# Local monodromy of Hilbert modular varieties

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## §1. Introduction and notation

The main theme of this article is the *variation* of the geometric local $p$-adic monodromy representation of modular varieties of PEL-type, where $p$ is a prime number fixed throughout this note. In this article we consider perhaps the simplest case, that of the ordinary locus of the Hilbert modular varieties and their compact versions, over an algebraically closed base field $k$ of characteristic $p$. In this case the targets of both the global and the local $p$-adic monodromy homomorphisms are abelian. To simplify the exposition, we consider only totally real number fields $F$ such that *$p$ is unramified in $F$*, and modular varieties $\mathcal{M}(B)$ attached to totally indefinite quaternion algebras $B$ over such a field $F$ which is split at all places above $p$. When the quaternion algebra $B$ is $\mathrm{M}_2(F)$, the associated variety $\mathcal{M}(B)$ is the Hilbert modular variety $\mathcal{M}(F)$.

For every closed point $x$ in a modular variety $\mathcal{M}(B)$ as above there is a local monodromy homomorphism $\rho_x$ whose target group is the product $\prod_{\wp|p} \mathcal{O}_\wp^\times$ of local units, where $\wp$ runs through all places of $F$ above $p$; see 4.3. There are three results in this article.

1. The first is a *constancy* result on local monodromy: for each place $\wp$ above $p$ in $F$ there is a closed subgroup $H_\wp \subseteq \mathcal{O}_\wp^\times$ with the following property. For each closed point $x$ of $\mathcal{M}(B)$, the image of $\rho_x$ is equal to the product of the groups $H_\wp$ where $\wp$ runs through the set of places of $F$ above $p$ such that $x$ lies in the irreducible component $D_\tau$ of the zero locus of the Hasse invariant attached to a ring homomorphism $\tau\colon \mathcal{O}_\wp \to k$. Moreover this subgroup $H_\wp$ depends only on the local field $F_\wp$ but not on $F$. See 4.9 and 4.10 for the precise statements. In particular, the image of $\rho_x$ is equal to $H_\wp$ for all $x \in W_\wp^0$, where $W_\wp$ denotes the union of the divisors $D_\tau$'s with $\tau$ running through all homomorphisms from $\mathcal{O}_\wp$ to $k$, and $W_\wp^0 := W_\wp - \cup_{\wp' \neq \wp} W_{\wp'}$ is the open subset of the divisor $W_\wp$ consisting of points outside of any other divisor $W_{\wp'}$ attached to a place $\wp' \neq \wp$ above $p$.

2. In 5.1 we show by a global argument that the subgroup $H_\wp$ is open in $\mathcal{O}_\wp^\times$.

3. Finally we show by a local computation that $H_\wp$ is equal to $\mathcal{O}_\wp^\times$; see 6.1 and 6.14.

The main ingredient of the proof of the constancy result is the purity of branch locus. The computation-free proof of 5.1 uses complex uniformization and a result on the abelianization of arithmetic subgroups. Thm. 6.14 generalizes Igusa's theorem in [14] on local monodromy of the modular curve at supersingular points. It also provides a local proof of the irreducibility of the Igusa tower, independent of Ribet's original arithmetic proof in [21] and [8]. and also independent of the method involving Hecke symmetry in [9], [10], [3], [4], [11], [5]. The method of this computational proof can be traced back to [12] and was used in [16] to compute

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the local monodromy of Picard modular varieties. Although the computation in §6 yields a statement stronger than 5.1, the proof of 5.1 offers perhaps a better perspective.

Most of our results can be generalized to the case when $p$ is possibly ramified in $F$. Global geometric information in [22] and [23] will be needed, especially the irreducibility of the zero locus of Hasse invariants associated to places of $F$ above $p$. The local computation in the ramified case is likely to be more complicated than what is done in §6. We hope to address this point in the future.

## Notation.

- $p$ denotes a fixed prime number.
- Let $k \supset \mathbb{F}_p$ be an algebraically closed field.
- $F$ be a totally real number field, $F \neq \mathbb{Q}$ such that *$p$ is unramified in $F$*.
- Denote by $\Sigma_{F,p}$ the set of all prime ideals of $\mathcal{O}_F$ above $p$.
- For each $\wp \in \Sigma_{F,p}$, let $I_\wp$ be the set of all ring homomorphisms $\bar{\iota}\colon \mathcal{O}_\wp \to k$, or equivalently the set of all ring homomorphisms $\iota\colon \mathcal{O}_\wp \to W(k)$.
- Let $I_F$ be the union of all $I_\wp$'s, or equivalently all ring homomorphisms from $\mathcal{O}_F$ to $W(k)$.
- Let $\epsilon\colon I_F \to \Sigma_{F,p}$ be the map which sends elements of $I_\wp$ to $\wp$.
- Let $\mathcal{L} = (L, L^+)$ be an invertible $\mathcal{O}_F$-module with a notion of positivity.
- In § 2–§ 5, $n \geq 3$ denotes a fixed positive integer not divisible by $p$.
- In § 6, $\mathcal{O} = W(\mathbb{F}_q)$, $q = p^n$, and $I := \mathrm{Hom}_{\mathrm{ring}}(W(\mathbb{F}_q), W(k)) \cong \mathrm{Hom}_{\mathrm{ring}}(\mathbb{F}_q, k)$.

# §2. Hilbert modular varieties and their compact siblings

In this section we review some basic facts about Hilbert modular varieties.

**(2.1)** Hilbert modular varieties
Let $\mathcal{M}(F, \mathcal{L}, n)$ be the Hilbert modular variety over $k$ attached to $F$ and $\mathcal{L}$ with level-$n$ structure; see [20],[7]. For any $k$-scheme $S$, the $S$-valued points of $\mathcal{M}(F)$ corresponds to isomorphism classes of quadruples $(A \to S, \alpha, \lambda, \psi)$, where

- $A \to S$ is a abelian scheme of relative dimension $[F : \mathbb{Q}]$,
- $\alpha : \mathcal{O}_F \to \mathrm{End}(A/S)$ is a ring homomorphism, $\lambda\colon L \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_F}(A, A^t)$ is an $\mathcal{O}_F$-linear homomorphism of $\mathcal{O}_F$-modules such that $\lambda$ induces an isomorphism $A \otimes_{\mathcal{O}_F} L \xrightarrow{\sim} A^t$ and positive elements in $L$ corresponds to $\mathcal{O}_F$-linear polarizations, and

- $\psi$ is a symplectic level-$n$ structure.

We will often write $\mathcal{M}(F)$ instead of $\mathcal{M}(F,\mathcal{L},n)$. It is known that $\mathcal{M}(F)$ is irreducible and smooth over $k$ of dimension $[F:\mathbb{Q}]$; see [20],[7].

**(2.2)** We have a stratification of $\mathcal{M}(F)$ by isomorphisms classes of the $(\mathcal{O}_F\otimes\mathbb{F}_p)$-linear polarized $p$-torsion subgroup schemes of the $\mathcal{O}_F$-linear polarized abelian varieties attached to points of $\mathcal{M}(F)$, called the *Ekedahl-Oort* stratification; see [19] and [13]. Each EO stratum is smooth, locally closed and *quasi-affine*; see [13]. In particular only the 0-dimensional EO stratum is closed. The ordinary locus of $\mathcal{M}_F^{\mathrm{ord}}$ is the open dense EO stratum. Those EO strata $D^0_{F,\tau}$ of codimension one are parametrized by the set $I_F$, consisting of all ring homomorphisms $\tau\colon\mathcal{O}_F\to k$. The closure $D_{F,\tau}$ of $D^0_{F,\tau}$ is a smooth divisor. The divisor $\sum_{\tau\in I_F}D_{F,\tau}$ is reduced with normal crossings, equal to the zero locus of the Hasse invariant on $\mathcal{M}(F)$. It is easy to see from the incidence relation among the EO strata that the prime-to-$p$ Hecke correspondences operate transitively on the set of irreducible components of $D^0_\tau$ for each $\tau\in I_F$.

**(2.3)** Consider the $(\mathcal{O}_F\otimes\mathbb{Z}_p)$-linear $p$-divisible group $A[p^\infty]\to\mathcal{M}(F)$ attached to the universal $\mathcal{O}_F$-linear polarized abelian scheme $A\to\mathcal{M}(F)$. The decomposition

$$\mathcal{O}_F\otimes\mathbb{Z}_p=\prod_{\wp\in\Sigma_{F,p}}\mathcal{O}_\wp$$

of the ring $\mathcal{O}_F\otimes\mathbb{Z}_p$ induces a decomposition of the $\left(\prod_{\wp\in\Sigma_{F,p}}\mathcal{O}_\wp\right)$-linear polarized $p$-divisible group $A[p^\infty]\to\mathcal{M}(F)$ into a fiber product

$$A[p^\infty]\to\mathcal{M}(F)=\prod_{\wp\in\Sigma_{F,p}}\left(A[\wp^\infty]\to\mathcal{M}(F)\right)$$

of $\mathcal{O}_\wp$-linear $\wp$-divisible groups over $\mathcal{M}(F)$.

**(2.4)** For any closed point $x$ in $\mathcal{M}(F)$, denote by $\mathrm{Def}(F,x,\wp)$ the universal deformation space of the $\mathcal{O}_\wp$-linear polarized abelian variety $(A_x,\alpha_x)[p^\infty]$, and let $X(F,x,\wp)\longrightarrow\mathrm{Def}(F,x,\wp)$ be the universal $\mathcal{O}_\wp$-linear polarized $p$-divisible group over $\mathrm{Def}(F,x,\wp)$. It is known that every $\mathcal{O}_\wp$-linear symmetric homomorphism from $A_x[\wp^\infty]$ to from $A_x^t[\wp^\infty]$ extends to an $\mathcal{O}_F$-linear symmetric homomorphism from $X(F,x,\wp)$ to $X(F,x,\wp)^t$; see [20] and [7].

**(2.5) Lemma.** *Notation as in 2.3. Let $x$ be a closed point of $\mathcal{M}(F)$, and let $\mathcal{M}(F)^{/x}$ be the formal completion of $\mathcal{M}(F)$ at $x$.*

(i) *The decomposition of the $\left(\prod_\wp\mathcal{O}_\wp\right)$-linear polarized $p$-divisible group $A[p^\infty]\to\mathcal{M}(F)$ in 2.3 induces a product decomposition*

$$\mathcal{M}(F)^{/x}=\prod_{\wp\in\Sigma_{F,p}}\mathrm{Def}(F,x,\wp)$$

*of $\mathcal{M}(F)$ at $x$.*

(ii) *For each* $\wp \in \Sigma_{F,p}$ *the* $\mathcal{O}_\wp$*-linear polarized* $\wp$*-divisible group* $A[\wp^\infty] \to \mathcal{M}(F)^{/x}$ *is the pull-back of* $X(F,x,\wp) \longrightarrow \mathrm{Def}(F,x,\wp)$ *by the projection* $\mathrm{pr}_\wp \colon \mathcal{M}(F)^{/x} \longrightarrow \mathrm{Def}(F,x,\wp)$.

(iii) *The formal scheme* $\mathrm{Def}(F,x,\wp)$ *is formally smooth of dimension* $[F_\wp : \mathbb{Q}_p]$ *for each* $\wp$.

Proof. Statements (i), (ii) are immediate from the Serre-Tate theorem. The statement (iii) can be proved by the crystalline deformation theory for $p$-divisible groups; see [20], [7]. □

# §3. Compact siblings of Hilbert modular varieties

**(3.1)** Let $B$ be a totally indefinite quaternion division algebra over $F$ which is split at all places $\wp$ above $p$. Let $*$ be a positive involution of $B$. Let $\mathcal{O}_B$ be a maximal order in $B$ stable under $*$; we have $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_\wp \cong \mathrm{M}_2(\mathcal{O}_\wp)$ for each $\wp \in \Sigma_{F,p}$.

Let $\mathcal{M}(\mathcal{O}_B, \mathcal{L}, n)$ be the "fake Hilbert modular variety" over $k$ attached to $\mathcal{O}_B$, $\mathcal{L}$ and $n$ as defined in [1]. Points of $\mathcal{M}(\mathcal{O}_B, \mathcal{L}, n)$ are isomorphism classes of quadruples $(A \to S, \alpha, \lambda, \psi)$, where

- $A \to S$ is a abelian scheme of relative dimension $2[F : \mathbb{Q}]$,
- $\alpha : \mathcal{O}_F \to \mathrm{End}(A/S)$ is a ring homomorphism,
- $\psi$ is a symplectic level-$n$ structure, $\lambda \colon L \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_B,*}(A, A^t)$ is an $\mathcal{O}_F$-linear homomorphism of $\mathcal{O}_F$-modules such that $A \otimes_{\mathcal{O}_F} L \xrightarrow{\sim} A^t$ and positive elements in $L$ corresponds to $\mathcal{O}_F$-linear polarizations.

Here $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_B,*}(A, A^t)$ is the etale sheaf over the base scheme $S$ consisting of symmetric $\mathcal{O}_F$-linear homomorphisms $h \colon A \to A^t$ such that $\alpha(b)^t \circ h = h \circ \alpha(b^*)$ for every $b \in \mathcal{O}_B$. We will often write $\mathcal{M}(B)$ instead of $\mathcal{M}(\mathcal{O}_B, \mathcal{L}, n)$. It is known that $\mathcal{M}(B)$ is proper smooth over $k$ and is irreducible, of dimension $[F : \mathbb{Q}]$. Denote by $A(B) \to \mathcal{M}(B)$ the universal $\mathcal{O}_B$-linear abelian scheme over $\mathcal{M}(B)$.

**(3.2)** We will need the fake Hilbert modular schemes in mixed characteristics. If in the above description of the moduli functor, we consider all scheme $S$ over the ring of Witt vectors $W(k)$, then the moduli functor is representable by a scheme $\mathbf{M}(\mathcal{O}_B, \mathcal{L}, n)$ which is proper and smooth over $W(k)$, whose closed fiber is naturally isomorphic to the modular variety $\mathcal{M}(\mathcal{O}_B, \mathcal{L}, n)$ over $k$. Let $\bar\eta$ be the geometric generic point of $\mathrm{Spec}(W(k))$. Then from the theory of complex uniformization of Shimura varieties one sees that the geometric fundamental group $\pi_1(\mathbf{M}(\mathcal{O}_B, \mathcal{L}, n)_{\bar\eta})$ is (non-canonically) isomorphic to the profinite completion of an arithmetic subgroup $\Lambda$ of $B_1$, the group of norm-one elements in $B^\times$.

**(3.3)** Morita equivalence. We know that $\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_\wp \cong \mathrm{M}_2(\mathcal{O}_\wp)$ for every $\wp \in \Sigma_{F,p}$ because $B$ is split above all $\wp \in \Sigma_{F,p}$. It is easy to see that for every closed point $x \in \mathcal{M}_F$, there exists a closed point $y \in \mathcal{M}(B)$ such that the universal polarized $(\mathcal{O}_B \otimes \mathbb{Z}_p)$-linear $p$-divisible group over the formal completion $\mathcal{M}_B^{/y}$ at $y$ is Morita equivalent to the universal polarized $(\mathcal{O}_F \otimes \mathbb{Z}_p)$-linear $p$-divisible group over $\mathcal{M}(F)^{/x}$. Similarly for every closed point $y \in \mathcal{M}(B)$

there exists a closed point $x \in \mathcal{M}_F$ such that the universal polarized $(\mathcal{O}_F \otimes \mathbb{Z}_p)$-linear $p$-divisible group over $\mathcal{M}(F)^{/x}$ is Morita equivalent to the universal polarized $(\mathcal{O}_B \otimes \mathbb{Z}_p)$-linear $p$-divisible group over $\mathcal{M}_B^{/y}$. If $x \in \mathcal{M}(F)$ and $y \in \mathcal{M}(B)$ are Morita equivalent in the above sense, then the deformation theory of $y \in \mathcal{M}(B)$ is isomorphic to the deformation theory of $x \in \mathcal{M}(F)$ as above: there exists an isomorphism between $\mathcal{M}(F)^{/x}$ and $\mathcal{M}(B)^{/y}$ which is compatible with the Morita equivalence between the universal $p$-divisible groups over these two deformation spaces. In particular the Ekedahl-Oort stratification on $\mathcal{M}(B)$ is isomorphic to the Ekedahl-Oort stratification locally in the etale topology. The codimension one EO-strata $D^0_{B,\tau}$ again are parametrized by $I_F$. The closure $D_{B,\tau}$ of $D^0_{B,\tau}$ is smooth for each $\tau$, and $\sum_{\tau \in I_F} D_{B,\tau}$ is a reduced divisor with normal crossings.

**(3.4) Proposition.** *Every non-supersingular EO-stratum in $\mathcal{M}(F)$ and $\mathcal{M}(B)$ is irreducible. In particular the divisors $D_{F,\tau}$ and $D_{B,\tau}$ are irreducible for every $\tau \in I_F$.*

Proof. It follows easily from the incidence relation of the EO-strata on $\mathcal{M}_F$ and $\mathcal{M}_B$ that the prime-to-$p$ Hecke correspondences operate transitively on the set of irreducible components of each EO-stratum. So the non-supersingular EO-strata are irreducible by the proof of [2, Prop. 4.5.4]. □

**(3.5)** The analog of the product decomposition 2.5 holds for the fake Hilbert modular variety $\mathcal{M}(B)$: For every closed point $y \in \mathcal{M}(B)$ there exists a canonical product decomposition

$$\mathcal{M}(B)^{/y} = \prod_{\wp \in \Sigma_{F,p}} \mathrm{Def}(B, y, \wp)\,.$$

This decomposition is compatible with the Morita equivalence explained in 3.3: Suppose that a closed point $x \in \mathcal{M}(F)$ is Morita equivalent to a closed point $y \in \mathcal{M}(B)$, and

$$\mathcal{M}(F)^{/x} = \prod_{\wp \in \Sigma_{F,p}} \mathrm{Def}(F, x, \wp)$$

is the decomposition of $\mathcal{M}(F)^{/x}$. Then $\mathrm{Def}(F, x, \wp)$ is Morita equivalent to $\mathrm{Def}(B, y, \wp)$ for each $\wp$, in the sense that we have an isomorphism induced by an Morita equivalence between the $\mathcal{O}_\wp$-linear $\wp$-divisible group $X(F, x, \wp) \longrightarrow \mathrm{Def}(F, x, \wp)$ and the $(\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_\wp)$-linear $\wp$-divisible group $X(B, y, \wp) \longrightarrow \mathrm{Def}(B, y, \wp)$.

# §4. Local $p$-adic monodromy

**(4.1)** Let $A(F)[p^\infty] \to \mathcal{M}(F)^{\mathrm{ord}}$ be the $(\mathcal{O}_F \otimes \mathbb{Z}_p)$-linear $p$-divisible group attached to the universal $\mathcal{O}_F$-linear abelian scheme over the ordinary locus $\mathcal{M}(F)^{\mathrm{ord}}$ in the Hilbert modular variety $\mathcal{M}(F)$ over $k$. The maximal etale quotient $A(F)[p^\infty]_{\mathrm{et}} \to \mathcal{M}(F)^{\mathrm{ord}}$ corresponds to a homomorphism

$$\rho_F \colon \pi_1(\mathcal{M}(F)^{\mathrm{ord}})_{\mathrm{ab}} \longrightarrow (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times = \prod_{\wp \in \Sigma_{F,p}} \mathcal{O}_\wp^\times\,,$$

where $\pi_1(\mathcal{M}(F)^{\mathrm{ord}})_{\mathrm{ab}}$ denotes the abelianized fundamental group $\pi_1(\mathcal{M}(F)^{\mathrm{ord}})$. The homomorphism $\rho_F$ is called the *global $p$-adic monodromy* of $\mathcal{M}(F)^{\mathrm{ord}}$. A theorem of Ribet says that $\rho_F$ is surjective; see [21], [8].

**(4.2)** Let $B$ be a totally indefinite quaternion division algebra which is split above all places above $p$. Consider the $(\mathcal{O}_B\otimes\mathbb{Z}_p)$-linear $p$-divisible group $A(B)[p^\infty] \to \mathcal{M}(B)^{\rm ord}$ as in § 3 and its maximal etale quotient $A(B)[p^\infty]_{et} \to \mathcal{M}(B)^{\rm ord}$. Since the ring of $(\mathcal{O}_B\otimes\mathbb{Z}_p)$-linear endomorphisms of the generic fiber of this $(\mathcal{O}_B\otimes\mathbb{Z}_p)$-linear $p$-divisible group is $\mathcal{O}_F\otimes\mathbb{Z}_p$, the global $p$-adic monodromy for $\mathcal{M}(B)^{\rm ord}$ is a homomorphism

$$\rho_B\colon \pi_1(\mathcal{M}(B)^{\rm ord})_{\rm ab} \longrightarrow (\mathcal{O}_F\otimes\mathbb{Z}_p)^\times = \prod_{\wp\in\Sigma_{F,p}} \mathcal{O}_\wp^\times\,.$$

The proof of Ribet's theorem applies to the situation of fake Hilbert modular variety as well: $\rho_B$ is surjective.

**(4.3)** For any closed point $x$ of the Hilbert modular variety $\mathcal{M}_F$ over $k$, denote by$Z(F,x)$ the spectrum of the formal completion $\mathcal{O}^\wedge_{x,\mathcal{M}(F)}$ of the local ring at $x$. Let

$$U(F,x) := Z(F,x)\times_{\mathcal{M}(F)}\mathcal{M}(F)^{\rm ord}$$

be the ordinary locus in the formal completion of $\mathcal{M}(F)$ at $x$. The composition

$$\rho_{F,x}\colon \pi_1(U(F,x))_{\rm ab} \longrightarrow \pi_1(\mathcal{M}(F)^{\rm ord})_{\rm ab} \xrightarrow{\rho_F} (\mathcal{O}_F\otimes\mathbb{Z}_p)^\times = \prod_{\wp\in\Sigma_{F,p}} \mathcal{O}_\wp^\times$$

will be called the *local $p$-adic monodromy homomorphism* at $x$. The local monodromy homomorphism

$$\rho_{B,y}\colon \pi_1(U(B,y))_{\rm ab} \longrightarrow \pi_1(\mathcal{M}(B)^{\rm ord})_{\rm ab} \xrightarrow{\rho_B} (\mathcal{O}_F\otimes\mathbb{Z}_p)^\times = \prod_{\wp\in\Sigma_{F,p}} \mathcal{O}_\wp^\times$$

for a closed point $y\in\mathcal{M}(B)$ is defined similarly, where

$$U(B,y) := Z(B,y)\times_{\mathcal{M}(B)}\mathcal{M}(B)^{\rm ord}\,,$$

and $Z(B,y) := \mathrm{Spec}\left(\mathcal{O}^\wedge_{y,\mathcal{M}(B)}\right)$. Note that the formation of the ordinary locus $U(F,x)$ (resp. $U(B,y)$) in $Z(F,x)$ (resp. $Z(B,y)$) is compatible with the product structure in 2.5.

**Remark** (i) Since we work over an algebraically closed base field $k$, the monodromy homomorphisms defined above are the *geometric* global and local monodromy homomorphisms respectively.

(ii) In a different but equivalent setup, the image of the geometric local monodromy homomorphisms are often called the *inertia groups*.

**(4.4) Lemma.** *Notation as above. Let $Z(F,x,\wp)$ (resp. $Z(B,y,\wp)$ be the spectrum of the coordinate ring of the local formal scheme* $\mathrm{Def}(F,x,\wp)$ *(resp.* $\mathrm{Def}(B,y,\wp)$*) so that*

$$Z(F,x) = \prod_{\wp\in\Sigma_{F,p}} Z(F,x,\wp) \quad\text{and}\quad Z(B,y) = \prod_{\wp\in\Sigma_{F,p}} Z(B,y,\wp)\,.$$

*Then there exist subschemes $U(F,x,\wp)\subset Z(F,x,\wp)$ and $U(B,y\wp)\subset Z(B,y,\wp)$ such that*

$$U(F,x) = \prod_{\wp\in\Sigma_{F,p}} U(F,x,\wp) \quad\text{and}\quad U(B,y) = \prod_{\wp\in\Sigma_{F,p}} U(B,y,\wp)\,.$$

Proof. We abuse notation and denote by $X(F,x,\wp) \longrightarrow Z(F,x,\wp)$ the $\mathcal{O}_\wp$-linear polarized $\wp$-divisible group over $Z(F,x,\wp)$ whose $p$-adic completion is the $\mathcal{O}_\wp$-linear polarized $\wp$-divisible group $X(F,x,\wp) \longrightarrow \mathrm{Def}(F,x,\wp)$ over $\mathrm{Def}(F,x,\wp)$. Here we have used the fact, from GFGA, that the category of finite locally free schemes over the formal scheme $\mathrm{Def}(F,x,\wp)$ is isomorphic to the category of finite locally free schemes over $Z(F,x,\wp)$. Because a $p$-divisible is an inductive system of finite locally free commutative group schemes, we can pass to the limit and obtain an isomorphism between the category of $p$-divisible groups over the formal scheme $\mathrm{Def}(F,x,\wp)$ and the category of $p$-divisible groups over $Z(F,x,\wp)$. Similarly denote by $X(B,y,\wp) \longrightarrow Z(B,y,\wp)$ the $(\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_\wp)$-linear polarized $\wp$-divisible group whose $p$-adic completion is $X(B,y,\wp) \longrightarrow \mathrm{Def}(B,y,\wp)$.

Define the open subscheme $U(F,x,\wp) \subset Z(F,x,\wp)$ (resp. $U(B,y,\wp) \subset Z(B,y,\wp)$) as the complement of the zero locus of the Hasse invariant corresponding to $\wp$, or equivalently the ordinary locus for $X(F,x,\wp) \longrightarrow Z(F,x,\wp)$ (resp. $X(B,y,\wp) \longrightarrow Z(B,y,\wp)$). It is easy to check that the required properties are satisfied. □

**(4.5)** The maximal etale quotient $X(F,x,\wp)_{\mathrm{et}} \longrightarrow U(F,x,\wp)$ of $X(F,x,\wp) \longrightarrow Z(F,x,\wp)$ defines a local $p$-adic monodromy homomorphism

$$\rho_{F,x,\wp} \colon \pi_1(U(F,x,\wp))_{\mathrm{ab}} \longrightarrow \mathcal{O}_\wp^\times .$$

Similarly the maximal etale quotient $X(B,y,\wp) \longrightarrow Z(B,y,\wp)$ of $X(B,y,\wp) \longrightarrow Z(B,y,\wp)$ defines a local $p$-adic monodromy homomorphism

$$\rho_{B,y,\wp} \colon \pi_1(U(B,y,\wp))_{\mathrm{ab}} \longrightarrow \mathcal{O}_\wp^\times .$$

Lemma 4.4 implies that the source $\pi_1(U(F,x))_{\mathrm{ab}}$ of $\rho_{F,x}$ is canonically isomorphic to the product $\prod_\wp \pi_1(F,x,\wp)_{\mathrm{ab}}$. Similarly $\pi_1(U(B,y))_{\mathrm{ab}} \cong \prod_\wp \pi_1(B,y,\wp)_{\mathrm{ab}}$

**(4.6) Lemma.** *The local $p$-adic monodromy $\rho_{F,x}$ is equal to the product $\prod_{\wp\in\Sigma_{F,p}} \rho_{F,x,\wp}$. Similarly the local $p$-adic monodromy $\rho_{B,y}$ is equal to $\prod_{\wp\in\Sigma_{F,p}} \rho_{B,y,\wp}$.*

The proof is obvious and is omitted.

**(4.7)** The global $p$-adic representation $\rho_F$ corresponds to a profinite etale $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$-torsor $\mathfrak{T}(F) \longrightarrow \mathcal{M}(F)^{\mathrm{ord}}$ which is irreducible by Ribet's theorem. For any open subgroup $N$ of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$, denote by $\mathfrak{T}(F)/N$ the push-out of $\mathfrak{T}(F)$ by the surjection

$$(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \twoheadrightarrow (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / N .$$

By construction $\mathfrak{T}(F)/N$ is an $((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/N)$-torsor over $\mathcal{M}(F)^{\mathrm{ord}}$. Let $(\mathfrak{T}(F)/N)^{\mathrm{norm}}$ be the normalization of $\mathfrak{T}(F)/N$ with respect to $\mathcal{M}(F)$ in the function field of $\mathfrak{T}(F)/N$.

Similarly, we have a profinite etale $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$-torsor $\mathfrak{T}(B) \longrightarrow \mathcal{M}(B)^{\mathrm{ord}}$ corresponding to $\rho_B$. For every open subgroup $N \subset (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ we have a push-out $((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/N)$-torsor $\mathfrak{T}(B)/N$ over $\mathcal{M}(B)^{\mathrm{ord}}$ Let $(\mathfrak{T}(B)/N)^{\mathrm{norm}}$ be the normalization of $\mathfrak{T}(B)/N$ with respect to $\mathcal{M}(B)$ in the function field of $\mathfrak{T}(B)/N$.

**(4.8) Proposition.** *Let $N$ be an open subgroup of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$. Let $x$ be a closed point of $\mathcal{M}(F)$, and let $y$ be a closed point of $\mathcal{M}(F)$.*

(1) *The finite $\mathcal{M}(F)$-scheme $(\mathfrak{T}(F)/N)^{\mathrm{norm}}$ is unramified above $x$ if and only if $N$ contains the image $\mathrm{Im}(\rho_{F,x})$ of the local monodromy $\rho_{F,x}$. Similarly the finite $\mathcal{M}(F)$-scheme $(\mathfrak{T}(B)/N)^{\mathrm{norm}}$ is unramified above $y$ if and only if $N \supseteq \mathrm{Im}(\rho_{B,y})$.*

(2) *If $(\mathfrak{T}(F)/N)^{\mathrm{norm}}$ is unramified above a point $x_1$ of the divisor $D_{F,\tau}$ for some $\tau \in I_F$, then $(\mathfrak{T}(F)/N)^{\mathrm{norm}}$ is unramified above every point of $D^0_{F,\tau}$. Similarly if $(\mathfrak{T}(B)/N)^{\mathrm{norm}}$ is unramified above a point $y_1$ of the divisor $D_{B,\tau}$ for some $\tau \in I_F$, then $(\mathfrak{T}(B)/N)^{\mathrm{norm}}$ is unramified above every point of $D^0_{B,\tau}$.*

Proof. The statement (1) is immediate from the construction of $(\mathfrak{T}(F)/N)^{\mathrm{norm}}$. The statement (2) is a consequence of the purity of branch locus: The branch locus of $(\mathfrak{T}(F)/N)^{\mathrm{norm}}$ is a union of irreducible components of the complements of the ordinary locus in $\mathcal{M}(F)$ which does not contain $x_1$ by assumption, hence it does not meet $D^0_{F,\tau}$ because $D^0_{F,\tau}$ is irreducible by 3.4. □

**(4.9) Proposition.** *Recall that $\epsilon \colon I_F \to \Sigma_{F,p}$ is the map which sends any element $\iota \colon \mathcal{O}_F \to \mathcal{O}_\wp \to W(k)$ in $I_F$ to $\wp$.*

(1) *The images $\mathrm{Im}(\rho_{F,x_1})$ and $\mathrm{Im}(\rho_{F,x_2})$ are equal if there exists an element $\tau \in I_F$ such that $x_1$ and $x_2$ belong to $D^0_{F,\tau}$. Denote by $G_{F,\tau}$ this common image.*

(2) *Similarly the images $\mathrm{Im}(\rho_{B,y_1})$ and $\mathrm{Im}(\rho_{B,y_2})$ are equal if there exists $\tau \in I_F$ such that $y_1$ and $y_2$ belong to $D^0_{B,\tau}$. Denote by $G_{B,\tau}$ this common image.*

(3) *If $\epsilon(\tau) = \wp$, then $G_{F,\tau} \subset \mathcal{O}_\wp^\times$ and $G_{B,\tau} \subset \mathcal{O}_\wp^\times$.*

(4) *We have $G_{F,\tau} = G_{B,\tau}$ for every $\tau \in I_F$.*

(5) *If $\epsilon(\tau_1) = \epsilon(\tau_2)$, then $G_{\tau_1} = G_{\tau_2}$.*

Proof. The statements (1) and (2) follow from 4.8. The statement (3) is consequence of the definition of the local $p$-adic monodromy and the product structure explained in 4.4. The statement (4) follows from the Morita equivalence between points on $\mathcal{M}(F)$ and $\mathcal{M}(B)$ explained in 3.3.

To prove (5), consider the base change of $\mathcal{M}(F)$ by the Frobenius morphism $\mathrm{Fr} \colon \mathrm{Spec}(k) \to \mathrm{Spec}(k)$, and the $p$-linear morphism $W_F \colon \mathcal{M}(F) \longrightarrow \mathcal{M}(F)$ coming from the $\mathbb{F}_p$-rational structure on $\mathcal{M}(F)$. Then $W_F$ maps $D^0_{F,\tau^{(p)}}$ to $D^0_{F,\tau}$, and the functoriality of the fundamental group implies that $G_{\tau^{(p)}} = G_\tau$. Since the Frobenius operates transitively on the set $\epsilon^{-1}(\wp)$ for each $\wp$, the statement (5) follows. □

**(4.10) Theorem.** *Notation as in 4.9. For any $\wp \in \Sigma_{F,p}$, define $H_\wp := G_\tau$ for any $\tau \in I_F$ such that $\epsilon(\tau) = \wp$; it is well-defined by 4.9 (5).*

(i) *Let $x$ be a closed point of $\mathcal{M}(F)$ not in the ordinary locus. Then $\mathrm{Im}(\rho_{F,x})$ is the subgroup generated by the $H_{\epsilon(\tau)}$'s, where $\tau$ runs through all elements of $I_F$ such that $x \in D_{F,\tau}$.*

(ii) *Let $y$ be a closed point of $\mathcal{M}(B)$ not in the ordinary locus. Then $\mathrm{Im}(\rho_{B,y})$ is the subgroup generated by all $H_{\epsilon(\tau)}$'s, where $\tau$ runs through all elements of $I_F$ such that $y \in D_{B,\tau}$.*

(iii) *The closed subgroup $H_\wp \subseteq \mathcal{O}_\wp^\times$ depends only on the local field $F_\wp$, and is independent of the global field $F$ such that $F_\wp$ occurs as a factor of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$.*

Proof. The statements (i) and (ii) are immediate from 4.8 and 4.9. The statement (iii) follows from the observation that $\rho_{F,x,\wp}$ depends only on the $\mathcal{O}_\wp$-linear polarized $p$-divisible group $A_x[\wp^\infty]$, where $A_x$ is the fiber over $x$ of the universal abelian scheme. □

# §5. Bound from characteristic zero

According to 4.10, the image of the local monodromy is completely determined by the closed subgroups $H_\wp \subset \mathcal{O}_\wp^\times$. In § 6 we will see that $H_\wp$ can be determined by a local computation at a superspecial point, that is a point where all the divisors $D_{F,\tau}$'s (or $D_{B,\tau}$'s) meet. In this section we show by a global argument that $H_\wp$ is a subgroup of finite index in $\mathcal{O}_\wp^\times$ *without* any computation.

**(5.1) Theorem.** *Assume that $[F:\mathbb{Q}] > 1$. Then the closed subgroup $H_\wp$ of $\mathcal{O}_\wp^\times$ is open and of finite index for each place $\wp$ of $F$ above $p$. Equivalently the product $\prod_{\wp \in \Sigma_{F,p}} H_\wp$ is of finite index in $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times = \prod_{\wp\in\Sigma_{F,p}} \mathcal{O}_\wp^\times$.*

Proof. Pick and fix a totally indefinite quaternion division algebra $B$ over $F$ which is split at all places of $F$ above $p$. It suffices to show that there exists a constant $C \geq 1$ such that $[(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times : N] \leq C$ for every open subgroup $N$ of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ which contains $\prod_{\wp\in\Sigma_{F,p}} H_\wp$.

Consider the scheme $V := (\mathfrak{T}(B)/N)^{\mathrm{norm}}$ over $\mathcal{M}(B)$ as in 4.7. We know from 4.8 (1) that $V$ is a finite etale Galois cover over $\mathcal{M}(B)$ whose Galois group is $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/N$. Let $\mathfrak{M}(B)$ be the $p$-adic completion of the moduli scheme $\mathbf{M}(\mathcal{O}_B, \mathcal{L}, n)$ over $W(k)$ in 3.2. Then $V$ extends uniquely to a finite etale Galois cover $\mathfrak{V}$ of $\mathfrak{M}(B)$ because the etale topology is insensitive to nilpotent extensions. The formal scheme $\mathfrak{V}$ is the $p$-adic completion of a scheme $\mathbf{V}$ finite etale over $\mathbf{M}(\mathcal{O}_B, \mathcal{L}, n)$ by GFGA.

By Grothendieck's theorem on specialization of the fundamental group, the Galois group of $V/\mathcal{M}(B)$ is a quotient of the fundamental group of the geometric generic fiber $\mathbf{M}(\mathcal{O}_B, \mathcal{L}, n)_{\bar\eta}$ of $\mathbf{M}(\mathcal{O}_B, \mathcal{L}, n)$. We saw in 3.2 that the latter fundamental group is the profinite completion of an arithmetic subgroup $\Lambda$ of $B^\times$. By 5.3 below, the abelianization $\Lambda_{\mathrm{ab}}$ of $\Lambda$ is a finite group. So $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times/N$ is a quotient of the finite group $\Lambda_{\mathrm{ab}}$, and we obtain $[(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times : N] \leq \mathrm{Card}(\Lambda_{\mathrm{ab}})$ as the required uniform upper bound. □

**(5.2) Remark.** We know that the subgroup $H_\wp \subset \mathcal{O}_\wp^\times$ depends only on the local field $F_\wp$, and every finite product of finite unramified extensions of $\mathbb{Q}_p$ can be realized as $E \otimes \mathbb{Q}_p$ for some totally real number field $E$. So the proof of 5.1 shows that the assertion in 5.1 holds in the case $F = \mathbb{Q}$ as well. Of course when $F = \mathbb{Q}$ a classical theorem of Igusa [14] states that the image of the local monodromy is $\mathbb{Z}_p^\times$, so the statement of 5.1 holds when $F = \mathbb{Q}$ anyway; see [15, Th. 2.3.1, p. 143] for a proof of Igusa's theorem..

**(5.3) Proposition.** *Let $G$ be a semisimple algebraic group over a number field $F$. Let $S$ be a finite set of places of $F$ which contains all archimedean places. Let $\Lambda$ be an $S$-arithmetic subgroup of $G(F)$. Assume that $G$ is $F$-simple and $\sum_{v\in S} \mathrm{rk}_{F_v}(G) \geq 2$. Then the maximal abelian quotient $\Lambda^{\mathrm{ab}} = \Lambda/\Lambda^{\mathrm{der}}$ of $\Lambda$ is finite.*

This is proved in [17], VIII 2.8 on p. 266.

# §6. Local computation

We have seen in 4.10 that to every finite unramified extension field $F_\wp$ of $\mathbb{Q}_p$, there is an associated closed subgroup $H_\wp \subset \mathcal{O}_\wp^\times$ which is the $\wp$-component of the image of the local monodromy homomorphism if $F_\wp$ is a local field of a totally real number field $F$ which is unramified above $p$. According to 5.1 $H_\wp$ is an open subgroup of $\mathcal{O}_\wp^\times$. It is natural to surmise that $H_\wp$ is equal to $\mathcal{O}_\wp^\times$. We will prove this by a direct computation.

**(6.1) Theorem.** *Notation as above. Then the image $H_\wp$ of the local monodromy homomorphism is equal to $\mathcal{O}_\wp^\times$.*

**(6.2)** Let $q = p^n$ be the number of elements of the residue field of $\mathcal{O}_\wp$, and write $\mathcal{O} = \mathcal{O}_\wp = W(\mathbb{F}_q)$. Let $X_0$ be an $\mathcal{O}$-linear $p$-divisible group over $k$ of height $2n$ which is superspecial in the sense that $M(X_0)/(FM + VM)$ is a free $\mathcal{O} \otimes_{\mathbb{Z}_p} k$-module of rank one. Here $M(X_0)$ denotes the Cartier module of $X_0$. It is known that $X_0$ is unique up to $\mathcal{O}$-linear isomorphism. Moreover $X_0$ admits an $\mathcal{O}$-linear principal polarization which is unique up to isomorphism.

Let $\mathrm{Spf}(R(X_0))$ be the equi-characteristic deformation space of $X_0$. Denote by $X$ the universal $\mathcal{O}$-linear $p$-divisible group over $\mathrm{Spf}(R(X_0))$. Since a $p$-divisible group is a limit of finite locally free group schemes, by GFGA the category of $p$-divisible groups over $\mathrm{Spf}(R(X_0))$ is isomorphic to the category of $p$-divisible groups over $\mathrm{Spec}(R(X_0))$. We will abuse notation and write $X \to \mathrm{Spec}(R(X_0))$ for the $\mathcal{O}$-linear polarized $p$-divisible group over $\mathrm{Spec}(R(X_0))$ attached to $X \to \mathrm{Spf}(R(X_0))$. Let $U(X_0) \subset \mathrm{Spec}(R(X_0))$ be the ordinary locus in $\mathrm{Spec}(R(X_0))$, a non-empty dense open subset of $\mathrm{Spec}(R(X_0))$. Let $\mathfrak{X}$ be the formal completion of $X$ along its zero section, a smooth formal group over $R(X_0)$. We know that the base change of $\mathfrak{X}$ to the ordinary locus $U(X_0)$ is a formal torus over $U(X_0)$, a fortiori the generic fiber $\mathfrak{X}_K$ of $\mathfrak{X}$ is a formal torus over $K$

**(6.3)** Let $I := \mathrm{Hom}(\mathcal{O}, W(k)) \cong \mathrm{Hom}(\mathbb{F}_q, k)$ be the set of all ring homomorphisms from $\mathcal{O}$ to $W(k)$. The set $I$ has a natural structure as a torsor over $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) = \mathbb{Z}/n\mathbb{Z}$, a cyclic group generated by the arithmetic Frobenius $\sigma\colon x \mapsto x^p$.

We have a $W(k)$-linear ring isomorphism $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O} \cong \prod_{\iota\in I} W(k)_\iota$, a product of copies of $W(k)$ indexed by $I$, such that the obvious embedding $\mathcal{O} \hookrightarrow W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}$ corresponds to the diagonal embedding $\mathcal{O} \hookrightarrow \prod_{\iota\in I} W(k)_\iota$, where the homomorphism from $\mathcal{O}$ to the component

$W(k)_\iota$ is given by $\iota$. Let $M_0 = M(X_0)$, with actions by $F, V$ and $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}$. The decomposition $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O} \cong \prod_{\iota\in I} W(k)_\iota$ of $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}$ gives us a decomposition $M_0 = \oplus_{\iota\in I} M_{0,\iota}$, where each $M_{0,\iota}$ is a free $W(k)$-module of rank two. Moreover we have $F(M_{0,\iota}) \subseteq M_{0,\iota+1}$ and $V(M_{0,\iota}) \subseteq M_{0,\iota-1}$ for all $\iota \in I$.

**(6.4)** We know from [13] that there exist $W(k)$-bases $e_\iota, f_\iota$ for $M_{0,\iota}$ such that

$$Fe_{\iota-1} = f_\iota = Ve_{\iota+1}, \quad Ff_{\iota-1} = pe_\iota = Ve_{\iota+1} \qquad \forall \iota \in I\,.$$

We also know that the local deformation space of the $\mathcal{O}$-linear polarized $p$-divisible group $X_0$ is formally smooth of dimension $n$, so that $R(X_0)$ is isomorphic to a formal power series ring over $k$ in $n$-variables. The argument in [18] applied to the present situation tells us that we can take $R(X_0)$ to be $k[[t_\iota]]_{\iota\in I}$ so that the Cartier module of the universal $\mathcal{O}$-linear $p$-divisible group to be the left module over the Cartier ring $\mathrm{Cart}_p(k[[t_\iota]])$ with generators $\{e_\iota : \iota \in I\}$, and relations

$$Fe_{\iota-1} - Ve_{\iota+1} - \langle t_\iota\rangle e_\iota = 0 \qquad \forall \iota \in I\,.$$

Here $\mathrm{Cart}_p(k[[t_\iota]])$ is the reduced Cartier ring for $k[[t_\iota]]$, and $\langle a\rangle$ denotes the Teichmüller representative of $a$ in $W(k[[t_\iota]])$ for all $a \in k[[t_\iota]]$. We refer to [25] for Cartier's theory.

**(6.5)** In the rest of this section, we choose and fix an element of $I$ and identify $I$ with $\mathbb{Z}/n\mathbb{Z}$ via the natural $(\mathbb{Z}/n\mathbb{Z})$-torsor structure of $I$. This induces an identification of $R(X_0)$ with $k[[t_0, t_1, \ldots, t_{n-1}]]$. Let $K$ be the fraction field of $R(X_0)$. Denote by $X^*(\mathfrak{X}_K)$ the character group of the $\mathcal{O}$-linear formal torus $\mathfrak{X}_K$ over $K$. We know that $X^*(\mathfrak{X}_K)$ is a free $\mathcal{O}$-module of rank one, with a $\mathcal{O}$-linear action by the Galois group $\mathrm{Gal}_K$ of $K$. So the Galois group $\mathrm{Gal}_K$ operates on $X^*(\mathfrak{X}_K)$ through a homomorphism

$$\rho_{X_0}\colon\ \mathrm{Gal}_K \to \mathcal{O}^\times\,.$$

Clearly $\mathrm{Gal}_K$ operates on $X^*(\mathfrak{X}_K)/p^3X^*(\mathfrak{X}_K)$ through the composition

$$\rho_{X_0}(\mathrm{mod}\ p^3)\colon\ \mathrm{Gal}_K \xrightarrow{\rho_{X_0}} \mathcal{O}^\times \twoheadrightarrow (\mathcal{O}/p^3\mathcal{O})^\times\,.$$

Since the action of $\mathrm{Gal}_K$ on $X^*(\mathfrak{X}_K)$ is isomorphic to the action of $\mathrm{Gal}_K$ on the maximal étale quotient $\mathfrak{X}_{K,\mathrm{et}}$ the $p$-divisible group $X_K$, Thm. 6.1 means that $\rho_{X_0}$ is surjective. Lemma 6.6 shows that the surjectivity of $\rho_{X_0}$ is equivalent to the surjectivity of $\rho_{X_0}(\mathrm{mod} p^3)$, so Thm. 6.1 follows from Thm. 6.7 below.

For any positive integer $a$, denote by $\mathcal{O}_a^\times$ the subgroup $1 + p^a\mathcal{O}$ of $\mathcal{O}^\times$.

**(6.6) Lemma.** *Let $U$ be a closed subgroup of $\mathcal{O}^\times$. Suppose that $\mathcal{O}^\times = U\cdot\mathcal{O}_3^\times$. Then $U = \mathcal{O}^\times$.*

Proof. Let $U_1 = U \cap \mathcal{O}_1^\times$. One knows that $(\mathcal{O}_1^\times)^p$ contains the subgroup $\mathcal{O}_3^\times$. The assumption on $U$ implies that $\mathcal{O}_1^\times = U_1\cdot\mathcal{O}_3^\times$. So $U_1 = \mathcal{O}_1^\times$ by Nakayama's lemma, hence $\mathcal{O}^\times = U\cdot\mathcal{O}_1^\times = U\cdot U_1 = U$. □

**Remark** (i) If $p > 2$, then $(\mathcal{O}_1^\times)^p = \mathcal{O}_2^\times$, and the conclusion of 6.6 under the weaker assumption that $\mathcal{O}^\times = U \cdot \mathcal{O}_2^\times$.

(ii) Analogs of 6.6 for non-commutative $p$-adic groups often hold under *weaker* assumptions. For instance suppose that $V$ is a closed subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$ with $p > 2$, such that (a) the image of $V$ in $\mathrm{GL}_m(\mathbb{F}_p)$ is $\mathrm{GL}_m(\mathbb{F}_p)$ and (b) the image of $V$ in $\mathrm{GL}_m(\mathbb{Z}/p^2\mathbb{Z})$ contains all diagonal matrices in $\mathrm{GL}_m(\mathbb{Z}/p^2\mathbb{Z})$, then $V$ is equal to $\mathrm{GL}_m(\mathbb{Z}_p)$.

**(6.7) Theorem.** *The homomorphism* $\rho_{X_0}(\mathrm{mod}\ p^3)\colon \mathrm{Gal}_K \to (\mathcal{O}/p^3\mathcal{O})^\times$ *is surjective.*

We will prove a more precise version of Thm. 6.7 in Thm. 6.14.

**(6.8)** Consider the commutative smooth formal group $\mathfrak{X}_K$ over $K$ and its base change $\mathfrak{X}_{K^{\mathrm{a}}}$ to the algebraic closure $K^{\mathrm{a}}$ of $K$. Denote by $M$ the Cartier module of $\mathfrak{X}_{K^{\mathrm{a}}}$. As a left module over the Cartier ring $\mathrm{Cart}_p(K^{\mathrm{a}})$, $M$ has generators $e_0, \ldots, e_{n-1}$ and relations

$$F e_{i-1} - V e_{i+1} - \langle t_i \rangle e_i = 0 \qquad \forall\, i \in \mathbb{Z}/n\mathbb{Z}\,. \tag{6.8.1}$$

Let $M_1$ be the Cartier module of the formal torus $\widehat{\mathbb{G}}_m$ over $K^{\mathrm{a}}$. It is a basic fact that $M_1$ is canonically isomorphic to the group of Witt vectors $W(K^{\mathrm{a}})$ with the standard action by $F$ and $V$. One knows from Cartier theory that we have a natural isomorphism

$$X^*(\mathfrak{X}_K)/p^3X^*(\mathfrak{X}_K) \xrightarrow{\ \beta\ } \mathrm{Hom}_{\mathrm{Cart}_p(K^{\mathrm{a}})}\big(M/V^3M, M_1/V^3M_1\big)$$

which is compatible with the action of $\mathrm{Gal}_K$. So the set $X^*(\mathfrak{X}_K)/p^3X^*(\mathfrak{X}_K)$ is in natural bijection with the set of all $n$-tuples $(\xi_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ in the group of truncated Witt vectors $W_3(K^{\mathrm{a}})$ indexed by $I \cong \mathbb{Z}/n\mathbb{Z}$ such that

$$F\,\xi_{i-1} = V\xi_{i+1} + \langle t_i\rangle\,\xi_i \qquad \forall i \in \mathbb{Z}/n\mathbb{Z}\,.$$

**(6.9)** Write $\xi_i = (x_i, y_i, z_i) \in W_3(K^{\mathrm{a}})$ for each $i \in \mathbb{Z}/n\mathbb{Z}$. We have

$$\begin{aligned} F\,\xi_{i-1} &= (x_{i-1}^p,\ y_{i-1}^p,\ z_{i-1}^p) \\ V\,\xi_{i+1} &= (\ 0,\ \ x_{i+1},\ \ y_{i+1}\ ) \\ \langle t_i\rangle\,\xi_i &= (t_i\,x_i,\ t_i^p\,y_i,\ t_i^{p^2}\,z_i) \end{aligned}$$

The condition $F\,\xi_{i-1} = V\xi_{i+1} + \langle t_i\rangle\,\xi_i$ on the $\xi_i$'s translates into the following system of equations in $\underline{x} = (x_0, \ldots, x_{n-1}), \underline{y} = (y_1, \ldots, y_{n-1}), \underline{z} = (z_0, \ldots, z_{n-1})$.

$$x_{i-1}^p = t_i\,x_i \qquad \forall i \in \mathbb{Z}/n\mathbb{Z} \tag{6.9.1}$$

$$y_{i-1}^p = t_i^p\,y_i + x_{i+1} \qquad \forall i \in \mathbb{Z}/n\mathbb{Z} \tag{6.9.2}$$

$$z_{i-1}^p = t_i^{p^2}\,z_i + y_{i+1} + \textstyle\sum_{a=1}^{p-1} C(p,a)\,t_i^{pa}\,x_{i+1}^{p-a}\,y_i^a \qquad \forall i \in \mathbb{Z}/n\mathbb{Z} \tag{6.9.3}$$

where

$$C(p,a) = -\frac{(p-1)!}{a!\,(p-a)!} \equiv \frac{(-1)^a}{a} \pmod p \qquad \text{for } a = 1, \ldots, p-1\,.$$

**(6.10)** We call a solution $(\underline{x}, \underline{y}, \underline{z}) = (x_0, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, z_0, \ldots, z_{n-1})$ of the system of equations (6.9) in $(K^{\mathrm{a}})^{3n}$ *primitive* if all of the $x_i$'s are non-zero (equivalently, one of the $x_i$'s is non-zero). It is easy to see from the considerations in 6.8 and 6.9 that the following assertions hold.

(i) The set of all primitive solutions of (6.9) is in natural bijection with the set of all generators of the $\mathcal{O}/p^3\mathcal{O}$-module $X^*(\mathfrak{X}_K)/p^2X^*(\mathfrak{X}_K)$.

(ii) Let $(\underline{x}, \underline{y}, \underline{z})$ be a primitive solution of (6.9). Then the field

$$K(\underline{x}, \underline{y}, \underline{z}) = K(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1})$$

is a finite abelian extension of $K$ whose Galois group is naturally isomorphic to the image of the homomorphism $\rho_{X_0} (\mathrm{mod}\ p^3)$. Note that the latter group is also the image of $H_\wp$ in $(\mathcal{O}/p^3\mathcal{O})^\times$.

(iii) Notation as in (ii). Then $K(\underline{x})$ (resp. $K(\underline{x}, \underline{y})$ is a finite abelian extension of $K$ whose Galois group is naturally isomorphic to the image of $H_\wp$ in $(\mathcal{O}/p\mathcal{O})^\times$ (resp. in $(\mathcal{O}/p^2\mathcal{O})^\times$).

**(6.11)** We will transform the equations in 6.9 to a system of equations of Kummer and Artin-Schreier type. The equations (6.9.1) reduce to one equation

$$x_{n-1}^{p^n} = t_0^{p^{n-1}}\, t_1^{p^{n-2}} \cdots t_{n-2}^{p}\, t_{n-1}\, x_{n-1}\,. \tag{6.11.1}$$

After choosing a $(p^n-1)$-th root for $t_0, t_1, \ldots, t_{n-1}$, a non-trivial solution of (6.9.1) is given by the following formula

$$x_i = \prod_{a=0}^{n-1} t_{i-a}^{p^a/(p^n-1)} \qquad \forall i \in \mathbb{Z}/n\mathbb{Z}\,. \tag{6.11.2}$$

From the systems of equations (6.9.2), we get

$$\begin{aligned} y_{n-1}^{p^n} = \left(t_0^{p^n}\, t_1^{p^{n-1}}\, t_2^{p^{n-2}} \cdots t_{n-1}^{p}\right) y_{n-1} \;+\; t_0^{p^n}\, t_1^{p^{n-1}} \cdots t_{n-2}^{p^2}\, x_0 \\ +\; t_0^{p^n}\, t_1^{p^{n-1}} \cdots t_{n-3}^{p^3}\, x_{n-1}^{p} \;+\; \cdots \;+\; t_0^{p^n}\, t_1^{p^{n-1}}\, x_3^{p^{n-3}} + t_0^{p^n}\, x_2^{p^{n-2}} + x_1^{p^{n-1}} \end{aligned} \tag{6.11.3}$$

More generally, we have

$$y_i^{p^n} = \left(\prod_{a=1}^{n} t_{i+1-a}^{p^a}\right) \cdot y_i \;+\; \sum_{a=0}^{n-1} \left(\prod_{b=0}^{n-2-a} t_{i+b+1}^{p^{n-b}}\right) \cdot x_{i+1-a}^{p^a} \qquad \forall\, i \in \mathbb{Z}/n\mathbb{Z} \tag{6.11.4}$$

Finally we turn to (6.9.3). Let

$$u_i = y_i + \sum_{a=1} C(p,a)\, t_{i-1}^{pa}\, x_{i1}^{p-a}\, y_{i-1}^{a} \tag{6.11.5}$$

So (6.9.3) becomes

$$z_{i-1}^{p} \;=\; t_i^{p^2}\, z_i \;+\; u_{i+1} \qquad \forall\, i \in \mathbb{Z}/n\mathbb{Z}$$

similar to (6.9.2), and we have

$$z_i^{p^n} = \left(\prod_{a=1}^{n} t_{i+1-a}^{p^{a+1}}\right) \cdot z_i \;+\; \sum_{a=0}^{n-1} \left(\prod_{b=0}^{n-2-a} t_{i+b+1}^{p^{n-b+1}}\right) \cdot u_{i+1-a}^{p^a} \qquad \forall\, i \in \mathbb{Z}/n\mathbb{Z} \tag{6.11.6}$$

**(6.12)** Suppose that $(x_0, \ldots, x_{n-1})$ is a non-zero solution of (6.11.1). We want to show that the Artin-Schreier equations (6.11.4) (resp. (6.11.6)) are irreducible over $K(\underline{x})$ (resp. over $K(\underline{x}, \underline{y})$). We will do so using a suitable valuation on $k[[t_0, \ldots, t_{n-1}]]$.

Let $\Gamma$ be the linearly order abelian group $(\mathbb{Z}^n, <)$, where the linear order is defined as follows. For any two elements $\underline{m} = (m_0, \ldots, m_{n-1})$ and $\underline{m}' = (m'_0, \ldots, m'_{n-1})$ in $\mathbb{Z}^n$, $\underline{m} < \underline{m}'$ if and only if

- either $|m| := m_0 + \cdots + m_{n-1} < m'_0 + \cdots + m'_{n-1} =: |m'|$, or
- $|m| = |m'|$ and there exists an integer $i$ with $0 \le i \le n-1$ such that $m_i < m'_i$ and $m_j = m'_j$ for all $0 \le j < i$.

Denote by $v$ the $\Gamma$-valued valuation on $K$ such that

$$v\left(\sum_{\underline{m}\in\mathbb{N}^n} a_{\underline{m}}\, \underline{t}^{\underline{m}}\right) = \min\{\, \underline{m} \colon a_{\underline{m}} \neq 0\,\}$$

for all non-zero elements $\sum a_{\underline{m}}\, \underline{t}^{\underline{m}}$ in $k[[t_0, \ldots, t_{n-1}]]$. Clearly $|v\,(\sum a_{\underline{m}}\, \underline{t}^{\underline{m}})|$ is the degree of the formal power series $\sum a_{\underline{m}}\, \underline{t}^{\underline{m}}$.

**(6.13)** Let $(\underline{x}, \underline{y}, \underline{z})$ be a primitive solution of (6.9) as before. Choose and fix an extension $w$ of the valuation $v$ to $K(\underline{x}, \underline{y}, \underline{z})$, so that the value group $\Gamma_3$ for $w$ is a subgroup of $\mathbb{Q}^n$ such that $[\Gamma_3 : \mathbb{Z}^n] < \infty$. Write the equation (6.11.3) as

$$y_{n-1}^{p^n} = A_{n-1}\, y_{n-1} + B_{n-1}\,, \tag{6.13.1}$$

where

$$A_{n-1} = t_0^{p^n}\, t_1^{p^{n-1}}\, t_2^{p^{n-2}} \cdots t_{n-1}^{p}$$

and

$$B_{n-1} = t_0^{p^n}\, t_1^{p^{n-1}} \cdots t_{n-2}^{p^2}\, x_0 \;+\; t_0^{p^n}\, t_1^{p^{n-1}} \cdots t_{n-3}^{p^3}\, x_{n-1}^{p} \;+\; \cdots \;+\; t_0^{p^n}\, t_1^{p^{n-1}}\, x_3^{p^{n-3}} + t_0^{p^n}\, x_2^{p^{n-2}} + x_1^{p^{n-1}}\,.$$

It is clear that $w(A_{n-1}) = (p^n, p^{n-1}, \ldots, p)$ and

$$w(B_{n-1}) = w(x_1^{p^{n-1}}) = \frac{p^{n-1}}{(p^n-1)}\left(p, 1, p^{n-1}, p^{n-2}, \ldots, p^2\right)\,,$$

so $|w(A_{n-1})| = \frac{p(p^n-1)}{p-1}$ and $|w(B_{n-1})| = p^{n-1}\,|w(x_1)| = \frac{p^{n-1}}{p-1}$.

A quick look at the equation (6.13.1) for $y_{n-1}$ with the above information on the degrees of the coefficients shows that

$$w(y_{n-1}) = \frac{1}{p^n} w(B_{n-1}) = \frac{1}{p} w(x_1) = \frac{1}{p}\left(p, 1, p^{n-1}, p^{n-2}, \ldots, p^2\right)\,.$$

The same argument shows that

$$w(y_i) = \frac{1}{p} w(x_{i+2}) \qquad \forall\, i \in \mathbb{Z}/n\mathbb{Z}\,,$$

and the coordinates of $w(y_i)$ is obtained from those of $w(y_{n-1})$ by a suitable cyclic permutation.

**(6.14) Theorem.** *Let $(\underline{x}, \underline{y}, \underline{z})$ be a primitive solution of* (6.9)*. Let $w$ be an extension of the valuation $v$ to the abelian extension $K(\underline{x}, \underline{y})$ of $K$, let $w_1$ (resp. $w_2$) be the restriction of $w$ to $K(\underline{x})$ (resp. to $K(\underline{x}, \underline{y})$. Let $\Gamma_3$ (resp. $\Gamma_1$, resp. $\Gamma_2$) be the value group for $w$ (resp. $w_1$, resp. $w_2$).*

(i) $\Gamma_1/\mathbb{Z}^n$ *is a cyclic group of order* $p^n - 1$, *and* $\mathrm{Gal}(K(\underline{x})/K) \cong (\mathcal{O}/p\mathcal{O})^\times$.

(ii) $\Gamma_2/\Gamma_1 \cong \mathbb{Z}^n/p\mathbb{Z}^n$, *and* $\mathrm{Gal}(K(\underline{x}, \underline{y})/K) \cong (\mathcal{O}/p^2\mathcal{O})^\times$.

(iii) $\Gamma_3/\Gamma_1 \cong \mathbb{Z}^n/p^2\mathbb{Z}^n$ *and* $\mathrm{Gal}(K(\underline{x}, \underline{y}, \underline{z})/K) \cong (\mathcal{O}/p^3\mathcal{O})^\times$.

Proof. It is immediate from the explicit formula for the $x_i$'s in 6.11 that $[\Gamma_1 : \mathbb{Z}^n] \geq p^n - 1$. By § 11 Thm. 19 on p. 55 of [24], we have

$$[K(\underline{x}) : K] \geq [\Gamma_1 \colon \Gamma] = p^n - 1 = \mathrm{Card}((\mathcal{O}/p\mathcal{O})^\times)$$

On the other hand the Galois group $\mathrm{Gal}(K(\underline{x})/K)$ is isomorphic to a subgroup of $(\mathcal{O}/p\mathcal{O})^\times$. The statement (i) follows.

**Claim 1.** $\Gamma_2 \otimes \mathbb{Z}_{(p)}$ contains $\frac{1}{p}\mathbb{Z}^n_{(p)}$, where $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal $(p)$.

Because the Galois group $\mathrm{Gal}(K(\underline{x}, \underline{y})/K)$ is isomorphic to a subgroup of $(\mathcal{O}/p^2\mathcal{O})^\times$ and $[K(\underline{x}) : K] = p^n - 1$, we have $[K(\underline{x}, \underline{y}) : K(\underline{x})] \leq p^n$. On the other hand by [24, § 11 Thm. 19] we have $[\Gamma_2 \colon \Gamma_1] \leq [K(\underline{x}, \underline{y}) : K(\underline{x})]$. So the assertion (ii) follows from Claim 1; moreover $\Gamma_2 \otimes \mathbb{Z}_{(p)} = \frac{1}{p}\mathbb{Z}^n_{(p)}$. Similarly, the assertion (iii) is a consequence of Claim 2 below.

**Claim 2.** $\Gamma_3 \otimes \mathbb{Z}_{(p)}$ contains $\frac{1}{p^2}\mathbb{Z}^n_{(p)}$.

Proof of Claim 1. The formula for $w(y_{n-1})$ at the end of 6.13 implies that $\Gamma_2$ contains an element $w(y_{n-1})$ which is congruent to $(0, -1/p, 0, \ldots, 0)$ modulo $\mathbb{Z}^n_{(p)}$. The other elements $w(y_i)$ satisfy similar congruences, hence $\Gamma_2 \otimes \mathbb{Z}_{(p)}$ contains $\frac{1}{p}\mathbb{Z}^n_{(p)}$. We have proved Claim 1.

Proof of Claim 2. To prove Claim 2, look at equations (6.11.6). It is easy to see that the valuation of the constant term of the Artin-Schreier equation for $z_i$ is $p^{n-1}w(y_{i+2}) = p^{n-2}w(x_{i+4})$. The same argument as before shows that

$$w(z_{n-1}) = \frac{1}{p^2}w(x_3) \equiv (0, 0, 0, -1/p^2, 0, \ldots, 0) \pmod{(1/p)\mathbb{Z}^n_{(p)}}\,,$$

and similar congruences hold for the valuation of other $z_i$'s. Claim 2 follows. We have proved 6.14 and 6.7. Theorem 6.1 follows from 6.7 in view of 6.6. □

**(6.15) Remark.** For $m \geq 1$, denote by $K_m$ the fixed field of $\rho^{-1}_{X_0}(\mathcal{O}^\times_m)$, and let $\Gamma_m$ be the value group of $K_m$. It is not difficulty to show by induction on $m$ using the argument 6.14 that $\Gamma_m = p^{1-m} \cdot \Gamma_1$ for all $m \geq 1$.

**(6.16) Remark.** (1) The proof of 6.14 is a generalization of the proof of Igusa's theorem, Thm. 2.3.1 on page 143 of [15], using Cartier's theory instead of formal group laws. The discrete valuation in [15, Thm. 2.3.1] is replaced with a rank-$n$ valuation on $k[[t_0,\dots,t_{n-1}]]$.

(2) It is possible to prove 6.14 using formal group laws as in [15]. To write down the group law for $\mathfrak{X}$, it is convenient to consider the smooth formal group $\mathbf{X}$ over $W(k)[[t_0,\dots,t_{n-1}]]$ whose Cartier module is given by (6.8.1). One can show that the logarithm $\underline{f}(\underline{u})$ of $\mathbf{X}$, a vector of formal power series in $n$ variables $u_0,\dots,u_{n-1}$ with coefficients in $W(k[[t_0,\dots,t_{n-1}]])$, satisfies the following functional equation

$$\underline{f}(\underline{u}) = \underline{u} + p^{-1}\,T\cdot \underline{f}^{(p)}\left(\underline{u}^{(p)}\right) + p^{-1}\,\underline{f}^{(p^2)}\left(\underline{u}^{(p^2)}\right)\,,$$

where

$$\underline{u} = \begin{bmatrix} u_0 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix}, \quad \underline{u}^{(p^a)} = \begin{bmatrix} u_0^{(p^a)} \\ u_2^{(p^a)} \\ \vdots \\ u_{n-1}^{(p^a)} \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & 0 & \cdots & \langle t_{n-1}\rangle \\ \langle t_0\rangle & 0 & 0 & \ddots & 0 \\ 0 & \langle t_1\rangle & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \langle t_{n-2}\rangle & 0 \end{bmatrix},$$

and $\underline{f}^{(p^a)}(\underline{u})$ is obtained from $\underline{f}(\underline{u})$ by changing its coefficients under the ring homomorphism from $W(k[[t_0,\dots,t_{n-1}]])$ to itself which is induced by the continuous $k$-algebra homomorphism

$$k[[t_0,\dots,t_{n-1}]] \to k[[t_0,\dots,t_{n-1}]]\,, \qquad t_i \mapsto t_i^{p^a}\,.$$

for $a = 1, 2$. This functional equation for the logarithm of $\mathbf{X}$ allows one to compute the logarithm $\underline{f}(\underline{u})$ and the multiplication-by-$p$ map $[p]_{\mathfrak{X}}(\underline{u})$ for $\mathfrak{X}$. From $[p]_{\mathfrak{X}}(\underline{u})$ we can write down a system of equations for the coordinates of the $p^3$-torsion points in $\mathfrak{X}_K$ as in [15, Thm. 2.3.1], and analyze the system of equations with the $\Gamma$-valued valuation used in 6.14.

(3) In theory it should be possible to prove 6.14 by looking at the Galois action on the cocharacter group $X_*(\mathfrak{X}_K)/V^m X_*(\mathfrak{X}_K)$. The latter group is in natural bijection with the set of all elements $\xi \in M/V^3M$ such that $F\xi = \xi$. If we write $\xi = \sum_{i\in\mathbb{Z}/n\mathbb{Z}, a=0,1,2} V^a\langle x_{a,i}\rangle e_i$ with $x_{a,i} \in K^{\mathrm{a}}$, then the equation $F\xi = \xi$ becomes a system of polynomial equations in the variables $x_{a,i}$ over $K$. Curiously, although this system equations for $x_{0,i}$'s and $x_{1,i}$'s can be analyzed by the same method as in 6.14, it becomes more difficult for the $x_{2,i}$'s. We can show that $[K(x_{0,i}, x_{1,i}) : K] = p^n(p^n - 1)$ for any primitive solution $(x_{a,i})$, which implies Thm. 6.1 when $p > 2$. However we have not been able to handle the case $p = 2$ with this alternative approach.

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