Elementary divisors of the base change conductor for tori

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Version 1.0, July 3, 2001

§1. Introduction

Let K be a local field, and let $\mathcal{O}$ be the ring of integers of K. In this paper we study some numerical invariants attached to tori over K, or equivalently, to integral Galois representations $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_d(\mathbb{Z})$ with finite image.

Let T be a torus over K. The definition of the numerical invariants

$$\underline{c}(T,K) = (c_1(T,K), \ldots, c_{\dim(T)}(T,K))$$

of T can be found in 2.4. The $c_i(T,K)$'s are non-negative rational numbers satisfying

$$c_1(T,K) \le \cdots \le c_{\dim(T)}(T,K)\,.$$

They come from the elementary divisors of the map from the Lie algebra of the Néron model $\underline{T}^{\mathrm{NR}}$ to the Lie algebra of the Néron model $\underline{T_L}^{\mathrm{NR}}$ over $\mathcal{O}$, where L is a finite Galois extension of K such that the torus T is split over L. One may think of $\underline{c}(T,K)$ as a measure for the failure of T to have semistable reduction over $\mathcal{O}$: A torus T extends to a torus over $\mathcal{O}$ if and only if $c_i(T,K) = 0$ for $i = 1, \ldots, \dim(T)$. We call the $c_i(T,K)$'s the *elementary divisors of the base change conductors*; their sum $c(T,K) := c_1(T,K) + \cdots + c_{\dim(T)}(T,K)$ is called the *base change conductor* of T.

According to [CYdS, 11.3, 12.1] or [dS], $c(T,K)$ is equal to one-half of the Artin conductor of the representation of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on the character group (or the cocharacter group) of T, if the residue field κ of $\mathcal{O}$ is perfect. In particular $c(T,K) = c(T_1,K)$ if T and T_1 are isogenous over K. In contrast, the $c_i(T,K)$'s tend to change under isogenies; see 5.2 for an example.

Using the correspondence between a torus T and its character group $X_*(T)$, one can regard $\underline{c}(T,K)$ as a numerical invariant $\underline{c}(\rho)$ attached to a continuous Galois representation $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_d(\mathbb{Z})$. Assume that $\mathrm{Char}(\kappa) = p > 0$. It is easy to see that if ρ_1 and ρ_2 are two Galois representations on $\mathbb{Z}^d$ which are conjugate in $\mathrm{GL}_d(\mathbb{Z}_{(p)})$, then $\underline{c}(\rho_1) = \underline{c}(\rho_2)$. So one can localize at p and define $\underline{c}(\rho)$ for any continuous Galois representation $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}_d(\mathbb{Z}_{(p)})$. In the next paragraph we describe some general estimates of these invariants in terms of the ramification of the Galois representation ρ.

Let $r \ge 0$ be the integer such that $c_r(T,K) = 0, c_{r+1}(T,K) > 0$, where $c_0(T,K) = 0$ by convention, and T is assumed to be non-trivial. Assuming that the residue field κ is perfect, and T is split over a finite Galois extension L of K. Let $\Gamma = \mathrm{Gal}(L/K)$ be the Galois group

[1]partially supported by grant DMS 9800609 from the National Science Foundation

of L/K, and let $\Gamma = \Gamma_{-1} \supseteq \Gamma_0 \supseteq \cdots$ be the lower numbering filtration of Γ. Denote by t be the integer such that $t \geq -1$ and $\Gamma = \Gamma_t \supsetneq \Gamma_{t+1}$. The results of this paper gives the following estimate:

$$\frac{t+1}{e(L/K)} \leq c_{r+1}(T,K) \leq c_{\dim T}(T,K) \leq \frac{\mathrm{ord}_K(\mathrm{disc}(L/K))}{e(L/K)} \qquad (\dagger)$$

See Cor. 4.3 and Cor. 4.5. In a sense these bounds cannot be improved: Denote by $\mathrm{R}_{L/M} := \mathrm{Res}_{L/K}(\mathbb{G}_{\mathrm{m}})$ the Weil restriction of $\mathbb{G}_{\mathrm{m}}$ from L to K. If L/K is totally ramified, then $r = 1$, $\dim(\mathrm{R}_{L/M}) = [L : K]$, $c_2(\mathrm{R}_{L/M}, K) = \frac{t+1}{e(L/K)}$, and $c_{[L:K]}(\mathrm{R}_{L/M}, K) = \frac{\mathrm{ord}_K(\mathrm{disc}(L/K))}{e(L/K)}$. On the other hand in the context of ($\dagger$), we do not have any non-trivial general estimate of $c_i(T,K)$ if i is in the "intermediate range" $r+1 \leq i \leq \dim(T) - 1$.

Here is a description the method used in this paper. In Thm. 3.1 we prove a series of inequalities relating $\underline{c}(T,K)$ to $\underline{c}(T',K)$ and $\underline{c}(T'',K)$, whenever there is an exact sequence of tori $1 \to T' \to T \to T'' \to 1$. In Thm. 4.1 we compute all the $c_i(\mathrm{R}_{L/K}, K)$'s for the Weil restriction $\mathrm{R}_{L/K}$, where L is a totally ramified separable extension of K. In Thm. 4.4 we compute the first invariant $c_1(\mathrm{R}'_{L/K}, K)$ for the norm-one subtorus

$$\mathrm{R}'_{L/K} := \ker\left(\mathrm{Nm}_{L/K} : \mathrm{R}_{L/K} \to \mathbb{G}_{\mathrm{m}}\right)$$

in $\mathrm{R}_{L/K}(\mathbb{G}_{\mathrm{m}})$, where L/K is assumed to be Galois and totally ramified. The estimate ($\dagger$) follows from these computation and inequalities provided by 3.1.

We would like to stress that the results in this paper about the invariants $c_i(T,K)$ is just a small step, under the benevolent assumption that the residue field is perfect. We have not even computed the invariants $c_i(\mathrm{R}'_{L/K}, K)$, $i \geq 2$ for the norm-one torus $\mathrm{R}'_{L/K}$. For a continuous Galois representation $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{GL}(\mathbb{Z}_{(p)})$, we do not know to what extent $\underline{c}(\rho)$ is related to $\underline{c}(\rho')$, where ρ' is the contragredient representation of ρ. If the residue field κ is not perfect, then the base change conductor $c(T,K)$ may change under K-isogenies; see 5.3 for such an example. It remains a challenge to determine which estimates about the invariants $c_i(T,K)$ proved under the assumption that κ is perfect remain true without that assumption.

§2. Notation and definitions

(2.1) In this paper $\mathcal{O}$ denotes a complete discrete valuation ring with fraction field K. Let κ be the residue field of $\mathcal{O}$, $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$, and let π be a generator of $\mathfrak{p}$. For a finite separable extension field L of K, denote by $\mathcal{O}_L$, κ_L, $\mathfrak{p}_L$, π_L the ring of integers of L, the residue field of $\mathcal{O}_L$, the maximal ideal of $\mathcal{O}_L$, and a generator of π_L respectively. The ramification index $e(L/K)$ is defined by $\pi\,\mathcal{O}_L = \pi_L^{e(L/K)}$. Let $\mathcal{O}^{\mathrm{sh}}$ be the strict henselization of $\mathcal{O}_L$. The fraction field K^{sh} of $\mathcal{O}^{\mathrm{sh}}$ is the maximal unramified extension of K, and the residue field of $\mathcal{O}^{\mathrm{sh}}$ is κ^{sep}, the separable closure of κ. We assume that the characteristic of the residue field κ is a prime number p. No generality is sacrificed by this assumption: Every finite extension of a local field is at most tamely ramified if the residue field has characteristic 0, in which case the questions treated in this paper are vastly simplified.

(2.2) Néron models Let T be a torus over K. We have two notions of Néron models of T. The lft Néron model $\underline{T}^{\mathrm{lft\ NR}}$ as defined in [BLR, Ch. 10], is a smooth group model of T locally of finite type over $\mathcal{O}$, and satisfies $\underline{T}^{\mathrm{lft\ NR}}(\mathcal{O}^{\mathrm{sh}}) = T(K^{\mathrm{sh}})$. We also have the open subgroup $\underline{T}^{\mathrm{ft\ NR}}$ of $\underline{T}^{\mathrm{lft\ NR}}$ such that $\underline{T}^{\mathrm{ft\ NR}}(\mathcal{O}^{\mathrm{sh}})$ is the maximal bounded subgroup of $T(K^{\mathrm{sh}})$; $\underline{T}^{\mathrm{lft\ NR}}$ is of finite type over $\mathcal{O}$. We abbreviate $\underline{T}^{\mathrm{ft\ NR}}$ to $\underline{T}^{\mathrm{NR}}$.

(2.3) Definition Let $\mathcal{O}_L$ be a discrete valuation ring with fraction field L. Let $N_1 \subset N_2$ be two free $\mathcal{O}_L$-modules of finite rank d. Let $h : N_1 \to N_2$ be an injective $\mathcal{O}_L$-homomorphism. Let j be a natural number such that $0 \le j \le d$. Define $\alpha_j(h; N_1, N_2)$ to be the largest natural number m such that the $\Lambda^j(h) \equiv 0 \pmod{\pi_L^m}$, where $\Lambda^j(h) : \Lambda^j_{\mathcal{O}_L}(N_1) \to \Lambda^j_{\mathcal{O}_L}(N_2)$ is the j-th exterior power of h. In other words, if $\pi_L^{e_1} \le \pi_L^{e_2} \ldots \le \pi_L^{e_{\dim(V)}}$ are the elementary divisors of the $\mathcal{O}_L$-module homomorphism h, then $\alpha_j(h; N_1, N_2) = e_1 + \cdots + e_j$. When $N_1 \subseteq N_2$ are two lattices in an d-dimensional vector space and $h : N_1 \to N_2$ is the inclusion map, then we abbreviate $\alpha_j(h; N_1, N_2)$ to $\alpha_j(N_1, N_2)$. For any integer $n \ge 0$, $\alpha_1(h; N_1, N_2) \ge n$ means that $h \equiv 0 \pmod{\pi_L^n}$, while $\alpha_d(h; N_1, N_d) - \alpha_{d-1}(h; N_1, N_2) \le n$ means that $\mathrm{coker}(h)$ is killed by π_L^n.

(2.4) Definition Let T be a torus over K, and let L be a finite Galois extension of K such that T is split over L. Let

$$\mathrm{can}_{T,L/K} : \underline{T}^{\mathrm{NR}} \times_{\mathrm{Spec}\ \mathcal{O}} \mathrm{Spec}\ \mathcal{O}_L \to \underline{T_L}^{\mathrm{NR}}$$

be the unique homomorphism which extends the identity map between the generic fibers, where $T_L := T \times_{\mathrm{Spec}\ K} \mathrm{Spec}\ L$. Let $\left(\mathrm{can}_{T,L/K}\right)_* : \mathrm{Lie}(\underline{T}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L \to \mathrm{Lie}(\underline{T_L}^{\mathrm{NR}})$ be the homomorphism between the Lie algebras induced by $\mathrm{can}_{T,L/K}$. Define

$$\underline{c}(T,K) = (c_1(T,K), \ldots, c_{\dim(T)}(T,K)) \in \mathbb{Q}^d_{\ge 0}$$

by

$$c_1(T,K) + \cdots + c_i(T,K) = \frac{1}{e(L/K)}\, \alpha_i\left(\left(\mathrm{can}_{T,L/K}\right)_*; \mathrm{Lie}(\underline{T}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L, \mathrm{Lie}(\underline{T_L}^{\mathrm{NR}})\right),$$

for $i = 1, \ldots, \dim(T)$. In other words, $c_1(T,K) \le \cdots \le c_{\dim(T)}(T,K)$, and

$$\pi_L^{e(L/K)c_1(T,K)}, \ldots, \pi_L^{e(L/K)c_{\dim(T)}(T,K)}$$

are the elementary divisors of the $\mathcal{O}_L$-module homomorphism $\left(\mathrm{can}_{T,L/K}\right)_*$. The above definition first appeared in [CYdS, 12.2]; it does note depend on the choice of L. The sum $c(T,K) := c_1(T,K) + \cdots + c_{\dim(T)}(T,K)$, called the *base change conductor* of T in [Ch1], was studied in [CYdS]. We call the $c_i(T,K)$'s the *elementary divisors of th base change conductor* of T.

(2.5) Let L be a finite separable extension of K. Denote by $\mathrm{R}_{L/K}$ the torus $\mathrm{Res}_{L/K}(\mathbb{G}_{\mathrm{m}})$, such $\mathrm{R}_{L/K}(A) = (A \otimes_K L)^\times$ for every K-algebra A. Its Néron model $\underline{\mathrm{R}_{L/K}}^{\mathrm{NR}}$ is equal to $\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}}(\mathbb{G}_{\mathrm{m}})$, the Weil restriction of $\mathbb{G}_{\mathrm{m}}$ from $\mathcal{O}_L$ to $\mathcal{O}$.

There are natural homomorphisms $w : \mathbb{G}_{\mathrm{m}} \to \mathrm{R}_{L/K}$ and $\mathrm{Nm}_{L/K} : \mathrm{R}_{L/K} \to \mathbb{G}_{\mathrm{m}}$. On the group of K-rational points they induce the inclusion $K^\times \hookrightarrow L^\times$ and the L/K-norm $\mathrm{Nm}_{L/K} : L^\times \to K^\times$ respectively. Let

$$\mathrm{R}'_{L/K} = \ker(\mathrm{Nm}_{L/K} : \mathrm{R}_{L/K} \to \mathbb{G}_{\mathrm{m}}), \qquad \mathrm{R}''_{L/K} = \mathrm{R}_{L/K}/w(\mathbb{G}_{\mathrm{m}})\,.$$

The character group of $X^*(\mathrm{R}_{L/K})$ is a free abelian group with basis elements $\{\chi_\sigma\}_{\mathrm{Hom}(L,K^{\mathrm{sep}})}$; the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ operates transitively on the χ_σ's. The restriction of the character $\chi_\sigma : \mathrm{R}_{L/K} \to \mathbb{G}_{\mathrm{m}}$ to $L^\times = \mathrm{R}_{L/K}(K)$ is equal to σ. The character group $X^*(\mathrm{R}''_{L/K})$ of $\mathrm{R}''_{L/K}$ is the subgroup of $X^*(\mathrm{R}_{L/K})$ generated by elements of the form $\chi_\sigma - \chi_{\sigma_0}$, where σ_0 is a fixed element of $\mathrm{Hom}(L, K^{\mathrm{sep}})$. The character group of $\mathrm{R}'_{L/K}$ is the quotient of $X^*(\mathrm{R}_{L/K})$ by the subgroup $\mathbb{Z} \cdot (\sum_{\sigma \in \mathrm{Hom}(L,K^{\mathrm{sep}})} \chi_\sigma)$.

(2.5.1) Let L be a finite Galois extension of K such that the κ_L/κ is separable. Let $\Gamma = \mathrm{Gal}(L/K)$. The lower numbering filtration of Γ defined in [S2, V §1] is a decreasing filtration

$$\Gamma = \Gamma_{-1} \supseteq \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots$$

indexed by integers ≥ -1 such that $\Gamma_n = \{1\}$ if n is sufficiently large. For each integer $m \geq 0$ and each $\sigma \in \Gamma$, $\sigma \in \Gamma_m$ if and only if σ operates trivially on $\mathcal{O}_L/\mathfrak{p}_L^{m+1}$. The function $i_\Gamma = i_{L/K} : \Gamma - \{1\} \to \mathbb{N}$, defined in [S2, V §1], is characterized by the following property: $i_\Gamma(\sigma) = m$ if and only if $\sigma \in \Gamma_{m+1}$ and $\sigma \notin \Gamma_{m+2}$.

§3. The basic estimates

(3.1) Theorem *Let $1 \to T' \to T \to T'' \to 1$ be a short exact sequence of tori over K.*

(i) *For all $1 \leq k \leq \dim(T')$ we have*

$$\sum_{1 \leq j \leq k} c_j(T, K) \leq \sum_{1 \leq j \leq k} c_j(T', K)\,.$$

(ii) *Assume that the residue field κ of $\mathcal{O}$ is perfect. Then*

$$\sum_{i+\dim(T') \leq j \leq \dim(T)} c_j(T, K) \geq \sum_{i \leq j \leq \dim(T'')} c_j(T'', K)$$

for all $1 \leq i \leq \dim(T'')$.

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
 & \mathrm{Lie}(\underline{T'}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d, "\mathrm{can}_{T',L/K_*}"'] & \mathrm{Lie}(\underline{T}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d, "\mathrm{can}_{T,L/K_*}"'] & \mathrm{Lie}(\underline{T''}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[d, "\mathrm{can}_{T'',L/K_*}"'] & \\
0 \arrow[r] & \mathrm{Lie}(\underline{T'_L}^{\mathrm{NR}}) \arrow[r] & \mathrm{Lie}(\underline{T_L}^{\mathrm{NR}}) \arrow[r] & \mathrm{Lie}(\underline{T''_L}^{\mathrm{NR}}) \arrow[r] & 0
\end{tikzcd}
$$

where the second row is exact and the first row is a complex which is not necessarily exact. To simplify the notation in the proof, we abbreviate the above diagram to

$$
\begin{tikzcd}
 & M' \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d] & M \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d] & M'' \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[d] & \\
0 \arrow[r] & M'_L \arrow[r] & M_L \arrow[r] & M''_L \arrow[r] & 0\,.
\end{tikzcd}
$$

Insert an exact middle row to obtain an enlarged commutative diagram

$$
\begin{tikzcd}
 & M' \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d] & M \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d, "="] & M'' \otimes_{\mathcal{O}} \mathcal{O}_L & \\
0 \arrow[r] & \widetilde{M'} \arrow[r] \arrow[d] & M \otimes_{\mathcal{O}} \mathcal{O}_L \arrow[r] \arrow[d] & \widetilde{M''} \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & M'_L \arrow[r] & M_L \arrow[r] & M''_L \arrow[r] & 0
\end{tikzcd}
$$

where $\widetilde{M'} = (M' \otimes_{\mathcal{O}} \mathcal{O}_L) \cap (M \otimes_{\mathcal{O}} \mathcal{O}_L)$, and $\widetilde{M''}$ is equal to the image of $M \otimes_{\mathcal{O}} \mathcal{O}_L$ in $M'' \otimes_{\mathcal{O}} \mathcal{O}_L$. Notice that all vertical arrows are injective homomorphisms between finite free $\mathcal{O}_L$-modules of the same rank. The second and the third row are exact, while the first row is a complex of $\mathcal{O}_L$-modules. Denote by r (resp. s) the rank of the free $\mathcal{O}$-module M' (resp. M''), so that M has rank $r + s$.

Let k be a natural number such that $0 \leq k \leq r = \dim(T')$. Then the natural map $\Lambda^k_{\mathcal{O}_L}(\widetilde{M'}) \to \Lambda^k_{\mathcal{O}_L}(M \otimes_{\mathcal{O}} \mathcal{O}_L)$ is a split injection. Therefore $\alpha_k(\widetilde{M'}, M'_L) \geq \alpha_k(M \otimes_{\mathcal{O}} \mathcal{O}_L, M_L)$. Combined with the trivial estimate $\alpha_k(M' \otimes_{\mathcal{O}} \mathcal{O}_L, M'_L) \geq \alpha_k(\widetilde{M'}, M'_L)$, we get $\alpha_k(M' \otimes_{\mathcal{O}} \mathcal{O}_L, M'_L) \geq \alpha_k(M \otimes_{\mathcal{O}} \mathcal{O}_L, M_L)$. Dividing the above inequality by $e(L/K)$ finishes the proof of 3.1 (i).

The proof of 3.1 (ii) is somewhat long and will be divided into several Lemmas for clarity.

(3.1.1) Lemma *Let* $r = \mathrm{rank}(M') = \dim(T')$, $s = \mathrm{rank}(M'') = \dim(T'')$ *as above. Then*

$$\alpha_r(M' \otimes_{\mathcal{O}} \mathcal{O}_L, \widetilde{M'}) = \alpha_s(\widetilde{M''}, M'' \otimes_{\mathcal{O}} \mathcal{O}_L)$$

Proof. According to [CYdS, 11.3, 12.1] or [dS] we have $c(T', K) + c(T'', K) = c(T, K)$, or equivalently,

$$\alpha_r(M' \otimes_{\mathcal{O}} \mathcal{O}_L, M'_L) + \alpha_s(M'' \otimes_{\mathcal{O}} \mathcal{O}_L, M''_L) = \alpha_{r+s}(M \otimes_{\mathcal{O}} \mathcal{O}_L, M_L)$$

On the other hand the exactness of the second and the third row in the digram (*) yields

$$\alpha_{r+s}(M \otimes_{\mathcal{O}} \mathcal{O}_L, M_L) = \alpha_r(\widetilde{M'}, M'_L) + \alpha_s(\widetilde{M''}, M''_L)\,.$$

The Lemma follows. ∎

(3.1.2) Lemma *For $i = 1, \ldots, s = \operatorname{rank}(M'')$, we have*

$$\alpha_i(\widetilde{M''}, M''_L) \leq \alpha_i(M'' \otimes_{\mathcal{O}} \mathcal{O}_L, M''_L) + \alpha_s(\widetilde{M''}, M'' \otimes_{\mathcal{O}} \mathcal{O}_L)$$

Proof. This is a special case of the following general statement in linear algebra: Suppose that $N_0 \subset N_1 \subset N_2$ are three $\mathcal{O}_L$-lattices in an s-dimensional L-vector space V, then

$$\alpha_i(N_0, N_2) \leq \alpha_i(N_1, N_2) + \alpha_s(N_0, N_1) \quad \forall\, i = 1, \ldots, s\,.$$

To prove this claim, choose an $\mathcal{O}_L$-basis $w_1, \ldots, w_s$ of N_1 such that $\pi_L^{e_1} w_1, \ldots, \pi_L^{e_s} w_s$ is an $\mathcal{O}_L$-basis of N_0. for suitable natural numbers $e_1, \ldots, e_s$. Let $v_1, \ldots, v_s$ be an $\mathcal{O}_L$-basis of N_2. Write $w_k = \sum_{l=1}^s a_{kl}\, v_l$ for $k = 1, \ldots, s$, so we have $\pi_L^{e_k} w_k = \sum_{l=1}^s \pi_L^{e_k} a_{kl}\, v_l$ for each k.

By definition, the determinant of every $i \times i$-minor of the $s \times s$-matrix $B = (\pi_L^{e_k} a_{kl}) \in \mathrm{M}_s(\mathcal{O}_L)$ is divisible by $\pi_L^{\alpha_i(N_0,N_2)}$. Hence the determinant of every $i \times i$-minor of the $s \times s$-matrix $A = (a_{kl}) \in \mathrm{M}_s(\mathcal{O}_L)$ is divisible by $\pi_L^{\alpha_i(N_0,N_2)} - \sum_{k=1}^s e_k$. This means that

$$\alpha_i(N_0, N_2) - \alpha_s(N_0, N_1) = \alpha_i(N_0, N_2) - \sum_{k=1}^{s} e_k \leq \alpha_i(N_1, N_2)\,.$$

The claim is proved. Notice that the argument also shows that we can replace $\alpha_s(N_0, N_1) = \sum_{k=1}^s e_k$ in the claim by

$$\max_{I \subseteq \{1, \ldots, s\},\, \mathrm{Card}(I) = i} \left(\sum_{k \in I} e_k \right)$$

in the claim. ∎

(3.1.3) Lemma *Notation as above, in particular $r = \operatorname{rank}(M')$. Then*

$$\alpha_{r+i}(M \otimes_{\mathcal{O}} \mathcal{O}_L, M_L) \leq \alpha_r(\widetilde{M'}, M'_L) + \alpha_i(\widetilde{M''}, M''_L)$$

for $i = 1, \ldots, s = \operatorname{rank}(M'')$.

Proof. This is a consequence of the fact that the natural map

$$\Lambda^r_{\mathcal{O}_L}(\widetilde{M'}) \otimes_{\mathcal{O}_L} \Lambda^i_{\mathcal{O}_L}(\widetilde{M''}) \to \Lambda^{r+i}_{\mathcal{O}_L}(M \otimes_{\mathcal{O}} \mathcal{O}_L)$$

is a split injection. ∎

Proof of 3.1 (ii), continued. Let i be a natural number such that $0 \leq i \leq s = \dim(T'')$. Then

$$\begin{aligned}
&\alpha_{r+i}(M \otimes_{\mathcal{O}} \mathcal{O}_L, M_L) \leq \alpha_r(\widetilde{M'}, M'_L) + \alpha_i(\widetilde{M''}, M''_L) \qquad \text{by Lemma 3.1.3}\\
&= \alpha_r(M' \otimes_{\mathcal{O}} \mathcal{O}_L, M'_L) - \alpha_r(M' \otimes_{\mathcal{O}} \mathcal{O}_L, \widetilde{M'}) + \alpha_i(\widetilde{M''}, M''_L)\\
&= \alpha_r(M' \otimes_{\mathcal{O}} \mathcal{O}_L, M'_L) - \alpha_s(\widetilde{M''}, M'' \otimes_{\mathcal{O}} \mathcal{O}_L) + \alpha_i(\widetilde{M''}, M''_L) \qquad \text{by Lemma 3.1.1}\\
&\leq \alpha_r(M' \otimes_{\mathcal{O}} \mathcal{O}_L, M'_L) + \alpha_i(M'' \otimes_{\mathcal{O}} \mathcal{O}_L, M''_L) \qquad \text{by Lemma 3.1.2}
\end{aligned}$$

Dividing the above inequality by $e(L/K)$ gives

$$\sum_{1 \leq j \leq \dim(T')+i} c_j(T, K) \leq c(T'K) + \sum_{1 \leq j \leq i} c_j(T'', K)$$

Since $c(T, K) = c(T', K) + c(T'', K)$, the above inequality is equivalent to

$$\sum_{\dim(T')+i+1 \leq j \leq \dim(T)} c_j(T, K) \geq \sum_{i+1 \leq j \leq \dim(T'')} c_j(T'', K)\,.$$

We have finished the proof of Thm. 3.1 (ii). ∎

(3.1.4) Remark Theorem 3.1 was asserted without proof in [Ch1] 8.5 (d). Unfortunately 3.1 (ii) appeared incorrectly there.

(3.2) Proposition *Let $1 \to T' \to T \to T'' \to 1$ be an exact sequence of tori over K. Assume that T' is a split torus over K.*

(i) *The induced complex*

$$0 \to \mathrm{Lie}(\underline{T'}^{\mathrm{NR}}) \to \mathrm{Lie}(\underline{T}^{\mathrm{NR}}) \to \mathrm{Lie}(\underline{T''}^{\mathrm{NR}}) \to 0$$

is a short exact sequence of $\mathcal{O}$-modules.

(ii) *For each $i = 1, \ldots, \dim(T'')$, we have $c_{\dim(T')+i}(T, K) = c_i(T'')$.*

Proof. The statement (i) is proved in [Ch1] 4.5 and 4.8. Statement (ii) follows from (i). ∎

(3.3) Proposition *Let T be a torus over K which is anisotropic over the maximal unramified extension K^{sh} of K.*

(i) *The neutral component of the closed fiber of $\underline{T}^{\mathrm{NR}}$ is a unipotent commutative algebraic group over κ.*

(ii) *We have $c_i(T, K) > 0$ for $i = 1, \ldots, \dim(T)$.*

PROOF. We may and do assume that $\mathcal{O}$ is strictly henselian. Since T is anisotropic, $T(K)$ is bounded: Every element $x \in T(K)$ corresponds to a Γ-equivariant homomorphism $h_x : X^*(T) \to L^\times$, and $\mathrm{ord}_L \circ h_x : X^*(T) \to \mathbb{Z}$ must be trivial because the Γ-coinvariant of $X^*(T)$ is finite. Therefore $\underline{T}^{\mathrm{NR}} = \underline{T}^{\mathrm{lft\,NR}}$.

Let $\underline{T}^{\mathrm{NR}\circ}_{\kappa}$ be the neutral component of $\underline{T}^{\mathrm{NR}} \times \mathrm{Spec}\,\mathcal{O}\,\mathrm{Spec}\,\kappa$. It is known that every connected commutative algebraic group over κ is canonically isomorphic to a product of a commutative unipotent group over κ and a torus over κ. Write $\underline{T}^{\mathrm{NR}\circ}_{\kappa} = U \times W$, where U is a commutative unipotent group over κ and W is a torus over κ. Let ℓ be a prime number different from $p = \mathrm{Char}(\kappa)$. For every natural number n, $[\ell^n]_U : U \to U$ is an isomorphism, $[\ell^n]_{\underline{T}^{\mathrm{NR}\circ}_{\kappa}}$ is étale, and $[\ell^n]_W : W \to W$ is an étale isogeny. The kernel–cokernel long exact sequence gives an isomorphism $\underline{T}^{\mathrm{NR}\circ}_{\kappa}[\ell^n] \xrightarrow{\sim} W[\ell^n]$ between the two finite étale group schemes over κ. Since $[\ell^n]_{\underline{T}^{\mathrm{NR}}} : \underline{T}^{\mathrm{NR}} \to \underline{T}^{\mathrm{NR}}$ is étale and $\mathcal{O}$ is strictly henselian, every element of $\underline{T}^{\mathrm{NR}\circ}_{\kappa}[\ell^n]$ can be uniquely lifted to an element of $\underline{T}^{\mathrm{NR}}[\ell^n](\mathcal{O})$. So if W is non-trivial, the ℓ-adic Tate module $\mathrm{V}(T)$ attached to T has a non-trivial Γ-invariant element. This is impossible because $\mathrm{V}(T)$ and $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(1)$ are isomorphic as Γ-modules, and $(X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q})^\Gamma$ is assumed to be trivial. We have proved (i).

Consider the map between Lie algebras $\left(\mathrm{can}_{T,L/K}\right)_*$ induced by the homomorphism $\mathrm{can}_{T,L/K} : \underline{T}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L \to \underline{T_L}^{\mathrm{NR}}$. Its reduction modulo $\mathfrak{p}_L$ is canonically identified with the tangent map of the restriction of $\mathrm{can}_{T,L/K}$ to the closed fibers. The neutral component of the closed fiber of $\underline{T}^{\mathrm{NR}}$ is unipotent by (i), and the closed fiber of $\underline{T_L}^{\mathrm{NR}}$ is a split torus by definition. So $\mathrm{can}_{T,L/K}$ is trivial on the neutral component of $\underline{T}^{\mathrm{NR}} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{O}_L$, and its tangent map is trivial. This proves (ii). ∎

(3.3.1) Remark (i) Here is an alternative proof of (ii) Suppose that L is a finite Galois extension of K such that T is split over K. We may and do assume that L is totally ramified over K, in the sense that κ_L is a purely inseparable extension of κ. By Prop. 3.4 (ii), it suffices to verify the statement of 3.3 for $T = \mathrm{R}'_{L/K}$. According to the description of $\mathrm{Lie}(\underline{T}^{\mathrm{NR}}$ in [dS, A1.7], we must show that if $u(x) = 1 + \sum_{n\geq 1} a_n\, x^n$ is a formal power series $\mathcal{O}_L[[x]]$ such that $\mathrm{Nm}_{L/K}(u(x)) = 1$, then $a_1 \in \mathfrak{p}_L$. Let b_n be the image of a_n in κ_L, and let $v(x) = 1+\sum_{n\geq 1} b_n\, x^n$ be the image of $u(x)$ in $\kappa_L[[x]]$. We have $1 = \mathrm{Nm}_{\kappa_L/\kappa}(v(x)) = v(x)^{[L:K]}$. Hence $v(x) = 1$. In particular $a_1 \in \mathfrak{p}_L$.

(ii) If κ_L is separable over κ, the same line of argument provides an explicit lower bound on $c_1(T, K)$; see Cor. 4.5.

(3.4) Proposition *Let T be a torus over K, and let L be a finite Galois extension of K such that T is split over K. Assume that T is anisotropic over K.*

(i) *There exists a short exact sequence of tori of the form*

$$1 \to T' \to (\mathrm{R}''_{L/K})^m \to T \to 1$$

(ii) *There exists a short exact sequence of tori of the form*

$$1 \to (\mathrm{R}'_{L/K})^n \to T \to T'' \to 1$$

PROOF. We first prove (i). The cocharacter group $X_*(T)$ has a natural structure as a module over the finite group $\Gamma = \mathrm{Gal}(L/K)$. The assumption that T is anisotropic over K means that $X_*(T)^\Gamma$, the submodule of all Γ-invariants in $X_*(T)$, is trivial. Choose a Γ-equivariant surjection $h : \mathbb{Z}[\Gamma]^{\oplus m} \twoheadrightarrow X_*(T)$. The submodule $(\mathbb{Z}\cdot\sum_{\sigma\in\Gamma}\sigma)^{\oplus m} \subset (\mathbb{Z}[\Gamma]^{\oplus m})^\Gamma$ maps to zero under h, because $X_*(T)^\Gamma = (0)$. The resulting Γ-equivariant surjection

$$\left(\mathbb{Z}[\Gamma]/\mathbb{Z}\cdot\sum_{\sigma\in\Gamma}\sigma\right)^{\oplus m} \twoheadrightarrow X_*(T)$$

gives a homomorphism of tori required in (i). The same argument, using the character group $X^*(T)$ instead of the cocharacter group $X_*(T)$, proves (ii). ∎

§4. Induced tori and norm-one tori

(4.1) Proposition *Let L be a totally ramified finite Galois extension of K such that $\kappa_L = \kappa$. Let $\Gamma = \mathrm{Gal}(L/K)$. Then for each $k = 1, \dots, e(L/K)$, we have*

$$c_1(\mathrm{R}_{L/K}, K) + \cdots + c_k(\mathrm{R}_{L/K}, K) = \min_{I\subseteq\Gamma,\,\mathrm{Card}(I)=k}\left(\frac{1}{2\,e(L/K)}\sum_{\substack{\sigma,\tau\in\Gamma\\ \sigma\neq\tau}} i_\Gamma(\sigma\,\tau^{-1})\right).$$

In particular, $c_1(\mathrm{R}_{L/K}, K) = 0$, and $\sum_{i=1}^{e(L/K)} c_i(\mathrm{R}_{L/K}, K) = \frac{1}{2}\,\mathrm{ord}_K(\mathrm{disc}(L/K))$.

PROOF. Let $\Gamma = \mathrm{Gal}(L/K) = \{1 = \sigma_1, \dots, \sigma_n\}$, $n = [L : K] = e(L/K)$. Let π_L be a generator of $\mathfrak{p}_L$. As an $\mathcal{O}$-algebra, $\mathcal{O}_L$ is generated by π_L. Hence $i_\Gamma(\sigma_i\,\sigma_j^{-1}) = \mathrm{ord}_L(\sigma_i(\pi_L) - \sigma_j(\pi_L))$ for any $1 \le i \neq j \le n$. Let A be the $n \times n$ matrix whose (i,j)-th entry is $\sigma_j(\pi_L^{i-1})$ for $i, j = 1, \dots, n$. For a natural number k, $1 \le k \le n$, $n\cdot(c_1(\mathrm{R}_{L/K}, K) + \cdots + c_k(\mathrm{R}_{L/K}, K))$ is equal to the minimum among the orders (measured by ord_L) of the determinant of all $k \times k$-minors of A; we have to show that it is equal to

$$\min_{I\subseteq\{1,\dots,n\},\,\mathrm{Card}(I)=k}\left(\sum_{\substack{i<j\\ i,j\in I}} \mathrm{ord}_L(\sigma_i(\pi_L) - \sigma_j(\pi_L))\right).$$

That each sum $\sum_{\substack{i<j\\ i,j\in I}} \mathrm{ord}_L(\pi_L^i - \pi_L^j)$ in the above displayed formula is equal to the order of the determinant of a $k \times k$-minor of A follows from the Vandermonde determinant. The following Lemma 4.1.1 finishes the proof.

(4.1.1) Lemma *Let m be a positive integer, and let $a_1, \ldots, a_m \geq 0$ be mutually distinct natural numbers. Consider the $m \times m$-matrix B with entries in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_m]$ whose (i,j)-th entry is equal to $x_j^{a_i}$ for $i, j = 1, \ldots, m$. Then* $\det(B)$ *is divisible by $\prod_{1<i,j<m}(x_i - x_j)$ in $\mathbb{Z}[x_1, \ldots, x_m]$*

PROOF OF 4.1.1. Clearly $\det(B)$ is divisible by $(x_i - x_j)$ if $i \neq j$. The Lemma follows because $\mathbb{Z}[x_1, \ldots, x_m]$ is a unique factorization domain. This finishes the proof of Lemma 4.1.1 and Prop. 4.1. ∎

(4.2) Corollary *Notation as above, and we assume that $L \neq K$.*

(i) *Let t be the natural number such that $\Gamma = \Gamma_t$, $\Gamma_t \neq \Gamma_{t+1}$. Then*

$$c_i(\mathrm{R}''_{L/K}, K) = c_{i+1}(\mathrm{R}_{L/K}, K) = \frac{i(t+1)}{e(L/K)}$$

for $i = 1, \ldots, [\Gamma : \Gamma_{t+1}] - 1$.

(ii) *We have $c_{e(L/K)-1}(\mathrm{R}''_{L/K}, K) = c_{e(L/K)}(\mathrm{R}_{L/K}, K) = \frac{\mathrm{ord}_K(\mathrm{disc}(L/K))}{e(L/K)}$. More generally,*

$$c_{e(L/K)-i}(\mathrm{R}''_{L/K}, K) = c_{e(L/K)-i+1}(\mathrm{R}_{L/K}, K) = \frac{\mathrm{ord}_K(\mathrm{disc}(L/K))}{e(L/K)} - \frac{(i-1)(t+1)}{e(L/K)}$$

for $i = 1, \ldots, [\Gamma : \Gamma_{t+1}]$.

PROOF. (i) Let $a = [\Gamma : \Gamma_{t+1}]$. We may and do assume that $\{1 = \sigma_1, \ldots, \sigma_a\}$ is a set of representatives of Γ/Γ_{t+1}. Then for $k = 1, \ldots, a\}$ the minimum in the formula in 4.1 for $c_1(\mathrm{R}_{L/K}, K) + \ldots + c_k(\mathrm{R}_{L/K}, K)$ is reached with I being any subset of $\{\sigma_1, \ldots, \sigma_a\}$ with k elements; it is equal to $\frac{k(k-1)(t+1)}{2e(L/K)}$. Hence $c_k(\mathrm{R}_{L/K}, K) = \frac{(k-1)(t+1)}{e(L/K)}$ for $k = 1, \ldots, a$.

(ii) For $i = 1, \ldots, [\Gamma : \Gamma_{t+1}]$, the minimum in the formula in 4.1 for $c_1(\mathrm{R}_{L/K}, K) + \ldots + c_{e(L/K)-i}(\mathrm{R}_{L/K}, K)$ is reached when $\Gamma - I$ is a subset of $\{\sigma_1, \ldots, \sigma_a\}$ with i elements, where $\{\sigma_1, \ldots, \sigma_a\}$ is a set of representatives of Γ/Γ_{t+1} as in (i). We fix a subset $J_0 = \{\tau_1, \ldots, \tau_i\}$ of Γ_s with i elements, let $I_0 = \Gamma - J_0$, and use I_0 to compute the minimum. One finds that

$$\begin{aligned}
&c_{e(L/K)-i+1}(\mathrm{R}_{L/K}, K) + \ldots + c_{e(L/K)}(\mathrm{R}_{L/K}, K) \\
&= \tfrac{i}{e(L/K)} \textstyle\sum_{1 \neq \sigma \in \Gamma} i_\Gamma(\sigma) - \tfrac{1}{e(L/K)} \textstyle\sum_{1 \leq k < l \leq i} i_\Gamma(\sigma_k \cdot \sigma_l^{-1}) \\
&= \tfrac{i}{e(L/K)} \,\mathrm{ord}_K(\mathrm{disc}(L/K)) - \tfrac{i(i-1)(t+1)}{2\,e(L/K)}\,.
\end{aligned}$$

The statement (ii) follows. ∎

(4.3) Corollary *Assume that the residue field κ of $\mathcal{O}$ is perfect. Let T be a torus over K which splits over a finite Galois extension L of K. Then*

$$c_{\dim(T)}(T, K) \leq \frac{\mathrm{ord}_K(\mathrm{disc}(L/K))}{e(L/K)}\,.$$

PROOF. This follows from Prop. 3.2, Prop. 3.4 (i), Thm. 3.1 (ii) and Cor. 4.2 (ii). ∎

(4.4) Theorem *Let L be a totally ramified finite Galois extension of K such that $\kappa_L = \kappa$. Let $\Gamma = \mathrm{Gal}(L/K)$, and let $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots$ be the lower-numbering filtration of Γ. Let t be the natural number such that $\Gamma = \Gamma_t \neq \Gamma_{t+1}$. Then $c_1(\mathrm{R}'_{L/K}, K) = \frac{t+1}{e(L/K)}$.*

PROOF. By Prop. 3.4 (ii), Thm. 3.1 (i) and Cor. 4.2 (i), we have

$$c_1(\mathrm{R}'_{L/K}, K) \leq c_1(\mathrm{R}''_{L/K}, K) = \frac{t+1}{e(L/K)}\,.$$

We will prove the inequality $c_1(\mathrm{R}'_{L/K}, K) \geq \frac{t+1}{e(L/K)}$ in the reversed direction by the method in [S2] V §3–§7; we have already used a part of it in the proof of 3.3.1. According to the description of $\mathrm{Lie}(\underline{T}^{\mathrm{NR}})$ in [dS, A1.7], the Lie algebra of $\underline{R'_{L/K}}^{\mathrm{NR}}$ can be identified with all elements $a_1 \in \mathcal{O}_L$ such that there exists a formal power series of the form

$$u(x) = 1 + a_1\, x + \sum_{n\geq 2} a_n\, x^n \;\in\; \mathcal{O}_L[[x]]^\times$$

such that $\mathrm{Nm}_{L/K}(u(x)) = 1$. We claim that $a_n \in \mathfrak{p}_L^{t+1}$ for all $n \geq 1$ for any $u(x)$ as above. Of course this claim implies that the inequality $c_1(\mathrm{R}'_{L/K}, K) \geq \frac{t+1}{e(L/K)}$ we need for the proof of 4.4. This will be accomplished in a series of lemmas below; the claim itself is the statement of Lemma 4.4.5.

We introduce some notation. Let $U_{L,0}$ be the subgroup of $\mathcal{O}_L[[x]]^\times$ consisting of all elements of the form $u(x) = 1 + \sum_{m\geq 1} a_m\, x^m$, $a_m \in \mathcal{O}_L$ for all $m \geq 1$. For each integer $b \geq 1$, denote by $U_{L,b}$ the subgroup of $U_{L,0}$ consisting of all elements of the form $u(x) = 1 + \sum_{m\geq 1} a_m\, x^m$ such that $a_m \in \mathfrak{p}_L^b$ for all $m \geq 1$. The quotient $U_{L,0}/U_{L,1}$ is canonically isomorphic to the subgroup $1 + x \cdot \kappa[[x]]$ of $\kappa[[x]]^\times$. Choose a generator π_L of $\mathfrak{p}_L$, we obtain an isomorphism $\beta_{b,\pi_L} : U_{L,b}/U_{L,b+1} \xrightarrow{\sim} x \cdot \kappa[[x]]$ for each $b \geq 1$. This map β_{b,π_L} sends an element $u(x) = 1 + \pi_L^b \sum_{m\geq 1} a_m\, x^m$ of $U_{L,b}$ to the element $\sum_{m\geq 1} \overline{a_m} x^m$ of $x \cdot \kappa[[x]]$, where $\overline{a_m}$ is the image of $a_m \in \mathcal{O}_L$ in κ.

(4.4.1) Lemma *Suppose that $\Gamma = \mathrm{Gal}(L/K)$ is cyclic of order ℓ, where ℓ is a prime number, and $\kappa_L = \kappa$. Let σ be a generator of Γ. Suppose that $i_\Gamma(\sigma) = t+1$, so that $\Gamma = \Gamma_t$, $\Gamma_{t+1} = \{1\}$. Then for each $n \geq 0$, we have*

$$\mathrm{Tr}_{L/K}(\pi_L^n \cdot x\mathcal{O}_L[[x]]) = \pi_K^r \cdot x\mathcal{O}_L[[x]] \quad \text{where} \quad r = \left\lfloor \frac{n + (t+1)(\ell-1)}{\ell} \right\rfloor\,.$$

In particular,

$$\begin{array}{lcll} \mathrm{Tr}_{L/K}(\mathfrak{p}_L^n\, x \cdot \mathcal{O}_L[[x]]) & \subseteq & \mathfrak{p}^{n+1}\, x \cdot \mathcal{O}[[x]] & \quad \text{for } n = 1, \ldots, t-1 \\ \mathrm{Tr}_{L/K}(\mathfrak{p}_L^t\, x \cdot \mathcal{O}[[x]]) & = & \mathfrak{p}^t\, x \cdot \mathcal{O}[[x]] & \end{array}$$

PROOF. This follows from [S2, V §3, Lemma 3]. ∎

(4.4.2) Lemma *Notation as in 4.4.1. Suppose that $\xi(x) \in \mathfrak{p}_L^n\, x \cdot \mathcal{O}_L[[x]]$, $n \geq 1$. Then*

$$\mathrm{Nm}_{L/K}(1+\xi(x)) \equiv 1 + \mathrm{Tr}_{L/K}(\xi(x)) + \mathrm{Nm}_{L/K}(\xi(x)) \pmod{\mathrm{Tr}_{L/K}(\mathfrak{p}_L^{2n})}\,.$$

PROOF. The argument in [S2, V §3 Lemma 5] works here as well.

(4.4.3) Lemma *Notation as in 4.4.1.*

(i) *We have $\mathrm{Nm}_{L/K}(U_{L,n}) \subseteq U_{K,n}$ for each $0 \leq n \leq t+1$.*

(ii) *Let $u(x)$ be an element of $U_{L,0}$. Let $\overline{u(x)}$ be the image of $u(x)$ in $1 + x \cdot \kappa[[x]] \cong U_{L,0}/U_{L,1}$. Then the image of $\mathrm{Nm}_{L/K}(u(x))$ in $1 + x \cdot \kappa[[x]] \cong U_{K,0}/U_{K,1}$ is $\overline{u(x)}^{\ell}$.*

(iii) *Let $u(x)$ be an element of $U_{L,n}$, $1 \leq n \leq t-1$. Let $\overline{\xi(x)} = \beta_{n,\pi_L}(u(x)) \in x \cdot \kappa[[x]]$. Then the image of $\mathrm{Nm}_{L/K}(u(x))$ in $x \cdot \kappa[[x]]$ under β_{n,π_K} is $\alpha_n\, \overline{\xi(x)}^{p}$, where α_n denotes the image of $\frac{\mathrm{Nm}_{L/K}(\pi_L^n)}{\pi_K^n} \in \mathcal{O}_L^{\times}$ in $\kappa^{\times}$.*

(iv) *The map $U_{L,t}/U_{L,t+1} \to U_{K,t}/U_{K,t+1}$ induced by $\mathrm{Nm}_{L/K}$ can be described as follows. Identify $U_{L,t}/U_{L,t+1}$ with $x \cdot \kappa[[x]]$ via β_{t,π_L}, and identify $U_{K,t}/U_{K,t+1}$ with $x \cdot \kappa[[x]]$ via β_{t,π_K}. Then there exists $\alpha, \gamma \in \kappa^{\times}$ such that $\mathrm{Nm}_{L/K}$ sends $\overline{\xi(x)} \in x \cdot \kappa[[x]] \cong U_{L,t}/U_{L,t+1}$ to $\alpha\, \overline{\xi(x)}^{p} + \gamma\, \overline{\xi(x)}$.*

PROOF. Use 4.4.2, 4.4.1 and the argument of [S2] V §3, Prop. 4 and Prop. 5. ∎

(4.4.4) Lemma *Let L be a finite Galois extension of K with group Γ, and assume that $\kappa_L = \kappa$. Suppose that $\Gamma = \Gamma_t \supsetneq \Gamma_{t+1}$.*

(i) *The L/K-norm satisfies $\mathrm{Nm}_{L/K}(U_{L,n}) \subset U_{K,n}$ for $0 \leq n \leq t+1$, and induces a homomorphism $\mathrm{Nm}_{L/K,n} : U_{L,n}/U_{L,n+1} \to U_{K,n}/U_{K,n+1}$ for $n = 0, 1, \ldots, t$.*

(ii) *Suppose that $1 \leq n \leq t$. Identify $U_{L,n}/U_{L,n+1}$ with $x \cdot \kappa[[x]]$ via β_{n,π_L}, and identify $U_{K,n}/U_{K,n+1}$ with $x \cdot \kappa[[x]]$ via β_{n,π_K}. Then*

$$\mathrm{Nm}_{L/K,n} : x \cdot \kappa[[x]] \xleftarrow[\beta_{n,\pi_L}]{\sim} U_{L,n}/U_{L,n+1} \to U_{K,n}/U_{K,n+1} \xrightarrow[\beta_{n,\pi}]{\sim} x \cdot \kappa[[x]]$$

can be described as follows. If $1 \leq n \leq t-1$, then there exists an element $\delta_n \in \kappa^{\times}$ such that

$$\mathrm{Nm}_{L/K,n} : \overline{\xi(x)} \mapsto \delta_n \cdot \overline{\xi(x)}^{[L:K]} \qquad \forall\, \overline{\xi(x)} \in x \cdot \kappa[[x]]\,.$$

(iii) *Notation as in* (ii). *For $n = t$, there exists an additive polynomial $P(Y) \in \kappa[Y]$ of degree $[L:K]$ and separable degree $[\Gamma : \Gamma_{t+1}]$ such that*

$$\mathrm{Nm}_{L/K,t} : \overline{\xi(x)} \mapsto P(\xi(x)) \qquad \forall\, \overline{\xi(x)} \in x \cdot \kappa[[x]]\,.$$

PROOF. The proof is similar to that of [S2] V §6 Prop. 9. One can build up the Galois extension L/K, starting from K, by a sequence of successive cyclic extensions. Apply Lemma 4.4.3 and use induction. ∎

(4.4.5) Lemma *Notation as in 4.4.4 Suppose that $u(x)$ is an element of $U_{L,0}$ such that $\mathrm{Nm}_{L/K}(u(x)) = 1$. Then $u(x) \in U_{L,t+1}$.*

Proof of 4.4.5 and Thm. 4.4. As explained at the beginning of the proof of 4.4, Lemma 4.4.5 implies Thm. 4.4. We have seen in 3.3.1 that $u(x) \in U_{L,1}$. Apply Lemma 4.4.4 (ii) repeatedly, we deduce that $u(x) \in U_{L,t}$. It remains to show that $u(x) \in U_{L,t+1}$. Let $\overline{\xi(x)} = \beta_{t,\pi_L}$. Apply Lemma 4.4.4 (iii) and use the notation there, we have $P(\overline{\xi(x)}) = 0$ in $\kappa[[x]]$, where $\overline{\xi(x)} \in x \cdot \kappa[[x]]$ is the image of $u(x)$ under $\beta_{t,\pi_L} : U_{L,t}/U_{L,t+1} \xrightarrow{\sim} x \cdot \kappa[[x]]$. We want to deduce, from the equation $P(\overline{\xi(x)}) = 0$, the desired conclusion that $\overline{\xi(x)} = 0$ in $\kappa[[x]]$.

There exists an additive polynomial $P_s(T) \in \kappa^{\mathrm{alg}}[T]$ of the form

$$P_s(T) = b_k\,T^{p^k} + \cdots + b_1\,T^p + a_0\,T \qquad \text{with } b_0 \in (\kappa^{\mathrm{alg}})^\times$$

such that $P(T) = P_s(T^{\mathrm{Card}(\Gamma_{t+1})})$. Notice that $\mathrm{Card}(\Gamma_{t+1})$ is a power of p. We know that $P_s(\overline{\xi(x)}^{\mathrm{Card}(\Gamma_{t+1})}) = P(\overline{\xi(x)}) = 0$. Therefore $\overline{\xi(x)}^{\mathrm{Card}(\Gamma_{t+1})} = 0$ by Hensel's Lemma. Hence $\overline{\xi(x)} = 0$. ∎

(4.5) Corollary *Let L be a finite Galois extension of K such that κ_L/κ_K is separable. Let T be a torus over K which is split over L. Assume that T is anisotropic over the maximal unramified extension of K. Then $c_1(T,K) \geq \frac{t+1}{e(L/K)}$, where t is the natural number such that $\Gamma = \Gamma_t \neq \Gamma_{t+1}$.*

Proof. This follows from Prop. 3.4 (ii), Thm. 3.1 and Prop. 4.4. ∎

§5. Examples

(5.1) Example Let L/K be a totally ramified extension such that $\Gamma = \mathrm{Gal}(L/K)$ is cyclic of order p. Assume that $\kappa_L = \kappa$ and $\mathrm{Char}(\kappa) = p$.

(i) For any two anisotropic torus T_1, T_2 over K of with $\dim(T_1) = \dim(T_2)$, there exists an isogeny $\alpha : T_1 \to T_2$ of degree prime to p. Consequently $c_i(T_1, K) = c_i(T_2, K)$ for each $i = 1, \ldots, \dim(T_1)$.

(ii) Let t be the integer such that $\Gamma = \Gamma_t$, $\Gamma_{t+1} = \{1\}$. Let T be an anisotropic torus over K of dimension $p-1$. Then $c_i(T,K) = \frac{i(t+1)}{p}$ for $i = 1, \ldots, p-1$.

Proof. (i) The quotient of the group ring $\mathbb{Z}[\Gamma]$ by the ideal generated by the element $\sum_{\sigma\in\Gamma} \sigma$ is isomorphic to $\mathbb{Z}[x]/(x^{p-1} + \ldots + x + 1)$, the ring of integers in the cyclotomic field $\mathbb{Q}(\mu_p)$. Hence the localization $R_{(p)}$ of the ring $R := \mathbb{Z}[\Gamma]/(\sum_{\sigma\in\Gamma} \sigma)$ at the ideal generated by p is a discrete valuation ring. For $i = 1, 2$, the character group $X_*(T_i)$ is naturally an R-module. Hence $X_*(T_1)_{(p)}$ is isomorphic to $X_*(T_2)_{(p)}$ as $R_{(p)}$-modules. In other words there exists a Γ-equivariant homomorphism $h : X_*(T_1) \to X_*(T_2)$ such that $\ker(h)$ and $\mathrm{coker}(h)$ are finite groups of order prime to p.

(ii) By (i) we may and do assume that $T = \mathrm{R}''_{L/K}$, so we can apply 4.2 (i). ∎

(5.2) Example Let $m \geq 2$ be a natural number. Let $K = \mathbb{Q}_2(\mu_{2^m})$, $M = \mathbb{Q}_2(\mu_{2^{m+1}})$, and $L = \mathbb{Q}_1(\mu_{2^{m+2}})$. Let $T' = \mathrm{R}_{M/K}$, $T = \mathrm{R}_{L/K}$, and let T'' be the quotient torus T/T'. Then

$$\underline{c}(T', K) = (0, 2^{m-1}), \quad \underline{c}(T'', K) = (2^{m-1}, 2^m), \quad \underline{c}(T, K) = (0, 2^{m-2}, 3 \cdot 2^{m-2}, 2^m)\,.$$

Proof. The Galois group $\Gamma = \mathrm{Gal}(L/K)$ is cyclic of order 4, canonically isomorphic to the subgroup of $\mathbb{Z}/2^{m+2}\mathbb{Z})^\times$ generated by the image of $1 + 2^m$; denote this element by σ. Let ζ be a primitive 2^{m+2}-th root of unity. Then

$$i_\Gamma(\sigma) = \mathrm{ord}_L(\sigma(\zeta) - \zeta) = \mathrm{ord}_L(\zeta \cdot (\sqrt{-1} - 1)) = 2^m$$

By 4.2 (i) we get $c_2(T, K) = 2^{m-2}$. From 4.2 (ii) we get $c_4(T, K) = 2^m$, and $c_3(T, K) = 2^m - 2^{m-2} = 3 \cdot 2^{m-2}$.

Let $\Delta = \mathrm{Gal}(M/K)$, generated by the image $\bar\sigma$ of σ in Δ. Then

$$\imath_\Delta(\bar\sigma) = \mathrm{ord}_M(\bar\sigma(\zeta^2) - \zeta^2) = \mathrm{ord}_M(\zeta^2 \cdot (-2)) = 2^m$$

and $c_1(T', K) = 2^{m-1}$ by 4.2 (i).

We still have to compute $\underline{c}(T'', K)$. By [Ch1] 4.5 and 4.8, we have a natural short exact sequence

$$0 \to \mathrm{Lie}(\underline{T'}^{\mathrm{NR}}) \to \mathrm{Lie}(\underline{T}^{\mathrm{NR}}) \to \mathrm{Lie}(\underline{T''}^{\mathrm{NR}}) \to 0$$

which is compatible with the map on Lie algebras induced by the base change map $\mathrm{can}_{L/K}$. We identify $\mathrm{Lie}(\underline{T'}^{\mathrm{NR}})$ with $\mathcal{O}_M$, and $\mathrm{Lie}(\underline{T}^{\mathrm{NR}})$ with $\mathcal{O}_L$. So $\mathrm{Lie}(\underline{T''}^{\mathrm{NR}})$ is identified with $\mathcal{O}_L/\mathcal{O}_M$. We identify $X^*(T)$ with $\mathbb{Z}[\Gamma]$, and the character group $X^*(T'')$ of T'') is identifies with the subgroup $\mathbb{Z} \cdot (\sigma^2 - 1) + \mathbb{Z} \cdot (\sigma^3 - \sigma)$ of $\mathbb{Z}[\Gamma]$. We use the above choice of basis of $X^*(T'')$ to identify $\mathrm{Lie}(\underline{T''_L}^{\mathrm{NR}})$ with $\mathcal{O}_L \oplus \mathcal{O}_L$. Hence the map

$$\mathrm{can}_{T'', L/K_*} : \mathrm{Lie}(\underline{T''}^{\mathrm{NR}}) \otimes_{\mathcal{O}} \mathcal{O}_L \to \mathrm{Lie}(\underline{T''_L}^{\mathrm{NR}})$$

is identified with the map from $(\mathcal{O}_L/\mathcal{O}_M) \otimes_{\mathcal{O}} \mathcal{O}_L \to \mathcal{O}_L \oplus \mathcal{O}_L$ which sends each element $\bar y \in \mathcal{O}_L/\mathcal{O}_M$ represented by $y \in \mathcal{O}_L$ to the element $(\sigma^2(y) - y, \sigma^3(y) - \sigma(y)) \in \mathcal{O}_L \oplus \mathcal{O}_L$. Using the $\mathcal{O}$-basis $\{\bar\zeta, \bar\zeta^3\}$ of $\mathcal{O}_L/\mathcal{O}_M$, one finds that the elementary divisors of $\mathrm{can}_{T'', L/K_*}$ are $(2, 4)$. In other words $c_1(T'', K) = \mathrm{ord}_K(2) = 2^{m-1}$, $c_2(T'', K) = \mathrm{ord}_K(4) = 2^m$. ∎

(5.3) Example This example shows that if the residue field κ of $\mathcal{O}$ is not perfect, then the base change conductor $c(T, K)$ for tori may change under K-isogenies.

The polynomial ring $\mathbb{Z}[u,v]$ contains subrings $\mathbb{Z}[x,v], \mathbb{Z}[u,y]$, and $\mathbb{Z}[x,uv]$, where $x = u^2, y = v^2$. Let $\mathcal{O}_K$ be the (2)-adic completion of $\mathbb{Z}[x,y]$, $\mathcal{O}_{M_1}$ be the (2)-adic completion of $\mathbb{Z}[u,y]$, $\mathcal{O}_{M_2}$ be the (2)-adic completion of $\mathbb{Z}[x,v]$, $\mathcal{O}_{M_3}$ be the (2)-adic completion of $\mathbb{Z}[x,uv]$, and let $\mathcal{O}_L$ be the (2)-adic completion of $\mathbb{Z}[u,v]$. These are complete discrete valuation rings, and 2 is a uniformizing element in each of them. The residue field of $\mathcal{O}_L$ (resp. $\mathcal{O}_{M_1}, \mathcal{O}_{M_2}, \mathcal{O}_{M_3}, \mathcal{O}_K$) is $\mathbb{F}_2(u,v)$ (resp. $\mathbb{F}_2(u,y), \mathbb{F}_2(x,v), \mathbb{F}_2(x,uv), \mathbb{F}_2(x,y)$.) Denote by K, M_1, M_2, M_3, L the fraction field of $\mathcal{O}_K, \mathcal{O}_{M_1}, \mathcal{O}_{M_2}, \mathcal{O}_{M_3}$ respectively. Let $T = \mathrm{R}_{L/K}$, and let $T_i = \mathrm{R}_{M_i/K}$, $T_i'' = T/T_i$, $i = 1,2,3$. Then

$$\underline{c}(T,K) = (0,1,1,2), \quad \underline{c}(T_i,K) = (0,1), \quad \underline{c}(T_i'',K) = (1,2)\,.$$

On the other hand, let $T_i' = \mathrm{R}'_{M_i/K} \cong \mathrm{R}''_{M_i/K}$, $i = 1,2,3$. Each T_i' is a one-dimensional torus over K, and $c(T_i',K) = 1$ for $i = 1,2,3$. Notice that T_1'' is isogenous to $T_2' \times T_3'$, $c(T_1'',K) = 3$, $c(T_2'' \times T_3'',K) = 2$.

Sketch of proof. The Galois group $\mathrm{Gal}(L/K)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Let σ (resp. τ) be the element of $\mathrm{Gal}(L/K)$ such that $\sigma(u) = -v, \sigma(v) = v$ (resp. $\tau(u) = u, \tau(v) = -v$), so $\mathrm{Gal}(L,K) = \{1,\sigma,\tau,\sigma\tau\}$. The set $\{1,u,v,uv\}$ is an $\mathcal{O}_K$-basis of $\mathcal{O}_L$. So the $c_i(T,K)$'s are the elementary divisors of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ u & -u & u & -u \\ v & v & -v & -v \\ uv & -uv & -uv & uv \end{pmatrix}.$$

A simple computation shows that $\underline{c}(T,K) = (0,1,1,2)$. A similar but simpler calculation shows that $\underline{c}(T_i,K) = (0,1)$, $i = 1,2,3$. The Lie algebra of $\underline{T_i''}^{\mathrm{NR}}$ is naturally isomorphic to the quotient of the Lie algebra of $\underline{T}^{\mathrm{NR}}$ by the Lie algebra of $\underline{T_i}^{\mathrm{NR}}$ according to [Ch1] 4.5 and 4.8, so $c_1(T_i'',K) + c_2(T_i'',K) = 3$. On the other hand $c_1(T_i'',K) > 0$ by 3.3. Therefore $\underline{c}(T_i'',K) = (1,2)$ because there is no other possibility. ∎

(5.4) Example We generalize 5.3 to odd primes. Let p be a prime number. Consider the polynomial ring $\mathbb{Z}[u,v]$ and its subring $\mathbb{Z}[u^p,v^p]$; let $x = u^p, y = v^p$. Let $\mathcal{O}_K$ and $\mathcal{O}_L$ be the (2)-adic completion of $\mathbb{Z}[x,y]$ and $\mathbb{Z}[x,y]$ respectively; denote the fraction fields by K, L. The field L is separable over K of degree p^2. Let $K_1 = K(\mu_p)$, $L_1 = L(\mu_p)$. Then L_1/K_1 is Galois with $\mathrm{Gal}(L_1/K_1) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Let $T = \mathrm{R}_{L/K}$, $T_1 = \mathrm{R}_{L_1/K_1}$. Then each $c_i(T,K)$ has the form $\frac{i}{p-1}$, where i is an integer, $0 \le i \le 2p-2$. For a natural number i with $0 \le i \le 2p-2$, the multiplicity of $\frac{i}{p-1}$ in $\underline{c}(T,K)$ is equal to $\min(i+1, 2p-1-i)$. Similarly each $c_i(T_1,K_1)$ is an integer i with $0 \le i \le 2p-2$. For a natural number i with $0 \le i \le 2p-2$, the multiplicity of i in $\underline{c}(T_1,K_1)$ is equal to $\min(i+1, 2p-1-i)$. The proofs of the above assertions are similar to the proof of 5.3, therefore omitted.

It is easy to compute the discriminants: $\mathrm{disc}(L/K) = (p)^{2p^2}$, while $\mathrm{disc}(L_1/K_1) = (p)^{2p^2} = (\zeta_p - 1)^{2(p-1)p^2}$. Notice that $c_{p^2}(T_1,K_1) = 2(p-1)$, and it coincides with the formula in 4.2 (ii) for $c_{\dim(T)}(T,K)$ for an induced torus T, if we interpret "$e(L/K)$" in 4.2 as $[L:K]$ when κ_L/κ_K is purely inseparable. This observation still holds if we change $\mathbb{Z}[u,v]/\mathbb{Z}[u^p,v^p]$ to $\mathbb{Z}[u_1,\ldots,u_m]/\mathbb{Z}[u_1^p,\ldots,u_m^p]$: Then $\mathrm{disc}(L_1/K_1) = n(p-1)p^n$, and $c_{p^n}(\mathrm{R}_{L_1/K_1},K_1) = n(p-1)$. Whether this is an accident, or is a special case of a general phenomenon, seems unclear at present.

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