Abelian varieties isogenous to a Jacobian

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ABSTRACT. We define a notion of *Weyl CM points* in the moduli space $\mathscr{A}_{g,1}$ of g-dimensional principally polarized abelian varieties and show that the André-Oort conjecture (or the GRH) implies the following statement: for any closed subvariety $X \subsetneq \mathscr{A}_{g,1}$ over $\mathbb{Q}^{\rm a}$ there exists a Weyl special point $[(B,\mu)] \in \mathscr{A}_{g,1}(\mathbb{Q}^{\rm a})$ such that B is *not* isogenous to the abelian variety A underlying any point $[(A,\lambda)] \in X$. The title refers to the case when $g \geq 4$ and X is the Torelli locus; in this case Tsimerman [32] has proved the statement unconditionally. The notion of Weyl special points is generalized to the context of Shimura varieties, and we prove a corresponding conditional statement with the ambient space $\mathscr{A}_{g,1}$ replaced by a general Shimura variety.[1]

§1. Introduction

This article was motivated by the following folklore question.[2]

(1.1) QUESTION. *Does there exist an abelian variety A over the field $\mathbb{Q}^{\rm a}$ of all algebraic numbers which is not isogenous to the Jacobian of a stable algebraic curve over $\mathbb{Q}^{\rm a}$?*

The above question deals with the closed Torelli locus $\mathscr{T}_g$ in the moduli space $\mathscr{A}_{g,1}$ of g-dimensional principally polarized abelian varieties. For $1 \leq g \leq 3$ we have $\mathscr{T}_g = \mathscr{A}_{g,1}$ and we see that every abelian variety of that dimension is even isomorphic to a Jacobian. However for $g \geq 4$ the answer to Question 1.1 is expected to be affirmative.

It turns out that one gains a better perspective by asking the same question for every closed subset $X \subsetneq \mathscr{A}_{g,1}$, which also makes the question somewhat easier. We are grateful to Bjorn Poonen for this suggestion.

We recall the definition of isogeny orbits and Hecke orbits before formulating the expected answer to the more general question. Let k be an algebraically closed field. For any point $x = [(A,\lambda)]$ in $\mathscr{A}_{g,1}(k)$ corresponding to a g-dimensional abelian variety A with a principal polarization λ over k, denote by $\mathscr{I}(x)$ (respectively by $\mathscr{H}(x)$) the *isogeny orbit* (respectively the *Hecke orbit*) of x in

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[1]Subject classification (MSC2010) Primary 14G35; Secondary 11G15, 14K22. Keywords: André-Oort conjecture, CM fields, GRH, Hecke orbit, Hilbert irreducibility, Hilbert modular variety, Jacobian, root system, Shimura (sub)variety, weak approximation, Weyl CM points, Weyl group, Weyl special points, Weyl subtorus.

[2]We thought that this question was first raised by Nick Katz, but Katz believes that Frans Oort first mentioned it.

$\mathscr{A}_{g,1}(k)$, consisting of all points $y = [(B,\nu)] \in \mathscr{A}_{g,1}(k)$ such that B is isogenous to A (resp. there exists a quasi-isogeny $\alpha\colon A \to B$ such that the pull-back $\alpha^*(\nu)$ of the principal polarization ν on B is equal to the principal polarization μ on A).

(1.2) Below are the expected answers to two versions of the generalization of the question 1.1, both depending on a chosen integer $g \in \mathbb{Z}_{\geq 1}$. The stronger version $\mathrm{sI}(k,g)$ specializes to the previous question when the closed subset X in $\mathscr{A}_{g,1}$ is the closed Torelli locus $\mathscr{T}_g$. The weaker version $\mathrm{I}(k,g)$ can be further extended to the context of Shimura varieties; see 5.2.

$\mathrm{I}(k,g)$ $\quad$ *$\forall$ closed subset $X \subsetneqq \mathscr{A}_{g,1}$ over k, $\exists$ a point $x = [(A,\lambda)] \in \mathscr{A}_{g,1}(k)$ such that $\mathscr{H}(x) \cap X = \emptyset$.*

$\mathrm{sI}(k,g)$ $\quad$ *$\forall$ closed subset $X \subsetneqq \mathscr{A}_{g,1}$ over k, $\exists$ a point $x = [(A,\lambda)] \in \mathscr{A}_{g,1}(k)$ such that $\mathscr{I}(x) \cap X = \emptyset$.*

The case $g = 1$ is easy: the statements $\mathrm{sI}(k,1)$ and $\mathrm{I}(k,1)$ hold for any algebraically closed field k because $\dim(\mathscr{A}_{g,1}) = 1$.

The case $k = \mathbb{C}$ is not hard either: $\mathrm{sI}(\mathbb{C},g)$ is true for all $g \geq 1$; see 3.11. More challenging are the cases when $k = \mathbb{Q}^{\mathrm{a}}$ or $\mathbb{F}$, where $\mathbb{F}$ denotes the algebraic closure of a finite prime field $\mathbb{F}_p$. We don't have much to say when $k = \mathbb{F}$, other than a very special case in §4.

(1.3) In the case when k is the field $\mathbb{Q}^{\mathrm{a}}$ of all algebraic numbers, we will prove that the property $\mathrm{sI}(\mathbb{Q}^{\mathrm{a}},g)$ follows from the André-Oort conjecture (AO); see 2.6 for the statement of the conjecture (AO). As this conjecture has been proved (conditionally, depending on the Generalized Riemann Hypothesis),

> $\mathrm{sI}(\mathbb{Q}^{\mathrm{a}},g)$ *and* $\mathrm{I}(\mathbb{Q}^{\mathrm{a}},g)$ *hold under* GRH *for all $g \geq 1$*;

see 3.1. In particular, granting GRH, there exists an abelian variety of any given dimension $g \geq 4$ over a number field which over $\mathbb{Q}^{\mathrm{a}}$ is not isogenous to a Jacobian.

We note that this result is true unconditionally, i.e. without assuming GRH, as was proved by Tsimerman, using and extending results of this paper in his Princeton PhD-thesis [32].

(1.4) Here is the idea of the proof of "(AO) $\Longrightarrow$ $\mathrm{I}(\mathbb{Q}^{\mathrm{a}},g)$". The André-Oort conjecture reduces the proof of $\mathrm{sI}(\mathbb{Q}^{\mathrm{a}},g)$ and $\mathrm{I}(\mathbb{Q}^{\mathrm{a}},g)$ to the following statement about *a special subset* in $\mathscr{A}_{g,1}$, i.e. a finite union of Shimura subvarieties.

> *For any* special subset *$Y \subsetneqq \mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$, there exists a CM-point $y \in \mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$ such that the isogeny orbit $\mathscr{I}(y)$ of y and the special subset Y are disjoint* ;

see Proposition 3.2. We find it convenient to take y to be a "sufficiently general CM point"[3], in the sense that the abelian variety A_y corresponding to y has the property that $\mathrm{End}^0(A_y)$ is a CM field L of degree $2g$ over $\mathbb{Q}$ such that the Galois group of the normal closure of $L/\mathbb{Q}$ is maximal, i.e. isomorphic to $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$. Such points y are called *Weyl CM points*[4] and they are abundant in $\mathscr{A}_{g,1}$; see 2.10 for the definition and 2.13 for their abundance. Weyl CM points enters the picture because of the following observation in Lemma 3.5:

[3] We used the adjective "sufficiently general" instead of "generic" because "generic point" has a specific technical meaning. Under any "reasonable" enumeration scheme for CM points of $\mathscr{A}_{g,1}$, the subset of Weyl CM points is expected to have density one for any "reasonable" definition of density.

[4] or *Weyl special points* because CM points are 0-dimensional special subsets.

An irreducible positive dimensional Shimura subvariety $S \subsetneqq \mathscr{A}_{g,1}$ which contains a Weyl CM point y as above is necessarily a Hilbert modular variety attached to the maximal totally real subfield of L.

The reason is that the root system $R(G)$ of the semi-simple group G attached to S is stable under the action of the Weyl group of Sp_{2g}; this property easily implies that $R(G)$ is the subset of all long roots in the root system for Sp_{2g}. Thus for any given special subset $Y \subsetneqq \mathscr{A}_{g,1}$, we only need to look at those irreducible components which are zero dimensional or are Hilbert modular varieties attached to totally real subfields of degree g. Pick any Weyl special point y with associate Weyl CM field L, such that L is not attached to any zero-dimensional irreducible component of Y and L does not contain any the totally real subfield associated to any of the Hilbert modular variety components of Y. Lemma 3.5 guarantees that $\mathscr{I}(y)$ is disjoint with Y.

The same argument also proves the following finiteness statement:

Assume (AO).
For any $g \geq 4$, there are only a finite number of Weyl CM Jacobians of dimension g;

see 3.7. This seems inaccessible by present technology without assuming (AO) or GRH.

(1.5) The notion of Weyl special points generalizes to the context of Shimura varieties. They are special points for which the Galois $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ action on the character group of the associated maximal torus contains the Weyl group of the reductive $\mathbb{Q}$-group for the Shimura variety; see 5.3 and 5.4. A maximal $\mathbb{Q}$-subtorus with the above property is said to be a Weyl subtorus.[5] As in the Siegel case, Weyl special points are abundant in any positive dimensional Shimura variety; see 5.11.

The main result 5.5 in this article in the context of Shimura varieties asserts the following.

For any special subset $Y \subsetneqq S$ in a Shimura variety S, there exists a Weyl special point y in S such that the Hecke orbit of y is disjoint from Y.

Lemma 6.7 provides the key property of Weyl special points in the Shimura variety situation. Below is a shorter version.

Let G be a connected and simply connected almost $\mathbb{Q}$-simple semisimple algebraic group over $\mathbb{Q}$ not of type G_2 or F_4, and $G = \mathrm{Res}_{F/\mathbb{Q}}(\tilde{G})$ for an absolutely almost simple semisimple algebraic group $\tilde{G}$ over a number field F. Suppose that H is a connected reductive $\mathbb{Q}$-subgroup of G which contains a Weyl subtorus T and $T \subsetneqq H \subsetneqq G$. Then

- *either $\tilde{G}$ is of type C_n and H has the form $H \cong \mathrm{Res}_{K/F}(\tilde{H})$ for an extension field K/F with $[K:F] = n$ and an absolutely simple semisimple algebraic group $\tilde{H}$ over K of type A_1, or*
- *there exists an integer $n \geq 3$ such that $\tilde{G}$ is of type B_n and $H = \mathrm{Res}_{F/\mathbb{Q}}(\tilde{H})$ for a semisimple subgroup $\tilde{H} \subset \tilde{G}$ over F of type D_n.*

[5] A gentle warning on the technical side: a product of Weyl special points in a product Shimura variety is not necessarily a Weyl special point; see 5.7 (b). One ramification of this phenomenon is that the proof of 5.5 cannot be directly reduced to the case when the reductive group in the Shimura input datum is almost $\mathbb{Q}$-simple

We indicate the idea of the proof of 5.5 when the reductive group G in the Shimura input datum is almost $\mathbb{Q}$-simple; a more detailed sketch is in 5.6. There is nothing to prove unless G is of type B_n or C_n. When G is of type C_n the proof is similar to the proof of 3.2, using totally real fields as obstructions for the Hecke orbit of a Weyl special point to meet a given special subset Y. When G is of type B_n we use the discriminant of quadratic forms as the source of obstruction; see 6.12.

(1.6) This article is organized as follows. In §2 we explain the notion of Weyl CM points in $\mathscr{A}_{g,1}$. A convenient version of Hilbert irreducibility with weak approximation, which guarantees an abundant supply of Weyl CM points in Shimura varieties is discussed in 2.14. The statement $\mathrm{sI}(g,\mathbb{Q}^{\mathrm{a}})$, after being reduced to 3.2 modulo (AO) or GRH is proved in §3. In §5 the notion of Weyl CM points and the analogue of $\mathrm{I}(g,\mathbb{Q}^{\mathrm{a}})$ are generalized to the context of Shimura varieties. See 5.1 for the more general version of $\mathrm{I}(g,\mathbb{Q}^{\mathrm{a}})$, 5.3 and 5.4 for the definition of Weyl tori and Weyl CM points in Shimura varieties, which are made explicit for classical groups in 5.15 – 5.18. The proof of Thm. 5.5, which is a generalization of 3.2 and arguably the main result of this article in a technical sense, is carried out in 6.13. As the proof of 5.5 is long, we provide in 5.6 an outline of the proof, including features not seen in the Siegel case. In §4 are some comments on the case when $k=\mathbb{F}$. A few questions about Weyl CM points are gathered in 3.13. Experts on Shimura varieties are urged to skip §§2–§4 and go directly to §5; others are advised to read only 5.1–5.6 and skip the rest of §5 and §6.

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§2. Definitions and preliminaries

(2.1) Let k be an algebraically closed field. Let $\mathscr{A}_{g,1}$ be the moduli space of g-dimensional principally polarized abelian varieties over k.

By a curve over k *of compact type*[6], we mean a complete stable curve C over k such that its (generalized) Jacobian variety is an abelian variety; in other words every irreducible component of C is smooth and the graph attached to C is a tree. Attached to every curve C of genus g of compact type over k is a principally polarized abelian variety $(\mathrm{Jac}(C),\lambda_C)$ over k, defined to be the product of the Jacobians of the irreducible components of C, with the product polarization.

We will use the term "Jacobian" to indicate the abelian variety underlying the principally polarized abelian variety $(\mathrm{Jac}(C),\lambda_C)$, and "canonically polarized Jacobian" when we need to consider the principal polarization of a Jacobian.

Consider the k-morphism

$$j\colon \mathscr{M}_g \to \mathscr{A}_{g,1}\,,$$

called the Torelli morphism, which associates to a curve its principally polarized Jacobian. Denote

[6]The adjective "compact" refers to the generalized Jacobian of the curve.

by $\mathscr{T}_g^0$ the image

$$j(\mathscr{M}_g) =: \mathscr{T}_g^0 \subset \mathscr{A}_{g,1},$$

of the Torelli morphism, called the *open Torelli locus*; this subset is locally closed in $\mathscr{A}_{g,1}$. Its closure in $\mathscr{A}_{g,1}$ is denoted by $\mathscr{T}_g \subset \mathscr{A}_{g,1}$, called the (closed) Torelli locus. A geometric point in the Torelli locus corresponds to a principally polarized abelian variety (A,λ) such that there exists a curve C of compact type whose canonically polarized Jacobian is (A,λ).

If $g \leq 3$ we have $\mathscr{T}_g = \mathscr{A}_{g,1}$ because $\dim(\mathscr{T}_g) = \dim(A_{g,1})$ for all $g \leq 3$, while $\mathscr{T}_g \subsetneqq \mathscr{A}_{g,1}$ if $g \geq 4$ because $\dim(\mathscr{T}_g) = 3g-3 < g(g+1)/2 = \dim(\mathscr{A}_{g,1})$ if $g \geq 4$.

(2.2) For any point $x = [(A,\lambda)] \in \mathscr{A}_{g,1}(k)$ we consider the symplectic Hecke orbit $\mathscr{H}(x)$, defined to be the set of all points $y = [(B,\mu)] \in \mathscr{A}_{g,1}$ such that there exists a quasi-isogeny $\alpha\colon B \to A$ which preserves the polarization.[7] Define the "isogeny orbit" $\mathscr{I}(x)$ as the set of all points $[(B,\mu)] = y \in \mathscr{A}_{g,1}(k)$ such that there exists an isogeny between A and B (without taking into account the polarizations). It is clear that $\mathscr{H}(x) \subset \mathscr{I}(x)$.

(2.3) Here is an example in which $\mathscr{H}^{\mathrm{GSp}}(x) \subsetneqq \mathscr{I}(x)$. Let L be a totally imaginary quadratic extension of a totally real number field F satisfying the following properties.

(a) There exists a CM type Φ for the CM field L which is *not* induced from a CM type (L',Φ') for any proper CM subfield $L' \subset L$.

(b) $(F\otimes\mathbb{R})_{>0} \cap (\mathrm{Nm}_{L/F}(L^\times)\cdot\mathbb{Q}^\times) \subsetneq (F\otimes\mathbb{R})_{>0} \cap F^\times$, where $(F\otimes\mathbb{R})_{>0}$ denotes the set of all totally positive elements in $F\otimes\mathbb{R}$.

Note that there exists a CM field L satisfying properties (a) and (b). Moreover the proof of 2.15 shows that for every finite quadratic extension field L_w of a finite extension F_v of $\mathbb{Q}_p$ satisfying $\mathrm{Nm}_{L_w/F_v}(L_w^\times)\cdot{\mathbb{Q}_p}^\times \subsetneqq F_v^\times$, there exists a totally imaginary quadratic extension L of a totally real field F such that $(L/F/\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{Q}_p \cong L_w/F_v/\mathbb{Q}_p$ and satisfies properties (a) and (b) above.

Start with a CM type (L,Φ) satisfying (a) and (b) above. From complex uniformization of abelian varieties, there exists a principally polarized abelian variety (A_1,λ_1) over $\mathbb{C}$ with action by (an order of) L such that $2\dim(A_1) = [L:\mathbb{Q}]$ and the CM type of (A_1,L) is (L,Φ). Condition (a) implies that $\mathrm{End}^0(A_1) = L$. It is well-known that (A_1,λ_1) is defined over the algebraic closure $\mathbb{Q}^{\mathrm{a}}$ of $\mathbb{Q}$ in $\mathbb{C}$. In terms of the complex uniformization, after fixing an L-linear isomorphism between $\mathrm{H}_1(A_1(\mathbb{C}),\mathbb{Q})$ with L, the Riemann form on $\mathrm{H}_1(A_1(\mathbb{C}),\mathbb{Q})$ corresponding to the principal polarization λ_1 has the form $(u,v) \mapsto \mathrm{Tr}_{L/\mathbb{Q}}(u\kappa\bar{v})$ for a suitable element $\kappa \in K^\times$ such that $-\kappa^2$ is totally positive. Condition (b) assures us that there is a totally positive element $\alpha \in F^\times$ such that $\alpha \notin \mathrm{Nm}_{L/F}(L^\times)\cdot\mathbb{Q}^\times$. Adjusting the Riemann form $(u,v) \mapsto \mathrm{Tr}_{L/\mathbb{Q}}(u\kappa\alpha\bar{v})$ by a suitable positive integer if necessary, we get a polarization λ_2 on A_1. Changing A_1 by a suitable isogeny, we get an L-linear principally polarized abelian variety (A_2,λ_2) over $\mathbb{Q}^{\mathrm{a}}$ and an L-linear isogeny $\beta\colon A_1 \to A_2$ over $\mathbb{Q}^{\mathrm{a}}$ such that $\mathrm{Hom}(A_1,A_2)\otimes\mathbb{Q} = L\cdot\beta$. Consider the two points $x_1 = [(A_1,\lambda_1)]$, $x_2 = [(A_1,\lambda_2)]$ in $\mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$. Clearly $x_2 \in \mathscr{I}(x_1)$. The condition (b) implies that $x_2 \notin \mathscr{H}^{\mathrm{GSp}}(x_1)$.

[7] If we use the group GSp_{2g} of all symplectic similitudes in $2g$ variable instead of Sp_{2g}, we will get a slightly bigger Hecke orbit $\mathscr{H}^{\mathrm{GSp}}(x)$; see 1.7 and 1.9 of [4]. We won't use it because the isogeny orbit $\mathscr{I}(x)$ is bigger.

(2.4) An abelian variety A over a field K is said to have *sufficiently many complex multiplication* (smCM for short) if $\mathrm{End}^0(A)$ contains a commutative semi-simple subalgebra L with $[L\colon \mathbb{Q}] = 2\dim(A)$. A point $x_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,1}(\mathbb{C})$ is said to be a *CM point* (or a *special point*) if the underlying abelian variety A_0 has smCM. Every CM point in $\mathscr{A}_{g,1}(\mathbb{C})$ is rational over $\mathbb{Q}^{\mathrm{a}}$.

Over $\mathbb{C}$, an equivalent condition for an abelian variety A over $\mathbb{C}$ to have smCM is that the *Mumford-Tate* group of (the Hodge structure attached to the first Betti homology group of) A is an algebraic torus over $\mathbb{Q}$. Recall that the Mumford-Tate group of A is the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(\mathrm{H}_1(A(\mathbb{C}),\mathbb{Q}))$ which contains the image of the $\mathbb{R}$-homomorphism $\rho_A\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow \mathrm{GL}(\mathrm{H}_1(A(\mathbb{C}),\mathbb{Q})_{\mathbb{R}})$ attached to the Hodge structure of $\mathrm{H}_1(A(\mathbb{C}),\mathbb{Q})$.

(2.5) We refer to [8] and [9] for basic properties of Shimura varieties. For us a Shimura variety is an algebraic variety of the form ${}_K\mathscr{M}_{\mathbb{C}}(G,X)$ in the notation of [9, 2.1.1], or one of its irreducible components; it has a natural structure as an algebraic variety over $\mathbb{Q}^{\mathrm{a}}$. Here (G,X) is a Shimura input datum as in [9, 2.1.1], and K is a compact open subgroup of $G(\mathbb{A}_f)$. A *special point* (or a CM point)[8] of a Shimura variety ${}_K\mathscr{M}_{\mathbb{C}}(G,X)$ is the image of a point $(x_\infty, g_f) \in X \times G(\mathbb{A}_f)$ where $x_\infty\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow G_{\mathbb{R}}$ is a point of X whose Mumford-Tate group is a torus over $\mathbb{Q}$.

A *special subset* of a Shimura variety $S = {}_K\mathscr{M}_{\mathbb{C}}(G,X)$ is a finite union of subvarieties S_j, where each S_j is an irreducible component of a "Hecke translate" by an element of $G(\mathbb{A}_f)$ of the image of a Shimura variety ${}_{K_j}\mathscr{M}_{\mathbb{C}}(G_j,X_j)$ under a morphism $h_j\colon {}_{K_j}\mathscr{M}_{\mathbb{C}}(G_j,X_j) \longrightarrow {}_K\mathscr{M}_{\mathbb{C}}(G,X)$ induced by a morphism of Shimura input data $(G_j,X_j) \longrightarrow (G,X)$. In particular each S_j contains a special point of S. It is clear that the image of a special subset under "conjugation" by an element of $G^{\mathrm{ad}}(\mathbb{Q})$ is again a special subset. A *special subvariety* (or a *Shimura subvariety*) is an irreducible special subset.

The modular variety $\mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$ is a Shimura variety, with Shimura input datum $(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})$, where $\mathbb{H}_g^{\pm}$ is the disjoint union of the Siegel upper-half space and the Siegel lower-half space.

(2.6) (AO) The André-Oort conjecture says: *Let S be a Shimura variety, and let Γ be a set of special points in S. The Zariski closure Γ^{Zar} of Γ is a special subset in S; in other words Γ^{Zar} is a finite union of Shimura subvarieties.* See Problem 1 on page 215 of [1], [22, 6A] and [23].

(2.7) We will use the term "Hilbert modular variety attached to a totally real field F" in a rather loose sense, namely an irreducible component of a closed subvariety of the form $\mathscr{A}_{g,1}^{\mathscr{O}} \subset \mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$. Here $\mathscr{O} \subseteq \mathscr{O}_F$ is an order in the totally real field F with $[F:\mathbb{Q}] = g$, and $\mathscr{A}_{g,1}^{\mathscr{O}}$ is the locus of all points $[(A,\lambda)]$ such that there exists an injective ring homomorphism $\mathscr{O} \hookrightarrow \mathrm{End}(A)$ which sends the unity element $1 \in \mathscr{O}$ to Id_A. Each Hilbert modular variety attached to a totally real field F with $[F:\mathbb{Q}] = g$ is a special subvariety of $\mathscr{A}_{g,1}$ over $\mathbb{Q}$ with Shimura input datum $(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2, (\mathbb{H}^{\pm})^g)$, where $\mathbb{H}^{\pm}$ is the union of the upper-half and lower-half plane.

In the rest of this section we define the notion of Weyl CM fields and Weyl CM points and explain some of their basic properties. Lemma 2.8 below is a preliminary remark on Galois groups of CM fields.

(2.8) **Lemma.** *Let F be a number field with $[F:\mathbb{Q}] =: g$, and let L be a quadratic extension field*

[8] We will use the terms "special point" and "CM point" interchangeably, following Deligne in [9, 2.2.4].

of F. Let M be the normal closure of L over $\mathbb{Q}$. The Galois group $\mathrm{Gal}(M/\mathbb{Q})$ *is isomorphic to a subgroup of* $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$*, the wreath product of the symmetric group* S_g *with* $\mathbb{Z}/2\mathbb{Z}$*, which is also the Weyl group of Dynkin diagrams* C_g *and* B_g*. In particular* $[M:\mathbb{Q}]$ *divides* $2^g \cdot g!$.

PROOF. Let $S' := \mathrm{Hom}_{\mathrm{ring}}(F, \mathbb{Q}^{\mathrm{a}})$ be the set of all embeddings of the field F to $\mathbb{Q}^{\mathrm{a}}$, and let $S := \mathrm{Hom}_{\mathrm{ring}}(L, \mathbb{Q}^{\mathrm{a}})$ be the set of all embeddings L to $\mathbb{Q}^{\mathrm{a}}$. The inclusion $F \hookrightarrow L$ induces a surjection $\mathrm{res}_F\colon S \twoheadrightarrow S'$. Let $\mathrm{Perm}(S') \cong S_g$ be the group of all permutations of S'. Let $\mathrm{Perm}(S/S')$ be the group of all permutations σ of S which respects the surjection $\mathrm{res}_F\colon S \to S'$, in the sense that there exists a (uniquely determined) permutation $\tau \in \mathrm{Perm}(S')$, of S' such that $\mathrm{res}_F \circ \sigma = \tau \circ \mathrm{res}_F$; the map $\sigma \mapsto \tau$ defines a surjective homomorphism $\pi\colon \mathrm{Perm}(S/S') \twoheadrightarrow \mathrm{Perm}(S')$. The kernel of π is the subgroup $\mathrm{Perm}_{S'}(S) \subset \mathrm{Perm}(S/S')$ consisting of all elements of $\mathrm{Perm}(S/S')$ which induce the identity on S'; it is naturally isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{S'}$. Every choice of a section $\varepsilon\colon S' \to S$ of $\mathrm{res}_F\colon S \to S'$ defines a section $\mathrm{Perm}(S') \to \mathrm{Perm}(S/S')$ of π: the stabilizer subgroup of $\varepsilon(S')$ in $\mathrm{Perm}(S/S')$ is isomorphic to $\mathrm{Perm}(S')$ via π.

Let M_1 be the normal closure of L_0. The natural faithful action of $\mathrm{Gal}(M/\mathbb{Q})$ on S induces an injective homomorphism $\rho_M\colon \mathrm{Gal}(M/\mathbb{Q}) \to \mathrm{Perm}(S/S')$. Restricting to the subfield M_1 gives an injection $\rho_{M_1}\colon \mathrm{Gal}(M_1/\mathbb{Q}) \to \mathrm{Perm}(S')$ compatible with ρ_M. □

(2.9) REMARK. Below are some properties of the group $\mathrm{Perm}(S/S')$. The proofs are left as exercises.

(a) The set of all unordered partitions of S' into a disjoint union of two subsets, each in bijection with S via the surjection $\mathrm{res}_F\colon S \to S'$, is in bijection with the set of all splittings of the surjective group homomorphism $\pi\colon \mathrm{Perm}(S/S') \twoheadrightarrow \mathrm{Perm}(S') \cong S_g$.

(b) The center Z of $\mathrm{Perm}(S/S')$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and contained in $\mathrm{Ker}(\pi) \cong (\mathbb{Z}/2\mathbb{Z})^{S'}$; it is the diagonally embedded copy of $\mathbb{Z}/2\mathbb{Z}$ in $(\mathbb{Z}/2\mathbb{Z})^{S'}$.

(c) The only non-trivial proper subgroups of $(\mathbb{Z}/2\mathbb{Z})^{S'}$ stable under the conjugation action of $\mathrm{Perm}(S')$ is the center Z of $\mathrm{Perm}(S/S')$ and the kernel $(\mathbb{Z}/2\mathbb{Z})_0^{S'}$ of the homomorphism

$$(\mathbb{Z}/2\mathbb{Z})^{S'} \to \mathbb{Z}/2\mathbb{Z}, \qquad (a_t)_{t\in S'} \mapsto \sum_{t\in S'} a_t\,.$$

(d) If a proper subgroup H of $\mathrm{Perm}(S/S')$ surjects to $\mathrm{Perm}(S')$ under π, then either H is isomorphic to $\mathrm{Perm}(S')$ under π, otherwise the surjection $\pi|_H : H \to \mathrm{Perm} S/S'$ makes H an extension of $\mathrm{Perm}(S')$ by $Z \cong \mathbb{Z}/2\mathbb{Z}$ or by $(\mathbb{Z}/2\mathbb{Z})_0^{S'}$. Such an extension of $\mathrm{Perm}(S')$ in the latter case is not necessarily a split extension, as one can see in the case $g = 2$.[9]

(2.10) DEFINITION. (a) A CM field L with $[L:\mathbb{Q}] = 2g$ is a *Weyl CM field* if the degree over $\mathbb{Q}$ of the normal closure $M/\mathbb{Q}$ of $L/\mathbb{Q}$ is equal to $2^g \cdot g!$, or equivalently if $\mathrm{Gal}(M/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$.

(b) A totally real number field F of degree $[F:\mathbb{Q}] = g$ is *of Weyl type* if the Galois group of the normal closure of $F/\mathbb{Q}$ is isomorphic to the symmetric group S_g.

[9] The group $\mathrm{Perm}(S/S')$ is isomorphic to the dihedral group with 8 elements when $g = 2$. Label the four elements of S by $1,2,3,4$ such that $\{1,3\}$ and $\{2,4\}$ are the two fibers of the surjection $S \to S'$. Then $\mathrm{Perm}(S/S')$ contains the subgroup H generated by the cyclic permutation $(1\,2\,3\,4)$, and H surjects to $\mathrm{Perm}(S')$ via π. The extension in question is isomorphic to the non-split extension $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

Remark. (1) For every totally real field F of Weyl type of degree g there exists a Weyl CM field L which is a quadratic extension of F.
(2) Let M_1 be the normal closure of the maximal totally real subfield F in a Weyl CM field L of degree $2g$. Then F is of Weyl type and $\mathrm{Gal}(M/M_1) \cong (\mathbb{Z}/2\mathbb{Z})^g$.

(2.11) Definition. A point $[(A,\lambda)] \in \mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$ is a *Weyl CM point* in $\mathscr{A}_{g,1}$ if the endomorphism algebra $\mathrm{End}^0(A)$ of A contains a Weyl CM field L with degree $[L\colon \mathbb{Q}] = 2g$. (Then $\mathrm{End}^0(A) = L$; see 2.12 (3) below.)

(2.12) Remarks on Weyl CM points.

(1) The only proper subfields of a Weyl CM field L are $\mathbb{Q}$ and the maximal totally real subfield of L. This statement amounts to the following fact about the group $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$:

> Suppose that $g \geq 2$ and H is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$ such that $(\mathbb{Z}/2\mathbb{Z})^{g-1} \rtimes S_{g-1} \subsetneqq H \subsetneqq \mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$, where $(\mathbb{Z}/2\mathbb{Z})^{g-1}$ is the wreath product of S_{g-1} and $\mathbb{Z}/2\mathbb{Z}$, embedded in $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$ in the standard way. Then $H = (\mathbb{Z}/2\mathbb{Z})^g \rtimes S_{g-1}$.

(2) If x is a Weyl CM point in $\mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$, so is every point in $\mathscr{I}(x)$.

(3) If $[(A,\lambda)]$ is a Weyl CM point in $\mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$, then $L := \mathrm{End}^0(A)$ is a Weyl CM field. In particular A is (absolutely) simple.

Proof. The endomorphism algebra $\mathrm{End}^0(A)$ contains a field L of degree $[L\colon \mathbb{Q}] = 2g$, so A is isogenous to B^n for some (absolutely) simple abelian variety B over $\mathbb{Q}^{\mathrm{a}}$. Suppose that $n > 1$. Then $E := \mathrm{End}^0(B)$ is a CM field, $\mathrm{End}^0(A) \cong \mathrm{M}_n(E)$, and L contains E, contradicting (1). So we have $L = E = \mathrm{End}^0(A)$. □

(4) A consequence of [6, Thm. 2.1] is the following. Suppose that $A \to U$ is an abelian scheme of relative dimension g over a geometrically irreducible variety $U_{/\mathbb{F}_q}$ over a finite field $\mathbb{F}_q$, such that the mod-ℓ geometric monodromy group is equal to $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ for all $\ell \gg 0$. Then the subset D of the set $|U|$ of all closed points of U consisting of all closed points $x \in |U|$ such that $\mathbb{Q}(\mathrm{Fr}_{A_x})$ is a Weyl CM field has density one.

(5) If $[(A,\lambda)] \in \mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$ is a *Weyl CM point* in $\mathscr{A}_{g,1}$, then the special Mumford-Tate group of the abelian variety A is $\mathrm{Ker}\left(\mathrm{Nm}_{L/F}\colon \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m\right) =: T_{L,1}$, where $L = \mathrm{End}^0(A)$ is the Weyl CM field attached to A and F is the maximal totally real subfield in L.

Proof. The special Mumford-Tate group of A is by definition contained in

$$\mathrm{Sp}\left(\mathrm{H}_1(A(\mathbb{C});\mathbb{Q}), \langle\cdot,\cdot\rangle\right) \cap \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m = T_{L,1},$$

where $\langle\cdot,\cdot\rangle$ is the perfect alternating pairing on $\mathrm{H}_1(A(\mathbb{C}),\mathbb{Q})$ induced by the principal polarization λ_0 on A. It is well-known that the standard representation of $(\mathbb{Z}/2\mathbb{Z})^g \rtimes S_g$ on $\mathbb{Q}^{2g}$ is irreducible, so the only non-trivial subtorus of $T_{L,1}$ is $T_{L,1}$ itself. □

(2.13) Lemma (Deligne, Ekedahl, Geyer). *Given a positive integer g and a number field E, there exists a Weyl CM field L with $[L : \mathbb{Q}] = 2g$ such that M is linearly disjoint from E.*

PROOF. This is an application of Hilbert irreducibility. We will use the version in [8, Lemma 5.13]; see also [12, Lemma 3.4] and [11, Thm. 1.3]. (Note that Lemma 5.13 in [8] follows right after the end of Lemma 5.1.2.) Consider the following extension of polynomial rings

$$\mathbb{Q}[s_1,\ldots,s_g] \longrightarrow \mathbb{Q}[x_1,\ldots,x_g] \longrightarrow \mathbb{Q}[u_1,\ldots,u_g]$$

where s_i is the i-th elementary symmetric polynomial for $i = 1,\ldots,g$ and $x_i = u_i^2$ for all $i = 1,\ldots,g$. Let $V = \mathrm{Spec}(\mathbb{Q}[s_1,\ldots,s_g])$, let $W_1 = \mathrm{Spec}(\mathbb{Q}[x_1,\ldots,x_g])$, and let $W = \mathrm{Spec}(\mathbb{Q}[u_1,\ldots,u_g])$. Let U be the open subset of $V(\mathbb{R})$, equal to the image in V of the open subset $U_1 \subset W_1(\mathbb{R})$ consisting of all $\mathbb{R}$-points $(a_1,\ldots,a_g) \in W_1(\mathbb{R})$ of W_1 such that $a_i \neq a_j$ for all $i \neq j$ and $a_i < 0$ for all $i = 1,\ldots,g$. By [8, Lemma 5.13], for any given number field E, there exists a $\mathbb{Q}$-rational point $v \in V(\mathbb{Q}) \cap U$ such that the inverse image of v in W is the spectrum of a field L of degree $2^g \cdot g!$ over $\mathbb{Q}$ and is linearly disjoint from E. Note that L is a CM field by construction. □

(2.14) REMARK. Here is a version of Hilbert irreducibility with weak approximation, slightly stronger than [8, Lemma 5.13], which will be used later. For a finite extension field F of $\mathbb{Q}$, the *product topology* on the ring $\mathbb{A}_F$ of all F-adeles is the topology induced by the natural inclusion $\mathbb{A}_F \hookrightarrow \prod_v F_v$ and the product topology on the infinite product $\prod_v F_v$, where v runs through all places of F and F_v is the completion of F at v. It is weaker than the adelic topology for $\mathbb{A}_F$.

Let E be a finite extension of a finite extension field F of $\mathbb{Q}$. Suppose that we have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\ f\ } & V \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \mathrm{Spec}(E) & \longrightarrow & \mathrm{Spec}(F) \end{array}$$

where

- *V is a non-empty Zariski open subset of an affine space*[10] *$\mathbf{A}^m$ over F,*
- *W is reduced and all geometric fibers of π are irreducible,*
- *f is quasi-finite and dominant.*

Suppose moreover that we are given a finite extension field E_1 of E and a non-empty open subset $U \subseteq V(\mathbb{A}_F)$ for the product topology on $\mathbb{A}_F$. Then there exists an element $v \in V(F)$ such that the image of v in $V(\mathbb{A}_F)$ lies in U, and the schematic inverse image $f^{-1}(v)$ is the spectrum of a finite extension of E which is linearly disjoint with E_1 over E.

The above statement follows from [12, Lemma 3.4], which asserts that every Hilbertian subset of $\mathbf{A}^m(F)$ of a Hilbertian F satisfies the weak approximation property for any given finite set of absolute values of F. It can also be deduced from [11, Thm. 1.3].

(2.15) COROLLARY. (1) *Let g be a positive integer. There exist infinitely many Weyl CM fields L with $[L\colon \mathbb{Q}] = 2g$.*

[10] We used $\mathbf{A}^m$ instead of the more standard $\mathbb{A}^m$ for an affine space $\mathrm{Spec}\,F[X_1,\ldots,X_m]$ over F, to avoid possible confusion with the notation $\mathbb{A}_F$ for the ring of adeles attached to F.

(2) *Let $g \geq 2$ be a positive integer. There exist infinitely many totally real number fields F of Weyl type with $[F : \mathbb{Q}] = g$.*

(3) *There exist infinitely many mutually non-isogenous Weyl CM points in $\mathscr{A}_{g,1}(\mathbb{Q})$.*

(4) *Suppose that $g \geq 2$. There exists a sequence of Weyl CM points $x_i = [(B_i, \mu_i)]$, $i \in \mathbb{N}$, such that the maximal totally real subfields F_i of the Weyl CM fields $L_i = \mathrm{End}^0(B_i)$ attached to x_i are mutually non-isomorphic.*

§3. Special subsets in $\mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$ and Weyl CM points

Recall that $\mathbb{Q}^{\mathrm{a}}$ is the field of all algebraic numbers in $\mathbb{C}$.

(3.1) THEOREM. *Suppose that the conjecture* (AO) *is true. Then for any $g \geq 1$ the statement* $\mathrm{sI}(\mathbb{Q}^{\mathrm{a}}, g)$ *is true. Consequently* $\mathrm{I}(\mathbb{Q}^{\mathrm{a}}, g)$ *is true as well.*

PROOF. Consider the set $C := \mathrm{CM}(\mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}}))$ of all CM points in $\mathscr{A}_{g,1}$ over number fields and let $C_X := C \cap X(\mathbb{Q}^{\mathrm{a}}) = \mathrm{CM}(X)$. Let Z be the Zariski closure of C_X. By (AO) we have $Z = \bigcup_j^N S_j$, a finite union of special varieties $S_j \subset X \subset \mathscr{A}_{g,1} \otimes \mathbb{Q}^{\mathrm{a}}$, $j = 1, \ldots, N$. Hence the theorem follows from Prop. 3.2 below, to be proved in 3.6. □

(3.2) PROPOSITION. *For any special subset $Y = \bigcup_j S_j$ with $S_j \subsetneqq \mathscr{A}_{g,1} \otimes \mathbb{Q}^{\mathrm{a}}$ there is a Weyl CM point $y \in \mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$, such that*

$$\mathscr{I}(y) \cap \left(\bigcup_{j}^{N} S_j(\mathbb{Q}^{\mathrm{a}}) \right) = \emptyset.$$

(3.3) REMARK. *Suppose the Generalized Riemann Hypothesis holds. Then* $\mathrm{sI}(\mathbb{Q}^{\mathrm{a}}, g)$ and $\mathrm{I}(\mathbb{Q}^{\mathrm{a}}, g)$ *are expected to be true.*

Indeed, in [18] and [33] a proof in announced that GRH implies the André-Oort conjecture.

(3.4) LEMMA. *Let L be a Weyl CM field with $[L : \mathbb{Q}] = 2g \geq 4$, and let F be the maximal totally real subfield of L. Let $T := T_{L,1} = \mathrm{Ker}\left(\mathrm{Nm}_{L/F} : \mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m) \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)\right)$ as in* 2.12 (5)*, a g-dimensional torus over $\mathbb{Q}$. Suppose that G is a connected closed algebraic subgroup over $\mathbb{Q}$ contained in Sp_{2g} which contains T as a closed algebraic subgroup over $\mathbb{Q}$ and $T \neq G$. Then either $G = \mathrm{Sp}_{2g}$ or the derived group G^{der} of G is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$.*

PROOF. Consider the adjoint action of the maximal torus T over $\mathbb{Q}$ of Sp_{2g} on the Lie algebras of G and Sp_{2g}. We get a subset $R(G,T)$ of the root system of Sp_{2g}, which is stable under the action of the Weyl group because the image of the action of the Galois group on the character group of T coincides with the Weyl group for (G,T) by the assumption on L. From basic Lie theory we know that the subset $R(G,T)$ has the following property.

(∗) *If α, β are elements of $R(G,T)$ such that $\alpha + \beta$ is a root for Sp_{2g}, then $\alpha + \beta \in R(G,T)$.*

In fact for two roots α, β in the root system of Sp_{2g}, the condition that $\alpha + \beta$ is again a root for Sp_{2g} means that $\alpha + \beta \neq 0$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, where $\mathfrak{g}_\gamma$ denotes the root space attached to γ for any root γ of Sp_{2g}; see for instance part (c) of the Proposition in 8.4, page 39 of [13]. The assertion $(*)$ follows.

There are two Weyl orbits in the root system C_g for Sp_{2g}, the subset of all short roots and the subset of all long roots. In standard coordinates, the short roots are $\pm x_i \pm x_j$ with $i \neq j$, $1 \leq i, j \leq g$, while the long roots are $\pm 2x_i$, $i = 1, \ldots, g$. We know that every long root is a sum of two distinct short roots; for instance $2x_1 = (x_1 + x_2) + (x_1 - x_2)$. If $R(G,T)$ contains all short roots then it must contain all long roots as well, by the property stated at the end of the previous paragraph. So $R(G,T)$ is the set of all long roots $\pm 2x_i$ if $G \neq \mathrm{Sp}_{2g}$. That means exactly that G is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$. □

(3.5) Lemma. *Let Y be an irreducible positive dimensional special subvariety of $\mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$. If $Y \neq \mathscr{A}_{g,1}$ and Y contains a Weyl CM point y_0 of $\mathscr{A}_{g,1}$, then Y is a Hilbert modular variety attached to the totally real subfield F of degree g over $\mathbb{Q}$ contained in the Weyl CM field attached to y_0.*

Proof. Let $[(B_0, \lambda_0)]$ be a g-dimensional principally polarized abelian variety over $\mathbb{Q}^{\mathrm{a}}$ with complex multiplication by a Weyl CM field L with $[L\colon \mathbb{Q}] = 2g$ contained in Y. Let G be the semi-simple algebraic subgroup group of Sp_{2g} over $\mathbb{Q}$ attached to Y. Then G contains a $\mathbb{Q}$-torus which is isomorphic to $T_{L,1} := \mathrm{Ker}\left(\mathrm{Nm}_{L/F}\colon \mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m) \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)\right)$, namely the special Mumford-Tate group of B_0; see 2.12(5). By 3.4, G is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$ because $S_j \subset Y \neq A_{g,1}$. This means that Y is a Hilbert modular variety attached to F. □

(3.6) Proof of Prop. 3.2. We may and do assume that $g \geq 2$. The given special subset Y is a union of irreducible components S_j, which we enumerate as follows.

(i) $S_j = [(A_j, \lambda_j)]$ is a point in $\mathscr{A}_{g,1}(\mathbb{Q}^{\mathrm{a}})$ for $j = 1, \ldots, a$,

(ii) S_j is a Hecke translate of a Hilbert modular variety associated to a totally real field F_j of Weyl type with $[F_j\colon \mathbb{Q}] = g$ for $j = a+1, \ldots, a+b$,

(iii) S_j is not of type (i) nor of type (ii) above for $j = a+b+1, \ldots, a+b+c$.

By 2.13, there exists a Weyl CM field L with $[L\colon \mathbb{Q}] = g$ such that the maximal totally real subfield F in L is not isomorphic to F_j for any $j = a+1, \ldots, a+b$ and L cannot be embedded in $\mathrm{End}^0(A_j)$ for any $j = 1, \ldots, a$.

Let (B_0, λ_0) be a g-dimensional principally polarized abelian variety such that $\mathrm{End}^0(B_0) \cong L$. Clearly $\mathscr{I}(x_0) := \mathscr{I}([(B_0, \lambda_0)]) \not\ni [(A_j, \lambda_j)]$ for all $j = 1, \ldots, a$. Suppose that there exists a point $y_0 \in \mathscr{I}(x_0)$ such that $y_0 \in S_{j_0}$ for some $j_0 > a$. We know from 3.5 that $a+1 \leq j_0 \leq a+b$. Let G_{j_0} be the derived group of the reductive algebraic subgroup of Sp_{2g} over $\mathbb{Q}$ attached to S_{j_0}, which is isomorphic to $\mathrm{Res}_{F_{j_0}/\mathbb{Q}}(\mathrm{SL}_2)$. However 3.5 tells us that it is also isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$. We know that the number field F is determined up to non-unique isomorphism by the $\mathbb{Q}$-group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$, namely it is the largest subfield of $\mathbb{Q}^{\mathrm{a}}$ fixed by the the stabilizer subgroup of any element of the finite set $\pi_0\mathscr{D}(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2))$ of all simple factors of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2) \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{Q}^{\mathrm{a}})$, under the transitive permutation representation of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on $\pi_0\mathscr{D}(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2))$.[11] We conclude that the number field F is isomorphic to F_{j_0}, which is a contradiction. We have proved that $\mathscr{I}(x_0) \cap Y = \emptyset$. □

[11] The notation $\pi_0\mathscr{D}$ means "the set of all connected components of the Dynkin diagram". When applied to the $\mathbb{Q}$-group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$, the $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$-set $\pi_0\mathscr{D}(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2))$ is equivariantly identified with $\mathrm{Hom}_{\mathrm{ring}}(F, \mathbb{Q}^{\mathrm{a}})$.

The proof of 3.2 provides a strong finiteness statement for Weyl CM points in the case when $g \geq 4$ and the closed subset $X \subsetneqq \mathscr{A}_{g,1}$ is $\mathscr{T}_g$.

(3.7) PROPOSITION. *Assume either* (AO) *or* GRH. *There are at most finitely many Weyl CM points in the Torelli locus $\mathscr{T}_g \subset \mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$ for any integer $g \geq 4$.*

PROOF. According to [16, Cor. 1.2], for a totally real number field E of degree $g = [E:\mathbb{Q}] \geq 4$ and a Hilbert modular variety M_E over $\mathbb{Q}^{\mathrm{a}}$ attached to E, the following holds.

(i) If $g \geq 5$, then Torelli locus $\mathscr{T}_g$ does not contain M_E.

(ii) If $g = 4$ and $\mathscr{T}_g$ contains M_E, then E is a quadratic extension of a real quadratic field.

Note that the Galois group $\mathrm{Gal}(\tilde{E}/\mathbb{Q})$ of the normal closure $\tilde{E}$ of a quartic field E as in (ii) is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ by 2.8 and not isomorphic to the symmetric group S_4. So E is not the maximal totally real subfield of a Weyl CM field.

If $\mathscr{T}_g$ contains infinitely many Weyl CM points, then it contains a Hilbert modular subvariety attached to a degree g totally real field of Weyl type, by (AO) and 3.5. That is a contradiction. □

(3.8) REMARK. We now know that for $g \geq 4$ the Torelli locus $\mathscr{T}_g$ does not contain any Hilbert modular variety associated to a totally real number field of degree g; the case $g = 4$ is settled in [2]. For further information see [20].

(3.9) REMARK. In [7, Conjecture 6] Coleman conjectured that for any $g \geq 4$ there are only a finite number of proper smooth curves of genus g over $\mathbb{C}$ with CM Jacobians. However, that conjecture is not correct, as has been shown by Shimura, see [27], and by de Jong and Noot, see [15]. Examples were given by families of cyclic covers of $\mathbb{P}^1$, producing a special subset of $\mathscr{T}_g$ of positive dimension. We have such examples for all $g \leq 7$. See [20] and [25] for a description of examples known at present, for a discussion, and for references. Whether Coleman's conjecture holds for any $g \geq 8$ seems unknown.

(3.10) REMARK. Suppose F is a totally real number field of Weyl type with $[F:\mathbb{Q}] = g \geq 2$. One may wonder whether the conclusion of 3.5 holds for a CM point associated with a totally imaginary quadratic extension L/F. Let M' be the normal closure of F and let M be the normal closure of L. By 2.9 (c), only the following three cases occur. (The two cases (1) and (3) coincide when $g = 2$.)

(1) $[M:M'] = 2$, (2) L is a Weyl CM field, (3) g is even[12] and $[M:M'] = 2^{g-1}$.

If $g > 1$ and we are in case (1), the conclusion of 3.5 need not hold in general: take an imaginary quadratic field E, let L be the compositum of F and E. A PEL Shimura variety associated with an action by an order in E contains a CM point associated with L; It is a special subvariety of positive dimension which is not a Hilbert modular variety.

[12] In this case the complex conjugation gives an element of $\mathrm{Gal}(M/\mathbb{Q})$ which belongs to the subgroup

$$\mathrm{Ker}\left(\mathrm{Gal}(M/\mathbb{Q}) \to \mathrm{Gal}(M'/\mathbb{Q}) \cong \mathrm{Perm}(S')\right) \cong (\mathbb{Z}/2\mathbb{Z})_0^{S'} \subset \mathrm{Perm}(S/S')$$

in the notation of 2.9 (c). We know that the complex conjugation corresponds to the element $(a_t)_{t\in S'}$ in $(\mathbb{Z}/2\mathbb{Z})^{S'}$ with $a_t = 1+2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z}$ for all $t \in S'$, and $\sum_{t\in S'} a_t \equiv 0 \pmod 2$ by the definition of $(\mathbb{Z}/2\mathbb{Z})_0^{S'}$, so $g = \mathrm{card}(S')$ is even.

However, if y is a CM point associated with a CM field L which is a quadratic extension of a totally real field F of Weyl type with $4 \leq [F:\mathbb{Q}] \equiv 0 \pmod 2$ such that the condition (3) above is satisfied, then the other conditions in 3.5 imply that the conclusion of 3.5 does hold, by the same argument. All one needs is that in the proof of 3.4 with the Weyl group of Sp_{2g} replaced by the index two subgroup $\mathrm{Gal}(M/\mathbb{Q})$, there are only two orbits for the action of $\mathrm{Gal}(M/\mathbb{Q})$ on the root system of Sp_{2g}.

(3.11) *We show that* $\mathrm{sI}(\mathbb{C},g)$ *holds for any* $g \geq 1$. Let $X \subsetneqq \mathscr{A}_{g,1} \otimes \mathbb{C}$ be a closed subset. Note that the set of points

$$\Lambda := \left\{ [(A,\lambda)] = x \in \mathscr{A}_{g,1}(\mathbb{C}) \mid \mathrm{End}(A) \neq \mathbb{Z} \right\}$$

has measure zero in $\mathscr{A}_{g,1}(\mathbb{C})$. Write $X^0 := X - (X \cap \Lambda)$. The union Λ'_X of all Hecke translates (for GSp_{2g}) of $X^0(\mathbb{C})$ is the same as the union of all isogeny translates of $X^0(\mathbb{C})$. Hence Λ'_X has measure zero in $\mathscr{A}_{g,1}(\mathbb{C})$ because it is a countable union of subsets with measure zero. So there exists a point $x \in \mathscr{A}_{g,1}(\mathbb{C})$ with $x \notin \Lambda \cup \Lambda'_X$. We have $\mathscr{I}(x) \cap (\Lambda \cup \Lambda'_X) = \emptyset$ as Λ and Λ'_X are both stable under translations by isogeny. So $\mathscr{I}(x) \cap X = \emptyset$ because $X \subset \Lambda \cup \Lambda'_X$. □

(3.12) EXPECTATION. *There is no Shimura subvariety of positive dimension over* $\mathbb{Q}^{\mathrm{a}}$ *contained in the closed Torelli locus* $\mathscr{T}_g$ *which meets the open Torelli locus* $\mathscr{T}_g^0$ *for* $g \gg 0$.

See Section 7 in [22]. Note that if this expectation holds for some value g_1 of g, and if (AO) holds, then there are only a finite number of proper smooth curves of genus g_1 with CM Jacobians.

(3.13) QUESTIONS.

1. Can one prove some special cases of 3.7 unconditionally? For instance, is there only a finite number of hyperelliptic curves with a given genus $g \geq 4$ (resp. smooth plane curves of degree $d \geq 5$) whose Jacobian is a Weyl CM point?

2. Given a closed (special) subset $X \subsetneqq \mathscr{A}_{g,1}$ over $\mathbb{Q}^{\mathrm{a}}$, can we find *explicitly* a point x, which is not a CM point, or a CM point which is not a Weyl CM point, such that $\mathscr{I}(x) \cap X(\mathbb{Q}^{\mathrm{a}}) = \emptyset$?

3. For which values of g does the open Torelli locus $\mathscr{T}_g^0$ contain CM points? For which values of g does the open Torelli locus $\mathscr{T}_g^0$ contains Weyl CM points? We do not know a single example of a Weyl CM Jacobian of dimension $g \geq 4$.

REMARK. (a) Dwork and Ogus wrote on [10, p. 112] "The question of constructing non-hyperelliptic curves of high genus with CM Jacobians remains quite mysterious; ..."

(b) The open Torelli locus $\mathscr{T}_g^0$ contains CM points for infinitely many values of g. For instance the Jacobian for any Fermat curve C_n defined by the equation $x^n + y^n = z^n$ has smCM and of dimension $g = (n-1)(n-2/2$. However the principally polarized Jacobian $J(C_n)$ of C_n is not a Weyl CM point for any $n \geq 4$ because any non-hyperelliptic curve with a non-trivial automorphism, or any hyperelliptic curve with more than 2 automorphisms, does not give a Weyl CM Jacobian.

Another series of examples are curves $C_{\ell,a}$ of genus $(\ell-1)/2$, where ℓ is an odd prime number; it is a cover of $\mathbb{P}^1$ over $\mathbb{Q}$ ramified over three points given by the equation $y^\ell = x^a(x-1)$ with $a \not\equiv 0 \pmod{\ell}$; see [34], pp.814/815 and [16, Example 1.4]. For more examples and references see [20] and [25].

§4. Over finite fields

In this section we work over the base field $\mathbb{F} := \overline{\mathbb{F}_p}$. There are several ways to formulate analogues of $\mathrm{sI}(\mathbb{F}, g)$ which reflect special features in characteristic p. Here we only record a positive result 4.1 in a very special case, and formulate a question in 4.5.

(4.1) PROPOSITION. *Let $g \in \mathbb{Z}_{\geq 2}$ and let $X \subset \mathscr{A}_{g,1} \otimes \mathbb{F}$ be a closed subset. Suppose X is irreducible of dimension at most equal to 1. Then there exists $[(A,\lambda)] = x \in \mathscr{A}_{g,1}(\mathbb{F})$ such that*

$$\mathscr{I}(x) \cap X = \emptyset.$$

(4.2) Weil numbers. For any simple abelian variety A over a finite field $\mathbb{F}_q$ with $q = p^n$, the geometric Frobenius $\mathrm{Fr}_{A,\mathbb{F}_q}$ gives rise to an algebraic integer π_A, called the Weil number of A, such that the absolute value $|\iota(\pi_A)| = \sqrt{q}$ for every embedding $\iota : \mathbb{Q}(\pi_A) \to \mathbb{C}$. Two Weil numbers π and τ (for possibly different values of q but in the same characteristic p) are said to be *similar* if a positive power of π is equal to a positive power of τ:

$$\pi \approx \tau \overset{\text{def}}{\iff} \exists s,t \in \mathbb{Z}_{>0} \text{ and } \exists \beta : \mathbb{Q}(\pi^s) \xrightarrow{\sim} \mathbb{Q}(\tau^t) \text{ such that } \beta(\pi^s) = \tau^t\,.$$

Note that the Honda-Tate theory implies that the set of all similarity classes of all Weil p^∞-numbers are in natural bijection with the set of all isogeny classes of simple abelian varieties over $\mathbb{F}$; see [30], [24]. Therefore the set of all isogeny classes of g-dimensional abelian varieties over $\mathbb{F}$ is in natural bijection with the set of all finite *unordered* sequences $(\pi_1,\ldots,\pi_m)$, where each π_i is a similarity class of Weil p^∞-numbers, and the sum of the dimensions of the corresponding isogeny classes of simple abelian varieties A_i over $\mathbb{F}$ is equal to g. Denote by $WN(\mathscr{A}_{g,1} \otimes \mathbb{F})$ the set of all such unordered sequences $(\pi_1,\ldots,\pi_m)$.

For any closed subset $X \subset A_{g,1} \otimes \mathbb{F}$, we write $WN(X)$ for the subset of $WN(\mathscr{A}_{g,1} \otimes \mathbb{F})$ arising from $\mathbb{F}$-points of X. It is clear that

$$WN(X) \subsetneqq WN(\mathscr{A}_{g,1} \otimes \mathbb{F}) \quad \iff \quad \exists x \in \mathscr{A}_{g,1}(\mathbb{F}) \text{ such that } \mathscr{I}(x) \cap X(\mathbb{F}) = \emptyset\,.$$

(4.3) LEMMA. *For any $g \in \mathbb{Z}_{>0}$ and any non-supersingular symmetric Newton polygon ξ,*

$$\#\Big(WN(W^0_\xi)\Big) = \infty.$$

PROOF. It suffices to verify the case when the Newton polygon ξ has only two slopes $m/(m+n)$ and $n/(m+n)$, where $m,n \in \mathbb{N}$, $\gcd(m,n) = 1$, $m \neq n$, and the two slopes both appear $m+n$ times. It is shown in both of the two proofs of [3, 4.9] that there exists infinitely many abelian varieties A_i over finite fields $\mathbb{F}_{q_i} \subset \mathbb{F}$ with ξ as Newton polygon such the Weil numbers π_{A_i} generate *distinct* imaginary quadratic fields $\mathbb{Q}(\pi_{A_i})$. □

(4.4) PROOF OF PROP. 4.1. Consider first the case when X does not contain any ordinary point. Then for any ordinary $x \in W^0_\rho(\mathbb{F})$ we have $\mathscr{I}(x) \cap X = \emptyset$, and we are done.

Suppose now that X contains an ordinary point. By a theorem by Grothendieck and Katz (see [17, Thm. 2.3.1]), it follows there is a dense open set $U \subset X$ consisting of all ordinary points in X. Because the dimension of X is at most one, the complement of U in X is a finite set of points. Because $g > 1$, there exists a symmetric Newton polygon ξ for $\mathscr{A}_{g,1}$ which is neither ordinary nor supersingular. The subset of points in $X(\mathbb{F})$ with Newton polygon equal to ξ is contained in $X \smallsetminus U$, therefore it is finite. Hence in this case the conclusion of 4.1 follows from 4.3. □

(4.5) Denote by $\mathscr{A}_{1,1}^{\mathrm{ord}}$ the ordinary locus of the j-line over $\mathbb{F}$. The following is the first non-trivial case of an analogue of $\mathrm{sI}(\mathbb{F},g)$ for reduction of Shimura varieties.

> (†) Suppose that $X \subset \left(\mathscr{A}_{1,1}^{\mathrm{ord}}\right)^2$ is a closed curve in the product of two copies of the $\mathscr{A}_{1,1}^{\mathrm{ord}}$ over $\mathbb{F}$. There exists a point $x = (x_1,x_2) \in \left(\mathscr{A}_{1,1}^{\mathrm{ord}}\right)^2(\mathbb{F})$ such that $(y_1,y_2) \notin X$ if E_{y_i} is isogenous to E_{x_i} for $i = 1,2$.

The case when $X = \{(x_1,x_2) \mid x_1,x_2 \in \mathscr{A}_{1,1}^{\mathrm{ord}},\ x_2 = x_1+1\}$ is already a challenge—we don't have a proof for this very special case.

§5. Special subsets in Shimura varieties

In this section we generalize Prop. 3.2 to the context of Shimura varieties. The main result is Theorem 5.5, with Proposition 5.1 as an immediate consequence. Corollary 5.2 is an analogue of $\mathrm{I}(g,\mathbb{Q}^{\mathrm{a}})$. An outline of the proof of 5.5 is provided in 5.6. The proof of 5.5 is in 6.13.

(5.1) PROPOSITION. *Let S be a Shimura variety over $\mathbb{Q}^{\mathrm{a}}$, and let $S_1,\ldots,S_m$ be a finite family of Shimura subvarieties of S such that $\dim(S_i) < \dim(S)$ for each $i = 1,\ldots,m$. Then there exists a special point $y \in S(\mathbb{Q}^{\mathrm{a}})$ such that $\mathscr{H}(y) \cap (\bigcup_i S_i(\mathbb{Q}^{\mathrm{a}})) = \emptyset$. Here $\mathscr{H}(y)$ denotes the* Hecke orbit *on the Shimura variety S, defined in terms of the reductive group G which is part of the input data for the Shimura variety S.*

(5.2) COROLLARY. *The following statement* $\mathrm{IS}(\mathbb{Q}^{\mathrm{a}})$ *holds modulo either* (AO) *or* GRH.

$\mathrm{IS}(\mathbb{Q}^{\mathrm{a}})$ *Let S be a Shimura variety over $\mathbb{Q}^{\mathrm{a}}$ and let $X \subset S$ be a closed subset over $\mathbb{Q}^{\mathrm{a}}$ of lower dimension. Then there exists a point $x \in S(\mathbb{Q}^{\mathrm{a}})$, which can be chosen to be a Weyl special point in S, such that $\mathscr{H}(x) \cap X = \emptyset$.*

(5.3) DEFINITION. Let G be a connected reductive linear algebraic group over $\mathbb{Q}$. A maximal $\mathbb{Q}$-subtorus $T \subset G$ is said to be a *Weyl subtorus* if the image of the natural action of the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on the character group of the image T^{ad} of T in the adjoint group G^{ad} of G contains the Weyl group $W(R(G^{\mathrm{ad}},T^{\mathrm{ad}}))$ of the root system of G^{ad} with respect to T^{ad}.

(5.4) DEFINITION. Let (G,X) be a Shimura input data as in [9, 2.1.1], where G is a connected reductive algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of $\mathbb{R}$-homomorphisms from $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ to $G_{\mathbb{R}}$.[13] A Weyl special point (or a Weyl CM point) in X is an $\mathbb{R}$-homomorphism $x_0 \colon \mathbb{S} \to G_{\mathbb{R}}$ which factor through a Weyl subtorus $T \subset G$. The image of a point $(x_0,g) \in X \times G(\mathbb{A}_f)$ in a Shimura variety ${}_K\mathscr{M}_{\mathbb{C}}(G,X)$ associated to a compact open subgroup $K \subset G(\mathbb{A}_f)$ is said to be a Weyl special point if x_0 is Weyl special point for (G,X).

(5.5) THEOREM. *Let S be a Shimura variety over $\mathbb{Q}^{\mathrm{a}}$, and let $S_1,\ldots,S_m$ be a finite family of Shimura subvarieties of S such that $\dim(S_j) < \dim(S)$ for each $j = 1,\ldots,m$. Then there exists a Weyl special point $y \in S(\mathbb{Q}^{\mathrm{a}})$ such that $\mathscr{H}(y) \cap \left(\bigcup_j S_j(\mathbb{Q}^{\mathrm{a}})\right) = \emptyset$.*

[13] We will assume conditions (2.1.1.1)–(2.1.1.3) of [9, 2.1.1]. In particular no $\mathbb{Q}$-simple factor of G^{ad} is compact. We also assume that G^{ad} is non-trivial; this will simplify future statements.

(5.6) Ingredients of the proof of 5.5.

(1) AN ABUNDANT SUPPLY OF WEYL SPECIAL POINTS in every Shimura variety.

This is a variant of Deligne's method in [8, § 5.1] for producing special points on Shimura varieties using Hilbert irreducibility. Our modified version produces Weyl special points, satisfying the weak approximation property. See 5.11 for the statement. In the set up of 5.11, the role of the Shimura reflex field $E(G,X)$ attached to a Shimura input datum (G,X) is replaced by a number field $E(G)$ which contains $E(G,X)$ and is finite Galois over $\mathbb{Q}$; see 5.9.

(2) CLASSIFICATION of connected closed subgroups H in a semi-simple almost $\mathbb{Q}$-*simple* group G which contains a Weyl subtorus T.

The point here is that in an irreducible root system, roots of the same length form a single orbit under the Weyl group. If the $\mathbb{Q}$-group G occurs in a Shimura input datum and H is not equal to G, the above fact implies that there aren't many possibilities for H: it has to be equal to T or to G unless G is of type C_n or B_n with $n \geq 2$.

In the C_n case, if $T \subsetneqq H \subsetneqq G$ then H is the restriction of scalars from a number field F to $\mathbb{Q}$ of a group of type A_1; see 6.4 and 6.5. We have seen such an example in §3, where $G = \mathrm{Sp}_{2g}$ over $\mathbb{Q}$ and H is the restriction of scalar of SL_2 over a totally real field of degree g.

In the B_n case, we can take G to be the restriction of scalars of the special orthogonal group $\mathrm{SO}(V,q)$ attached to a non-degenerate quadratic space (V,q) over a number field F and $\dim_F(V) = 2n+1$. If $T \subsetneqq H \subsetneqq G$ then H is the restriction of scalars of a D_n-type group $\mathrm{SO}(V'^{\perp}, q_{V'^{\perp}})$, where V' is a one-dimensional anisotropic subspace of V fixed by the Weyl subtorus T; see 6.6.

(3) PRODUCT SITUATIONS.

There is no surprise here. Suppose that a semi-simple $\mathbb{Q}$-group G is part of a Shimura input datum, and $G = G_1 \times \cdots \times G_N$ where each factor G_i is almost $\mathbb{Q}$-simple. Suppose moreover that $T = T_1 \times \cdots \times T_N$ is a Weyl subtorus of G, where T_i is a Weyl subtorus of G_i for each $i = 1,\ldots,N$, and H is a closed subgroup of G which contains T. Then H is a product: $H = H_1 \times \cdots \times H_N$ where each H_i can be only T_i or G_i if G_i is not of type C_n or B_n. If G_i is of type C_n or B_n with $n \geq 2$, then there is a third possibility for H_i as described in (2) above.

(4) NUMBER FIELDS AS OBSTRUCTION.

Given a semi-simple $\mathbb{Q}$-group $G = G_1 \times \cdots \times G_N$ as in (3), and m subgroups

$$H_a = H_{a,1} \times \cdots \; H_{a,N} \subsetneqq G_1 \times \cdots \times G_N\,, \qquad a = 1,\ldots,m$$

of G, each of the type described in (3), we need to produce a compact Weyl $\mathbb{Q}$-subtorus T of G which is not contained in any $G(\mathbb{Q})$-conjugate of H_a for any $1 \leq a \leq m$.

Ingredient (1) allows us to produce a compact Weyl $\mathbb{Q}$-subtorus T such that the number field K_T fixed by the kernel of the representation of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on the character group of T is linearly disjoint with any given number field $\tilde{E}$ over a small number field attached to G. Choosing a large enough finite Galois extension $\tilde{E}$ over $\mathbb{Q}$, we can ensure that the Weyl subtorus T is not contained in any $G(\mathbb{Q})$-conjugate of H_a, unless for every index i such that $H_{a,i} \subsetneqq G_i$, the group G_i is of type B_n with $n \geq 2$.

The idea is simple and has already been used in §3. If a factor $H_{a,i}$ of H_a is a torus, we get a number field E_a from the Galois representation on the character group of $H_{a,i}$ in the same way as above. If a G_i is of type C_n and $H_{a,i}$ is the restriction of scalar of a type A_1 group from a field F to $\mathbb{Q}$, again we get a number field $E_a = F$. If the field $\tilde{E}$ contains all Galois conjugates of E_a, then we have successfully *obstructed* the Weyl subtorus T from being contained in any $G(\mathbb{Q})$-conjugate of H_a. See 6.8 for the C_n case.

(5) DISCRIMINANTS AS OBSTRUCTION.

Notation as in (4). Suppose that for some a between 1 and m, the group G_i is of type B_{n_i} for every index i such that $H_{a,i} \subsetneqq G_i$, and the subgroup $H_{a,i}$ is as described in (2). In this situation we use another invariant, the discriminant of the quadratic space $(V'^{\perp}, q_{V'^{\perp}})$ in the notation of (2); see 6.10. This discriminant is an element of $F^{\times}/F^{\times 2}$, and we can obstruct the Weyl torus T in (4) from being contained in any $G(\mathbb{Q})$-conjugate of H_a by imposing local conditions on T at *any* single prime number p; see 6.12. Here the weak approximation property in (2) comes very handy, as we need to enforce the obstructions for a finite number of subgroups $H_a \subset G$.

REMARK. (a) In the case when the adjoint group G^{ad} of the reductive group G in the input datum for the Shimura variety S in 5.5 is $\mathbb{Q}$-simple, the proof of 5.5 becomes a little shorter: it follows from 5.11, 6.5, 6.6, 6.8 and 6.12.
(b) It is tempting to try to prove Thm. 5.5 by reducing it to the case when the semi-simple group G in the Shimura input datum of the ambient Shimura variety S is adjoint and $\mathbb{Q}$-simple. But the truth of Thm. 5.5 in the $\mathbb{Q}$-simple case does *not* (seem to) formally imply the more general case when G is a product of $\mathbb{Q}$-simple groups.[14]

(5.7) Remarks on Weyl tori. (a) Clearly, a maximal $\mathbb{Q}$-subtorus of G is a Weyl subtorus of G if and only if its image in the adjoint group G^{ad} of G is a Weyl subtorus of G^{ad}.

It is also clear that being a Weyl torus is stable under central $\mathbb{Q}$-isogeny: suppose that $\alpha\colon G_1 \to G_2$ is a central isogeny between connected semi-simple algebraic groups over $\mathbb{Q}$ and that T_1, T_2 are maximal $\mathbb{Q}$-tori in G_1 and G_2 respectively, such that α induces in isogeny $\alpha|_{T_1}: T_1 \to T_2$. Then T_1 is a Weyl subtorus of G_1 if and only if T_2 is a Weyl subtorus of G_2.

[14] The problem here has to do with 5.7 (b) below: if x_1 and x_2 are Weyl special points in Shimura varieties S_1 and S_2, the point (x_1, x_2) is not necessarily a Weyl special point of $S_1 \times S_2$. In some sense the proof in 6.13 of 5.5 goes by reducing the latter to $\mathbb{Q}$-simple factors of G^{ad} at the level of 5.11, the production machinery for Weyl subtori. In other words one can formulate a statement which incorporates part of 5.11, is stronger than 5.5, and can be proved by reducing to the case when G^{ad} is almost $\mathbb{Q}$-simple. However that statement is long and we have opted for the shorter one in 5.5. We formulate this statement below using the notation in 5.9–5.11; e.g. V is the scheme of regular elements of $\mathrm{Lie}(G)$ and $f\colon W \to V$ is the finite étale Galois cover of V in 5.10.

There exists a non-empty open subset U in $V(\mathbb{A}_{\mathbb{Q}})$ for the product topology on $\mathbb{Q}$ and a finite extension field $\tilde{E}_{/E(G)}$ of the Weyl reflex field $E(G)$, with the following property: Suppose we have

(a) *an element $v \in V(\mathbb{Q}) \cap U$ such that $\mathrm{Stab}_G(v)$ is a Weyl subtorus T_v in G, and $f^{-1}(v)$ is the spectrum of a field K_v linearly disjoint with $\tilde{E}$ over $E(G)$,*

(b) *an $\mathbb{R}$-homomorphism $\mathbb{S} \to T_v$ such that the composition $\tilde{y}\colon \mathbb{S} \to T \hookrightarrow G$ is a Weyl special point for (G,X).*

Then the Hecke orbit $\mathscr{H}(y)$ in S is disjoint from $\cup_j S_j(\mathbb{Q}^{\mathrm{a}})$, where y is the image of $\tilde{y}$ in $S(\mathbb{Q}^{\mathrm{a}})$.

(b) Suppose that $(G,T) = (G_1,T_1) \times_{\mathrm{Spec}(\mathbb{Q})} (G_2,T_2)$, where G_1 and G_2 are connected semi-simple groups over $\mathbb{Q}$. If T is a Weyl subtorus of G, then T_i is a Weyl subtorus of G_i for $i = 1,2$. However the assumption that T_i is a Weyl subtorus of G_i for both $i = 1$ and $i = 2$ does *not* imply that T is a Weyl subtorus of the product group G. What one needs is a condition on linear disjointness. More precisely, T is a Weyl subtorus if and only if T_i is a Weyl subtorus for G_i for $i = 1,2$, and the three natural maps below are bijective.[15]

- $K_1 \otimes_{E_1} (E_1 \cdot E_2) \xrightarrow{\sim} K_1 \cdot E_2$
- $(E_1 \cdot E_2) \otimes_{E_2} K_2 \xrightarrow{\sim} E_1 \cdot K_2$
- $(K_1 \cdot E_2) \otimes_{E_1} (E_1 \cdot E_2)) \otimes_{E_1 \cdot E_2} (E_1 \cdot K_2) \xrightarrow{\sim} K_1 \cdot K_2$

In the above, K_1, K_2 are finite Galois extension of $\mathbb{Q}$, both contained in an algebraic closure $\mathbb{Q}^{\mathrm{a}}$ of $\mathbb{Q}$, and E_i is a subfield of K_i for $i = 1,2$, defined as follows. For $i = 1,2$, $K_i/\mathbb{Q}$ is the finite Galois extension such that the linear action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on the cocharacter group $X_*(T_i)$ of T_i factors through $\mathrm{Gal}(K_i/\mathbb{Q})$ and induces a faithful action ρ_i of $\mathrm{Gal}(K_i/\mathbb{Q})$ on $X_*(T_i)$. The subfields E_i of K_i are defined by the property that ρ_i induces an isomorphism from $\mathrm{Gal}(K_i/E_i)$ to the Weyl group $W(R(G_i,T_i))$ for $i = 1,2$.

(c) See 5.15–5.18 for explicit descriptions of Weyl subtori in classical groups.

(d) The notion of Weyl subtori generalizes immediately to all connected reductive groups over an arbitrary field F, for instance any global field: replace $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ by $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ in the definition 5.3.

(5.8) Remark on Weyl special points. Notation as in 5.4. Let $\pi\colon G \to G^{\mathrm{ad}}$ be the canonical map. Let $x_0\colon \mathbb{S} \to G_{\mathbb{R}}$ be a Weyl special point, and let $T^{\mathrm{ad}} \subset G^{\mathrm{ad}}$ be a Weyl subtorus in G^{ad} which contains the image of the homomorphism $\pi \circ x_0\colon \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$. Then the image in G^{ad} of the Mumford-Tate group of x_0 is equal to the Weyl subtorus $T^{\mathrm{ad}} \subset G^{\mathrm{ad}}$ itself.

PROOF. After modifying G by a central isogeny, we may assume that (G,T) is a product: $(G,T) = \prod_{i=1}^{N}(G_i,T_i)$, where each G_i is almost $\mathbb{Q}$-simple. As the image in G^{ad} of the Mumford-Tate group of x_0 is the Mumford-Tate group of $\pi \circ x_0$, we may assume that $G = G^{\mathrm{ad}}$ and $T = T^{\mathrm{ad}}$. Because the irreducible factors of the $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$-module $X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ are exactly the $X^*(T_i)\otimes_{\mathbb{Z}}\mathbb{Q}$'s for $i = 1,\ldots,N$, there exists a subset $J \subset \{1,2,\ldots,N\}$ such that the Mumford-Tate group of x_0 is equal to $\prod_{i\in J} T_i$. Since the G_i-component of x_0 is non-trivial for each $i = 1,\ldots,N$, the Mumford-Tate group of x_0 must be equal to T. □

(5.9) We will generalize the argument in [8, Thm. 5.1] to show the existence of Weyl subtori in any connected reductive group G over $\mathbb{Q}$. Moreover there are plenty of them so that a weak approximation statement holds.

[15] Equivalently, $(K_1 \cdot E_2) \otimes_{E_2} K_2 \xrightarrow{\sim} K_1 \cdot K_2 \xleftarrow{\sim} K_1 \otimes_{E_1} (E_1 \cdot K_2)$.

We set up notation following [8, § 5.1]. Let G be a reductive group over $\mathbb{Q}$ whose adjoint group G^{ad} is non-trivial. Fix a maximal $\mathbb{Q}^{\mathrm{a}}$-torus T_0 in G. Let $X_0 = X^*(T_0) := \mathrm{Hom}(T_{0/\mathbb{Q}^{\mathrm{a}}},\ \mathbb{G}_{m/\mathbb{Q}^{\mathrm{a}}})$ be the character group of T_0 and let $X_0^\vee = X_*(T_0) = \mathrm{Hom}(\mathbb{G}_{m/\mathbb{Q}^{\mathrm{a}}}, T_{0/\mathbb{Q}^{\mathrm{a}}})$, be the cocharacter group of T_0. Let

$$(R_0 := R(G,T), X_0, R_0^\vee = R(G,T)^\vee, X_0^\vee)$$

be the (absolute) root system attached to (G,T). Pick a basis D_0 of R_0, with R_0^+ the corresponding system of positive roots in R; let $D_0^\vee$ be the basis of $R_0^\vee$ dual to D_0. Then

$$\mathscr{D}_0 := (X_0, D_0, X_0^\vee, D_0^\vee)$$

is a *based root datum* for $(G_{/\mathbb{Q}^{\mathrm{a}}}, T_{0/\mathbb{Q}^{\mathrm{a}}})$ according to the terminology in [28, p. 271], where $G_{/\mathbb{Q}^{\mathrm{a}}}$ is short for $G \times_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}(\mathbb{Q}^{\mathrm{a}})$ and similarly for $T_{0/\mathbb{Q}^{\mathrm{a}}}$.

Let T_1 be the split torus over $\mathbb{Q}$ with character group X_0. Let $\tilde{Y} = \tilde{Y}(G)$ be the moduli scheme over $\mathbb{Q}$ of *maximal tori rigidified by the based root system* $\mathscr{D}_0$. This means that for every $\mathbb{Q}$-scheme S, the set $\tilde{Y}(S)$ of all S-points of $\tilde{Y}$ is the set of all sextuples

$$\big(T, X^*(T), D, X_*(T), D^\vee, \psi\big)\ ,$$

where

- T is a maximal torus of $G \times_{\mathrm{Spec}\mathbb{Q}} S$ over S,
- $(X^*(T), D, X_*(T), D^\vee)$ is a based root datum for $(G \times_{\mathrm{Spec}(\mathbb{Q})} S, T)$, and
- $\psi\colon T_0 \times_{\mathrm{Spec}(\mathbb{Q})} S \xrightarrow{\sim} T$ is an isomorphism of tori over S which induces an isomorphism

  $$\mathscr{D}_{0/S} = (X_0, D_0, X_0^\vee, D_0^\vee)_{/S} \xrightarrow{\sim} (X^*(T), D, X_*(T), D^\vee)$$

  of based root data over S. In particular T is a split torus of $G \times_{\mathrm{Spec}\mathbb{Q}} S$ over S.

Here $X^*(T)$ and $X_*(T)$ are regarded as sheaves of locally free $\mathbb{Z}$-modules of finite rank for the étale topology of S, while D and $D^\vee$ are their global sections over S.

Some remarks are in order:
(a) The group G operates naturally on the left of $\tilde{Y}$ by conjugation.
(b) The choice of the maximal torus T_0 and the base root datum $\mathscr{D}_0$ of $(G_{/\mathbb{Q}^{\mathrm{a}}}, T_{0/\mathbb{Q}^{\mathrm{a}}})$ is of course harmless: two different choices of $\mathscr{D}_0$ give two $\tilde{Y}$'s connected by an isomorphism compatible with the natural G-actions.
(c) The maximal torus T_0 and the base root datum $\mathscr{D}_0$ defines a "geometric base point" $\tilde{y}_0 \in \tilde{Y}(\mathbb{Q}^{\mathrm{a}})$ in $\tilde{Y}$ corresponding to the sextuple $(T_{0/\mathbb{Q}^{\mathrm{a}}}, X_0, D_0, X_0^\vee, D_0^\vee, \mathrm{Id}_{T_{0/\mathbb{Q}^{\mathrm{a}}}})$.

The quotient $Y = Y(G) := G \backslash \tilde{Y}$ is a 0-dimensional scheme over $\mathbb{Q}$. Let y_0 be the image of $\tilde{y}_0$ in $Y(\mathbb{Q}^{\mathrm{a}})$. Let $Y_0 = Y_0(G)$ be the connected component of Y such that $Y_0(\mathbb{Q}^{\mathrm{a}})$ contains the geometric point y_0 of Y. So $Y_0(G)$ is isomorphic to the spectrum of a number field $E(G)$, which is determined by G up to non-unique isomorphisms. This number field $E(G)$ is the analogue of the Shimura reflex field in our present situation, and can be described explicitly in terms of the *indexed root datum*; our terminology here follows [28, p. 271].

We have a natural representation $\tau\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathscr{D}_0)$ of the Galois group as symmetries of the *based* root system $\mathscr{D}_0$ of (G,T); see 15.5.2 on pp. 265–266 of [28]. (When G is semi-simple, this is the natural action of the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on the absolute Dynkin diagram of G.) As remarked in [28, p. 271], a different choice of T_0 leads to an isomorphic Galois action, up to conjugation by an element of $\mathrm{Aut}(\mathscr{D}_0)$. For any element $\gamma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$, γ fixes the element $y_0 \in Y_0(\mathbb{Q}^{\mathrm{a}})$ if and only if $\gamma\cdot\tilde{y}_0$ lies in the $G(\mathbb{Q}^{\mathrm{a}})$-orbit of $\tilde{y}_0 \in \tilde{Y}(\mathbb{Q}^{\mathrm{a}})$, which means according to the definition of the action of $\tau\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathscr{D}_0)$ that $\gamma \in \mathrm{Ker}(\tau)$. In other words, $E(G)$ is the largest subfield of $\mathbb{Q}^{\mathrm{a}}$ fixed by $\mathrm{Ker}(\tau)$. In particular $E(G)$ is a finite Galois extension field of $\mathbb{Q}$, and it contains the splitting field of the $\mathbb{Q}$-torus $Z(G)^0$, the neutral component of the center $Z(G)$ of G. We will call $E(G)$ the *Weyl reflex field* of G.

Let $\tilde{Y}_0 = \tilde{Y}(G)$ be the inverse image of Y_0 in $\tilde{Y}$. It is easy to see that the natural morphism $\tilde{Y}_0 \to Y_0$ is smooth, and is a homogeneous space for the G-action, as a sheaf for the flat topology. Similarly $\tilde{Y}$ is smooth over Y, and G operates transitively on the fibers of $\tilde{Y}_0 \to Y_0$.

(5.10) Let V' be the affine space over $\mathbb{Q}$ associated to $\mathrm{Lie}(G)$, and let $V = V(G) := \mathrm{Lie}(G)_{\mathrm{reg}} \subset V'$ be the dense open subscheme of V consisting of all *regular* elements; i.e. for every extension field $F/\mathbb{Q}$, $V(F)$ is the subset of $V'(F)$ consisting of all regular elements in the Lie algebra $\mathrm{Lie}(G)\otimes_{\mathbb{Q}}F$. Let $W = W(G)$ be the moduli space over $\mathbb{Q}$ such that for every $\mathbb{Q}$-scheme S, $W(S)$ is the set of all septuples

$$\big(T, X^*(T), D, X_*(T), D^\vee, \psi, v\big)$$

where

- $(T, X^*(T), D, X_*(T), D^\vee, \psi)$ is an element of $\tilde{Y}_0(S)$, and
- $v \in \Gamma(S, \mathrm{Lie}(T)\otimes_{\mathbb{Q}} \mathscr{O}_S) = \mathrm{Lie}(T)\otimes_{\mathbb{Q}}\Gamma(S,\mathscr{O}_S)$ is a global section of the sheaf of Lie algebras $\underline{\mathrm{Lie}}_{G_S/S}$ of the group scheme $G\times_{\mathrm{Spec}(\mathbb{Q})}S \to S$ such that every fiber v_s of v is a regular element of the Lie algebra $\mathrm{Lie}(G)\otimes_{\mathbb{Q}} \kappa(s)$ for every point $s \in S$ (i.e. $\mathrm{ad}(v_s)$ has maximal rank in the adjoint representation of $\mathrm{Lie}(G)$.)

Notice that $T = \mathrm{Z}_G(v)$ for each point $(T, X^*(T), D, X_*(T), D^\vee, \psi, v)$ of W. The group G operates naturally on the left of the scheme $W = W(G)$ by conjugation. We also have a natural *right* action of the absolute Weyl group $W(R_0)$ of the root system $(R_0, X_0, R_0^\vee, X_0^\vee)$ underlying $\mathscr{D}_0$, by "changing the marking", which commutes with the left G-action on the scheme W. The definition of this right action of $W(R_0)$ on the scheme W is as follows. Suppose that w is an element of the Weyl group $W(R_0)$, which induces an automorphism w_1 of the split torus T_1 with cocharacter group $X_0^\vee$. Then w sends an S-point $(T, X^*(T), D, X_*(T), D^\vee, \psi, v)$ of the scheme $W(G)$ to the point

$$\big(T, X^*(T), D\cdot w, X_*(T), D^\vee\cdot w, \psi\circ w_1, v\big)\,,$$

where the basis $D\cdot w$ and its dual $D^\vee\cdot w$ in the root system $R(G,T)$ are uniquely determined so that the isomorphism $\psi\circ w_1\colon T_1\times_{\mathrm{Spec}(\mathbb{Q})}S \xrightarrow{\sim} T$ induces an isomorphism

$$(X_0, D_0, X_0^\vee, D_0^\vee)_{/S} \xrightarrow{\sim} (X^*(T), D\cdot w, X_*(T), D^\vee\cdot w)$$

of based root data.

We have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\beta} \\ Y_0 & \xrightarrow[\text{can}]{} & \operatorname{Spec}\mathbb{Q} \end{array}$$

where

- f is the finite étale morphism given by

$$f\colon\ (T, X^*(T), D, X_*(T), D^\vee,\ \psi, v) \mapsto v\,,$$

- π is the morphism which sends a septuple

$$(T, X^*(T), D, X_*(T), D^\vee,\ \psi, v)$$

  to the image in Y_0 of the sextuple $(T, X^*(T), D, X_*(T), D^\vee,\ \psi)$ in $\tilde{Y}_0$,

- β is the structural morphism for V,
- can is the structural morphism for Y_0.

Clearly the morphism f is G-equivariant. The above diagram factors as

$$\begin{array}{ccccc} W & \xrightarrow{g} & V \times_{\operatorname{Spec}\mathbb{Q}} Y_0 & \xrightarrow{\alpha} & V \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow \\ Y_0 & \xrightarrow[=]{} & Y_0 = \operatorname{Spec} E(G) & \xrightarrow[\text{can}]{} & \operatorname{Spec}\mathbb{Q} \end{array}$$

where the right-half is the fiber product of β with the morphism "can".

(5.11) PROPOSITION. *Notation and assumption as in* 5.9–5.10.

(1) *The scheme W is smooth over Y_0, and all geometric fibers of π are irreducible.*

(2) *The right action of the Weyl group $W(R_0)$ on the scheme W makes W a right $W(R_0)$-torsor over $V_{E(G)}$. In other words $g\colon W \longrightarrow V \times_{\operatorname{Spec}(\mathbb{Q})} \operatorname{Spec}(E(G))$ is a finite étale Galois cover with the Weyl group of $(G_{/\mathbb{Q}^{\mathrm{a}}}, T_{0/\mathbb{Q}^{\mathrm{a}}})$ as its Galois group.*

(3) *Suppose we are given*

  - *a finite extension field $\tilde{E}$ of $E(G)$,*
  - *a finite subset Σ_1 of places of $\mathbb{Q}$ including the infinite place and a non-empty open subset $U_\wp \subset V(\mathbb{Q}_\wp)$ for each $\wp \in \Sigma_1$.*

  There exists an element $v \in V(\mathbb{Q})$ such that the follow statements hold.

  (a) *The image of v in $V(\mathbb{Q}_\wp)$ is in $U_\wp$ for every $\wp \in \Sigma_1$.*

(b) *The inverse image $f^{-1}(v)$ of v in W, which is a torsor for $W(R_0)$ by* (1)*, is the spectrum of a finite extension field $K_v/E(G)$ which is linearly disjoint from $\tilde{E}$ over $E(G)$.*

(c) *The centralizer subgroup $T_v := \mathrm{Z}_G(v)$, of v in G is a maximal $\mathbb{Q}$-subtorus in G such that the action of the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/E(G))$ on the character group $X^*(T_v)$ of T_v gives an isomorphism $\mathrm{Gal}(K_v/E(G)) \xrightarrow{\sim} W(G,T_v)$. In particular T_v is a Weyl subtorus of G.*

Note that the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/E(G))$ operates trivially on the character group of the $\mathbb{Q}$-subtorus $Z(G)^0 \subset T_v$.

PROOF. In the present set-up, the proof of 5.11 is essentially identical with the proof of [8, Thm. 5.1], except that Deligne used the moduli space of triples (T,s,v) where $s\colon \mathbb{G}_m \to G$ is a one-parameter subgroup in G, and we used the moduli space $W = W(G)$ of septuples defined in 5.10.

The second part of (1) that the fibers of π are geometrically irreducible is a consequence of the following facts in Lie theory.

(i) Over $\mathbb{C}$, every maximal torus in $G_{\mathbb{C}}$ is conjugate to $T_{0,\mathbb{C}}$.

(ii) Over $\mathbb{C}$, every point in $V(\mathbb{C})$ is conjugate to a regular element $V(\mathbb{C}) \cap \mathrm{Lie}(T_0) \otimes \mathbb{C}$ of the Lie algebra $\mathrm{Lie}(T_0) \otimes \mathbb{C}$, unique up to the action of the Weyl group $W(R_0)$.

(iii) The Weyl group $W(R_0)$ operates freely on the set $V(\mathbb{C}) \cap \mathrm{Lie}(T_0) \otimes \mathbb{C}$ of all regular elements of $\mathrm{Lie}(T_0) \otimes \mathbb{C}$.

These facts also imply that the Weyl group $W(R_0)$ operates simply transitively on each geometric fiber of $g : W \longrightarrow V \times_{\mathrm{Spec}(\mathbb{Q})} Y_0$, and the statement (2) follows. The first part of (1) that W is smooth over Y_0 follows from (2). Another way to prove the statement (1) is to consider the natural projection morphism $\mathrm{pr}\colon W \to \tilde{Y}_0$, which is a smooth surjective affine morphism whose fibers are schemes attached to the regular loci of the Lie algebra of maximal tori in G. These properties of pr imply the statement (1).

The statement (3) follows from the proof of Lemma 5.13 in [8]; one only has to replace the Hilbert irreducibility statement quoted there by the version of Hilbert irreducibility with weak approximation in 2.14; see also [12, Lemma 3.4] or [11, Thm. 1.3]. □

(5.12) REMARK. Given a Shimura input datum (G,X), Prop. 5.11 gives an easy way to produce lots of Weyl special points: choose a suitable open subset $U_\infty \subset \mathrm{Lie}(G_{\mathbb{R}})$ to make sure that the (set of $\mathbb{R}$-points of the) Weyl torus $Z_G(v)$ is compact modulo the central subtorus $Z(G)^0$, the neutral component of the center $Z(G)$ of G. Take a suitable $\mathbb{R}$-homomorphism $\mu\colon \mathbb{S} \to Z_G(v)$ such that the composition $j \circ \mu$ of μ with the inclusion $j\colon Z_G(v) \hookrightarrow G$ belongs to the hermitian symmetric space X. Then the composition $x_0 := j \circ \mu\colon \mathbb{S} \to G$ is a Weyl special point. Notice that Prop. 5.11 allows us to specify that the splitting field of the Weyl torus $Z_G(v)$ attached to the Weyl special point x_0 is linearly disjoint with any given finite extension field of the "generalized reflex field" $E(G)$, and satisfies specified local conditions at all finite places $\wp \in \Sigma_1$ encoded by the open subsets $U_\wp \subset V(\mathbb{Q}_p)$.

Lemma 5.13 below is a generalization of 2.8; it plays a role in the explicit description of Weyl subtori for classical groups.

(5.13) LEMMA. *Let F be a finite extension field of $\mathbb{Q}$, and let L be a finite extension field of F, both contained in an algebraic closure $\mathbb{Q}^{\mathrm{a}}$ of $\mathbb{Q}$. Let $\tilde{F}$ be the normal closure of $F/\mathbb{Q}$ in $\mathbb{Q}^{\mathrm{a}}$, and let $\tilde{L}$ be the normal closure of $L/\mathbb{Q}$ in $\mathbb{Q}^{\mathrm{a}}$. Let $\Phi := \mathrm{Hom}_{\mathrm{ring}}(F,\mathbb{Q}^{\mathrm{a}})$, and let $\Psi := \mathrm{Hom}_{\mathrm{ring}}(L,\mathbb{Q}^{\mathrm{a}})$. Let $\pi\colon \Psi \to \Phi$ be the natural surjection induced by "restriction to F".*

(a) *There is a natural map from* $\mathrm{Gal}(\tilde{L}/\mathbb{Q}) \hookrightarrow \mathrm{Perm}(\Psi)$ *to* $\mathrm{Gal}(\tilde{F}/\mathbb{Q}) \hookrightarrow \mathrm{Perm}(\Phi)$. *In particular* $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$ *is a subgroup of* $\mathrm{Perm}(\Psi/\Phi)$, *where* $\mathrm{Perm}(\Psi/\Phi)$ *is the subgroup of* $\mathrm{Perm}(\Psi)$ *consisting of all permutations* $\tau \in \mathrm{Perm}(\Psi)$ *such that there exists an element* $\sigma \in \mathrm{Perm}(\Phi)$ *with* $\pi\circ\tau = \sigma\circ\pi$. *Moreover we have* $\mathrm{Gal}(\tilde{L}/\tilde{F}) \hookrightarrow \mathrm{Perm}_{\Phi}(\Psi)$, *where* $\mathrm{Perm}_{\Phi}(\Psi)$ *is the subgroup of* $\mathrm{Perm}(\Psi)$ *consisting of all permutations* $\tau \in \mathrm{Perm}(\Psi)$ *preserving all fibers of* π. *In other words* $\mathrm{Perm}_{\Phi}(\Psi) = \mathrm{Ker}(\mathrm{Perm}(\Psi/\Phi) \to \mathrm{Perm}(\Phi))$.

(b) *Suppose that L_0/F is a subextension of L/F with $[L : L_0] = 2$. Let* $\Psi_0 := \mathrm{Hom}_{\mathrm{ring}}(L_0,\mathbb{Q})$, *and let $\pi_1\colon \Psi \twoheadrightarrow \Psi_0$ and $\pi_0\colon \Psi_0 \twoheadrightarrow \Phi$ be the natural surjections. Then we have a natural inclusion* $\mathrm{Gal}(\tilde{L}/\mathbb{Q}) \hookrightarrow \mathrm{Perm}(\Psi/\Psi_0/\Phi)$, *where* $\mathrm{Perm}(\Psi/\Psi_0/\Phi)$ *is the subgroup of* $\mathrm{Perm}(\Psi)$ *consisting of all elements* $\tau \in \mathrm{Perm}(\Psi)$ *such that there exist elements* $\tau_0 \in \mathrm{Perm}(\Psi_0)$ *and* $\sigma \in \mathrm{Perm}(\Phi)$ *satisfying* $\pi_1\circ\tau = \tau_0\circ\pi_1$ *and* $\pi_0\circ\tau_0 = \sigma\circ\pi_0$. *Moreover*

$$\mathrm{Gal}(\tilde{L}/\tilde{F}) \hookrightarrow \mathrm{Perm}_{\Phi}(\Psi/\Psi_0) := \mathrm{Ker}\left(\mathrm{Perm}(\Psi/\Psi_0/\Phi) \to \mathrm{Perm}(\Phi)\right).$$

(5.14) REMARK. Suppose that $[L : F] = g$ in 5.13 (a), then $\mathrm{Perm}_{\Phi}(\Psi) \cong (S_g)^{[F:\mathbb{Q}]}$. Similarly if $[L : F] = 2n$ in 5.13 (b), then $\mathrm{Perm}_{\Phi}(\Psi/\Psi_0) \cong ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$.

Weyl tori in classical groups.

In the rest of this section we illustrate the definition of Weyl tori and provide explicit descriptions of Weyl tori in semi-simple almost simple classical groups over $\mathbb{Q}$ except those of trialitarian type D_4. Here G being almost simple means that G is semi-simple and the only positive dimensional closed normal subgroup of G over $\mathbb{Q}$ is G itself. It is equivalent to saying that the adjoint group G^{ad} attached to G is simple over $\mathbb{Q}$. Equivalently, the Lie algebra $\mathrm{Lie}(G)$ of G is simple, in the sense that the only non-trivial Lie ideal of $\mathrm{Lie}(G)$ over $\mathbb{Q}$ is $\mathrm{Lie}(G)$ itself.

In view of 5.7 (a), we have the freedom of modifying any connected $\mathbb{Q}$-almost simple group up to central $\mathbb{Q}$-isogeny. In 5.15 to 5.18 below is a list of connected almost simple classical groups over $\mathbb{Q}$ up to central isogeny. Weyl subtori are given explicit descriptions in each case. Our reference is [19], especially Ch. VI §26 of [19]. Every central isogeny class of almost simple groups over $\mathbb{Q}$, except those of trialitarian D_4 type, appears in this list. The only overlaps are in the low-rank cases, described in [19, Ch. IV §15]. We refer to [31], [28, Ch. 17] and [19, Ch. VI §26] for proofs and further information.

REMARK. Semi-simple almost simple groups of trialitarian type D_4 are related to twisted composition algebras of octonion type. Their *indices* are of type ${}^3D_4$ or ${}^6D_4$ in the notation of [31, p. 58]. For more information we refer to [28, §17.9], Ch. VIII §36 and Ch. X §§42–44 of [19], and [29, Ch. 4].

(5.15) TYPE A_n, $n \geq 1$.

^1A_n Let F be a finite extension field of $\mathbb{Q}$ and let B be a central simple algebra over F with $\dim_F(B) = (n+1)^2$. The linear algebraic group SL_B over F attached to B is a form of SL_{n+1} over F whose F-points consists of all elements in $B^\times$ with reduced norm 1. Take G to be $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_B$. Then G is semi-simple and almost simple over $\mathbb{Q}$ of type A_n.

Every maximal $\mathbb{Q}$-subtorus T in G comes from a unique maximal commutative semi-simple subalgebra $L \subset B$ with $[L\colon F] = n+1$, such that $T(\mathbb{Q})$ consists of all elements $x \in L^\times$ with $\mathrm{Nm}_{L/F}(x) = 1$. More precisely $T = \mathrm{Res}_{F/\mathbb{Q}}\left(\mathrm{Ker}\left(\mathrm{Nm}_{L/F}\colon \mathrm{Res}_{L/F}\mathbb{G}_m \longrightarrow \mathbb{G}_{m/F}\right)\right)$, the Weil restriction of scalar of the F-torus $\mathrm{Ker}\left(\mathrm{Nm}_{L/F}\colon \mathrm{Res}_{L/\mathbb{F}}\mathbb{G}_m \longrightarrow \mathbb{G}_{m/F}\right)$. Equivalently

$$T = \mathrm{Ker}\left(\mathrm{Nm}_{L/F}\colon \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m\right).$$

A maximal torus T over $\mathbb{Q}$ attached to a maximal commutative semi-simple subalgebra L in B as in the previous paragraph is a Weyl subtorus of G if and only if L is a field and the Galois group $\mathrm{Gal}(\tilde{L}/\tilde{F})$ is isomorphic to $S_{n+1}^{[F:\mathbb{Q}]}$, the product of $[F:\mathbb{Q}]$-copies of the symmetric group S_{n+1}. Here $\tilde{L}$ is the normal closure of $L/\mathbb{Q}$, and $\tilde{F}$ is the normal closure of $F/\mathbb{Q}$. Note that $\mathrm{Gal}(\tilde{L}/\tilde{F})$ is naturally identified with a subgroup of $\mathrm{Perm}_\Phi(\Psi) \cong S_{n+1}^{[F:\mathbb{Q}]}$ by 5.13, where $\Phi = \mathrm{Hom}(F, \mathbb{Q}^{\mathrm{a}})$ and $\Psi = \mathrm{Hom}(L, \mathbb{Q}^{\mathrm{a}})$. The Weyl group for (G,T) is naturally isomorphic to $\mathrm{Perm}_\Phi(\Psi)$. The Weyl reflex field $E(G)$ is $\tilde{F}$.

^2A_n Let F be a finite extension field of $\mathbb{Q}$, E/F is a quadratic extension field of F, and let B be a central simple algebra over E with $\dim_K(B) = (n+1)^2$, and τ is an involution[16] of B whose restriction to E is the generator of $\mathrm{Gal}(E/F)$. The group $\mathrm{SU}(B,\tau)$ is an outer form of SL_{n+1} over F. Take G to be $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SU}(B,\tau)$; it is semi-simple and almost simple over $\mathbb{Q}$ of type A_n.

Every maximal $\mathbb{Q}$-torus T in G comes from a maximal commutative semi-simple E-subalgebra $L \subset B$ with $[L\colon E] = n+1$ which is stable under the involution τ, as follows. Denote by σ the automorphism of L induced by τ, and let $L_0 := L^\sigma$ be the F-subalgebra of L consisting of all elements of L fixed by τ. The maximal torus T is related to L by

$$T = \mathrm{Ker}\left((\mathrm{Nm}_{L/L_0}, \mathrm{Nm}_{L/E})\colon \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{L_0/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m\right).$$

In particular $T(\mathbb{Q})$ is the subgroup of $L^\times$ consisting of all elements $x \in L^\times$ such that

$$x \cdot \sigma(x) = \sigma(x) \cdot x = 1 \quad \textit{and} \quad \mathrm{Nm}_{L/E}(x) = 1.$$

A maximal $\mathbb{Q}$-subtorus $T \subset G$ as above is a Weyl subtorus if and only if L is a field and the Galois group $\mathrm{Gal}(\tilde{L}/\tilde{E})$ is isomorphic to the $S_{n+1}^{[F:\mathbb{Q}]}$, product of $[F:\mathbb{Q}]$-copies of the symmetric group S_{n+1}. Here $\tilde{L}$ is the normal closure of $L/\mathbb{Q}$ and $\tilde{E}$ is the normal closure of $E/\mathbb{Q}$. Notice that $L = L_0 \cdot E$, the compositum of L_0 with the quadratic extension E/F, and $\tilde{L}$ is equal to the compositum of $\tilde{E}$ with the normal closure M_0 of $L_0/\mathbb{Q}$. We have

$$\mathrm{Gal}(\tilde{L}/\tilde{E}) = \mathrm{Gal}(M_0 \cdot \tilde{E}/\tilde{E}) \hookrightarrow \mathrm{Gal}(M_0/\tilde{F}) \hookrightarrow \mathrm{Perm}_{\Phi_0}(\mathrm{Hom}(L_0, \mathbb{Q}^{\mathrm{a}})) \cong S_{n+1}^{[F:\mathbb{Q}]}$$

[16]We have followed the terminology in [19], so $\tau(xy) = \tau(y)\cdot\tau(x)$ for all $x,y \in B$ and $\tau\circ\tau = \mathrm{Id}_B$. Both "involution" and "anti-involution" are used in the literature for such anti-automorphism of rings.

by 5.13, where $\Phi_0 = \mathrm{Hom}(F, \mathbb{Q}^{\mathrm{a}})$. The Weyl group for (G,T) is naturally identified with $\mathrm{Perm}_{\Phi_0}(\mathrm{Hom}(L_0, \mathbb{Q}^{\mathrm{a}}))$. The condition for $T \subset G$ to be a Weyl subtorus is that both inclusions in the above displayed formula are equalities. Equivalently, the condition means that

$$\mathrm{Gal}(\tilde{L}/\tilde{F}) \cong \mathrm{Gal}(\tilde{L}/M_0) \times \mathrm{Gal}(\tilde{L}/\tilde{E}) \cong \mathrm{Gal}(\tilde{E}/\tilde{F}) \times \mathrm{Gal}(M_0/\tilde{F}) \cong \mathrm{Gal}(\tilde{E}/\tilde{F}) \times S_{n+1}^{[F:\mathbb{Q}]}\,.$$

The Weyl reflex field $E(G)$ in the present ${}^2A_n$ case is $\tilde{E}$.

PROOF. We will prove the ${}^2A_n$ case; the proof of the ${}^1A_n$ case is omitted because it is similar but simpler. First we show that every maximal $\mathbb{Q}$-subtorus T of G comes from a commutative semi-simple E-subalgebra L free of rank $n+1$ over E. This statement is easy to see after base change from F to $\mathbb{Q}^{\mathrm{a}}$; therefore it follows from descent. It remains to verify the stated necessary and sufficient condition for the maximal $\mathbb{Q}$-subtorus T to be a Weyl subtorus.

Recall that the character group of $\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m$ (resp. of $\mathrm{Res}_{L_0/\mathbb{Q}}\mathbb{G}_m$, resp. $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$) is

$$\mathbb{Z}^{\mathrm{Hom}_{\mathrm{ring}}(L,\mathbb{Q}^{\mathrm{a}})} \quad (\text{resp. } \mathbb{Z}^{\mathrm{Hom}_{\mathrm{ring}}(L_0,\mathbb{Q}^{\mathrm{a}})}, \quad \text{resp. } \mathbb{Z}^{\mathrm{Hom}_{\mathrm{ring}}(E,\mathbb{Q}^{\mathrm{a}})})$$

with the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ induced from the natural Galois action on

$$\Psi := \mathrm{Hom}_{\mathrm{ring}}(L, \mathbb{Q}^{\mathrm{a}}), \ \ (\text{resp. } \Psi_0 := \mathrm{Hom}_{\mathrm{ring}}(L_0, \mathbb{Q}^{\mathrm{a}}), \quad \text{resp. } \Phi := \mathrm{Hom}_{\mathrm{ring}}(E, \mathbb{Q}^{\mathrm{a}}))\,.$$

So the character group of the $\mathbb{Q}$-torus T is the quotient of $\mathbb{Z}^{\Psi}$ by the $\mathbb{Z}$-submodule generated by

$$\left\{ \sum_{\alpha\in\Psi,\ \alpha|_{L_0}=\alpha_0} \alpha \;\middle|\; \alpha_0 \in \Psi_0 \right\} \cup \left\{ \sum_{\beta\in\Psi,\ \beta|_{E}=\delta} \beta \;\middle|\; \delta \in \Phi \right\}.$$

It is clear that the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on the character group $X^*(T)$ of T factors through the finite quotient $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$ of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$. Moreover the $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\tilde{F})$ is the subgroup of $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$ of $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$ consisting of all elements of $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$ which stabilize every $\mathbb{Q}^{\mathrm{a}}$-simple factor of G. Let $\Phi_0 := \mathrm{Hom}_{\mathrm{ring}}(F, \mathbb{Q}^{\mathrm{a}})$. We have $\Psi \cong \Psi_0 \times_{\Phi_0} \Phi$, the fiber product of Ψ and Φ over Φ_0. Moreover we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gal}(\tilde{L}/M_0) \times \mathrm{Gal}(\tilde{L}/\tilde{E}) & \longrightarrow & \mathrm{Gal}(\tilde{E}/\tilde{F}) \times \mathrm{Gal}(M_0/\tilde{F}) \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Gal}(\tilde{L}/\tilde{F}) & & \mathrm{Perm}_{\Phi_0}(\Phi) \times \mathrm{Perm}_{\Phi_0}(\Psi_0) \xrightarrow{\ \simeq\ } (\mathbb{Z}/2\mathbb{Z})^{[F:\mathbb{Q}]} \times S_{n+1}^{[F:\mathbb{Q}]}
\end{array}$$

where all arrows are natural injections. The Weyl group $W(R(G,T))$ is naturally isomorphic to $\mathrm{Perm}_{\Phi_0}(\Psi_0)$. The inverse image of $\{1\} \times \mathrm{Perm}_{\Phi_0}(\Psi_0)$ in $\mathrm{Gal}(\tilde{L}/\tilde{F})$ is $\mathrm{Gal}(\tilde{L}/\tilde{E})$; it is the subgroup of $\mathrm{Gal}(\tilde{L}/\tilde{F})$ consisting of all elements of $\mathrm{Gal}(\tilde{L}/\tilde{F})$ (or of $\mathrm{Gal}(\tilde{L}/\mathbb{Q})$) whose action on $X^*(T)$ is induced by some element of the Weyl group $W(R(G,T))$ of (G,T). The condition for T to be a Weyl subtorus of G is equivalent to the condition that $\mathrm{Gal}(\tilde{L}/\tilde{E})$ has the same size as $W(R(G,T))$. The latter condition is clearly equivalent to

$$\mathrm{Gal}(\tilde{L}/\tilde{E}) \xrightarrow{\sim} \mathrm{Gal}(M_0/\tilde{F}) \xrightarrow{\sim} \mathrm{Perm}_{\Phi_0}(\Psi_0) \xrightarrow{\sim} S_{n+1}^{[F:\mathbb{Q}]}\,.$$

When these equivalent conditions hold, we have

$$\mathrm{Gal}(\tilde{L}/M_0) \times \mathrm{Gal}(\tilde{L}/\tilde{E}) \xrightarrow{\sim} \mathrm{Gal}(\tilde{L}/\tilde{F}) \xrightarrow{\sim} \mathrm{Gal}(\tilde{E}/\tilde{F}) \times \mathrm{Gal}(M_0/\tilde{F})\,. \quad \square$$

(5.16) TYPE B_n, $n \geq 1$.

Let F to be a finite extension field of $\mathbb{Q}$, V be a $2n+1$-dimensional vector space over F and let $q\colon V \to F$ be a non-degenerate quadratic form on V. Let $\mathrm{SO}(V,q)$ be the special orthogonal group over F attached to the quadratic space (V,q). Take G to be $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(V,q)$. Then G is semi-simple and almost simple over $\mathbb{Q}$ of type B_n.

Every maximal $\mathbb{Q}$-torus $T \subset G$ is related to a maximal commutative semi-simple F-subalgebra $L' \subset \mathrm{End}_F(V)$ of the form $L' = L \times F'$ such that

- $\dim_F(L) = 2n$,
- L is stable under the involution $\tau = \tau_q$ for the quadratic space (V,q),
- L is free of rank 2 over L_0, where $L_0 = \{x \in L \mid \tau(x) = x\}$,
- F' is a one-dimensional subspace of $\mathrm{End}_F(V)$ fixed by τ, and
- the image of any non-zero element of F' is a one-dimensional *anisotropic* subspace V' of (V,q).

The $\mathbb{Q}$-torus T attached to the commutative semi-simple subalgebra $L' \subset \mathrm{End}_F(V)$ above is

$$T = \mathrm{Ker}\big(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{L_0/\mathbb{Q}}\mathbb{G}_m\big) \times \{1\} \subset \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m = \mathrm{Res}_{L'/\mathbb{Q}}\mathbb{G}_m\,.$$

In particular $T(\mathbb{Q}) = \{x \in L^\times \mid x \cdot \sigma(x) = 1\}$, where σ is the restriction to L of the involution τ.

Such a maximal $\mathbb{Q}$-torus T attached to a commutative semi-simple subalgebra $L' \subset \mathrm{End}_F(V)$ is a Weyl subtorus of G if and only if L is a field and the Galois group $\mathrm{Gal}(\tilde{L}/\tilde{F})$ is isomorphic to $((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$, where $\tilde{L}$ is the normal closure of $L/\mathbb{Q}$ and $\tilde{F}$ is the normal closure of $F/\mathbb{Q}$. This condition means that the natural inclusion

$$\mathrm{Gal}(\tilde{L}/\tilde{F}) \hookrightarrow \mathrm{Perm}_\Phi(\Psi/\Psi_0) \cong ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$$

is an equality, where $\Phi = \mathrm{Hom}(F,\mathbb{Q}^{\mathrm{a}})$, $\Psi = \mathrm{Hom}(L,\mathbb{Q}^{\mathrm{a}})$ and $\Psi_0 = \mathrm{Hom}(L_0,\mathbb{Q}^{\mathrm{a}})$. The Weyl reflex field $E(G)$ is equal to $\tilde{F}$.

(5.17) TYPE C_n, $n \geq 1$.

Let F to be a finite extension field of $\mathbb{Q}$, and let B be a central simple F-algebra with $\dim_F(B) = 4n^2$, and let τ be a symplectic involution of B inducing id_F on F. The symplectic group $\mathrm{Sp}(B,\tau)$ is a form of Sp_{2n} over F whose F-points consists of all elements $x \in B^\times$ such that $x \cdot \tau(x) = \tau(x) \cdot x = 1$. Here we have followed the notation in [19, Ch. VI §23] so that

$$\mathrm{Sp}(B,\tau)(R) = \big\{\, b \in (B \otimes_F R)^\times \mid b \cdot \tau_R(b) = 1 \,\big\}$$

for all commutative F-algebra R. Take G to be $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}(B,\tau)$. Then G is semi-simple and almost simple over $\mathbb{Q}$ of type C_n.

Every maximal $\mathbb{Q}$-torus T is associated to a commutative semi-simple F-subalgebra $L \subset B$ with $\dim_F(L) = 2n$ stable under the involution τ such that L is a free rank 2 algebra over L_0, where $L_0 = \{x \in L \mid \tau(x) = x\}$. The maximal $\mathbb{Q}$-torus T attached to L is

$$T = \mathrm{Ker}\left(\mathrm{Nm}_{L/L_0}\colon \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{L_0/\mathbb{Q}}\mathbb{G}_m\right)\,.$$

For a maximal $\mathbb{Q}$-torus subtorus $T \subset G$ as above to be a Weyl subtorus of G, it is necessary and sufficient that $\mathrm{Gal}(\tilde{L}/\tilde{F})$ is isomorphic to $((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$, where $\tilde{L}$ is the normal closure of $L/\mathbb{Q}$ and $\tilde{F}$ is the normal closure of $F/\mathbb{Q}$. This condition means that the natural inclusion

$$\mathrm{Gal}(\tilde{L}/\tilde{F}) \hookrightarrow \mathrm{Perm}_{\Phi}(\Psi/\Psi_0) \cong ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$$

is an equality, where $\Phi = \mathrm{Hom}(F, \mathbb{Q}^{\mathrm{a}})$, $\Psi = \mathrm{Hom}(L, \mathbb{Q}^{\mathrm{a}})$, and $\Psi_0 = \mathrm{Hom}(L_0, \mathbb{Q}^{\mathrm{a}})$. The Weyl reflex field $E(G)$ is equal to $\tilde{F}$.

(5.18) TYPE D_n, $n \geq 2$, non-trialitarian.

Let F be a finite extension field of $\mathbb{Q}$, let B be a central simple F-algebra with $\dim_F(B) = 4n^2$, and let τ be an orthogonal involution on B which induces id_F on the center F of B. The orthogonal group $\mathrm{O}^+(B,\tau)$ attached to (B,τ) is semi-simple and absolutely almost simple over F of non-trialitarian type D_n; it is the neutral component of the F-group $\mathrm{O}^+(B,\tau)$ whose F-points consists of all elements $x \in B^\times$ such that $x \cdot \tau(x) = \tau(x) \cdot x = 1$. Take G to be $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{O}^+(B,\tau)$. Then G is semi-simple and almost simple over $\mathbb{Q}$ of type D_n.

Similar to the C_n case, every maximal $\mathbb{Q}$-torus T is associated to a commutative semi-simple F-subalgebra $L \subset B$ with $\dim_F(L) = 2n$ stable under the orthogonal involution τ such that L is a free rank-2 algebra over L_0, where $L_0 = \{x \in L \mid \tau(x) = x\}$. The maximal $\mathbb{Q}$-torus T attached to L is

$$T = \mathrm{Ker}\left(\mathrm{Nm}_{L/L_0} \colon \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m \longrightarrow \mathrm{Res}_{L_0/\mathbb{Q}} \mathbb{G}_m\right) .$$

If L is a field, denote by $\tilde{L}$ the normal closure of $L/\mathbb{Q}$ and let $\tilde{F}$ be the normal closure of $F/\mathbb{Q}$, We know from 5.13 that $\mathrm{Gal}(\tilde{L}/\tilde{F})$ is a subgroup of $\mathrm{Perm}_{\Phi}(\Psi/\Psi_0) \cong ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$, where $\Phi = \mathrm{Hom}(F, \mathbb{Q}^{\mathrm{a}})$, $\Psi_0 = \mathrm{Hom}(L_0, \mathbb{Q}^{\mathrm{a}})$ and $\Psi = \mathrm{Hom}(L, \mathbb{Q}^{\mathrm{a}})$.

A maximal $\mathbb{Q}$-torus $T \subset G$ as above is a Weyl subtorus if and only if L is a field and the Galois group $\mathrm{Gal}(M/\tilde{F})$ *contains* the subgroup $\left((\mathbb{Z}/2\mathbb{Z})_0^n \rtimes S_n\right)^{[F:\mathbb{Q}]}$ where $(\mathbb{Z}/2\mathbb{Z})_0^n$ is the kernel of the "summing coordinates homomorphism" from $(\mathbb{Z}/2\mathbb{Z})^n$ to $\mathbb{Z}/2\mathbb{Z}$ as in 2.9.

REMARK. The index of $\mathrm{O}^+(B,\tau)$ is either of type ${}^1D_n$ or ${}^2D_n$, depending on whether the *discriminant* of (B,τ) is the trivial element of $F^\times/F^{\times 2}$. See [19, Ch. II §7] for the definition of the discriminant of an involution of the first kind on a central simple algebra of even degree. We recall that the index of a type D_n group over F is of type ${}^1D_n$ (resp. ${}^2D_n$) in the notation of [31] if and only if the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/F)$ operates trivially (resp. non-trivially) on the Dynkin diagram of type D_n (which is the absolute Dynkin diagram for $\mathrm{O}^+(B,\tau)$). For a Weyl subtorus T in a type D_n group G as above, the Galois group $\mathrm{Gal}(M/\tilde{F}) \hookrightarrow ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{[F:\mathbb{Q}]}$ is equal to $\left((\mathbb{Z}/2\mathbb{Z})_0^n \rtimes S_n\right)^{[F:\mathbb{Q}]}$ if the index of G is of type ${}^1D_n$. The Weyl reflex field $E(G)$ is the normal closure over $\mathbb{Q}$ of the field number $F\left(\sqrt{\mathrm{disc}(B,\tau)}\right)$; it is equal to $\tilde{F}$ in the ${}^1D_n$ case.

§6. Subgroups containing a Weyl subtorus and obstructions

(6.1) The title of this section refers to a general phenomenon about Weyl subtori: if G is a connected semi-simple algebraic group over $\mathbb{Q}$ and T is a Weyl subtorus in G, then up to conjugation by $G^{\mathrm{ad}}(\mathbb{Q})$ there are very few closed $\mathbb{Q}$-subgroups H of G which contains T. If G is almost $\mathbb{Q}$-simple and is part of a Shimura input datum, then H can only be T or G unless G is of type C_n or B_n; see 6.3. In the cases when G is of type C_n or B_n, there is a collection of semi-simple subgroups H, of type A_1 or D_n respectively, which may contain Weyl subtori of G; see 6.4 and 6.5 for the C_n case, and 6.6 for the B_n case.

The question we need to address is this. Given a finite number of such subgroups $H_1, \ldots, H_r$, we want to produce a Weyl subtorus T in G such that no $G^{\mathrm{ad}}(\mathbb{Q})$-conjugate of T is contained in any of the subgroups H_i's. One way to solve this question is to find a convenient invariant $\mathrm{inv}(H_i)$, attached to the subgroups H_i's and use it as an obstruction in the following way: construct a Weyl subtorus T such that any subgroup H of G of the same type as the subgroups H_i's which contains a $G^{\mathrm{ad}}(\mathbb{Q})$-conjugate of T will have $\mathrm{inv}(H)$ different from all the $\mathrm{inv}(H_i)$'s. In the C_n case the invariant is a number field; we have seen this in §3 when G is a split symplectic group; see 6.8. The invariant in the B_n case turns out to be a *discriminant*, an element of $F^\times/F^{\times 2}$ for some number field F; see 6.9–6.12. The proof of Thm. 5.5, which uses these obstructions, is in 6.13.

(6.2) LEMMA. *Let R be an irreducible root system in a finite dimensional Euclidean vector space.*

(i) *The orbit of the Weyl group $W(R)$ of an element $\alpha \in R$ is the set of all elements in R of the same length as α.*

(ii) *There is only one Weyl orbit in R if R is of type A_n $(n \geq 1)$, D_n $(n \geq 2)$, E_6, E_7 or E_8.*

(iii) *There are two Weyl orbits in R if R is of type B_n $(n \geq 2)$, C_n $(n \geq 2)$, G_2 or F_4.*

PROOF. See [14, p. 40] for a proof of (i). The statements (ii) and (iii) follow from (i) and the classification of irreducible root systems. □

(6.3) LEMMA. *Let G be a connected semi-simple linear algebraic group over $\mathbb{Q}$ which is almost simple over $\mathbb{Q}$. Suppose that H is a non-trivial connected closed $\mathbb{Q}$-subgroup of G which contains a Weyl maximal torus T over $\mathbb{Q}$ but is not equal to T. If G is of type A_n, D_n, E_6, E_7 or E_8, then $H = G$.*

PROOF. The T-roots of the subgroup H is a subset $\Phi(H,T)$ of the set of all roots $R(G,T)$ of (G,T) which is stable under the natural action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$, hence is stable under the Weyl group $W(G,T)$ of (G,T). We know that $R(G,T)$ decomposes into a disjoint union of root systems of the same type: $R(G,T) = \sqcup_{i=1}^r R_i$, where all R_i's have the same Dynkin diagram. Moreover the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ induces a transitive action on the set of all connected components of $R(G,T)$ because G is almost simple over $\mathbb{Q}$.

Since the Weyl group $W(G,T)$ is the product of the Weyl groups $W(R_i)$ of the irreducible components of $R(G,T)$, every subset of $R(G,T)$ stable under $W(G,T)$ is the disjoint union of subsets of R_i stable under $W(R_i)$. In particular $\Phi(H,T) = \sqcup_{i=1}^r \Phi_i$, where each Φ_i is a subset of R_i stable under the action of $W(R_i)$, hence each Φ_i is either empty or equal to R_i. Since $\Phi(H,T)$ is stable under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ operates transitively on the set of connected components of $R(G,T)$, we conclude that $\Phi(H,T)$ is either empty or equal to $R(G,T)$. The assumption that $H \neq T$ means that $\Phi(H,T) \neq \emptyset$. It follows that $\Phi(H,T) = R(G,T)$, therefore $H = G$. □

REMARK. Lemma 6.3 leaves the cases when G is almost simple groups over $\mathbb{Q}$ of type B_n $(n \geq 2)$, C_n $(n \geq 2)$, G_2 or F_4. Since factors of type G_2, F_4 or E_8 do not appear for hermitian symmetric spaces, we will not discuss the G_2 or the F_4 case here.

(6.4) LEMMA. *Let G be a connected semi-simple simply connected algebraic group over $\mathbb{Q}$, which is almost $\mathbb{Q}$-simple and of type C_n, $n \geq 2$, as in* 5.17. *In other words $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{Sp}(B,\tau)$, where (B,τ) is a central simple algebra over a number field F and τ is a symplectic involution of the first kind on B. Suppose that $H \subsetneqq G$ is a positive dimensional closed subgroup of G over $\mathbb{Q}$ which contains a Weyl $\mathbb{Q}$-subtorus T of G. We know from* 5.17 *that T is attached to a subfield $L \subset B$ containing F and stable under the involution τ, with $[L:F] = 2n$. Let $L_0 = \{x \in L \mid \tau(x) = x\}$. If $H \neq T$, then H is isomorphic to the Weil restriction of scalars $\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H})$ of a connected semi-simple simply connected almost L_0-simple algebraic group $\mathcal{H}$ of type A_1 over L_0.*

PROOF. Recall that the roots in a split Sp_{2n} in standard coordinates are

$$\left\{\pm x_i \pm x_j \mid 1 \leq i < j \leq n\right\} \cup \left\{\pm 2x_i \mid 1 \leq i \leq n\right\}.$$

There are two Weyl orbits, the set of all long roots $2x_i$'s and the set of all short roots $x_i \pm x_j$'s. It is clear that each long root is the sum of two short roots; for instance $2x_1 = (x_1 + x_2) + (x_1 - x_2)$.

Write the root system $R(G,T)$ as a disjoint union of connected components: $R(G,T) = \sqcup_{i=1}^r R_i$, where each irreducible component R_i is an irreducible root system of type C_n. The Galois group $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ operates on $R(G,T)$ and the induced $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$-action on $\pi_0(R(G,T))$ is transitive.

The set of all T-roots $\Phi(H,T)$ is a subset of $R(G,T)$ stable under the action of $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$. Standard Lie theory tells us that if the sum of two elements $\alpha, \beta \in \Phi(H,T)$ belongs to $R(G,T)$, then $\alpha + \beta$ is an element of $\Phi(H,T)$; cf. the proof of 3.4.

Write $\Phi(H,T) = \sqcup_{i=1}^r \Phi_i$ with $\Phi_i \in R_i$. Then each Φ_i is stable under the action of the Weyl group $W(R_i)$. So there are four possibilities for each subset $\Phi_i \subset R_i$: (1) $\emptyset$, (2) R_i, (3) all long roots in R_i, (4) all short roots in R_i. Since $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ operates transitively on $\pi_0(R(G,T))$, one of the four possibilities for Φ_i must hold simultaneously for all $i = 1, \ldots, r$. The property of $\Phi(H,T)$ reviewed in the previous paragraph implies that case (4) does not happen. The assumption that $T \subsetneqq H \subsetneqq G$ implies that $\Phi(H,T)$ is the set of all long roots in $R(G,T)$. That means that H is of the form $\mathrm{Res}_{L_0/\mathbb{Q}}\mathcal{H}$ as described in the statement of 6.4. □

(6.5) COROLLARY. *Notation as in* 6.4. *Let G^{ad} be a connected semi-simple adjoint $\mathbb{Q}$-simple group of type C_n over $\mathbb{Q}$, $n \geq 2$. In other words $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}}\mathcal{G}$ for a connected semi-simple absolutely simple group $\mathcal{G}$ of type C_n over a finite extension field F of $\mathbb{Q}$. Let $\tilde{F}$ be the normal closure of $F/\mathbb{Q}$. Let T^{ad} be a Weyl $\mathbb{Q}$-subtorus of G^{ad}, so that the action of* $\mathrm{Gal}(\mathbb{Q}^a/\tilde{F})$ *on $X^*(T^{\mathrm{ad}})$ gives a surjection $\rho_{T^{\mathrm{ad}},\tilde{F}} \colon \mathrm{Gal}(\mathbb{Q}^a/\tilde{F}) \twoheadrightarrow W(G^{\mathrm{ad}}, T^{\mathrm{ad}})$. Suppose that T^{ad} is contained in a closed subgroup H^{ad} in G^{ad} and $T^{\mathrm{ad}} \subsetneqq H^{\mathrm{ad}} \subsetneqq G$. Then H^{ad} is isomorphic to the Weil restriction of scalars $\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H}^{\mathrm{ad}})$ of a connected adjoint semi-simple L_0-simple algebraic group $\mathcal{H}$ of type A_1 over a number field L_0 such that*

(i) *L_0 is isomorphic to a subfield of $\tilde{L}$, the finite Galois extension of $\tilde{F}$, which is the the largest subfield of $\mathbb{Q}^a$ fixed by the kernel of the Galois representation*

$$\rho_{T^{\mathrm{ad}},\tilde{F}} \colon \mathrm{Gal}(\mathbb{Q}^a/\tilde{F}) \twoheadrightarrow W(G^{\mathrm{ad}}, T^{\mathrm{ad}}).$$

(ii) *$\tilde{L}$ is the normal closure over $\mathbb{Q}$ of a quadratic extension field L/L_0, and the $\mathbb{Q}$-torus T^{ad} is isogenous to* $\mathrm{Ker}\left(\mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \longrightarrow \mathrm{Res}_{L_0/\mathbb{Q}}\mathbb{G}_m\right)$.

(iii) $[L_0:\mathbb{Q}] = n\cdot\frac{\dim(G^{\mathrm{ad}})}{(2n+1)n} = \frac{\dim(G^{\mathrm{ad}})}{2n+1}$.

Note that $(2n+1)n$ is the dimension of the symplectic group over $\mathbb{C}$ in $2n$ variables.

PROOF. Let $\pi\colon G\to G^{\mathrm{ad}}$ be the simply connected cover of G^{ad} over $\mathbb{Q}$. Let T be the Weyl $\mathbb{Q}$-subtorus of G such that π induces an isogeny $\pi|_T\colon T\to T^{\mathrm{ad}}$. Let H be the connected closed $\mathbb{Q}$-subgroup of G such that π induces a central isogeny from H to H^{ad}. We see from 6.4 that H is of the form $H=\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H})$, where $\mathcal{H}$ is a connected simply connected almost L_0-simple group of type A_1. So $H^{\mathrm{ad}}\cong\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H}^{\mathrm{ad}})$, where $\mathcal{H}^{\mathrm{ad}}=\mathcal{H}/Z(\mathcal{H})$ is the adjoint group over L_0 associated to $\mathcal{H}$. Clearly L_0 is a subfield of $\tilde{L}$ and $[L_0:\mathbb{Q}]=n\cdot[F:\mathbb{Q}]$ in the notation of 6.4. From $G=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}(B,\tau)$ we see that $\dim(G)=[F:\mathbb{Q}]\cdot(2n+1)n$. So $[L_0:\mathbb{Q}]=n\cdot\frac{\dim(G)}{(2n+1)n}=\frac{\dim(G^{\mathrm{ad}})}{2n+1}$. ∎

(6.6) LEMMA. *Let G be a connected semi-simple almost $\mathbb{Q}$-simple group of type B_n as described in 5.16. In other words $G=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(V,q)$, where (V,q) is a non-degenerate quadratic space over F with $\dim_F(V)=2n+1$ and F is a finite extension field of $\mathbb{Q}$. Suppose that $H\subsetneqq G$ is a positive dimensional connected closed subgroup of G over $\mathbb{Q}$ which contains a Weyl $\mathbb{Q}$-subtorus T of G but is not equal to T. Let $L'=L\times F'$ be the commutative semi-simple algebra over F attached to T as in 5.16. In particular L is a field, $[L:F]=2n$, $T(\mathbb{Q})$ is naturally identified with a subgroup of $L^\times\times\{1\}\subset(L\times F')^\times$, and the subspace of V fixed by all elements of $T(\mathbb{Q})$ is a one-dimensional anisotropic subspace V' of V over F. Then H is equal to $G=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(V'^{\perp},q|_{V'^{\perp}})$, where $V'^{\perp}$ is the orthogonal complement to V' in V.*

PROOF. Recall that the roots in a split $\mathrm{SO}(2n+1)$ in standard coordinates are

$$\left\{\pm x_i\pm x_j \mid 1\le i<j\le n\right\}\cup\left\{\pm x_i \mid 1\le i\le n\right\}.$$

There are two Weyl orbits, the set of all long roots $\pm x_i\pm x_j$ and the set of all short roots $\pm x_i$. It is clear that each long root is a sum of two short roots; for instance x_1-x_2 is the sum of x_1 and $-x_2$.

Write the root system $R(G,T)$ as a disjoint union of connected components: $R(G,T)=\sqcup_{i=1}^r R_i$. We have a natural action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on $R(G,T)$ which induces a transitive action on the set of all connected components of $R(G,T)$.

The set of all T-roots $\Phi(H,T)$ for H is a subset of $R(G,T)$ stable under the natural action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$, therefore also stable under the action of the Weyl group $W(G,T)$. As in 6.3 and 6.4 we have $\Phi(H,T)=\sqcup_{i=1}^r\Phi_i$, where Φ_i is a subset of R_i stable under the Weyl group $W(R_i)$. We also know from standard Lie theory that if α,β are two elements of $\Phi(H,T)$ such that $\alpha+\beta\in R(G,T)$, then $\alpha+\beta\in\Phi(H,T)$; cf. the proof of 3.4.

Elementary argument as in 6.3 shows that each Φ_i is either empty, or equal to R_i, or equal to the set of all long roots in R_i, or equal to the set of all short roots in R_i. Because each long root in R_i is a sum of two short roots, if Φ_i contains all short roots of R_i it must also contain all long roots of R_i. Together with the fact that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ operates transitively on $\pi_0(R(G,T))$, we see that there are only three possibilities for $\Phi(H,T)$: $\emptyset$, $R(G,T)$, and the set of all long roots in $R(G,T)$. The assumption that $T\subsetneqq H\subsetneqq G$ rules out the first two possibilities. The last possibility means exactly that H is a type D_n subgroup as described in the statement of 6.6 ∎

(6.7) LEMMA. *For each $i = 1,\dots,N$ let G_i be a connected almost $\mathbb{Q}$-simple semi-simple algebraic group over $\mathbb{Q}$ not of type G_2 or F_4. Let $G = \prod_{i=1}^N G_i$. Let T be a Weyl $\mathbb{Q}$-subtorus of G, so that $T = \prod_{i=1}^N T_i$, where T_i is a Weyl $\mathbb{Q}$-subtorus for each i. Let H be a connected reductive subgroup of G which contains T.*

(1) *Suppose that G_i is a special orthogonal group of the form $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(V,q)$ as in 6.6 for each i such that G_i is of type B_n with $n \geq 2$, and that G_i is a symplectic group of the form $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}(B,\tau)$ as in 6.4 for each i such that G_i is of type C_n with $n \geq 2$. Then $H = \prod_{i=1}^N H_i$ is a product subgroup of G, where the factor H_i is a reductive $\mathbb{Q}$-subgroup of G_i for each i, and the following properties hold.*

   - *If G_i is of type A_n, D_n, E_6, E_7 or E_8, then H_i is equal to either T_i or G_i*
   - *If G_i is of type B_n with $n \geq 2$, then H_i is equal to T_i or G_i, or is a type D_n subgroup of the form $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(V'^{\perp}, q|_{V'^{\perp}})$ as in 6.6.*
   - *If G_i is of type C_n with $n \geq 2$, then H_i is equal to T_i or G_i, or is a type A_1 subgroup of the form $\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H})$ as in 6.4.*

(2) *Suppose that G_i has the form $G_i = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{PGO}^+(V,q))$ as in 6.6 for each factor G_i of type B_n with $n \geq 2$, and G_i is of the form $G_i = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGSp}(B,\tau)$ where (B,τ) is as in 6.4[17] for each i such that G_i is of type C_n with $n \geq 2$. Then $H = \prod_{i=1}^N H_i$ is a product subgroup of G, where H_i is a reductive $\mathbb{Q}$-subgroup of G_i for each i such that the following properties hold.*

   - *If G_i is of type A_n, D_n, E_6, E_7 or E_8, then H_i is equal to either T_i or G_i*
   - *If G_i is of type B_n with $n \geq 2$, then H_i is equal to T_i or G_i, or is a type D_n subgroup of the form $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGO}^+(V'^{\perp}, q|_{V'^{\perp}})$ with the notation in 6.6.*
   - *If G_i is of type C_n with $n \geq 2$, then H_i is equal to T_i or G_i, or is a type A_1 subgroup of G_i of the form $\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H}^{\mathrm{ad}})$, where $\mathcal{H}^{\mathrm{ad}}$ is the adjoint group of the group $\mathcal{H}$ in 6.4.*

PROOF. This is a consequence of 6.3, 6.4, 6.5 and 6.6. □

In 6.8 and 6.12 below, we show how to construct a $\mathbb{Q}$-Weyl subtorus T in a semi-simple almost $\mathbb{Q}$-simple group G which is not $G^{\mathrm{ad}}(\mathbb{Q})$-conjugate to a subgroup in any of a given finite family $\{H_j\}$ of proper subgroups of G. We begin with the C_n-case similar to the proof of Thm. 3.1, where the obstruction comes from the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})$ on the set of geometrically connected components $\pi_0^{\mathrm{geom}}(H_j)$ of the subgroups H_j. The B_n case is treated in 6.12, where each of the subgroups H_j is of type D_n and we use the discriminant class of quadratic forms as an obstruction.

(6.8) LEMMA. *Let G^{ad} be a connected semi-simple adjoint group over $\mathbb{Q}$ which is $\mathbb{Q}$-simple and of type C_n, $n \geq 2$. Let $\{H_j \mid j = 1,\dots,s\}$ be a finite family of connected closed $\mathbb{Q}$-subgroups of G^{ad}. Suppose that for each $j = 1,\dots,s$, there is a number field $L_{0,j}$ and a connected semi-simple group $H_{1,j}$ over $L_{0,j}$ satisfying the following properties.*

(i) *$H_{1,j}$ is adjoint and $L_{0,j}$-simple of type A_1 for all $j = 1,\dots,r$.*

[17] We refer to [19, Ch. VI §23] for the definition and general properties of the groups $\mathrm{PGO}^+(V,q)$ and $\mathrm{PGSp}(B,\tau)$.

(ii) $[L_{0,j}:\mathbb{Q}] = \frac{\dim(G^{\mathrm{ad}})}{2n+1}$ *for all* $j = 1,\ldots,r$.

(iii) H_j *is isomorphic to* $\mathrm{Res}_{L_{0,j}}(H_{1,j})$ *for all* $j = 1,\ldots,s$.

Let $\tilde{F}$ *be the normal closure over* $\mathbb{Q}$ *of the number field* F. *Let* $\tilde{E}$ *be the smallest finite Galois extension of* $\mathbb{Q}$ *which contains the number fields* $L_{0,j}$ *for all* $j = 1,\ldots,s$. *Let* U_∞ *be a non-empty open subset of the set* $(\mathrm{Lie}(G^{\mathrm{ad}})\otimes_{\mathbb{Q}}\mathbb{R})_{\mathrm{reg}}$ *of all regular elements in* $(\mathrm{Lie}(G^{\mathrm{ad}})\otimes_{\mathbb{Q}}\mathbb{R})$. *Suppose that* v *is a regular element of* $\mathrm{Lie}(G^{\mathrm{ad}})$ *with the following properties. .*

(a) *The image of* v *in* $\mathrm{Lie}(G^{\mathrm{ad}})\otimes_{\mathbb{Q}}\mathbb{R}$ *lies in the open subset* U_∞.

(b) *The schematic inverse image* $f^{-1}(v)$ *of* v *in the scheme* W *under the morphism* $f\colon W \to V$ *in* 5.11 *is isomorphic to the spectrum of a field* K_v.

(c) *The field* K_v *in* (b) *and the compositum* $\tilde{E}\cdot\tilde{F}$ *of the fields* $\tilde{E}$ *and* $\tilde{F}$ *are linearly independent over* $\tilde{F}$.

Then the centralizer subgroup of v *in* G^{ad} *is a Weyl* $\mathbb{Q}$*-subtorus* T^{ad} *of* G^{ad}*, and* T^{ad} *is not contained in any* $G^{\mathrm{ad}}(\mathbb{Q})$*-conjugate of* H_j *for any* $j = 1,\ldots,s$. *Note that the existence of a regular element of* $\mathrm{Lie}(G^{\mathrm{ad}})$ *satisfying conditions* (a), (b) *and* (c) *follows from* 5.11.

PROOF. We keep the notation in 6.4 and 6.5, so the simply connected cover G of G^{ad} is of the form $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Sp}(B,\tau)$, where (B,τ) is a central simple algebra over a number field F and τ is a symplectic involution of the first kind on B. Note that $\tilde{F}$ is equal to the Weyl reflex field $E(G^{\mathrm{ad}})$.

Let $T^{\mathrm{ad}} = T_v$ be the centralizer subgroup of v in G^{ad}; it is a maximal $\mathbb{Q}$-torus because v is regular. The assumption (b) implies that the natural Galois representation $\rho_{T^{\mathrm{ad}},\tilde{F}}\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\tilde{F}) \xrightarrow{\sim} W(G^{\mathrm{ad}},T^{\mathrm{ad}})$ is a surjective group homomorphism whose kernel is $\mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/K_v)$. So T^{ad} is a Weyl $\mathbb{Q}$-subtorus in G^{ad}. The assumption (c) implies that none of the fields $L_{0,j}$ can be embedded in K_v. So the Weyl $\mathbb{Q}$-subtorus T^{ad} is not contained in any $G^{\mathrm{ad}}(\mathbb{Q})$-conjugate of H_j by 6.5. □

(6.9) Suppose that (V,q) is a non-degenerate quadratic form over $\mathbb{Q}$, $\dim_F(V)$ is odd, and $\dim_F(V) \geq 3$.[18] Let $\mathcal{G} := \mathrm{SO}(V,q)$, $G := \mathrm{Res}_{F/\mathbb{Q}}(\mathcal{G})$, $\mathcal{G}^{\mathrm{ad}} = \mathrm{PGO}^+(V,q)$ be the adjoint group of $\mathcal{G}$, and let $G^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}}(\mathcal{G}^{\mathrm{ad}})$ be the adjoint group of G.

We know that every maximal $\mathbb{Q}$-torus T of G has the form $T = \mathrm{Res}_{F/\mathbb{Q}}(\mathcal{T})$, where $\mathcal{T}$ is a maximal F-torus of $\mathcal{G}$. Moreover for every maximal F-torus $\mathcal{T}$ of $\mathcal{G}$, the F-subspace $V' = V^{\mathcal{T}}$ of V consisting of all elements of V fixed by $\mathcal{T}$ is a one-dimensional F-vector subspace of V which is anisotropic for the quadratic form q. Let $V'^{\perp}$ be the orthogonal complement of $V'^{\perp}$ with respect to q. An easy calculation shows that the discriminant of the quadratic space $(V'^{\perp}, q|_{V'^{\perp}})$, as an invariant of the maximal F-torus S, does not change under conjugation by elements of $\mathrm{GO}(V,q)(F)$; see 6.10 below. Since $\mathcal{G}^{\mathrm{ad}}$ is naturally isomorphic to the neutral component of $\mathrm{PGO}(V,q)$, the above invariant does not change under conjugation by elements of $\mathcal{G}^{\mathrm{ad}}(F)$.

Suppose that the maximal $\mathbb{Q}$-subtorus $T \subset G$ comes from a commutative semi-simple subalgebra $L' = L \times F' \subset \mathrm{End}_F(V)$ as in 5.16 and L *is a field*. Recall that L is stable under the involution $\tau = \tau_q$ on $\mathrm{End}_F(V)$ attached to the quadratic form q, $\dim_F(F') = 1$, the image of any non-zero element of F' is the one-dimensional anisotropic subspace $V' \subset V$, and $T(\mathbb{Q}) = \mathcal{T}(F)$ is isomorphic to

[18] We will use 6.10–6.12 only when $\dim_F(V) \geq 7$, corresponding to Dynkin diagrams of type B_n with $n \geq 3$.

$\{x \in L^\times \mid x \cdot \tau(x) = 1\}$. In 6.11 we show that the discriminant of $(V'^{\perp}, q|_{V'^{\perp}})$ can be easily "read off" from the field extensions $L/L_0/F$, where L_0 is the subfield of L consisting of all elements of L fixed by the involution σ. This fact will not be needed in the rest of this article.

(6.10) LEMMA. *Notation as in 6.9. Suppose that $\gamma \in \mathrm{GL}_F(V)$ is an F-linear automorphism of V and c is an element of $F^\times$ such that*

$$q(\gamma(v)) = c \cdot q(v) \quad \text{for all } v \in V\,.$$

Let $u \in V'$ be a non-zero element of V'. Then $c = q(\gamma(u)) \cdot q(u)^{-1} \in F^{\times 2}$ and

$$\mathrm{disc}(\gamma(V')^{\perp}, q|_{\gamma(V')^{\perp}}) = \mathrm{disc}(V'^{\perp}, q|_{V'^{\perp}}) \in F^\times/F^{\times 2}.$$

PROOF. The assumption on γ implies that $\mathrm{disc}(V,q) = c^{\dim_F(V)}\,\mathrm{disc}(V,q)$. We know that $\dim_F(V)$ is odd, so $c \in F^{\times 2}$. It follows that

$$\mathrm{disc}(\gamma(V')^{\perp}, q|_{\gamma(V')^{\perp}}) = c^{\dim_F(V)-1} \cdot \mathrm{disc}(V'^{\perp}, q|_{V'^{\perp}}) = \mathrm{disc}(V'^{\perp}, q|_{V'^{\perp}}) \in F^\times/F^{\times 2}. \quad \square$$

(6.11) REMARK. Notation as in the last paragraph of 6.9; see also 5.16. Let L_0 be the subfield of the field L fixed by the involution σ of L induced by τ. Write $L = L_0(\sqrt{D})$ with $D \in L_0$. Then the discriminant class of the quadratic space $(V'^{\perp}, q|_{V'^{\perp}})$ over F is the element $\mathrm{Nm}_{L_0/F}(D) \cdot F^{\times 2}$ in $F^\times/F^{\times 2}$.

PROOF. We follow the sign convention for the *discriminant* in [19, 7A. p. 80], where the discriminant $\mathrm{disc}(U,f)$ of a non-singular quadratic space (U,f) over a field k with $\mathrm{char}(k) \neq 2$ as follows. Let $b\colon U \times U \to k$ be the symmetric bilinear form attached to f, given by $b(x,y) := f(x+y) - f(x) - f(y)$. Let $(u_1, \ldots, u_m)$ be an arbitrary k-basis of U. Then

$$\mathrm{disc}(U,f) = (-1)^{m(m-1)/2} \det\big(b(u_i,u_j)\big)_{1 \le i,j \le m} \in k^\times/k^\times 2\,.$$

According to [19, Prop. 7.3 (3)], when $m = 2n$ is even the discriminant $\mathrm{disc}(U,f)$ of the quadratic space (U,f) coincides with the discriminant $\mathrm{disc}(\mathrm{End}_F(U), \tau_f) \in k^\times/k^{\times 2}$ of the central simple algebra with involution $(\mathrm{End}_F(U), \tau_f)$, where $\mathrm{disc}(\mathrm{End}_F(U), \tau_f)$ is defined as

$$\mathrm{disc}(\mathrm{End}_F(U), \tau_f) = (-1)^n\,\mathrm{Nrd}_{\mathrm{End}_F(U)/F}(a) \in k^\times/k^{\times 2}$$

for any element $a \in \mathrm{End}_F(U)^\times$ satisfying $\tau_f(a) = -a$.

Apply the above to $(U,f) = (V'^{\perp}, q|_{V'^{\perp}})$ and the skew-symmetric element

$$\sqrt{D} \in L^\times \subset \mathrm{End}_F(U'^{\perp})^\times,$$

we see that the discriminant class of $(V'^{\perp}, q|_{V'^{\perp}})$ is represented by the element

$$(-1)^n \mathrm{Nrd}_{\mathrm{End}(V'^{\perp})/F}(\sqrt{D}) = (-1)^n\,\mathrm{Nm}_{L/F}(\sqrt{D}) = (-1)^n\,\mathrm{Nm}_{L_0/F}(-D) = \mathrm{Nm}_{L_0/F}(D)$$

in $F^\times$. $\square$

Let G be a semi-simple $\mathbb{Q}$-simple adjoint algebraic group over $\mathbb{Q}$ of type B_n, $n \geq 2$. In other words there exists a non-degenerate quadratic space (V,q) of dimension $2n+1$ over a number field F such that $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{PGO}^+(V,q))$. Write $\mathcal{G} := \mathrm{SO}(V,q)$, and let $\tilde{G} := \mathrm{Res}_{F/\mathbb{Q}}(\mathcal{G})$. Denote by $\mathrm{Lie}(G)_{\mathrm{reg}}$ the affine $\mathbb{Q}$-scheme such that $\mathrm{Lie}(G)_{\mathrm{reg}}(E)$ is the set of all regular elements of $\mathrm{Lie}(G)\otimes_{\mathbb{Q}}E$ for every extension field $E/\mathbb{Q}$. Similarly, denote by $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}$ the affine F-scheme such that $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(E)$ is the set of all regular elements of $\mathrm{Lie}(\mathcal{G})\otimes_F E$ for every extension field E/F. Note that we have a natural isomorphism $\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}) \cong \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F)$.

(6.12) PROPOSITION. *Notation as above. Let $V_1',\dots,V_r'$ be one-dimensional anisotropic subspaces of V over F. For each $i=1,\dots,r$, let $H_i = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGO}^+(V_i'^{\perp}, q|_{V_i'^{\perp}})$ be a standardly embedded adjoint $\mathbb{Q}$-simple subgroup of G of type D_n. Let $\wp_1,\dots,\wp_r$ be r distinct finite places of F. There exist non-empty open subsets $U_{\wp_i} \subset \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_i})$ for $i=1,\dots,r$ satisfying the following condition.*

> *Suppose that $v \in \mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}) = \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F)$ is a regular element of the Lie algebra of G such that the image of v in $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_i})$ lies in the open subset $U_{\wp_i} \subset \mathrm{Lie}(\mathcal{G})(F_{\wp_i})$ for every $i=1,\dots,r$ and the centralizer subgroup $Z_G(v)$ of v in G is a maximal $\mathbb{Q}$-subtorus T_v in G. Then no $G(\mathbb{Q})$-conjugate of T_v is contained in the subgroup H_i for any $i=1,\dots,r$.*

Note that the existence of an element $v \in \mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q})$ such that image of v in $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_i})$ lies in $U_{\wp_i}$ for all $i=1,\dots,r$ and $Z_G(v)$ is a Weyl subtorus of G follows from 5.11.

PROOF. Pick generators v_i of the one-dimensional subspace V_i' for $i=1,\dots,r$. Let

$$a_i := q(v_i)\cdot F^{\times 2} \in F^{\times}/F^{\times 2}.$$

Then we have

$$\mathrm{disc}(V_i'^{\perp}, q|_{V_i'^{\perp}}) = a_i \cdot \mathrm{disc}(V,q) \in F^{\times}/F^{\times 2}.$$

It is well known that every non-degenerate quadratic form in at least 5 variables over a p-adic local field is isotropic; see [21, 63:22] or [26, Ch. 6, Thm. 4.2]. Therefore for every $i=1,\dots,r$, every element of $F_{\wp_i}^{\times}$ is the norm of some element of $V\otimes_F F_{\wp_i}$ under the quadratic form q. Let u_i be an anisotropic element of $V\otimes_F F_{\wp_i}$ such that $q(u_i) \not\equiv q(v_i) \pmod{F_{\wp_i}^{\times 2}}$.

For each $i=1,\dots,r$, let $\mathcal{T}_i$ be a maximal $F_{\wp_i}$-subtorus of $\mathrm{SO}\left((F_{\wp_i}u_i)^{\perp}, q|_{(F_{\wp_i}u_i)^{\perp}}\right)$. Then $\mathcal{T}_i$ is a maximal $F_{\wp_i}$-subtorus of $\mathcal{G}\times_{\mathrm{Spec}(F)}\mathrm{Spec}(F_{\wp_i})$ which fixes u_i, because the rank of $\mathrm{SO}\left((F_{\wp_i}u_i)^{\perp}, q|_{(F_{\wp_i}u_i)^{\perp}}\right)$ is equal to $(\dim_F(V)-1)/2$, which is the same as the rank of $\mathrm{SO}(V,q)$. Pick a non-zero regular element $v_i \in \mathrm{Lie}(\mathcal{T}_i)(F_{\wp_i})$ for each $i=1,\dots,s$. Let $U_{\wp_i}$ be a sufficiently small neighborhood of v_i in $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_i})$ such that the one-dimensional subspace of $V\otimes_F F_{\wp_i}$ fixed by the centralizer $Z_{\mathcal{G}}(w)$ of any element $w \in U_{\wp_i}$ is generated by an element u_w with

$$q(u_w) \equiv q(u_i) \pmod{F_{\wp_i}^{\times 2}}.$$

Such an open subset $U_{\wp_i}$ exists because the above congruence is an open condition on $w \in \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_i})$.

For $i = 1, \dots, r$, let $\tilde{H}_i$ be the inverse image of H_i in $\tilde{G}$. Suppose that v is an element of $\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}) = \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F)$ as in the statement of 6.12. In other words the image of v in $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_i})$ lies the open subset $U_{\wp_i}$ for every $i = 1, \dots, r$ and the centralizer subgroup $Z_G(v)$ of v is a maximal $\mathbb{Q}$-subtorus $\tilde{T}_v$ in $\tilde{G}$. Then the one-dimensional subspace of V fixed by $\tilde{T}_v$ is generated by an element u such that

$$q(u) \equiv q(u_i) \not\equiv q(v_i) \pmod{F_{\wp_i}^{\times 2}} \qquad \forall i = 1, \dots, r.$$

We know that any $\mathcal{G}^{\mathrm{ad}}(F)$-conjugate of $\tilde{H}_i$ fixes a one-dimensional F-linear subspace $\tilde{V}_i$ of V, and any generator $\tilde{v}_i$ of $\tilde{V}_i$ satisfies $q(\tilde{v}_i) \cong q(v_i) \pmod{F^{\times 2}}$ by lemma 6.10. We conclude that the maximal $\mathbb{Q}$-torus $\tilde{T}_v$ is not contained in any $\mathcal{G}^{\mathrm{ad}}(F)$-conjugate of $\tilde{H}_i$ for any $i = 1, \dots, r$, because otherwise we would have $q(u) \equiv q(v_i) \pmod{F^{\times 2}}$, a contradiction. Note that the image T_v of $\tilde{T}_v$ in G is the centralizer subgroup $Z_G(v)$ of v in G, and the above conclusion means that T_v is not contained in any $G(\mathbb{Q})$-conjugate of H_i for any $i = 1, \dots, r$. □

(6.13) Proof of Thm. 5.5.
Step 1. Let $S = {}_K\mathscr{M}_{\mathbb{C}}(G, X)$, where (G, X) is a Shimura input datum as in [9, 2.1.1] and K is a compact open subgroup of the group of finite adelic points $G(\mathbb{A}_f)$ of G. Let $G^{\mathrm{ad}} := G/Z(G)$. Then G^{ad} decomposes into a product $G^{\mathrm{ad}} \cong \prod_{i=1}^N G_i$, where each G_i is a connected semi-simple adjoint $\mathbb{Q}$-simple group over $\mathbb{Q}$. Choose a compact open subgroup $K' \subseteq K \subset G(\mathbb{A}_f)$ and compact open subgroups $K_i \in G_i(\mathbb{A}_f)$, such that K' is contained in the inverse image of $\prod_{i=1}^N K_i$ under the natural surjective homomorphism $\alpha\colon G \twoheadrightarrow \prod_{i=1}^N G_i$ and the arithmetic subgroup $G^{\mathrm{ad}} \cap \prod_{i=1}^N K_i$ of $G^{\mathrm{ad}}(\mathbb{Q})$ is torsion free. For each $i = 1, \dots, N$ let X_i be the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class of $\mathbb{R}$-homomorphisms $\mathbb{S} \to G_{i,\mathbb{R}}$ induced by the composition of the $\mathbb{R}$ conjugacy class of $\mathbb{R}$-homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ in X with the base change to $\mathbb{R}$ of the $\mathbb{Q}$-homomorphism $G \to G_i$. Then we have a morphism $(G, X) \longrightarrow \prod_{i=1}^N (G_i, X_i)$ between Shimura input data. It is clear that the statement for the Shimura variety S in 5.5 follow from the statement for the Shimura variety

$${}_{(\prod_i K_i)}\mathscr{M}_{\mathbb{C}}(G_1 \times \cdots \times G_N, X_1 \times \cdots \times X_N) = \prod_{i=1}^N {}_{K_i}\mathscr{M}_{\mathbb{C}}(G_i, X_i).$$

So we may and do assume that $G = \prod_{i=1}^N G_i$ and each factor G_i of G is adjoint and $\mathbb{Q}$-simple.

After re-indexing, we may assume that among the irreducible Shimura subvarieties $S_1, \dots, S_m$ of S, the first n subvarieties $S_1, \dots, S_n$ are *weak product Shimura subvarieties* of $\prod_{i=1}^N {}_{K_i}\mathscr{M}_{\mathbb{C}}(G_i, X_i)$'s, in the sense that for each $a = 1, \dots, n$, the subvariety S_a of S is a Hecke translate of a the Shimura subvariety attached to a Shimura input datum of the form

$$(H_{a,1}, Y_{a,1}) \times \cdots \times (H_{a,N}, Y_{a,N}) \subsetneqq (G_1, X_1) \times \cdots \times (G_N, X_N),$$

where each factor $(H_{a,i}, Y_{a,i})$ is a Shimura input subdatum of (G_i, X_i) for all $i = 1, \dots, N$. The rest of the subvarieties $S_{a+1}, \dots, S_N$ are assumed not to be weak product Shimura subvarieties. The assumption that $S_i \subsetneqq S$ for all $i = 1, \dots, m$ implies that for each $a = 1, \dots, n$, there exists i_a with $1 \le i_a \le N$ such that $H_{a,i_a} \subsetneqq G_{i_a}$.

Step 2. Skip step 2 and go to step 3 if $n = 0$. Assume that $n \ge 1$ in the rest of step 2.

Pick n distinct prime numbers $p_1,\dots,p_n$. For each $a=1,\dots,n$, exactly one of the four case below happens, and we assign

- a finite extension field E_a of $\mathbb{Q}$, and
- a non-empty open subset $U_a \subset \mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}_{p_a})$ of the set of all regular elements of the Lie algebra $\mathrm{Lie}(G)\otimes_{\mathbb{Q}}\mathbb{Q}_{p_a}$ which is stable under conjugation by $G(\mathbb{Q}_{p_a})$

for each $a\in\{1,\dots,n\}$ according to the following scheme, depending on which of the four cases occurs for the subgroup $H_{a,1}\times\cdots\times H_{a,N} \subsetneqq G_1\times\cdots\times G_N$.

1. There exists an index i_a with $1\le i_a\le N$, such that H_{a,i_a} is a torus.

   Let $\rho_{H_{a,i_a}}\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{a}}/\mathbb{Q})\to \mathrm{GL}_{\mathbb{Z}}(X^*(H_{a,i_a}))$ be the natural Galois representation on the character group of the $\mathbb{Q}$-torus $H_{a,i}$. We take E_a to be the fixed subfield in $\mathbb{Q}^{\mathrm{a}}$ of $\mathrm{Ker}\left(\rho_{H_{a,i_a}}\right)$, and set $U_a=(\mathrm{Lie}(G)\otimes_{\mathbb{Q}}\mathbb{Q}_{p_a})_{\mathrm{reg}}$ in case 1.

2. None of the $H_{a,i}$'s is a torus, but there exists an index i_a with $1\le i_a\le N$ such that G_{i_a} is neither of type C_ℓ nor of type B_ℓ for any integer $\ell\ge 2$, and $H_{a,i_a}\subsetneqq G_{i_a}$.

   We take $E_a=\mathbb{Q}$ and $U_a=(\mathrm{Lie}(G)\otimes_{\mathbb{Q}}\mathbb{Q}_{p_a})_{\mathrm{reg}}$ in case 2.

3. None of the factor subgroups $H_{a,i}$ is a torus, G_i is of type C_ℓ for some $\ell\ge 1$ or B_ℓ for some $\ell\ge 3$ for every i such that $H_{a,i}\subsetneqq G_i$,[19] and there exists an index i_0 with $1\le i_0\le N$ such that $H_{a,i_a}\subsetneqq G_{i_a}$ and the factor G_{i_0} is of type C_ℓ for some integer $\ell\ge 2$. Choose such an index i_0 and call it i_a.

   SUBCASE 3A. The group H^{ad}_{a,i_a} is of the form $H^{\mathrm{ad}}_{a,i_a}=\mathrm{Res}_{L_0/\mathbb{Q}}(\mathcal{H})$ as in 6.4 and 6.5, where L_0 is a number field and $\mathcal{H}$ is an adjoint simple group of type A_1 over L_0

   We take E_a to be the normal closure of L_0 in $\mathbb{Q}^{\mathrm{a}}$ and again set $U_a=(\mathrm{Lie}(G)\otimes_{\mathbb{Q}}\mathbb{Q}_{p_a})_{\mathrm{reg}}$ in subcase 3A.

   SUBCASE 3B. The group H^{ad}_{a,i_a} is not of the form described in 3A above.

   We take $E_a=\mathbb{Q}$ and $U_a=(\mathrm{Lie}(G)\otimes_{\mathbb{Q}}\mathbb{Q}_{p_a})_{\mathrm{reg}}$ in subcase 3B

4. None of the factor subgroups $H_{a,i}$ is a torus, G_i is of type B_ℓ for some integer $\ell\ge 3$ for every index i such that $H_{a,i}\subsetneqq G_i$.

   In case 4 there exists an index i_a with $1\le i_a\le N$ such that $H_{a,i_a}\subsetneqq G_{i_a}$, and G_{i_a} is of type B_{ℓ_a} for some $\ell_a\ge 2$. We know that there is a quadratic space (V,q) over a number field F such that $G_{i_a}=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGO}^+(V,q)$. Write $\mathcal{G}:=\mathrm{SO}(V,q)$.

   SUBCASE 4A. There exists a one-dimensional anisotropic F-linear subspace $V'\subset V$ such that $H_{a,i_a}=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGO}^+\left(V'^{\perp},q|_{V'^{\perp}}\right)$.

   Let $\wp_1$ be a place of the number field F above p_a. Apply 6.12 with $r=1$ to the present situation with $V_1'=V'$. Let $U_{\wp_1}$ be an open subset of $\mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_1})$ satisfying the condition

[19] The Dynkin diagrams C_2 and B_2 are the same.

specified in 6.12. Let $\{\wp_1,\dots,\wp_d\}$ be the set of all places of F above p. Let $U'_{\wp_1}$ be the union of all $\mathcal{G}^{\mathrm{ad}}(F_{\wp_1})$-conjugates of $U_{\wp_1}$.

We set $E_a=\mathbb{Q}$ and take U_a to be the product

$$\begin{array}{rcl} U_a & = & U'_{\wp_1}\times \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_2})\times\cdots\times \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_d}) \\ & \subset & \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_1})\times \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_2})\times\cdots\times \mathrm{Lie}(\mathcal{G})_{\mathrm{reg}}(F_{\wp_d}) \\ & = & \mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}_{p_a}) \end{array}$$

in subcase 4A.

SUBCASE 4B. The subgroup H_{a,i_a} of G_{i_a} is not of the form described in case 4A.

We take $E_a=\mathbb{Q}$ and $U_a=(\mathrm{Lie}(G)\otimes_{\mathbb{Q}}\mathbb{Q}_{p_a})_{\mathrm{reg}}$ in subcase 4B.

Step 3. Let U_∞ be a non-empty open subset of $\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{R})$ such that the centralizer in $G\times_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(\mathbb{R})$ of any element $v_\infty\in U_\infty$ is a compact maximal subtorus in $G\times_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(\mathbb{R})$. Denote by $\tilde{E}$ the smallest Galois extension of $\mathbb{Q}$ which contains the number fields $E_1,\dots,E_n$ and also the number field $E(G)$ attached to G in the notation of 5.9 – 5.11.

By Prop. 5.11, there exists a regular element $v\in\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q})$ such that the following statements hold.

- The image of v in $\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{R})$ lies in the open subset $U_\infty\subset\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{R})$.
- The image of v in $\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}_{p_a})$ lies in the open subset $U_a\subset\mathrm{Lie}(G)_{\mathrm{reg}}(\mathbb{Q}_{p_a})$ for every $a=1,\dots,n$.
- The centralizer subgroup $Z_G(v)$ of v in G is a Weyl $\mathbb{Q}$-subtorus of G.
- In the notation of Prop. 5.11, the scheme-theoretic inverse image of v in the scheme W is isomorphic to the spectrum of a number field K_v which is linearly disjoint with $\tilde{E}$ over $E(G)$.

Let $\mu:\mathbb{S}\to Z_G(v)_{\mathbb{R}}$ be an $\mathbb{R}$-homomorphism such that the composition of μ with the base change to $\mathbb{R}$ of the inclusion map $Z_G(v)\hookrightarrow G$ is an element x_0 of the hermitian symmetric space $X:=X_1\times\cdots\times X_N$. Recall that X is the $G(\mathbb{R})$-conjugacy class of the $\mathbb{R}$-homomorphism $x_0:\mathbb{S}\to G_{\mathbb{R}}$ and may not be connected. But we can change x_0 to a suitable $G(\mathbb{Q})$-conjugate of x_0 to ensure that the new x_0 lies in a given connected component of $X_1\times\cdots\times X_N$.

The image y in S of the point $(x_0,1)\in X\times G(\mathbb{A}_f)$ is a Weyl special point of S by construction. By lemma 6.7, the Hecke orbit $\mathscr{H}(y)$ of y does not meet $S_{n+1}\cup\cdots\cup S_m$. We have chosen the number field $\tilde{E}$ to ensure that $\mathscr{H}(y)$ is disjoint from S_a for any a between 1 and n such that cases 1, 2 or 3 occur for S_a; this is a consequence of 6.7 and 6.8. Similarly we see from 6.7 and 6.12 that the Hecke orbit $\mathscr{H}(y)$ of y is disjoint from any S_a, with $1\le a\le n$, such that case 4 occurs for S_a. We have proved that the Hecke orbit $\mathscr{H}(y)$ is disjoint from the special subset $S_1\cup\cdots\cup S_m$. □

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