q}_\ell)$ as $-\mathrm{id} \oplus \mathrm{id}$:

$$h^0(X, \mathbb{Q}_\ell) = 1, \quad h^1(X, \mathbb{Q}_\ell) = h^3(X, \mathbb{Q}_\ell) = 2, \quad h^2(X, \mathbb{Q}_\ell) = 2, \quad h^4(X, \mathbb{Q}_\ell) = 1\,.$$

When considering deformation of X and its Picard scheme $\mathcal{P}ic(X)$, one needs the de Rham cohomology. It turns out that the de Rham cohomology of an Igusa surface strongly resembles the de Rham cohomology of an abelian surface.

(1.2) Proposition *Let X be an Igusa surface as above.*

(i) *The cotangent sheaf Ω^1_X of X is isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$. The canonical invertible sheaf $K_X = \Omega^2_X$ is isomorphic to $\mathcal{O}_X$. The tangent sheaf $\Theta_X = \Omega^1_X(-K_X)$ is isomorphic to $\mathcal{O}_X \oplus \mathcal{O}_X$.*

(ii) *The cup product induces isomorphisms*

$$H^i(X, \mathcal{O}_X) \otimes H^0(X, \Omega^j_X) \xrightarrow{\sim} H^i(X, \Omega^j)\,, \qquad \bigwedge^2 H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^2(X, \mathcal{O}_X)\,.$$

(iii) *The Hodge numbers* $h^{ij} := \dim H^j(X, \Omega^i_X)$ *are given as follows:*

$$h^{00} = h^{22} = 1, \quad h^{01} = h^{10} = h^{12} = h^{21} = 2, \quad h^{02} = h^{20} = 1, \quad h^{11} = 4\,.$$

Moreover $\dim H^1(X, \Theta_X) = 4$.

PROOF. The cotangent sheaf $\Omega^1_{E_1\times E_2}$ for the abelian variety $E_1\times E_2$ is trivial. Moreover the involution σ operates on

$$H^0(E_1\times E_2, \Omega^1_{E_1\times E_2}) \cong H^0(E_1, \Omega^1_{E_1}) \oplus H^0(E_2, \Omega^1_{E_2})$$

as $-\mathrm{id}\oplus\mathrm{id} = \mathrm{id}\oplus\mathrm{id}$, because k has characteristic 2. So every global section of $\Omega^1_{E_1\times E_2}$ descends to a global section of Ω^1_X, and the statement (i) follows. The first isomorphism in (ii) follows from (i); the second isomorphism in (ii) follows from the first part of (ii) and the Serre duality.

The Hodge numbers of X are recorded in [2], Example 6, p. 22. We provide a proof here. By (ii), we only need to show that $h^{00} = 1$, $h^{02} = 1$ and $h^{01} = 2$. The first equality is clear, while the second follows from the Serre duality and (i). So we only need to show that $h^{01} = 2$.

Consider the Leray spectral sequence

$$\mathrm{E}_2^{ij} = H^i(E_3, R^jf_*\mathcal{O}_X) \implies H^{i+j}(X, \mathcal{O}_X)$$

for $f\colon X \to E_3$. It is easy to see that $\dim \mathrm{E}_2^{01} = \dim \mathrm{E}_2^{10} = 1$, so $\dim H^1(X, \mathcal{O}_X) \le 2$.

To get a lower bound of $\dim H^1(X, \mathcal{O}_X)$, look at the Hochschild-Serre spectral sequence

$$\mathrm{E}_2^{ij} = H^i(G, H^j(E_1\times E_2, \mathcal{O}_{E_1\times E_2})) \implies H^{i+j}(X, \mathcal{O}_X)$$

where $G \cong \mathbb{Z}/2\mathbb{Z}$ operates trivially on the cohomology groups $H^j(E_1\times E_2, \mathcal{O}_{E_1\times E_2})$. We have $\mathrm{E}_2^{i0} \cong k$, $\mathrm{E}_2^{i1} \cong k^2$, $\mathrm{E}_2^{i2} \cong k$ for all $i \ge 2$. So $\dim H^1(X, \mathcal{O}_X) \ge 2$. □

Remark. It is not difficult to see that the Hochschild-Serre spectra sequence at the end of the proof of 1.2 degenerates at $\mathrm{E}_3^{\bullet\bullet}$. The differentials $d_2^{i,1} : \mathrm{E}_2^{i,1} \cong k^2 \to \mathrm{E}_2^{i+2,0} \cong k$ are surjective, while differentials $d_2^{i,2} : \mathrm{E}_2^{i,2}] \cong k \to \mathrm{E}_2^{i+2,1} \cong k^2$ are injective. We have short exact sequences

$$0 \longrightarrow \mathrm{E}_2^{i,2} \xrightarrow{d_2} \mathrm{E}_2^{i+2,1} \xrightarrow{d_2} \mathrm{E}_2^{i+4,0} \longrightarrow 0$$

for all $i \ge 0$.

(1.3) Proposition *Let X be an Igusa surface as above.*

(1) *The morphism $f\colon X \longrightarrow E_3$ is an Albanese morphism for X.*

(2) *The Picard scheme $\mathcal{P}ic(X)$ is a one-dimensional non-reduced group scheme over k; its Lie algebra is two dimensional. Pulling back by $f\colon X \to E_3$ induces an isomorphism from $\mathcal{P}ic^0(E_3) \cong E_3$ to $\mathcal{P}ic^0(X)_{\mathrm{red}}$.*

(3) *We have a short exact sequence*

$$0 \longrightarrow \mathcal{P}ic^0 X \longrightarrow \mathcal{P}ic^\tau(X) \longrightarrow E_1[2]_{\mathrm{et}} \longrightarrow 0\,,$$

where $E_1[2]_{\mathrm{et}}$ is the maximal etale quotient of the finite group scheme

$$E_1[2] := \mathrm{Ker}([2]\colon E_1 \to E_1)$$

over k. Notice that $E_1[2]_{\mathrm{et}}$ is trivial if E_1 is supersingular, and is canonically isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if E_1 is ordinary.

Proof. The statements (1) and (2) are due to Igusa; see [1]. Notice that the Lie algebra of $\mathcal{P}ic(X)$ is $H^1(X, \mathcal{O}_X)$.

The Hochschild-Serre spectral sequence for $\pi\colon E_1 \times E_2 \longrightarrow X = (E_1 \times E_2)/G$ gives us an exact sequence

$$1 \longrightarrow H^1(\{G, \bar{k}^\times) \longrightarrow \mathcal{P}ic^\tau(X)(\bar{k}) \longrightarrow H^0(G, \mathcal{P}ic^0(E_1 \times E_2)(\bar{k})) \longrightarrow H^2(G, \bar{k}^\times)\,.$$

Since $\bar{k}$ is perfect, both $H^1(G, \bar{k}^\times) = \mathrm{Hom}(G, \bar{k}^\times)$ and $H^2(G, \bar{k}^\times)$ are trivial. The automorphism σ acts on $\mathcal{P}ic^0(E_1 \times E_2) = \mathcal{P}ic^0(E_1) \times \mathcal{P}ic^0(E_2)$ as $-\mathrm{id}_{\mathcal{P}ic^0(E_1)} \times \mathrm{id}_{\mathcal{P}ic^0(E_2)}$. Therefore

$$\mathcal{P}ic^\tau(X)(k) \xrightarrow{\sim} \mathcal{P}ic^0(E_1)[2](\bar{k}) \times \mathcal{P}ic^0 E_3(\bar{k})\,,$$

and the $\bar{k}$-points of the finite etale group scheme $\mathcal{P}ic^\tau(X)/\mathcal{P}ic^0(X)$ is canonically isomorphic to the 2-torsion points of $\mathcal{P}ic^0(E_1)[2]_{\mathrm{et}} \cong E_1[2]_{\mathrm{et}}$. The statements (3) follows. ∎

(1.4) Proposition *We have a canonical short exact sequence of commutative group schemes*

$$0 \longrightarrow \mathcal{P}ic^0(X)_{\mathrm{red}} \longrightarrow \mathcal{P}ic^\tau(X) \longrightarrow \mathcal{P}ic^0(E_1)[2] \cong E_1[2] \longrightarrow 0\,.$$

Proof. Let $\alpha\colon E_3 \to \mathrm{Spec}(k)$ and $\beta = \alpha \circ \pi\colon X \to \mathrm{Spec}(k)$ be the structural morphisms. The Leray spectral sequence for the morphism $f\colon X \to E_3$ with respect to the flat topology gives an exact sequence

$$0 \longrightarrow R^1\alpha_{\mathrm{fl},*}\mathbb{G}_m \longrightarrow R^1\beta_{\mathrm{fl},*}\mathbb{G}_m \longrightarrow \alpha_{\mathrm{fl},*}R^1\pi_{\mathrm{fl},*}\mathbb{G}_m \longrightarrow R^2\alpha_{\mathrm{fl},*}\mathbb{G}_m\,.$$

The last term $R^2\alpha_{\mathrm{fl},*}\mathbb{G}_m$ vanishes by Tsen's theorem. Restricting to $\mathcal{P}ic^\tau$ gives the desired short exact sequence. ∎

(1.5) Proposition *Both the Hodge-de Rham spectral sequence*

$$\mathrm{E}_1^{ij} = H^j(X, \Omega^i_X) \implies H^{i+j}(X, \Omega^\bullet_X)$$

and the conjugate spectral sequence

$$\mathrm{E}_2^{ij} = H^i(X^{(p)}, \mathcal{H}^j(\Omega^\bullet_{X^{(p)}})) \implies H^{i+j}(X, \Omega^\bullet_X)$$

degenerate.

Proof. The Poincaré duality for de Rham cohomology and the Hodge-de Rham spectral sequence give $\dim H^0(X, \Omega^\bullet) = \dim H^4(X, \Omega^\bullet) = 1$, and upper bounds

$$\dim H^1(X, \Omega^\bullet) = \dim H^3(X, \Omega^\bullet \le 4, \quad \dim H^2(X, \Omega^\bullet) \le 6\,.$$

As in the proof of 1.2, we will use the Hochschild-Serre spectral sequence

$$\mathrm{E}_2^{ij} = H^i(G, H^j(E_1 \times E_2, \Omega^\bullet_{E_1 \times E_2})) \implies H^{i+j}(X, \Omega^\bullet_X)$$

to get lower bounds for $\dim H^1(X, \Omega^\bullet)$ and $\dim H^2(X, \Omega^\bullet)$. The group $G \cong \mathbb{Z}/2\mathbb{Z}$ operates trivially on the cohomology groups $H^j(E_1 \times E_2, \Omega^\bullet_{E_1 \times E_2}))$'s because the field k has characteristic 2. So

$$\mathrm{E}_2^{i0} \cong k, \quad \mathrm{E}_2^{i1} \cong k^4, \quad \mathrm{E}_2^{i2} \cong k^6, \quad \mathrm{E}_2^{i3} \cong k^4, \quad \mathrm{E}_2^{i4} \cong k \quad \forall\, i \ge 0\,.$$

It follows immediately that $\dim \mathrm{E}_3^{01} = \dim \mathrm{E}_\infty^{01} \ge 3$, so $\dim H^1(X, \Omega^\bullet_X) \ge 4$. Combined with the upper bound from the Hodge-de Rham spectral sequence, we conclude that $\dim H^1(X, \Omega^\bullet) = 4$. It follows that $\dim H^3(X, \Omega^\bullet) = 4$ by Poincare duality.

Now we examine the Hodge-de Rham spectral sequence

$$\mathrm{E}_1^{ij} = H^j(X, \Omega_X^i) \implies H^{i+j}(X, \Omega_X^\bullet)\,.$$

What we just proved implies that

$$\dim\mathrm{E}_2^{01} = \dim\mathrm{E}_\infty^{01} = 2 \qquad \begin{matrix} \dim\mathrm{E}_2^{12} = \dim\mathrm{E}_\infty^{12} = 2 \\ \dim\mathrm{E}_2^{10} = \dim\mathrm{E}_\infty^{10} = 2 \end{matrix} \qquad \dim\mathrm{E}_2^{21} = \dim\mathrm{E}_\infty^{21} = 2$$

So the differentials

$$d_1^{ij} : \mathrm{E}_1^{i,j} \longrightarrow \mathrm{E}_1^{i,j+1} \qquad 0 \le i \le 1, 0 \le j \le 2$$

in the E_1-term of the Hodge-de Rham spectral sequence are all equal to 0, because their source or target involves one of E_2^{ij} with $i + j = 1$ or 3. Next we look at the differentials

$$d_2^{ij} : E_2^{ij} \longrightarrow E_2^{i+2,j-1}\,.$$

The only two possibly nontrivial differentials are d_2^{01} and d_2^{02}, and they vanish for the same reason. So the Hodge-de Rham spectral sequence degenerates. The degeneration of the conjugate spectral sequence follows. □

Remark. It is not difficult to deduce from the degeneration of Hodge-de Rham spectral sequence that the Hochschild-Serre spectral sequence

$$\mathrm{E}_2^{ij} = H^i(G, H^j(E_1 \times E_2, \Omega^\bullet_{E_1\times E_2})) \implies H^{i+j}(X, \Omega_X^\bullet)$$

degenerates at E_3. Moreover,

- The maps $d_2^{i,1} \colon \mathrm{E}_2^{i,1} \cong k^4 \longrightarrow \mathrm{E}_2^{i+2,1} \cong k$ are surjective for all $i \ge 0$.
- The maps $d_2^{i,2} \colon \mathrm{E}_2^{i,2} \cong k^6 \longrightarrow \mathrm{E}_2^{i+2,1} \cong k^4$ have rank 3 for all $i \ge 0$.
- The maps $d_2^{i,3} \colon \mathrm{E}_2^{i,3} \cong k^4 \longrightarrow \mathrm{E}_2^{i+2,2} \cong k^6$ have rank 3 for all $i \ge 0$.
- The maps $d_2^{i,4} \colon \mathrm{E}_2^{i,4} \cong k \longrightarrow \mathrm{E}_2^{i+2,2} \cong k^4$ are injective for all $i \ge 0$.

They imply for instance the exactness of

$$0 \longrightarrow \mathrm{E}_2^{i,4} \xrightarrow{d_2} \mathrm{E}_2^{i+2,3} \xrightarrow{d_2} \mathrm{E}_2^{i+4,2} \xrightarrow{d_2} \mathrm{E}_2^{i+6,1} \xrightarrow{d_2} \mathrm{E}_2^{i+8,0} \longrightarrow 0$$

for all $i \ge 0$.

Recall that Hodge-de Rham spectral sequence (resp. the conjugate spectral sequence) induces the *Hodge filtration* $\mathrm{F}_{\mathrm{Hodge}}$ (resp. the *conjugate filtration* $\mathrm{F}_{\mathrm{con}}$) on $H^\bullet(X, \Omega_X^\bullet)$.

(1.6) Proposition *Assume that the elliptic curve E_1 over $k \supset \mathbb{F}_2$ is ordinary.*

(i) *The relative Frobenius morphism $\mathrm{Fr}_X \colon X \to X^{(p)}$ induces an isomorphism*

$$\mathrm{Fr}_X^* \colon H^i(X^{(p)}, \mathcal{O}_{X^{(p)}}) \longrightarrow H^i(X, \mathcal{O}_X)$$

for every $i \ge 0$.

(ii) *We have*

$$\mathrm{F}^1_{\mathrm{Hodge}} H^1(X,\Omega^\bullet_X) \cap \mathrm{F}^1_{\mathrm{con}} H^1(X,\Omega^\bullet_X) = 0$$

and

$$\mathrm{F}^2_{\mathrm{Hodge}} H^2(X,\Omega^\bullet_X) \cap \mathrm{F}^1_{\mathrm{con}} H^2(X,\Omega^\bullet_X) = 0\,.$$

PROOF. Since both E_1 and E_2 are ordinary, the relative Frobenius morphism $\mathrm{Fr}_{A/k} : A \to A^{(p)}$ induces an isomorphism

$$\mathrm{Fr}^*_{A/k} \colon H^i(A^{(p)}, \mathcal{O}_{A^{(p)}}) \xrightarrow{\sim} H^i(A, \mathcal{O}_A)$$

for all $i \geq 0$, where $A = A_1 \times A_2$. We have $\pi^{(p)} \circ \mathrm{Fr}_{A/k} = \mathrm{Fr}_{X/k} \circ \pi$ by the functoriality of the relative Frobenius morphism. This gives a relative Frobenius map

$$\mathrm{Fr}^* \colon \left(\mathrm{E}^{i,j}_{2,\pi^{(p)}} = H^i(G, H^j(A^{(p)}, \mathcal{O}_{A^{(p)}})\right)_{i,j\geq 0} \longrightarrow \left(\mathrm{E}^{i,j}_{2,\pi} = H^i(G, H^j(A, \mathcal{O}_A))\right)_{i,j\geq 0}\,,$$

where $\left(\mathrm{E}^{i,j}_{2,\pi^{(p)}}\right)_{i,j\geq 0}$ is the Hochschild-Serre spectral sequence for $(\pi^{(p)} \colon A^{(p)} \to X^{(p)}, \mathcal{O}_{X^{(p)}})$, and $\left(\mathrm{E}^{i,j}_{2,\pi}\right)_{i,j\geq 0}$ is the Hochschild-Serre spectra sequence for $(\pi \colon A \to X, \mathcal{O}_X)$. Since this relative Frobenius map

$$\mathrm{Fr}^* \colon \mathrm{E}^{i,j}_{2,\pi^{(p)}} \longrightarrow \mathrm{E}^{i,j}_{2,\pi}$$

is an isomorphism for all i, j, the relative Frobenius map

$$\mathrm{Fr}^*_{X/k} \colon H^i(X^{(p)}, \mathcal{O}_{X^{(p)}}) \longrightarrow H^i(X, \mathcal{O}_X)$$

is an isomorphism for every $i \geq 0$ by the 5-lemma. We have proved (i).

It follows from (i) that the relative Frobenius morphism induces a p-linear automorphism on $\mathrm{F}^1_{\mathrm{con}} H^1(X,\Omega^\bullet_X)$ and on $\mathrm{F}^2_{\mathrm{con}} H^2(X,\Omega^\bullet_X)$. On the other hand, the relative Frobenius morphism induces the zero map on $\mathrm{F}^1_{\mathrm{Hodge}} H^1(X,\Omega^\bullet_X)$ and $\mathrm{F}^1_{\mathrm{Hodge}} H^2(X,\Omega^\bullet_X)$. The statement (ii) follows. □

§2. Deformation theory

In this section X is an Igusa surface as in § 1, and *assume that* E_1 *is ordinary.* We assume for simplicity that the base field $k \supset \mathbb{F}_2$ is *algebraically closed.* We explain how to produce a deformation X_1 over an Artinian ring R_1 such that $\mathcal{P}ic^\tau(X_1/R_1)$ is not flat over R_1.

(2.1) Proposition *Let Y be a proper smooth irreducible scheme over k such that the Hodge-de Rham spectral sequence for Y degenerates. Then first Chern class map defines an injection*

$$c_1 \colon \mathcal{P}ic(X)(k) \otimes_{\mathbb{Z}} \mathbb{F}_p \hookrightarrow \mathrm{F}^1_{\mathrm{Hodge}} H^2(X,\Omega^\bullet_X) \cap \mathrm{F}^1_{\mathrm{con}} H^2(X,\Omega^\bullet_X)\,.$$

If the first Chern class $c_1(\mathcal{L})$ of an invertible $\mathcal{O}_X$-module lies in either $\mathrm{F}^2_{\mathrm{Hodge}} H^2(X,\Omega^\bullet_X)$ or in $\mathrm{F}^2_{\mathrm{con}} H^2(X,\Omega^\bullet_X)$, then $c_1(\mathcal{L})$ lies in $\mathrm{F}_{\mathrm{Hodge}} \cap \mathrm{F}^2_{\mathrm{con}}$.

PROOF. See [4, Cor. 1.4]. □

(2.2) Corollary *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $[\mathcal{L}]$ belongs to $\mathcal{P}ic^\tau(X)(k)$ and its image in $\mathcal{P}ic^\tau(X)(k)/\mathcal{P}ic^0(X)(k)$ is non-trivial. Then the image of the first Chern class $c_1(\mathcal{L})$ in $\mathrm{gr}^1_{\mathrm{F}_{\mathrm{Hodge}}} H^2(X,\Omega^\bullet_X) \cong H^1(X,\Omega^1_X)$ is non-zero.*

PROOF. Immediate from 2.1 and 1.6. □

(2.3) Let $s \in \mathcal{P}ic^\tau(X)(k)$ be a point of $\mathcal{P}ic^\tau(X)$ corresponding to an invertible $\mathcal{O}_X$-module $\mathcal{L}_s$. Let $\mathcal{O}_s$ be the local ring of $\mathcal{P}ic^\tau(X)$ at s, let $\mathfrak{m}_s$ be the maximal ideal of $\mathcal{O}_s$, and let $R_0 := \mathcal{O}_s/\mathfrak{m}_s^2$. Let $I_0 = \mathfrak{m}_s/\mathfrak{m}_s^2$ be the maximal ideal of R_0.

Let I_1 be the k-linear dual space of $H^1(X, \Theta_X)$. Let $R_1 := k \oplus I_1$ be the commutative local k-algebra such that $I_1^2 = (0)$. Let $R = R_0 \otimes_k R_1 = k \oplus I_0 \oplus I_1$, an Artinian k-algebra with maximal ideal $I := I_0 \oplus I_1$ such that $I^2 = (0)$. Let $S_0 := \mathrm{Spec}(R_0)$, $S_1 := \mathrm{Spec}(R_1)$, and $S := \mathrm{Spec}(R) = S_2 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R_1)$. Let $i\colon S_0 \hookrightarrow S$ be the closed immersion corresponding to the surjection $R \twoheadrightarrow R_0$ with kernel I_1, and let $\mathrm{pr}\colon S \to S_1$ be the projection morphism corresponding to the inclusion $R_1 \hookrightarrow R$.

Let $X_1 \to S_1$ be the flat deformation of X whose Kodaira-Spencer class $\gamma \in H^1(X, \Theta_X) \otimes_k I_1$ is equal to $\mathrm{id}_{H^1(X,\Theta_X)}$, i.e. $X_1 \to S_1$ is the first order universal equi-characteristic deformation of X. Let $\widetilde{X} \to S$ be the pull-back of $X_1 \to S_1$ by $\mathrm{pr}\colon S \to S_1$. The pull-back of $\widetilde{X} \to S$ by $i\colon S_0 \hookrightarrow S$ is naturally isomorphic to $X \times_{\mathrm{Spec}(k)} S_0$. The inclusion $X \times S_0 \subset \widetilde{X}$ is defined by the ideal I_1 for $S_0 \subset S$.

Let $\mathcal{M}_s$ be an invertible $\mathcal{O}_{X\times S_0}$-module, isomorphic to the restriction to $S_0 \subset \mathcal{P}ic^\tau(X)$ of a universal line bundle on $X \times \mathcal{P}ic^\tau(X)$. In particular the restriction of $\mathcal{M}_s$ to X is isomorphic to $\mathcal{L}$. Consider the problem of lifting the invertible sheaf $\mathcal{M}_s$ on $X \times S_0$ to an invertible sheaf on $\widetilde{X}$. The obstruction class $o(\mathcal{M}_s)$ is an element of $H^2(X, \mathcal{O}_X) \otimes_k I_1$.

(2.4) Theorem *Notation as in 2.3. Recall that E_1 and E_2 are both ordinary elliptic curves over $k \supset \mathbb{F}_2$.*

(i) *The obstruction $o(\mathcal{M}_s)$ for lifting $\mathcal{M}$ to $\widetilde{X}$ is given by*

$$o(\mathcal{M}_s) = \mathrm{gr}^1_{\mathrm{Hodge}}(c_1(\mathcal{L}_s)) \cup \gamma \in H^2(X, \mathcal{O}_X) \otimes_k I_1 \,,$$

where $\mathrm{gr}^1_{\mathrm{Hodge}}(c_1(\mathcal{L}_s)) \in H^1(X, \Omega^1_X)$ is the image of $c_1(\mathcal{L}_s)$ under the projection

$$\mathrm{gr}^1_{\mathrm{Hodge}}\colon \mathrm{F}^1_{\mathrm{Hodge}} \twoheadrightarrow \mathrm{F}^1_{\mathrm{Hodge}}/\mathrm{F}^2_{\mathrm{Hodge}} \cong H^1(X, \Omega^1_X)\,.$$

(ii) *The obstruction class $o(\mathcal{M}_s)$ vanishes if and only if $[\mathcal{L}_s] \in \mathcal{P}ic^0(X)(k)$.*

(iii) *The group scheme $\mathcal{P}ic^\tau(X_1/S_1)$ is not flat over S.*

Proof. The formula for the obstruction $o(\mathcal{M}_s)$ in (i) is standard; see for instance [4, Cor. 1.14]. We know from Cor. 2.2 that $\mathrm{gr}^1_{\mathrm{Hodge}}(c_1(\mathcal{L}_s))$ is a non-zero element in $H^1(X, \Omega^1_X)$ if and only if s is not an element of $\mathcal{P}ic^0(X)(k)$. The cup product pairing

$$H^1(X, \Omega^1_X) \times H^1(X, \Theta_X) \longrightarrow H^2(X, \mathcal{O}_X) \cong k$$

is non-degenerate by 1.2. Since γ corresponds to the identity map on $H^1(X, \Theta_X)$, the obstruction class $o(\mathcal{M}_s)$ does not vanish if $\mathrm{gr}^1_{\mathrm{Hodge}}(c_1(\mathcal{L}_s)) \neq 0$. We have proved (ii)

Suppose that the k-point $s \in \mathcal{P}ic^\tau(X)$ does not belong to $\mathcal{P}ic^0(X)(k)$. Since $H^2(X, \mathcal{O}_X)$ is a one-dimensional vector space over k, there is a unique one-dimensional k-linear subspace $J \subset I_1$ such that $o(\mathcal{M}_s) \subset H^2(X, \mathcal{O}_X) \otimes_k J$. Then $\operatorname{Spec}(S/J)$ is the largest closed subscheme $T \subset S$ such that $\mathcal{M}$ can be extended to an invertible sheaf on $\widetilde{X} \times_S T$. This means that T *is* the tangent space of $\mathcal{P}ic^\tau(X_1/S_1)$ at the point s. In particular the dimension of the tangent space T_s of $\mathcal{P}ic^\tau(X_1/S_1)$ at s is equal to $\dim(I_0) + \dim(I_1) - 1 = 2 + 4 - 1 = 5$; the natural map from T_s to the tangent space T_{S_1} of S_1 is *not* surjective.

On the other hand, suppose that s is point of $\mathcal{P}ic^0(X)$, then the same argument shows that the tangent space T_s of $\mathcal{P}ic^\tau(X_1/S_1)$ at s is 6-dimensional, and the natural map from T_s to the tangent space T_{S_1} of S_1 is surjective.

The group scheme $\mathcal{P}ic^\tau(X_1/S_1)$ is a disjoint union of $\mathcal{P}ic^0(X_1/S_1)$ with another connected component $\mathcal{P}'$. Suppose that $\mathcal{P}ic^\tau(X_1/S_1)$ is flat over S_1. Then $\mathcal{P}'$ is a $\mathcal{P}ic^0(X_1/S_1)$-torsor over S_1. The different behavior of the tangent spaces at these two connected components explained in the previous two paragraphs lead to a contradiction. So $\mathcal{P}ic^\tau(X_1/S_1)$ is *not* flat over S_1. □

Remark. (1) The deformation argument in the proof of 2.4 shows that the structural morphism $\mathcal{P}' \to S_1$ factors through a closed subscheme $T_1 \subset S_1$ defined by a non-zero ideal in R_1. This gives a slightly different proof of 2.4.

(2) One can show that $\mathcal{P}ic^0(X_1/S_1)$ in 2.4 is flat over S_1 using the local criterion of flatness and the surjectivity of $T_s \to T_{S_1}$ for every $s \in \mathcal{P}ic^0(X)(k)$.

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