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Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli

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Abstract. We prove that any ordinary symplectic separable isogeny class in the moduli space of principally polarized abelian varieties over a field of positive characteristic is dense in the Zariski topology.

Contents

Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let $g \geqq 1$ be a positive integer and let $\mathscr{A}_g$ be the moduli stack of principally polarized abelian varieties of dimension g. Suppose that $x \in \mathscr{A}_g(k)$ corresponds to an abelian variety A_x. Let D be a positive integer not divisible by p. Consider the set $\mathscr{G}^{(p)}(x) \subseteq \mathscr{A}_g(k)$ (resp. $\mathscr{G}_D(x) \subset \mathscr{A}_g(k)$) consisting of all points $y \in \mathscr{A}_g(k)$ such that there exists an isogeny $\phi : A_y \to A_x$ with $\phi^*(\mathrm{pol}_{A_x}) = m \cdot \mathrm{pol}_{A_y}$ for some positive integer m, $(m, p) = 1$ (resp. $m \mid D^N$ for some non-negative integer N). We shall refer to $\mathscr{G}^{(p)}(x)$ (resp. $\mathscr{G}_D(x)$) as the prime-to-

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p (resp. D-power) Hecke orbit of x in $\mathscr{A}_g$. This paper was started by the following question:

When is the countable subset $\mathscr{G}^{(p)}(x)$ (resp. $\mathscr{G}_D(x)$) Zariski dense in $\mathscr{A}_g(k)$?

Clearly a necessary condition for either question is that x has to be ordinary. Otherwise $\mathscr{G}^{(p)}(x)$ and $\mathscr{G}_\ell(x)$ will be contained in the zero locus of the determinant of the Hasse-Witt matrix. The question is whether this condition is also sufficient. Lacking any counterexample, an optimist would be inclined to form the opinion after some cautious thought that the answer should be 'yes'. We formulate this expectation as

Question (Q 1). Given an ordinary principally polarized abelian variety A_0 over an algebraically closed field k of characteristic $p > 0$, denote by $\mathscr{G}^{(p)}(A_0)$ the set of all principally polarized abelian varieties A' over k such that there exists an isogeny $\beta : A' \to A_0$ which preserves the principal polarizations up to a positive integer prime to p. Is the set $\mathscr{G}^{(p)}(A_0)$ Zariski dense in the moduli stack $\mathscr{A}_g/k$ classifying principally polarized abelian varieties of dimension g over k?

Notice that in (Q 1) we can substitute "isogeny" by "prime-to-p quasi-isogeny" without changing the set $\mathscr{G}^{(p)}(A_0)$. By prime-to-p quasi-isogenies we mean the groupoid generated by prime-to-p isogenies. There is a several-prime version of (Q 1):

Question (Q 1)$_D$. Given an ordinary principally polarized abelian variety A_0 over an algebraically closed field k of characteristic $p > 0$. Let D be a positive integer not divisible by p. Denote by $\mathscr{G}_D(A_0)$ the set of all principally polarized abelian varieties A' over k such that there exists an isogeny $\beta : A_0 \to A'$ preserving the principal polarizations up to a divisor of a power of D. Is the set $\mathscr{G}_D(A_0)$ Zariski dense in the moduli stack $\mathscr{A}_g/k$ classifying principally polarized abelian varieties of dimension g over k?

Notice that we can replace "isogeny" by "D-power quasi-isogeny in (Q 1)$_D$". Here "D-power quasi-isogenies" is the groupoid generated by isogenies whose kernels are killed by some power of D.

The main result of this article confirms the optimists' prediction about (Q 1). Namely for any ordinary A_0, the countable subset $\mathscr{G}^{(p)}(A_0)$ is Zariski dense in $\mathscr{A}_g$. We actually prove a strong form of (Q 1)$_D$: for any prime number $\ell \neq p$ the ℓ-power Hecke orbit of any ordinary point in $\mathscr{A}_g(k)$ is Zariski dense in $\mathscr{A}_g$. The same is also true for general polarization types. See Theorem 2 at the end of this paper. Of course one could have asked the same questions in characteristic zero. But then the answers are readily available via complex analytic uniformization: for any point τ in the period domain, $\mathrm{Sp}_{2g}(\mathbb{Z}[1/\ell]) \cdot \tau$ is already dense in period domain with respect to the metric topology by strong approximation.

One can also consider the question on whether a prime-to-p Hecke orbit in characteristic p is dense in the more general context of Shimura varieties. Suppose that G is a connected reductive linear algebraic group over $\mathbb{Q}$, X is a $G(\mathbb{R})$-conjugacy class of $\mathbb{R}$-homomorphisms $h : \mathbb{S} \to G$. Here $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the Weil restriction of scalars of $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$, and the pair (G, X) satisfies Deligne's axioms in [6, 2.1.1]. Let $Sh(G, X)$ be the canonical model of the Shimura variety associated to (G, X), defined over the Shimura field $E = E(G, X)$. If $\mathfrak{p}$ is a finite place of $E(G, X)$ lying over a prime number p, and suppose that the group $G(\mathbb{Q}_p)$ has a hyperspecial maximal compact subgroup K_p. Under these conditions Langlands and Rapoport [20] conjectured that $Sh(G, X)/K_p$ has good reduction at $\mathfrak{p}$, and Milne [23, 24] has given a conjectural description of its canonical integral model. For the sake of discussion, let us assume that the canonical integral model exists. Then $G(\mathbb{A}_f^{(p)}) = \prod'_{\ell \neq p} G(\mathbb{Q}_\ell)$ operates on the mod $\mathfrak{p}$ fiber $(Sh(G, X)/K_p) \otimes_{\mathcal{O}_E} \kappa(\mathfrak{p})$ of the canonical integral model of $Sh(G, X)/K_p$ via algebraic correspondences. It is expected that $(Sh(G, X)/K_p) \otimes_{\mathcal{O}_E} \kappa(\mathfrak{p})$ has a natural stratification, which is preserved by the $G(\mathbb{Q}_\ell)$-action. Unless a given stratum has a natural fibration structure compatible with the Hecke action coming from $G(\mathbb{A}_f^{(p)})$, one expects that the Zariski closure of any point in a given stratum contains an irreducible component of the whole stratum. Especially, if $(Sh(G, X)/K_p) \otimes_{\mathcal{O}_E} \kappa(\mathfrak{p})$ has ordinary points, one expects that the Zariski closure of the $G(\mathbb{A}_f^{(p)})$-orbit of an ordinary point is actually equal to $(Sh(G, X)/K_p) \otimes_{\mathcal{O}_E} \kappa(\mathfrak{p})$. Here we have to clarify the meaning of "ordinary points". Conjecturally, over $(Sh(G, X)/K_p) \otimes_{\mathcal{O}_E} \kappa(\mathfrak{p})$ there is a natural family of F-crystals $\mathscr{E}_\rho$ attached to any $\mathbb{Q}$-rational representation ρ of G. We say that a point x of $(Sh(G, X)/K_p) \otimes_{\mathcal{O}_E} \kappa(\mathfrak{p})$ is ordinary if the Newton polygon of the F-crystal $\mathscr{E}_{\rho,x}$ coincides with its Hodge polygon for every $\mathbb{Q}$-rational representation ρ. This more general question is far from being solved. The method we used applies to other Shimura varieties of PEL-type as well. We intend to explore this in another paper.

There is a local version of question (Q 1):

Question (Q 2). Let (A_1, λ_1) be a principally polarized supersingular abelian variety of dimension g over an algebraically closed field k of characteristic $p > 0$. To say that A_1 is supersingular means that A_1 is isogenous to the product of g copies of a supersingular elliptic curve E_s over k, or equivalently that all slopes of the Newton polygon of A_1 are equal to 1/2. Let $\mathfrak{S}_{pol,1}$ be the equicharacteristic deformation space of (A_1, λ_1), which by the Serre-Tate theorem is the same as the equicharacteristic deformation space of the formal completion $(\hat{A}_1, \hat{\lambda}_1)$ of (A_1, λ_1) along the zero section. The compact p-adic group $\mathrm{Aut}(\hat{A}_1, \hat{\lambda}_1)$ of automorphisms of $(\hat{A}_1, \hat{\lambda}_1)$ operates naturally on $\mathfrak{S}_{pol,1}$. If $\mathfrak{Z} \subseteq \mathfrak{S}_{pol,1}$ is a closed formal subscheme of $\mathfrak{S}_{pol,1}$ stable under an open subgroup of $(\hat{A}_1, \hat{\lambda}_1)$ such that the Hasse invariant (= determinant of the Hasse-Witt matrix of the universal abelian scheme over $\mathfrak{S}_{pol,1}$) is not a zero-divisor on $\mathfrak{Z}$, is $\mathfrak{Z}$ necessarily equal to $\mathfrak{S}_{pol,1}$ itself?

We first approached the global question (Q 1) by reducing it to (Q 2), a question in deformation theory. Unfortunately (Q 2) has resisted all my attempts to unravel it. The mysterious action of $\mathrm{Aut}(\hat{A}_1, \hat{\lambda}_1)$ on $\mathfrak{S}_{pol,1}$ remains to be understood. At this moment I can only prove a much weakened version of a special case of (Q 2) by a revoltingly complicated calculation. Therefore it seems preferable to wait until better theorems are available, rather than publishing this hopelessly incomplete result prematurely. However it may be worthwhile to formulate this local question in a more general setting. For any p-divisible formal group G_0 over a field of characteristic p, there is a natural action of the group of automorphisms of G_0 on the deformation space of G_0. As far as I know, the paper [21] by Lubin and Tate is the first to discuss group actions of this sort. The action on the Lubin-Tate moduli space on the generic fiber was closely studied in the recent paper [14]. They also appeared in the important recent work of Rapoport and Zink. It seems that this action is quite fundamental and deserves further investigation. Even in the case when G_0 is a one-dimensional formal group of finite height, this action is not at all well-understood. For instance the equivariant cohomology groups of the group of units of the deformation space contain deep information about the stable homotopy group of spheres, because they are closely related to the chromatic spectral sequence. The analogue of question (Q 2) is also unresolved in this case, see the end of [5] for a precise formulation of the question. Hopefully the question (Q 2) will be eventually solved as our knowledge about these actions accumulates.

I would like to express my gratitude to J. Milne, whose conjecture on the existence of canonical integral models of Shimura varieties motivated me to ask (Q 1); to G. Faltings for the idea of reduction to supersingular points; to M. Larsen for many hours of stimulating and enjoyable discussion on (Q 1), and also for relating Faltings' idea to me; to F. Oort for patiently going through the proof and pointing out mistakes, and also for stimulating comments and encouragement; to B. Gross and M. Hopkins, for their inspiring paper [14] and also for discussions and email communications about the p-adic period map; to P. Deligne for pointing out a slip in an earlier version; to A. Neeman and W. Messing for useful conversations and comments; and to the referee for a very careful reading of the manuscript and many useful suggestions. It is a great pleasure to acknowledge my intellectual debt to all of them. A part of the work on this paper was done while the author was on sabbatical leave from the University of Pennsylvania during the year 1992/93. I would like to thank Academia Sinica and the Mathematical Sciences Research Institute for their hospitality and for providing excellent working environment.

This paper is organized as follows. The strategy of the proof of the main results Theorems 1 and 2 is described in §1. There are three kinds of local calculations involved: at the boundary of $\mathscr{A}_g$, at ordinary points over finite fields and at supersingular points. These are carried out in §2, §4 and §5 respectively. Another ingredient is a trick of reduction to the Hilbert-Blumenthal moduli space. This is explained in §3. Readers in a hurry can read §1 and the statements of propositions and theorems to get the idea of the proof.

1. Strategy and methods of the proof

Let k be an algebraically closed field of characteristic $p > 0$. Let $g \geqq 1$ be a positive integer. Let $\mathscr{A}_g = \mathscr{A}_g/k$ be the moduli stack classifying principally polarized abelian varieties of dimension g over $\mathrm{Spec}(k)$. For a point $x \in \mathscr{A}_g(k)$ corresponding to a principally polarized abelian variety (A_x, λ_x), the expectation (Q 1) in the introduction states that

(Q 1). The countable set

$$\mathscr{G}^{(p)}(x) = \left\{ y \in \mathscr{A}_g(k) \;\middle|\; \begin{array}{l} \exists \text{ an isogeny } \phi : A_y \to A_x \text{ and } m \in \mathbb{N}, \\ \text{s. t. } (m, p) = 1 \text{ and } \phi^*(\lambda_x) = m\lambda_y \end{array} \right\}$$

is Zariski dense in $\mathscr{A}_g$ if A_x is ordinary.

One problem in dealing with problems like (Q 1) is that the usual machinery of algebraic geometry is not designed to deal with problems about finding the Zariski closure of a countable set of points. At a first look, it is not even clear whether there exists any ordinary point $x \in \mathscr{A}_g(k)$ such that $\mathscr{G}^{(p)}(x)$ is dense if $g \geqq 2$. When $g = 1$, it is easy to see that for any prime number $\ell \neq p$, $\mathscr{G}_\ell(x)$ is infinite if and only if A_x is an ordinary elliptic curve, say by using the theory of canonical liftings. (Later on in this section we shall see that for any $g \geqq 1$, $\mathscr{G}^{(p)}(x)$ is finite if and only if x is supersingular.) Thus (Q 1)$_\ell$ holds for every $\ell \neq p$ in this case for the reason of dimension.

We first give an example, due to Michael Larsen, of a point $x \in \mathscr{A}_g(k)$ such that $\mathscr{G}_\ell(x)$ is Zariski dense in $\mathscr{A}_g$.

Example (M. Larsen) Let E be an ordinary elliptic curve over k, with its natural principal polarization λ_E. Let $(A, \lambda) = (E, \lambda_E)^{\oplus g}$, $g \geqq 2$, and let $x \in \mathscr{A}_g(k)$ be the corresponding point. Then $\mathscr{G}_\ell(x)$ is Zariski dense in $\mathscr{A}_g$ for any prime number $\ell \neq p$.

Proof of Example First of all, we show that x is a smooth point of the Zariski closure Z_ℓ of $\mathscr{G}_\ell(x)$. The ℓ-power Hecke operates on $\mathscr{A}_g$ via algebraic correspondences. Although this is not exactly a group action, it comes from the group $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$, and $\mathscr{G}_\ell(x)$ is very much like an orbit of x under $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$. The usual argument which shows that any orbit for a connected algebraic group acting on an algebraic variety is smooth applies in this case: The smooth locus of Z_ℓ is a nonempty open subscheme $Z_{\ell,sm}$ of Z_ℓ, which is stable under ℓ-power Hecke correspondences and contains points of $\mathscr{G}_\ell(x)$ by definition. Hence $x \in Z_{\ell,sm}$.

Let $\mathcal{O}$ be the endomorphism ring of E. It is well-known that $\mathcal{O}$ is an order in an imaginary quadratic number field, and $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. The semigroup

$$S = \mathrm{GL}_g(\mathcal{O}[1/\ell]) \cap \mathrm{M}_{g\times g}(\mathcal{O}) \cap \mathrm{GSp}_{2g}(\mathbb{Q})$$

operates on $(A, \lambda) = (E, \lambda_E)^{\oplus g}$ as endomorphisms, which are ℓ-power isogenies and preserve the polarization up to ℓ-power multiples. Also the elements in the semigroup S induce ℓ-power Hecke correspondences on $\mathscr{A}_g/k$. Consequently the formal completion $Z_{\ell,x}^{\wedge} \subseteq \mathscr{A}_{g,x}^{\wedge}$ of Z_ℓ at x is stable under the natural action of S on the formal completion $\mathscr{A}_{g,x}^{\wedge}$ of $\mathscr{A}_g/k$ at x. Since $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$. $\mathrm{GL}_g(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \cap \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ is isomorphic to $\mathrm{GL}_g(\mathbb{Z}_p)$, and S is dense in $\mathrm{GL}_g(\mathbb{Z}_p)$ under the isomorphism. Therefore the formal subscheme $Z_{\ell,x}^{\wedge}$ of $\mathscr{A}_{g,x}^{\wedge}$ is stable under the action of $\mathrm{GL}_g(\mathbb{Z}_p)$.

We will only use a tiny part of the above information at the level of the tangent space, namely that the tangent space $\mathrm{T}_x Z_\ell$ of Z_ℓ at x is stable under the natural action of $\mathrm{GL}_g(\mathbb{Z}_p)$ on $\mathrm{T}_x(\mathscr{A}_g/k)$. But we know that the action of $\mathrm{GL}_g(\mathbb{Z}_p)$ on $\mathrm{T}_x(\mathscr{A}_g/k)$ factorizes through the natural surjection $\mathrm{GL}(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_g(\mathbb{F}_p)$. Moreover as a representation of $\mathrm{GL}_g(\mathbb{F}_p)$ the tangent space $T_x(\mathscr{A}_g/k)$ is isomorphic to the subspace of symmetric elements in $k^{\oplus g} \otimes_k k^{\oplus g}$. Now it is well known that the second symmetric product of the standard representation of $\mathrm{GL}_g(\mathbb{F}_p)$ is absolutely irreducible if $p > 2$. (This can be verified either directly, or using general results on representations of Chevalley groups due to Curtis. See for instance [2], especially Theorem 6.4 and Corollary 7.3.) If $p > 2$, the invariant subspace $\mathrm{T}_x Z_\ell$ being nonzero, it has to be equal to $\mathrm{T}_x(\mathscr{A}_g/k)$, and we conclude that $Z_\ell = \mathscr{A}_g/k$ when $p > 2$.

When $p = 2$, one can still conclude that $Z_\ell = \mathscr{A}_g/k$. Since the purpose of this example is to illustrate a general principle to be explained next, we shall only indicate how a proof can be supplied. There are at least two ways to do this. The first way is to use the stronger information that $Z_{\ell,x}$ is stable under the action of $\mathrm{GL}_g(\mathbb{Z}_p)$, and use Serre-Tate coordinates on $\mathscr{A}_{g,x}$ to perform computation with higher order deformations. This method of using Serre-Tate coordinates will be explored in §4. Another way is to observe that Z_ℓ contains the diagonally embedded modular curve, therefore the Zariski closure Z_ℓ^* of Z_ℓ in the minimal compactification $\mathscr{A}_g^*/k$ of $\mathscr{A}_g/k$ contains the 0-dimensional cusp "$\sqrt{-1}\,\infty \cdot I_g$". The result of §2 then implies that $Z_\ell = \mathscr{A}_g/k$. □

A close examination of the proof of this example reveals a general method of getting information about the Zariski closure Z of $\mathscr{G}^{(p)}(x)$ and about the Zariski closure Z_ℓ of $\mathscr{G}_\ell(x)$. This is also the only method we know of. Let Z^* be the Zariski closure of Z in the minimal compactification $\mathscr{A}_g^*$ of $\mathscr{A}$, or equivalently the Zariski closure of $\mathscr{G}^{(p)}(x)$ in $\mathscr{A}_g^*$. Clearly if a point $y \in Z^*(k) \subseteq \mathscr{A}_g^*(k)$ is stable under an algebraic correspondence γ coming from $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$, then the formal completion $Z_y^{*\wedge} \subseteq \mathscr{A}_{g,y}^*$ of Z^* at y is stable under γ as well. Hence if a point $y \in Z^{*\wedge}(k)$ has a large "stabilizer subgroup" in $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$, we get information by analyzing the action of the stabilizer subgroup on $\mathscr{A}_{g,y}^{*\wedge}$. When $y \in \mathscr{A}_g(k)$, this is a problem in deformation theory about arbitrarily high order deformations. When y lies in the boundary of $\mathscr{A}_g$, we shall need the theory of degeneration as explained in [12].

For a 'general' point $y \in \mathscr{A}_g(k)$, we cannot expect its stabilizer subgroup in the the group $\mathrm{GSp}_{2g}(\mathbb{A}_f^{(p)})$ (resp. $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$) to be bigger than $(\mathbb{A}_f^{(p)})^\times$ (resp. $\mathbb{Q}_\ell^\times$). The phenomenon of having large stabilizer subgroups occurs only for points which are somewhat special. First, points of $\mathscr{A}_g^*$ at the boundary tend to have large stabilizer subgroups. For instance the 0-dimensional cusp

"$\sqrt{-1}\ \infty \cdot I_g$" contains a maximal parabolic subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ with Levi factor $\mathrm{GL}_g(\mathbb{Q}_\ell)$, even with level structure thrown in. For a point y in the interior, having a large stabilizer subgroup means that the abelian variety A_y has more endomorphisms than it is entitled to. This happens for example if y is rational over a finite field. In this case A_y has sufficiently many complex multiplication, namely $\mathrm{End}_k(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a commutative semisimple algebra of dimension 2g over $\mathbb{Q}$. An extreme case is when A_y is supersingular, in which case we have $\mathrm{End}_k(A_y) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong \mathrm{M}_{2g \times 2g}(\mathbb{Q}_\ell)$.

We shall prove at the end of this section that for a point $y \in \mathscr{A}_g(k)$, its prime-to-p Hecke orbit $\mathscr{G}^{(p)}(y)$ (resp. ℓ-power Hecke orbit $\mathscr{G}_\ell(y)$) is a finite set if and only if A_y is supersingular. Combined with the computation at the boundary of $\mathscr{A}_g$ and the results in [10], this implies that Z_ℓ always contains supersingular points for every prime number $\ell \neq p$. More precisely, in [10] it is shown that $\mathscr{A}_g$ has a natural stratification (called "canonical stratification" in *loc. cit.*), such that each stratum is quasi-affine and stable under all prime-to-p Hecke correspondences. The open stratum corresponds to ordinary points, and the 0-dimensional stratum corresponds to superspecial points. If Z_ℓ^* meets the 0-dimensional cusp of $\mathscr{A}_g^*$, the calculation in §2 implies that $Z_\ell = \mathscr{A}_g$. This calculation can be generalized to other cusps at the boundary of $\mathscr{A}_g$, and an induction on g gives $Z_\ell = \mathscr{A}_g$ if Z_ℓ^* meets the boundary of $\mathscr{A}_g$. So we may assume that Z_ℓ is proper over Spec(k). For any stratum $\mathscr{S}$ of $\mathscr{A}_g$, unless the intersection of Z_ℓ with $\mathscr{S}$ is finite, Z_ℓ will have to contain a point in a lower stratum. This argument can be repeated, so eventually there exists a stratum $\mathscr{S}$ such that $\mathscr{S} \cap Z_\ell$ is non-empty and 0-dimensional. Then every point in $\mathscr{S} \cap Z_\ell$ has finite ℓ-power Hecke orbit, therefore is supersingular. Applying our general principle, one sees that an affirmative answer to the question (Q 2) implies that Z_ℓ is equal to $\mathscr{A}_g$ for every prime number $\ell \neq p$. Since (Q 2) turns out to be difficult, instead we proceed with a sequence of applications of the general method of using points with large stabilizer subgroups.

Let Z be the Zariski closure of the prime-to-p Hecke orbit of an ordinary point of $\mathscr{A}_g(k)$ as before, and let Z^* be the Zariski closure of Z in $\mathscr{A}_g^*$. The logical structure of our proof is as follows:

Step 1. Show that if Z^* meets the 0-dimensional cusp of $\mathscr{A}_g$, then $Z = \mathscr{A}_g$.

The proof of step 1 consists of a direct computation. This is possible because the structure of the completed local ring at the 0-dimensional cusp of $\mathscr{A}_g^*$ is known, and the action of the stabilizer subgroup on the completed local ring is given by classical formulas. This step is explained in §2.

Step 2. Reduction of (Q 1) to its analogue for the Hilbert-Blumenthal moduli spaces.

Since Z contains ordinary points of $\mathscr{A}_g$ which are defined over finite fields, one may assume that A_x is defined over a finite field $\mathbb{F}_q$. To simplify the exposition, let us assume that all simple factors of A_x are isogenous. Choose a suitable totally real subfield $F \subseteq \mathrm{End}_k(A_x) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is fixed by the Rosati involution such that $[F : \mathbb{Q}] = \dim(A)$. Such an F always exists, see Lemma 4 in section 3. To further simplify the exposition, we assume that $\mathcal{O}_F \subseteq \mathrm{End}_k(A_x)$ and that the principal polarization of A is $\mathcal{O}_F$-linear. Then there is a Hilbert-Blumenthal moduli space $\mathcal{M}_F$, a morphism $\mathcal{M}_F \to \mathcal{A}_g$ sending the boundary of $\mathcal{M}_F$ to the 0-dimensional cusp of $\mathcal{A}_g$, and an ordinary point $\tilde{x} \in \mathcal{M}_F(k)$ mapping to $x \in \mathcal{A}_g(k)$. The Hecke correspondences on $\mathcal{M}_F$ come from the reductive $\mathbb{Q}$-algebraic group $\mathrm{GL}_2(F)$. Moreover the morphism $\mathcal{M}_F \to \mathcal{A}_g$ is compatible with the Hecke actions with respect to a homomorphism of groups $\mathrm{GL}(2, \mathbb{A}_{f,F}^{(p)})^{\mathrm{red}} \to \mathrm{GSp}_{2g}(\mathbb{A}_{f,\mathbb{Q}}^{(p)})$. Here $\mathrm{GL}(2, \mathbb{A}_{f,F}^{(p)})^{\mathrm{red}}$ is the group of all elements in $\mathrm{GL}(2, \mathbb{A}_{f,F}^{(p)})$ with determinants in $\mathbb{A}_{f,\mathbb{Q}}^{(p)\times}$. Since there are two different groups and Hecke correspondence coming from them, we introduce the following definition to avoid possible confusion. We shall call the Hecke orbit of a point $\tilde{x}$ in $\mathcal{M}_F$ for the group $\mathrm{GL}(2, \mathbb{A}_{f,F}^{(p)})^{\mathrm{red}}$ the *reduced prime-to-p F-Hecke orbit* of $\tilde{x}$ in $\mathcal{M}_F$. Similarly we define the group $\mathrm{GL}_2(\mathbb{Q}_\ell)^{\mathrm{red}}$ to be the group of all elements in $\mathrm{GL}_2(\mathbb{Q}_\ell)$ with determinant in $\mathbb{Q}_{\ell}^{\times}$, and call the $\mathrm{GL}_2(\mathbb{Q}_\ell)^{\mathrm{red}}$-Hecke orbit the *reduced ℓ-power F-Hecke orbit*. If we can prove that the Zariski closure of the reduced prime-to-p F-Hecke orbit of $\tilde{x}$ is equal to $\mathcal{M}_F$, then Z^* will contain the 0-dimensional cusp of $\mathcal{A}_g$. Hence $Z = \mathcal{A}_g$ by Step 1. This step is explained in §3.

Now let $\tilde{x}$ be an ordinary point in $\mathcal{M}_F(\overline{\mathbb{F}}_p)$ and let $\widetilde{Z}$ be the Zariski closure of the reduced prime-to-p F-Hecke orbit of $\tilde{x}$ in $\mathcal{M}_F$. We must show that $\widetilde{Z} = \mathcal{M}_F$.

Step 3. Use Serre-Tate coordinates at ordinary points of $\widetilde{Z}(\overline{\mathbb{F}}_p)$ to limit $\widetilde{Z}$ to a finite number of possibilities.

For any smooth ordinary point $\tilde{y} \in \mathcal{M}_F(\overline{\mathbb{F}}_p)$, the formal subscheme $\widetilde{Z}_{\tilde{y}}^{\wedge}$ of $\mathcal{M}_{F,\tilde{y}}^{\wedge}$ is stable under the action of the stabilizer subgroup of $\tilde{y}$, which contains prime-to-p isogenies coming from a totally imaginary quadratic extension of F because $A_{\tilde{y}}$ is defined over a finite field. Using the Serre-Tate coordinates on $\mathcal{M}_{F,\tilde{y}}$, the computation in §4 shows that there are only a finite number of possibilities for $\widetilde{Z}_{\tilde{y}}^{\wedge}$. In any case $\widetilde{Z}_{\tilde{y}}^{\wedge}$ must be a "coordinate subspace" of $\mathcal{M}_{F,\tilde{y}}$, determined by a subset of the set of all prime ideals in $\mathcal{O}_F$ lying above p. By faithfully flat descent, this result can be globalized to a statement about tangent spaces of the smooth ordinary locus of $\widetilde{Z}$.

Step 4. Examine the action of the stabilizer subgroup on the completed local ring of supersingular points of $\mathcal{M}_F$ to eliminate all other possibilities, and conclude that $\widetilde{Z} = \mathcal{M}_F$.

If the closure of $\widetilde{Z}$ in $\mathcal{M}_F^*$ meets the boundary of $\mathcal{M}_F^*$, a similar but simpler calculation as in Step 1 shows that $\widetilde{Z} = \mathcal{M}_F$. Otherwise $\widetilde{Z}$ is proper over $\mathrm{Spec}(k)$, and the results in [E-O] implies that $\widetilde{Z}$ contains supersingular points by an argument similar to one used before. The point is that results in Step 3 allows us to conclude that $\widetilde{Z} = \mathcal{M}_F$ from very weak information about the deformation theory at supersingular points. In fact the deformation space has a natural product structure, and all we need is that none of the projections of the formal completion of $\widetilde{Z}$ to the individual factors is fixed under the action of the stabilizer subgroup.

Step 5. Examine the action of the stabilizer subgroup of supersingular points of $\mathcal{M}_F$ on the formal completion of $\mathcal{M}_F$ to show that the Zariski closure of the reduced ℓ-power F-Hecke orbit of an ordinary point in $\mathcal{M}_F$ is actually stable under all reduced prime-to-p F-Hecke correspondences 'at supersingular points'. This forces the reduced ℓ-power F-Hecke orbit of any ordinary point of $\mathcal{M}_F$ to be dense by what we already know.

To conclude this section, we classify polarized abelian varieties over k with finite prime-to-p Hecke orbits. Let (A_0, λ_0) be a polarized abelian variety over k of dimension $g \geqq 1$. Recall that k is algebraically closed of characteristic $p > 0$. Let A_0^t be the dual abelian variety of A_0. For any prime ℓ' different from p, let

$$\mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) = \mathrm{H}^1_{\acute{e}t}(A_0^t,\ \mathbb{Z}_{\ell'}(1)),$$

a free $\mathbb{Z}'_\ell$-module of rank $2g$. For $\ell' = p$, let

$$\mathrm{H}_1(A_0,\ \mathbb{Z}_p) = \mathrm{H}^1_{\mathrm{crys}}(A_0^t/W(k),\ \mathcal{O}_{A_0^t/W(k)}) \otimes_{W(k)} \mathbb{Z}_p(1),$$

a free $W(k)$-module of rank $2g$ with a natural F-crystal structure. Here $\mathbb{Z}_p(m)$ denotes $\mathrm{H}^2_{\mathrm{crys}}(\mathbb{P}^1_k/W(k),\ \mathcal{O}_{\mathbb{P}^1_k/W(k)})^{\otimes(-m)}$ for any $m \in \mathbb{Z}$. Let

$$\mathrm{H}_1(A_0,\ \mathbb{Q}_{\ell'}) = \mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) \otimes_{\mathbb{Z}_{\ell'}} \mathbb{Q}_{\ell'},\ \text{ if } \ell' \neq p\,;$$

$$\mathrm{H}_1(A_0,\ \mathbb{Q}_p) = \mathrm{H}_1(A_0,\ \mathbb{Z}_p) \otimes_{W(k)} B(k),\ \text{ where } B(k) = \mathit{frac}(W(k)).$$

For any prime ℓ' the polarization λ_0 induces a $\mathbb{Z}_{\ell'}$-linear map

$$\lambda_{0,\ell'} : \mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) \xrightarrow{\sim} \mathrm{H}_1(A_0^t,\ \mathbb{Z}_{\ell'}),$$

which becomes an isomorphism when tensored with $\mathbb{Q}_{\ell'}$. The Poincaré sheaf on the product $A_0 \times_{Spec(k)} A_0^t$ induces the nondegenerate Weil pairing

$$\mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) \otimes \mathrm{H}_1(A_0^t,\ \mathbb{Z}_{\ell'}) \longrightarrow \mathbb{Z}_{\ell'}(1).$$

The Weil pairing and the polarization together give a pairing

$$\lambda_{0,\ell'} : \bigwedge^2 \mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) \to \mathbb{Z}_{\ell'}(1),$$

which is nondegenerate after tensoring with $\mathbb{Q}_{\ell'}$. For any prime ℓ' the endomorphism ring $\mathrm{End}_k(A_0)$ (resp. $\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$) operates on $\mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'})$ (resp. $\mathrm{H}_1(A_0,\ \mathbb{Q}_{\ell'})$) by functoriality, and the action extends by linearity to $\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell'}$ (resp. $\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell'}$). The algebra $\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite dimensional semisimple algebra over $\mathbb{Q}$ with a positive Rosati involution $*$ given by the polarization λ_0. This semisimple algebra with involution $(\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q},\ *)$ defines a reductive linear algebraic group $\mathcal{H}$ over $\mathbb{Q}$ whose $\mathbb{Q}$-rational points $\mathcal{H}(\mathbb{Q})$ consists of all elements $h \in \mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $h^* \cdot h = h \cdot h^* = c(h) \cdot \mathrm{id}$ for some $c(h) \in \mathbb{Q}^\times$. The lattice $\mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) \subseteq \mathrm{H}_1(A_0,\ \mathbb{Q}_{\ell'})$ defines a compact subgroup $K_{\ell'} \subseteq \mathcal{H}(\mathbb{Q}_{\ell'})$ consisting of all elements $h \in \mathcal{H}(\mathbb{Q}_{\ell'})$ such that $h \cdot \mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'}) = \mathrm{H}_1(A_0,\ \mathbb{Z}_{\ell'})$. $K_{\ell'}$ is a hyperspecial maximal compact subgroup of $\mathcal{H}(\mathbb{Q}_{\ell'})$ for all but finitely many primes ℓ'. For ℓ' different from p let $G_{\ell'}$ denotes the group of all $\mathbb{Q}_{\ell'}$-linear automorphisms of $H_1(A_0, \mathbb{Q}_{\ell'})$ which preserve the $\mathbb{Q}_{\ell'}$-polarization $\lambda_{0,\ell'}$ on $H_1(A_0, \mathbb{Q}_{\ell'})$ up to $\mathbb{Q}_{\ell'}^{\times}$. The group $G_{\ell'}$ actually comes from a unique reductive linear algebraic group $\mathfrak{G}_{\ell'}$ over $\mathbb{Q}_{\ell'}$ such that $\mathfrak{G}_{\ell'}(\mathbb{Q}_{\ell'}) = G_{\ell'}$. We have a natural inclusion $\mathscr{H}(\mathbb{Q}_{\ell'}) \subseteq G_{\ell'}$, induced by a natural inclusion $\mathscr{H} \times_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}\mathbb{Q}_{\ell'} \subseteq \mathfrak{G}_{\ell'}$.

Using (A_0, λ_0) as a base point, the non-p Hecke orbit of A_0 is parameterized by the double coset

$$(\mathscr{H}(\mathbb{Q}) \cap K_p) \setminus \prod_{\ell' \neq p}{}' G_{\ell'} / \prod_{\ell' \neq p} K_{\ell'} .$$

This is standard: The set

$$\prod_{\ell' \neq p}{}' G_{\ell'} / \prod_{\ell' \neq p} K_{\ell'}$$

parameterizes the set of isomorphism classes of triples (β, A_x, λ_x), where (A_x, λ_x) is a principally polarized abelian variety, and $\beta : A_x \longrightarrow A_0$ is a prime-to-p quasi-isogeny such that

$$\beta^*(\lambda_0) = \frac{a}{b}\lambda_x$$

for some positive integers a, b which are prime to p. If $(\beta_1, A_1, \lambda_1)$ and $(\beta_2, A_2, \lambda_2)$ are two such triples, and

$$\xi : (A_1, \lambda_1) \xrightarrow{\sim} (A_2, \lambda_2)$$

is an isomorphism, then ξ determines an element $h \in \mathscr{H}(\mathbb{Q})$ such that $h \circ \beta_1 = \beta_2 \circ \xi$. This defines a map from the prime-to-p Hecke orbit $\mathscr{G}^{(p)}(A_0, \lambda_0)$ of A_0 to the double coset

$$\mathscr{H}(\mathbb{Q}) \cap K_p \setminus \prod_{\ell' \neq p}{}' G_{\ell'} / \prod_{\ell' \neq p} K_{\ell'} .$$

It is easy to see that this is indeed a bijection. Similarly the ℓ-power Hecke orbit $\mathscr{G}_\ell(A_0, \lambda_0)$ of A_0 is parameterized by

$$(\mathscr{H}(\mathbb{Q}) \cap \prod_{\ell' \neq \ell} K_{\ell'}) \setminus G_\ell / K_\ell .$$

Notice that if $\mathscr{H}(\mathbb{Q}_\ell) = G_\ell$ for one prime number $\ell \neq p$, then $\mathscr{H}(\mathbb{Q}_{\ell'}) = G_{\ell'}$ for all ℓ' different from p, and by the finiteness of generalized class numbers for reductive groups over global fields the ℓ-power Hecke orbit of A_0 is finite.

Proposition 1. *Assume that ℓ is a prime number which is different from $p =$ char.(k). If the ℓ-power Hecke orbit $\mathscr{G}_\ell(A_0, \lambda_0)$ of a polarized abelian variety (A_o, λ_0) over k is finite, and dim$(A_0) \geqq 1$, then A_0 is a supersingular abelian variety. Conversely if (A_0, λ_0) is a supersingular polarized abelian variety, then its prime-to-p Hecke orbit is finite.*

Proof. Assume that A_0 is supersingular. Then $\mathscr{H}(\mathbb{Q}_{\ell'}) = G(\mathbb{Q}_{\ell'})$ for all $\ell' \neq p$, because the dimension of the semisimple algebra $\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell'}$ is equal to $4\dim(A)^2$, which is equal to the dimension of $\mathrm{End}_{\mathbb{Q}_{\ell'}}(\mathrm{H}_1(A_0, \mathbb{Q}_{\ell'}))$. The prime-to-$p$ Hecke orbit of A_0 is also finite, again by the finiteness of class numbers. In fact if A_0 is a supersingular, then $\mathscr{H}$ is a form of GSp_{2g}, its derived group $\mathscr{H}^{\mathrm{der}}$ is simply connected, and $\mathscr{H}/\mathscr{H}^{\mathrm{der}} \cong \mathbb{G}_m$. Since $\mathscr{H}^{\mathrm{der}}(\mathbb{R})$ is compact, the class number in general is bigger than one. Information about the class numbers is available but is not needed here.

Conversely if the ℓ-power Hecke orbit of A_0 is finite for a prime number $\ell \neq p$, then the homogeneous space $G_\ell/\mathscr{H}(\mathbb{Q}_\ell)$ is compact. This implies that $\mathscr{H} \times_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}\mathbb{Q}_\ell = \mathfrak{G}_\ell$ by Wang's generalization of the Borel density theorem, therefore $\mathscr{H}(\mathbb{Q}_\ell)$ is to G_ℓ. For the reader's convenience, we recall a special case of Wang's theorem: If G is a connected semisimple algebraic group defined over a nondiscrete locally compact field K without nontrivial K-anisotropic factors, and if $\Gamma \subset G(K)$ is a closed subgroup in the Hausdorff topology such that $G(K)/\Gamma$ has a finite $G(K)$-invariant measure, then Γ is Zariski dense in G. See [34], Cor. 1.4.

Clearly $\mathscr{H}(\mathbb{Q}_\ell) \subseteq G_\ell$ is induced by

$$(\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,\ *) \subseteq (\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}_1(A_0,\ \mathbb{Q}_\ell)),\ *).$$

Since the representation of $\mathscr{G}_\ell$ on $\mathrm{H}_1(A_0, \mathbb{Q}_\ell)$ is absolutely irreducible, $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}_1(A_0,\ \mathbb{Q}_\ell))$ is the $\mathbb{Q}_\ell$-linear span of $\mathscr{G}_\ell$. It follows that

$$\dim_{\mathbb{Q}_\ell}(\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell}(\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}_1(A_0,\ \mathbb{Q}_\ell))) = 4g^2 \quad ,$$

Hence $\dim_{\mathbb{Q}}(\mathrm{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}) = 4g^2$. This implies that A_0 is a supersingular abelian variety. Proposition 1 is proved. □

2. Calculation at the 0-dimensional cusp

Let Z be the Zariski closure of the prime-to-p Hecke orbit of an ordinary point $x \in \mathscr{A}_g(k)$ as in §1. In this section, we assume that the closure Z^* of Z in the minimal compactification $\mathscr{A}_g^*$ contains the 0-dimensional cusp of $\mathscr{A}_g$. Therefore Z^* is a Zariski closed subscheme of $\mathscr{A}_g^*$, stable under the ℓ-power Hecke correspondences for a prime number $\ell \neq p$, and contains the 0-dimensional cusp. In this section we shall show that these assumptions force Z to be equal to $\mathscr{A}_g$.

The proof itself is an illustration of the general principle explained in §1. The point is that the the 0-dimensional cusp has a large "stabilizer subgroup" in the ℓ-power Hecke operators: The ℓ-power Hecke correspondences come from the group $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$, and the stabilizer subgroup of the 0-dimensional cusp contains a maximal parabolic subgroup P, which is a semidirect product of its Levi factor L and its unipotent radical U. The Levi component L is isomorphic to $\mathrm{GL}_g(\mathbb{Q}_\ell)$; the unipotent radical is commutative, and as a representation of $\mathrm{GL}_g(\mathbb{Q}_\ell)$ it is isomorphic to the second divided power product of the standard representation of $\mathrm{GL}_g(\mathbb{Q}_\ell)$. In this section we shall follow the notation in [12]. The free abelian group X of rank g used in [12], chapters 4 and 5 corresponds to the dual of the standard representation of our Levi factor GL_g here. Of course this depends on the choice of an isomorphism from the Levi factor to GL_g. In terms of the standard block form realization of Sp_{2g} and the maximal parabolic subgroup P which we use below, the isomorphism is induced by sending an upper-triangular block to the entry A in the upper-left corner of the block.

As before ℓ is a prime number different from p, where $p > 0$ is the characteristic of the algebraically closed field k. For any positive integer n prime to p and a choice of a primitive n-th root of unity, let $\mathscr{A}_{g,n}$ denote the moduli stack of principally polarized abelian varieties over k with symplectic level-n-structure; let $\mathscr{A}^*_{g,n}$ denotes the minimal compactification of $\mathscr{A}_{g,n}$. We choose a trivialization

$$\mathbf{e}_\ell \cdot \mathbb{Z}[1/\ell]/\mathbb{Z} \xrightarrow{\sim} \mu_{\ell^\infty}(k)$$

of μ_{ℓ^∞} over k. Then we can consider the projective system

$$\varprojlim_{m \in \mathbb{N}} \mathscr{A}_{g,\ell^m},$$

the ℓ-adic tower of moduli spaces of principally polarized abelian varieties over k with symplectic ℓ-power level-structure. On each individual $\mathscr{A}_{g,\ell^m}$, the group

$$\mathrm{Sp}_{2g}(\mathbb{Z}/\ell^m\mathbb{Z}) = \left\{ \gamma \in M_{2g}(\mathbb{Z}/\ell^m\mathbb{Z}) \;\middle|\; \langle \gamma v, \gamma w\rangle = \langle v, w\rangle \quad \forall v, w \in (\mathbb{Z}/\ell^m\mathbb{Z})^{2g} \right\}$$

operates on $\mathscr{A}_{g,\ell^m}/\mathscr{A}_g$. On the ℓ-adic tower, of course the group

$$\mathrm{Sp}_{2g}(\mathbb{Z}_\ell) = \varprojlim \mathrm{Sp}_{2g}(\mathbb{Z}/\ell^m\mathbb{Z})$$

operates "as covering transformations over $\mathscr{A}_g$". Here and below $\langle\ ,\ \rangle$ denotes the standard symplectic pairing on the $2g$-dimensional column vectors. In other words,

$$\left\langle \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\rangle = {}^t a_1 \cdot b_2 - {}^t b_1 \cdot a_2 \quad \forall a_1, b_1, a_2, b_2 \in R^g,$$

where R is any commutative ring, and R^g denotes the space of g-dimensional column vectors with entries in R. Moreover, this action of $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ extends to an action of the much bigger group

$$\mathrm{GSp}_{2g}(\mathbb{Q}_\ell) = \left\{ \gamma \in M_{2g}(\mathbb{Q}_\ell) \;\middle|\; \begin{array}{l} \exists\ c(\gamma) \in \mathbb{Q}_\ell^\times \text{ such that} \\ \langle \gamma v, \gamma w\rangle = c(\gamma)\langle v, w\rangle \quad \forall v, w \in \mathbb{Q}_\ell^{2g} \end{array} \right\}$$

on the tower $\varprojlim \mathscr{A}_{g,\ell^m}$. This action of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ on the ℓ-adic tower $\varprojlim \mathscr{A}_{g,\ell^m}$ gives the ℓ-power Hecke correspondences on $\mathscr{A}_g$.

For our purpose it is convenient to use the following subgroup

$$\begin{aligned} \mathscr{G}_\ell = \{ &\gamma \in M_{2g}(\mathbb{Z}[1/\ell]) \mid \exists\ c(\gamma) \in \ell^{\mathbb{Z}} \text{ s.t. } \langle \gamma v, \gamma w\rangle = c(\gamma)\langle v, w\rangle \\ &\forall v, w \in \mathbb{Z}[1/\ell]^{2g} \} \end{aligned}$$

of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ when considering ℓ-power Hecke operators. The assumption on Z means that the inverse image of Z in the ℓ-power tower $\varprojlim \mathscr{A}_{g,\ell^m}$ is stable under $\mathscr{G}_\ell$. Consequently the inverse image of Z^* in $\varprojlim \mathscr{A}^*_{g,\ell^m}$ is also stable under the action of $\mathscr{G}_\ell$.

Recall that our goal is to deduce $Z = \mathscr{A}_g$ from the hypothesis that Z is a Zariski closed subscheme of $\mathscr{A}_g$ over k which is stable under the ℓ-power Hecke correspondences and such that the Zariski closure Z^* in $\mathscr{A}^*_g$ contains the 0-dimensional cusp. This implies that Z contains a nonempty open subscheme whose geometric points are all ordinary. Since the problem is trivial for $g \leqq 1$, we may assume that $g \geqq 2$.

If $m' > m \geqq 0$, the natural morphism

$$\mathscr{A}^*_{g,\ell^{m'}} \longrightarrow \mathscr{A}^*_{g,\ell^m}$$

sends the standard 0-dimensional cusp of $\mathscr{A}^*_{g,\ell^{m'}}$ to the standard 0-dimensional cusp of $\mathscr{A}^*_{g,\ell^m}$. If we denote by $\mathrm{Spf}(R_{\ell^m})$ the formal completion of $\mathscr{A}^*_{g,\ell^m}$ at the standard 0-dimensional cusp, then we have a projective system

$$\varprojlim_m \mathrm{Spf}(R_{\ell^m}) \; .$$

The inverse image of Z^* in $\mathrm{Spf}(R_{\ell^m})$ is a closed formal subscheme Spf (R_{ℓ^m}/I_{ℓ^m}), and the projective limit

$$\varprojlim_m \mathrm{Spf}(R_{\ell^m}/I_{\ell^m}) \subseteq \varprojlim_m \mathrm{Spf}(R_{\ell^m})$$

is stable under the stabilizer subgroup in $\mathscr{G}_\ell$ for the projective limit of the standard 0-dimensional cusp.

Some explanation is in order. First we clarify what we meant by "the standard 0-dimensional cusp". In [12], the minimal compactification $\mathscr{A}^*_{g,\ell^m}$ of $\mathscr{A}_{g,\ell^m}$ is constructed by blowing down any smooth arithmetic toroidal compactification $\overline{\mathscr{A}}_{g,\ell^m}$ using sections of powers of the Hodge line bundle $\underline{\omega}$. We follow the notation in [12] Chapter 4. Let $X = \mathbb{Z}^g$, $X_{\ell^m} = Y_{\ell^m} = (1/\ell^m)X$. Let $C = C(X)$ be the convex cone of all positive semi-definite symmetric bilinear forms on $X_{\mathbb{R}}$ whose radicals are defined over $\mathbb{Q}$. We choose a smooth $GL(X)$-admissible polyhedral cone decomposition $\{\sigma_\alpha\}_{\alpha\in J}$ of $C(X)$. The part of the toroidal compactification given by the cone decomposition $\{\sigma_\alpha\}_{\alpha\in J}$ of $C(X)$ over the standard standard 0-dimensional cusp can be described as follows. Give $C(X)$ the integral structure attached to the lattice of integral valued symmetric bilinear forms on $(\ell^m X) \times (\ell^m X)$. Then over

$$\mathrm{Spec}\, k\,[X_{\ell^m} \otimes X_{\ell^m}/\mu \otimes v - v \otimes \mu]_{\mu,v\in X_{\ell^m}}$$

we have a universal bimultiplicative function

$$b_{\ell^m} : Y_{\ell^m} \times X_{\ell^m} \;\; \rightarrow \left(k\,[X_{\ell^M} \otimes X_{\ell^m}/\mu \otimes v - v \otimes \mu]_{\mu,v\in X_{\ell^m}}\right)^\times .$$

Now the construction in [12] Chapter 4, §§3, 6 gives us a semiabelian scheme

$$^{\heartsuit}G \rightarrow \overline{E}^{\wedge}/\mathrm{GL}(X)(\ell^m)$$

over the $\mathrm{GL}(X)(\ell^m)$-quotient of the formal completion of the torus embedding E determined by the cone decomposition $\{\sigma_\alpha\}_{\alpha\in J}$ with integral structure given by $(1/\ell^m)\,\Gamma_2(X^*)$, along the union of strata corresponding to those cones lying in the interior of C. Here $\Gamma_2(X^*)$ denotes the second divided power product of X^*. The generic fiber of the semiabelian scheme ${}^{\heartsuit}G \to \overline{E}^{\wedge}/\mathrm{GL}(X)(\ell^m)$ is abelian, and the abelian part has a symplectic level-ℓ^m-structure. The level-structure comes from the natural identification of Y_{ℓ^m}/Y with $(\mathbb{Z}/\ell^m\mathbb{Z})^g$ and our choice of ℓ^m-th root of unity $\mathbf{e}_\ell(1/\ell^m)$. Roughly, b_{ℓ^m} determines a map from $Y_{\ell^m} = X_{\ell^m}$ to $\tilde{G}_{\ell^m}(K)$, where K is the fraction field of the ring

$$k\,[X_{\ell^M} \otimes X_{\ell^m}/\mu\otimes\nu - \nu\otimes\mu]_{\mu,\nu\in X_{\ell^m}}$$

and $\tilde{G}_{\ell^m}$ is the split torus with character group X_{ℓ^m}. On G_{ℓ^m} = "$\tilde{G}_{\ell^m}/Y$", we have a polarization of degree ℓ^{2mg} which is the pull-back from the principal polarization on "$\tilde{G}_{\ell^m}/Y_{\ell^m}$". This polarization descends to a principal polarization on the quotient of G_{ℓ^m} by the ℓ^m-torsion points of its multiplicative part. The abelian part of this quotient has a natural level-ℓ^m-structure. This semiabelian scheme, together with its principal polarization and level-ℓ^m-structure on the abelian part, descends first to $\overline{E}^{\wedge}$ and then to $\overline{E}^{\wedge}/\mathrm{GL}(X)(\ell^m)$. This gives the semiabelian scheme ${}^{\heartsuit}G \to \overline{E}^{\wedge}/\mathrm{GL}(X)(\ell^m)$ above, together with its principal polarization and level-ℓ^m-structure on the abelian part.

The rings R_{ℓ^m}'s are calculated in [12], Chapter 5, §2: There is a natural isomorphism

$$R_{\ell^m} \cong k[[q^\lambda]]^{\Gamma_{\ell^m}}_{\lambda\geqq 0}\,,$$

the ring of Γ_{ℓ^m}-invariants in $k[[q^\lambda]]_{\lambda\geqq 0}$, where λ runs through all positive semidefinite elements in $(1/\ell^m)\cdot \mathrm{S}^2(X)$ (i.e. elements in $(1/\ell^m)\cdot\mathrm{S}^2(X)$ having non-negative values on $C(X)$) and Γ_{ℓ^m} is the principal congruence subgroup $\mathrm{GL}(X)(\ell^m)$ of $\mathrm{GL}(X)$ of level ℓ^m. Taking the inverse limit, we get the ring

$$R = R(\ell) \overset{\text{def}}{=} \bigcup_n R_{\ell^m}.$$

Our previous discussion tells us that inside the ring R we have an ideal

$$I = I(\ell) \overset{\text{def}}{=} \bigcup_n I_{\ell^m}.$$

which is stable under the the stabilizer subgroup in $\mathscr{G}_\ell$ for the projective limit of the standard 0-dimensional cusp.

This stabilizer subgroup $\mathscr{P}_\ell$ in $\mathscr{G}_\ell$ for the standard 0-dimensional cusp of $\varprojlim \mathscr{A}_{g,\ell^m}$ can be explicitly described in block form; it is

$$\mathscr{P}_\ell \overset{\text{def}}{=} \left\{\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\;\middle|\; A,B,D\in \mathrm{M}_g(\mathbb{Z}[1/\ell]),\ A\cdot{}^tD\in \ell^{\mathbb{Z}}\cdot Id,\ A\cdot{}^tB = B\cdot{}^tA\right\}.$$

We shall prove that if $I \neq (0)$, then $\mathrm{Spf}(R/I)$ is contained in the boundary. Since the minimal compactification $\mathscr{A}^*_{g,\ell^m}$ has nasty singularities at the boundary, it is not so easy to prove this directly. Instead we prove that the inverse image of $\mathrm{Spf}(R/I)$ in a toroidal compactification is contained in the boundary. In other words we use the toroidal desingularization of the minimal compactification $\mathscr{A}^*_{g,\ell^m}$ to handle the singularity.

The group $\mathscr{P}_\ell$ operates on the rings R_{ℓ^m} by pulling back formal functions. We shall only need its intersection with $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$, namely the subgroup

$$\mathscr{P}'_\ell \stackrel{\mathrm{def}}{=} \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \;\middle|\; A,B,D \in \mathrm{M}_g(\mathbb{Z}[1/\ell]),\ A \cdot {}^tD = Id,\ A \cdot {}^tB = B \cdot {}^tA \right\}.$$

When the formal functions are represented as formal power series as above, the action of the elements of $\mathscr{P}'_\ell$ is given by classical formulas. If $f = \sum_\lambda a_\lambda \cdot q^\lambda$ is an element of R, then the pull-back of f by an element of $\mathscr{P}'_\ell$ is given by

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : f = \sum_\lambda a_\lambda \cdot q^\lambda \longmapsto \sum_\lambda \mathbf{e}\left((\lambda, B \cdot {}^t A)\right) a_\lambda q^{\lambda * A}.$$

The expressions $(\lambda, B \cdot {}^t A)$ and $\lambda * A$ have to be explained. We identify the second divided power product $\Gamma_2(X^*)$ of X^* with the group of all $g \times g$ symmetric matrices with integer entries; similarly after change of rings. In the formula above $(\lambda, B \cdot {}^t A)$ is the value of the pairing between $\lambda \in \mathrm{S}^2(X \otimes \mathbb{Z}[1/\ell])$ and the symmetric matrix

$$B \cdot {}^t A \in \mathrm{M}_{g\times g}(\mathbb{Z}[1/\ell]) = \Gamma_2(X^* \otimes \mathbb{Z}[1/\ell]).$$

In the traditional treatment of Siegel modular forms $\mathrm{S}^2(X)$ is identified with the group of all $g \times g$ half-integral matrices with integral diagonal entries, $\Gamma_2(X^*)$ is identified with the group of all $g \times g$ symmetric integral matrices, and the pairing between them becomes the composition of the matrix product with the trace function. As to the expression $\lambda * A$, recall our decree that X^* corresponds to the standard representation of $\mathrm{GL}_g(\mathbb{Z})$. The expression $\lambda * A$ in the formula above is the image of λ under the transpose of the action of A on $\Gamma_2(X^*)$. In classical notation, if λ is given by a symmetric even half-integral matrix L as we explained above, then $\lambda * A$ corresponds to the symmetric matrix ${}^tALA$.

As $\mathscr{P}'_\ell$ is a semi-direct product of its unipotent radical $\mathscr{U}_\ell$ (those with $A = D = \mathrm{Id}$) and its Levi factor $\mathscr{L}_\ell$ (those with $B = 0$), we also give the formula for the action of these two subgroups.

$$\mathscr{U}_\ell = \left\{ \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix} \;\middle|\; B \in \mathrm{M}_g(\mathbb{Z}[1/\ell]),\ B = {}^tB \right\};$$

$$\mathscr{U}_\ell \ni \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix} : f = \sum_\lambda a_\lambda \cdot q^\lambda \longmapsto \sum_\lambda \mathbf{e}((\lambda, B))\, a_\lambda q^\lambda, \quad f \in R.$$

$$\mathscr{L}_\ell = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \;\middle|\; A,D \in M_g(\mathbb{Z}[1/\ell]),\ A \cdot {}^tD = \mathrm{Id} \right\};$$

$$\mathscr{L}_\ell \ni \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : f = \sum_\lambda a_\lambda \cdot q^\lambda \longmapsto \sum_\lambda a_\lambda q^{\lambda * A}, \quad f \in R.$$

We are in a position to state the main result of this section.

Proposition 2. *If* $(0) \neq I_{\ell^m}$ *is an ideal in* R_{ℓ^m} *such that* $I_{\ell^m} R$ *is stable under the action of* $\mathscr{P}'_\ell$. *Then* I_{ℓ^m} *contains a power of the maximal ideal of* R_{ℓ^m}. *Consequently, if* Z *is a closed subscheme stable under the* ℓ*-power Hecke correspondences, and if the Zariski closure* Z^* *of* Z *in* $\mathscr{A}_g^*$ *contains the* 0*-dimensional cusp of* $\mathscr{A}_g^*$, *then* $Z = \mathscr{A}_g$.

Before giving the proof, we would like to explain the idea and outline the proof. We have explicit formulas for the formal completion of $\mathscr{A}^*_{g,\ell^m}$ at the standard 0-dimensional cusp (i.e. $\mathrm{Spf}(R_{\ell^m})$), the stabilizer subgroup of the standard 0-dimensional cusp in the projective system $\varprojlim \mathrm{Spf}(R_{\ell^m})$, and the action of the stabilizer subgroup on $\varprojlim \mathrm{Spf}(R_{\ell^m})$. Thus it seems reasonable that one should be able to prove Proposition 2 by direct computation. This is only partly true, for the rings R_{ℓ^m} are difficult. But the toroidal resolution comes to our rescue.

Let $\overline{\mathscr{A}}_{g,\ell^m}$ be the toroidal compactification over k given by a chosen smooth cone decomposition $\{\sigma_\alpha\}_{\alpha \in J}$ of $C(X)$. The inverse image in $\overline{\mathscr{A}}_{g,\ell^m}$ of the standard 0-dimensional cusp has a stratification $\sqcup_\alpha W_\alpha$ with the strata parameterized by

$$\{\alpha | \sigma_\alpha \subset C(X)^\circ\} / \mathrm{GL}(X)(\ell^m) \ .$$

We assume that $\ell^m \geqq 3$ for convenience. The formal completion of $\overline{\mathscr{A}}_{g,\ell^m}$ along the inverse image of the standard 0-dimensional cusp is covered by affine open formal subschemes S_α, again parameterized by $\{\alpha | \sigma_\alpha \subset C(X)^\circ\} / \mathrm{GL}(X)(\ell^m)$. Let $\sigma_\alpha \subset C(X)^\circ$ be such a cone. The coordinate ring of S_α is naturally isomorphic to the formal completion of

$$\bigoplus_{\substack{\lambda \in \frac{1}{\ell^m} \mathrm{S}^2(X) \\ \lambda \geqq 0 \text{ on } \sigma}} k \cdot q^\lambda$$

along the ideal

$$\bigoplus_{\substack{\lambda \in \frac{1}{\ell^m} \mathrm{S}^2(X) \\ \lambda > 0 \text{ on } \overline{\sigma} \cap C(X)^\circ}} k \cdot q^\lambda \ .$$

Call this completed ring R_{σ,ℓ^m} and let J_{σ,ℓ^m} be the ideal in R_{σ,ℓ^m} generated by the ideal above. Since the cone compactification $\sigma_\alpha \subset C(X)^\circ$ is smooth, J_{σ,ℓ^m} is a principal ideal.

If

$$f = \textstyle\sum_\lambda a_\lambda q^\lambda \in R_{\sigma,\ell^m}$$

is an element of R_{σ,ℓ^m} such that there exists a λ_1 with the following properties

(a) $a_{\lambda_1} \neq 0$.

(b) $\lambda \in \lambda_1 \cdot J_{\sigma,\ell^m}$ for any λ with $a_\lambda \neq 0$. Equivalently, $\lambda - \lambda_1 > 0$ on every face τ of σ such that $\overline{\tau} \subset C(X)^\circ \cup \{0\}$.

Then we say that f has $a_{\lambda_1} q^{\lambda_1}$ as the *leading term*, with respect to the ideal J_{σ,ℓ^m}. The condition (b) means that f is congruent to $a_{\lambda_1} q^{\lambda_1}$ modulo $1 + J_{\sigma,\ell^m}$ in the multiplicative sense. Clearly if f has leading term $a_{\lambda_1} q^{\lambda_1}$ with $a_{\lambda_1} \neq 0$, then $f \cdot R_{\sigma,\ell^m} = q^{\lambda_1} \cdot R_{\sigma,\ell^m}$ as ideals of R_{σ,ℓ^m}.

From the above discussion, we see that if an ideal Q of R_{ℓ^m} contains an element f_σ with a leading term $a_{\lambda_\sigma} q^{\lambda_\sigma}$ with respect to J_{σ,ℓ^m} for every cone σ in a system of representatives of

$$\{\sigma_\alpha\}_{\alpha \in J}/\mathrm{GL}(X)(\ell^m) ,$$

then $\mathrm{Spf}(R_{\ell^m}/Q)$ is contained in the formal completion of the boundary of $\mathscr{A}^*_{g,\ell^m}$ at the standard 0-dimensional cusp. This follows from Grothendieck's formal function theorem, applied to the morphism from $\overline{\mathscr{A}}_{g,\ell^m} \to \mathscr{A}^*_{g,\ell^m}$ over the standard 0-dimensional cusp.

Similarly, if Q is a finitely generated ideal of R such that for every cone σ in the cone decomposition $\{\sigma_\alpha\}_{\alpha \in J}$, Q contains an element f_σ which has a leading term $a_{\lambda_\sigma} q^{\lambda_\sigma}$ with respect to the ideal

$$J_\sigma = \bigcup_n J_{\sigma,\ell^m} .$$

Then $\mathrm{Spf}(R/Q)$ is contained in the the boundary of $\varprojlim \mathscr{A}^{*\wedge}_{g,\ell^m}$, the projective limit of the completion of the boundary of $\mathscr{A}^*_{g,\ell^m}$ at the standard 0-dimensional cusp.

We have to produce sufficiently many elements in the ideal $I \subset R$ which have leading terms. In order to get elements with leading terms, one only has to "shave off" undesirable terms in the Fourier expansion of any given nonzero element in I. For this we use the action of $\mathscr{U}_\ell$. This allows us to "shave off" finitely many terms in the Fourier expansion of a given non-zero element of I. Then we show that for cones σ in the cone decomposition such that $\overline{\sigma} \subset C(X)^\circ \cup \{0\}$, just throwing away a finite number of terms is enough to produce elements with leading terms. For cones "touching the boundary of $C(X)$", things get a little more complicated, but the same idea works. We formulate these parts as separate lemmas:

Lemma 1. *For any nonzero element*

$$0 \neq f = \sum_\lambda a_\lambda \cdot q^\lambda \in I$$

in I, and for any finite subset $\{\lambda_0, \lambda_1, \ldots, \lambda_N\} \subseteq$ support(f), there exists an element $g \in I$ such that

$$\lambda_0 \in \mathit{support}(g) \subseteq \mathit{support}(f) - \{\lambda_1, \ldots, \lambda_N\} .$$

Here support(f) denotes the set of all λ's such that $a_\lambda \neq 0$, similarly for support(g). In other words, $g = \sum_\lambda b_\lambda \cdot q^\lambda$ is such that $b_{\lambda_0} \neq 0$, $b_{\lambda_1} = \ldots = b_{\lambda_N} = 0$, and $b_\lambda \neq 0 \Rightarrow a_\lambda \neq 0$.

Lemma 1 is an immediate consequence of the formula for the $\mathscr{U}_\ell$-action and the linear independence of characters of an abelian group. In fact we can choose g to be a suitable linear combination of elements in the $\mathscr{U}_\ell$-orbit of f.

Lemma 2. *Let $b \in C(X)^\circ$ be a positive definite symmetric bilinear form on $X_{\mathbb{R}}$. Let M be a given positive real number. Then there are only finitely* *many integral elements λ in the dual cone of positive semidefinite elements in $S^2(X_{\mathbb{R}})$ such that $(\lambda, b) \leqq M$.*

Lemma 2 is certainly well-known, and it holds in the more general situation of the polyhedral reduction theory for self-adjoint cones as in [1], Chap. 2. We supply a direct elementary proof for the convenience of the reader. We shall identify $S^2(X)$ with the group of all half-integral symmetric $g \times g$ matrices with integral diagonal entries, $\Gamma_2(X^*)$ with symmetric integral $g \times g$ matrices, and the pairing between them is the composition of the matrix product and the trace function. The element $b \in C(X)^\circ$ corresponds to a positive definite symmetric $g \times g$ matrix B. Clearly there exists a positive constant $c > 0$ such that $c \cdot B \gg Id$, in other words $c \cdot B$ dominates Id as positive definite symmetric $g \times g$ matrices. Therefore if L is an half-integral positive semi-definite symmetric matrix with integral diagonal entries such that $tr(L \cdot B) \leqq M$, then the absolute values of the diagonal entries of L are all bounded by cM. It follows that the absolute values of all off-diagonal entries of L are also bounded by cM. Therefore there are only finitely many possible L's. This proves Lemma 2.

Proof of Proposition 2. Of course the key is to produce enough elements in the ideal $I_{\ell^m}R$ with leading terms. Since $I_{\ell^m} \neq (0)$ there exists a nonzero function $f = \sum_\lambda a_\lambda q^\lambda$ in the ideal I_{ℓ^m}. Let σ be a cone in the cone decomposition $\{\sigma_\alpha\}$ such that $\overline{\sigma} \subset C(X)^\circ \cup \{0\}$. Since $f \neq 0$ and $I_{\ell^m} \neq (0)$, there exists a λ_0 such that $a_{\lambda_0} \neq 0$ and $\lambda_0 \neq 0$. By Lemma 2, there exists a finite set $\Lambda \subset \text{support}(f)$ such that $\lambda - \lambda_0 \ > 0$ on $\overline{\sigma} - \{0\}$, for every $\lambda \in \text{support}(f) - \Lambda$. By Lemma 1, there exists an element $f_1 \in I$ with leading term q^{λ_0}.

We only have to generalize the above argument a little in order to deal with the the general case of a cone $\sigma \subset C(X)^\circ$ in the cone decomposition. Again let $f = \sum_\lambda a_\lambda q^\lambda \in R_{\ell^m}$ be a nonzero function in the ideal I_{ℓ^m}. Using Lemma 2 and Lemma 1, for any positive integer $N > 0$, we can find a λ_0 such that $a_{\lambda_0} \neq 0$, and a suitable linear combination $g_{_N}$ of elements in the $\mathcal{U}_\ell$-orbit of f such that

$$g_N \in q^{\lambda_0} + J^N_{\sigma,\ell^m} \subset R_{\ell^m}.$$

The point is that whether a monomial q^λ with $\lambda \in (1/\ell^m)S^2(X)$ is in J^N_{σ,ℓ^m} can be detected by the $\mathbb{R}$-valued function given by the restriction of λ to the largest closed face $\overline{\tau}$ of $\overline{\sigma}$ such that $\overline{\tau} \subset C(X)^\circ \cup \{0\}$. Also, $g_{_N} \in R_{\ell^m}$ because $\text{support}(g_N) \subseteq \text{support}(f)$. Since $g_N \in I_{\ell^m}R \subset I_{\ell^m}R_\sigma := \bigcup_n R_{\sigma,\ell^n}$, and $\bigcup_n R_{\sigma,\ell^n}$ is faithfully flat over R_{σ,ℓ^m}, $g_{_N} \in I_{\ell^m}R_{\sigma,\ell^m}$. Thus we see that $q^{\lambda_0} \in IR_{\sigma,\ell^m} + J^N_{\sigma,\ell^m}$ for every N. This implies that $q^{\lambda_0} \in IR_{\sigma,\ell^m}$ since the ideal $q^{\lambda_0} \in I_{\ell^m}R_{\sigma,\ell^m}$ is closed in the J_{σ,ℓ^m}-adic topology. So we have found an element in $I_{\ell^m}R_{\sigma,\ell^m}$ with a leading term, for any cone $\sigma \subset C(X)^\circ$. This proves Proposition 2. □

Notice that in the proof above we only needed the $\mathcal{U}_\ell$-action. Thus we have actually proved a strengthened form of Proposition 2:

Proposition 2′. *If $(0) \neq I_{\ell^m}$ is an ideal in R_{ℓ^m} such that $I_{\ell^m}R$ is stable under the action of $\mathcal{U}_\ell$. Then I_{ℓ^m} contains a power of the maximal ideal of R_{ℓ^m}.*

3. Reduction to the Hilbert-Blumenthal case

In this section we reduce the question (Q 1) to an analogous question for the Hilbert-Blumenthal moduli spaces over k. The underlying idea is very simple. We want to show that every Zariski closed subscheme Z of $\mathscr{A}_g$ over $\operatorname{Spec} k$ which is stable under all prime-to-p Hecke correspondences and which contains ordinary points is necessarily equal to $\mathscr{A}_g$. Since Z is of finite presentation over $\operatorname{Spec} k$ and $\mathscr{A}_g$ is defined over $\mathbb{F}_p$, there exists a subring $D \subset k$ of finite type over $\mathbb{F}_p$, a scheme $\mathscr{Z}^*$ proper and flat over D, a closed embedding $\mathscr{Z}^*_{/D} \hookrightarrow \mathscr{A}^*_{g/D}$ over D of $\mathscr{Z}^*$ into the minimal compactification, such that every fiber of $\mathscr{Z}^* \to \operatorname{Spec} D$ contains points corresponding to ordinary abelian varieties, and such that $\mathscr{Z}^* \times_{\operatorname{Spec} D} \operatorname{Spec} k$ is equal to Z^*, the Zariski closure of Z in $\mathscr{A}^*_g$. Since the generic fiber of $\mathscr{Z}^*$ is stable under all prime-to-p Hecke correspondences, so is $\mathscr{Z}^*$. To show that Z is equal to $\mathscr{A}_g$, it suffices to show that the fibers of $\mathscr{Z}^* \to \operatorname{Spec} D$ over finite fields are equal to $\mathscr{A}^*_g$. Therefore we may assume without loss of generality that Z contains a point x_0 defined over some finite field $\mathbb{F}_q$. The point x_0 corresponds to a principally polarized ordinary abelian variety (A_0, λ_0) defined over $\mathbb{F}_q$. It is well-known that every simple abelian variety over a finite field has complex multiplication by a CM field of degree twice that of the dimension of the abelian variety, therefore has real multiplication by the totally real subfield of the CM field. A reference is [33]. In fact, the endomorphism algebra $\operatorname{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ of an absolutely simple ordinary abelian variety A_0 defined over a finite field is a CM field, generated by a Frobenius element. Therefore if A_0 is absolutely simple, then $\operatorname{End}_k(A_0)$ is an order in a CM field L. Let $F \subset L$ be the totally real subfield of L. If in addition the ring of integers $\mathcal{O}_F$ in F is contained in $\operatorname{End}_k(A_0)$, x_0 is in the image of a morphism from a Hilbert-Blumenthal moduli scheme $\mathscr{M}_F$ to $\mathscr{A}_g$ over k. Moreover, this morphism is compatible with Hecke operators. Hence if we can prove that the reduced prime-to-p F-Hecke orbit of $[A_0]$ in $\mathscr{M}_F$ is dense, Z contains a family degenerating to a torus. In other words, Z^* contains the 0-dimensional cusp in the minimal compactification $\mathscr{A}^*_g$. Proposition 2 then implies that $Z = \mathscr{A}_g$.

To get around the extra assumptions we placed on A_0, we shall show that one can pass from A_0 to an abelian variety A_1 isogenous to it, and change the polarization by the isogeny, without changing the question (Q 1). Also we will have to deal with the case that the polarization is not necessarily a product. For this we use an easy algebraic lemma.

The following lemma is well-known. We include a proof for the convenience of the reader.

Lemma 3. *Let A be an ordinary absolutely simple abelian variety defined over a finite field $\mathbb{F}_q$, and let $K = \operatorname{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then K is a CM field, $[K : \mathbb{Q}] = 2\dim(A)$, and $K = \operatorname{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let F be the fixed field of the complex conjugation in K, and let $\pi_q = \operatorname{Frob}_{A/\mathbb{F}_q}$. Then $K = \mathbb{Q}(\pi_q)$, $F = \mathbb{Q}(\pi_q + q\pi_1^{-1})$, and every prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ lying over p splits in K.*

Proof. For every valuation v of $\mathbb{Q}(\pi_q)$ lying over p, $v(\pi_q)$ is equal to either $v(q)$ or 0 since A is ordinary. This implies by the the main theorem of [33] that the Brauer invariant of the central divisional algebra $K/\mathbb{Q}(\pi_q)$ at every place over p vanishes. By [33] Example a), the field $\mathbb{Q}(\pi_q)$ is totally imaginary, otherwise A_0 is a supersingular elliptic curve. Since the central division algebra $K/\mathbb{Q}(\pi_q)$ splits at all finite places prime to p, we see that $K = \mathbb{Q}(\pi_q)$. Again by [33], $[K:\mathbb{Q}] = 2\dim(A)$. Example b) of [33] shows that K is a totally imaginary quadratic extension of the totally real field $F = \mathbb{Q}(\pi_q + q\pi_1^{-1})$. Since π_q and $q\pi_q^{-1}$ are complex conjugate, exactly one of them is a unit for v. Hence every place $\mathfrak{p}$ of F lying over p splits in K. □

Lemma 4. *Let K be a CM field and let $*$ be a a positive definite involution of the second kind on $M_{n\times n}(K)$. Then there exists a maximal subfield L in $M_{n\times n}(K)$ which is stable under $*$. Moreover L is a CM field and $[L:\mathbb{Q}] = [K:\mathbb{Q}]\cdot n$.*

Proof. Consider the subspace of $\mathfrak{g}$ of $\mathrm{M}_{n\times n}(K)$ consisting of elements $x \in \mathrm{M}_{n\times n}(K)$ such that $x^* = -x$. In other words, $\mathfrak{g}$ is the Lie algebra of the unitary group associated to $(\mathrm{M}_{n\times n}(K), *)$. We know that $\mathfrak{g}$ is a Lie algebra over the totally real subfield F of K, and $\mathfrak{g} \otimes_F \overline{F} = \mathfrak{gl}_n(\overline{F})$. The standard representation of $\mathrm{M}_{n\times n}(K)$ can be regarded as a $2n$-dimensional representation of $\mathfrak{g}$ over F. The characteristic polynomial for this representation defines a polynomial $f(X,T)$ of degree $2n$ in the variable T, with coefficients in the ring $\mathrm{S}^\bullet(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}$. It is easy to see that that $f(T)$ is irreducible of degree $2n$, but becomes a product of two geometrically irreducible polynomials of degree n in $\mathrm{S}^\bullet(\mathfrak{g}^*)[T] \otimes_F K$,. By the Hilbert irreducibility theorem, there exists an element $x \in \mathfrak{g}$ such that $f(x,T)$ is irreducible over F. This means that the F-subalgebra $F[x]$ of $\mathrm{M}_{n\times n}(K)$ is a field of degree $2n$ over F, therefore is equal to $K[x]$. Take $L = F[x] = K[x]$. Clearly the subfield L is stable under the involution $*$ and hence is a CM field because $*$ is a positive definite involution. This proves lemma 4. □

Remark. Lemma 4 can certainly be generalized. Group theoretically,it essentially asserts the existence of an anisotropic maximal torus in the special unitary group associated to the simple algebra with involution $(\mathrm{M}_{n\times n}(K), *)$.

Let (A_0, λ_0) be an ordinary principally polarized abelian variety of dimension g over $\overline{\mathbb{F}_p}$. Then A_0 is isogenous to a product $\prod_{i=1}^m B_i^{n_i}$, where each B_i is a simple ordinary abelian variety over $\overline{\mathbb{F}_p}$, and B_i is not isogenous to B_j if $i \neq j$. The last condition implies that

$$\mathrm{Hom}_{\overline{\mathbb{F}_p}}(B_i, B_j) = (0) = \mathrm{Hom}_{\overline{\mathbb{F}_p}}(B_i, B_j^t) \quad \text{if } i \neq j,$$

where B_j^t denotes the dual abelian variety of B_j. Let $K_i = \mathrm{End}_{\overline{\mathbb{F}_p}}(B_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Lemma 3 tells us that K_i is a CM field. Clearly

$$\mathrm{End}_{\overline{\mathbb{F}_p}}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{i=1}^m \mathrm{End}_{\overline{\mathbb{F}_p}}(B_i^{n_i}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{i=1}^m \mathrm{M}_{n_i\times n_i}(K_i).$$

The pull-back of λ_0 to $\prod_{i=1}^m B_i^{n_i}$ is necessarily a product $\prod_{i=1}^m \lambda_i$, where λ_i is a polarization of $B_i^{n_i}$. Therefore the Rosati involution for (A_0, λ_0) on

$\mathrm{End}_{\overline{\mathbb{F}}_p}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ can be identified with the product $\prod_{i=1}^m *_i$, where $*_i$ is the Rosati involution of $(B_i^{n_i}, \lambda_i)$. By Lemma 2, for each $i = 1, \dots, m$, there exists a CM field $L_i \subseteq \mathrm{End}_{\overline{\mathbb{F}}_p}(B_i^{n_i}) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is stable under the Rosati involution λ_i. It is well known that for each $i = 1, \dots, m$, there exists an abelian variety A_i over $\overline{\mathbb{F}}_p$ which is isogenous to $B_i^{n_i}$ and such that $\mathrm{End}_{\overline{\mathbb{F}}_p}(A_i)$ contains the ring $\mathcal{O}_{L_i}$ of all algebraic integers in L_i. We conclude that there exist abelian varieties A_i over $\overline{\mathbb{F}}_p$, $i = 1, \dots, m$, an isogeny $h : \prod_{i=1}^m A_i \to A_0$ and CM fields $L_i \subseteq \mathrm{End}_{\overline{\mathbb{F}}_p}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$, $i = 1, \dots, m$ such that

(i) $\mathrm{Hom}_{\overline{\mathbb{F}}_p}(A_i, A_j) = (0) = \mathrm{Hom}_{\overline{\mathbb{F}}_p}(A_i, A_j^t)$ if $i \neq j$;
(ii) the pull-back of the polarization λ_0 on A_0 to $\prod_{i=1}^m A_i$ is a product $\prod_{i=1}^m \lambda_i$, where λ_i is a polarization of A_i, $i = 1, \dots, m$;
(iii) $[L_i : \mathbb{Q}] = 2\dim(A_i)$;
(iv) the Rosati involution induced by λ_i sends L_i to itself and induces the complex conjugation on L_i;
(v) $\mathcal{O}_{L_i} \subseteq \mathrm{End}_{\overline{\mathbb{F}}_p}(A_i)$.

Actually property (i) will not be used in what follows.

There exist positive integers $d_1, \dots, d_g$, with $d_1 | \cdots | d_g$, such that

$$Ker\left(\prod_{i=1}^m \lambda_i\right) \cong \left(\prod_{j=1}^g (\mathbb{Z}/d_j\mathbb{Z})\right) \times \left(\prod_{j=1}^g \mu_{d_j}\right)$$

over $\overline{\mathbb{F}}_p$. Let $\delta = (d_1, \dots, d_g)$ and let $K(\delta) = \left(\prod_{j=1}^g (\mathbb{Z}/d_j\mathbb{Z})\right) \times \left(\prod_{j=1}^g \mu_{d_j}\right)$. Let $\mathscr{A}_{g,\delta}$ denote the moduli stack of polarized abelian schemes (A, λ) over a scheme S over $\mathrm{Spec}(\mathbb{F}_p)$ such that $Ker(\lambda)$ is locally isomorphic to $K(\delta)$ in the flat topology. Let $\mathscr{A}_{g,\delta}^{or}$ be the open algebraic substack of $\mathscr{A}_{g,\delta}$ whose objects consist of pairs $(A, \lambda) \to S$ in $\mathscr{A}_{g,\delta}(S)$ such that the all the fibers A_s are ordinary abelian varieties. Clearly $\mathscr{A}_{g,\delta}^{or}$ is equal to $\mathscr{A}_{g,\delta}$ if $d_1 > 1$. By [28] Thm 3.1, p 431, $\mathscr{A}_{g,\delta}^{or}$ is always dense in $\mathscr{A}_{g,\delta}$, see also [19] Thm 7, p 113 for the case when p does not divide d_g. As a consequence of the result on p-adic monodromy in [9] or [12] V.7.1, $\mathscr{A}_{g,\delta}$ is irreducible, see [16]. Let $\mathscr{A}_g^{or}$ be the open algebraic substack of ordinary principally polarized abelian varieties in characteristic p. Again by [28], $\mathscr{A}_g^{or}$ is dense in $\mathscr{A}_g$.

From the construction of $\left(\prod_{i=1}^m A_i, \prod_{i=1}^m \lambda_i\right)$ we know that $Ker(h) \subseteq Ker\left(\prod_{i=1}^m \lambda_i\right)$ is a maximal totally isotropic subgroup of of $Ker\left(\prod_{i=1}^m \lambda_i\right)$ with respect to the Weil pairing $\langle\ ,\ \rangle_{\prod_i \lambda_i}$. Let $\mathscr{A}_{g,1,\delta}^{or}(Ker(h) \subseteq Ker(\prod_i \lambda_i))$ be the moduli stack of isogenies $\phi : (B, \lambda_B) \to (A, \lambda_A)$ over a scheme S of characteristic p such that (A, λ_A) is an ordinary principally polarized abelian variety, λ_B is the pull-back $\phi^*(\lambda_A)$ of λ_A, and $\left(Ker(\lambda_B), Ker(\phi), \langle\ ,\ \rangle_{\lambda_B}\right)$ is locally isomorphic to $\left(Ker(\prod_i \lambda_i), Ker(h), \langle\ ,\ \rangle_{\prod_i \lambda_i}\right)$ in the flat topology. There are natural morphisms

$$pr_1 : \mathscr{A}_{g,1,\delta}^{or}(Ker(h) \subseteq Ker(\textstyle\prod_i \lambda_i)) \longrightarrow \mathscr{A}_{g,\delta}^{or}\ ,$$

$$pr_2 : \mathscr{A}_{g,1,\delta}^{or}(Ker(h) \subseteq Ker(\textstyle\prod_i \lambda_i)) \longrightarrow \mathscr{A}_{g}^{or}.$$

Both morphisms pr_1, pr_2 are finite flat and surjective. Moreover there are prime-to-p Hecke correspondences coming from $GSp_{2g}(\mathbb{A}_f^{(p)})$ which act on $\mathscr{A}^{or}_{g,1,\delta}(Ker(h) \subseteq Ker(\prod_i \lambda_i))$ and $\mathscr{A}^{or}_{g,\delta}$ as algebraic correspondences, such that the morphisms pr_1, pr_2 are compatible with the prime-to-p Hecke correspondences. Consequently, if $\phi : (B, \lambda_B) \to (A, \lambda_A)$ is a geometric point of $\mathscr{A}^{or}_{g,1,\delta}(Ker(h) \subseteq Ker(\prod_i \lambda_i))$, then the prime-to-$p$ Hecke orbit (resp. ℓ-power Hecke orbit) of the point $[(B, \lambda)]$ is Zariski dense in $\mathscr{A}^{or}_{g,\delta}$ if and only if the prime-to-p Hecke orbit (resp. ℓ-power Hecke orbit) of the point $[(A, \lambda_A)]$ is Zariski dense in $\mathscr{A}^{or}_g$. Therefore the prime-to-p Hecke orbit (resp. ℓ-power Hecke orbit) of the point $[(B, \lambda)]$ is Zariski dense in $\mathscr{A}_{g,\delta}$ if and only if the prime-to-p Hecke orbit (resp. ℓ-power Hecke orbit) of the point $[(A, \lambda_A)]$ is Zariski dense in $\mathscr{A}_g$. The same consideration also leads to

Proposition 2″. Let k be an algebraically closed field of characteristic $p > 0$. Let ℓ be a prime number different from p. Let $d_1, \ldots, d_g$ be positive integers such that $d_1 | \cdots | d_g$, and let $\delta = (d_1, \ldots, d_g)$. Denote by $\mathscr{A}^{or}_{g,\delta}$ the moduli stack of polarized ordinary abelian varieties over k as above. Suppose that $Z \subseteq \mathscr{A}^{or}_{g,\delta}$ is a closed subscheme of $\mathscr{A}^{or}_{g,\delta}$ which is stable under all ℓ-power Hecke correspondences. Assume furthermore that there exists a nonconstant morphism $C \to Z$ from a smooth open curve C over k to Z, such that the pull-back of the universal abelian scheme over C degenerates to a torus at some boundary point of C. Then $Z = \mathscr{A}_{g,\delta}$.

Proof. The hypothesis implies that the Zariski closure W^* of $W = pr_2(pr_1^{-1}(Z))$ in $\mathscr{A}^*_g$ contains the standard 0-dimensional cusp and is stable under all prime-to-p Hecke correspondences. Therefore $W = \mathscr{A}^{or}_g$ by Proposition 2. This implies that $Z = \mathscr{A}_{g,\delta}$. □

The above discussion clearly applies to the isogeny $h : (\prod_{i=1}^m A_i, \prod_{i+1}^m \lambda_i) \to (A_0, \lambda_0)$. From Proposition 2″, we see that the Zariski closure Z of the prime-to-p Hecke orbit of the point $[(A_0, \lambda_0)]$ in $\mathscr{A}_g$ is equal to $\mathscr{A}_g$ if and only if the Zariski closure Z' of the prime-to-p orbit of the point $[(\prod_{i=1}^m A_i, \prod_{i+1}^m \lambda_i)]$ in $\mathscr{A}_{g,\delta}$ contains a 1-parameter family of abelian varieties degenerating to a torus. This will clearly be the case if for each i, the Zariski closure of the prime-to-p Hecke orbit of the point $[(A_i, \lambda_i)]$ in $\mathscr{A}_{g_i,\delta_i}$ contains a one-parameter family of abelian varieties degenerating to a torus. Here $g_i = \dim(A_i)$, and $K(\delta_i) \cong Ker(\lambda_i)$ geometrically. So far we have reduced our original question (Q 1) to a similar question on the Zariski closure of the prime-to-p Hecke orbit of an ordinary polarized abelian variety (A_1, λ_1) in $\mathscr{A}_{g,\delta_1}(\overline{\mathbb{F}}_p)$, where (A_1, λ_1) has the property that there exists a CM field L with $[L : \mathbb{Q}] = 2\dim(A_1)$ and an embedding $\mathcal{O}_L \subseteq \mathrm{End}_{\overline{\mathbb{F}}_p}(A_1)$ such that the Rosati involution on $\mathrm{End}_{\overline{\mathbb{F}}_p}(A_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by λ_1 sends L to L and induces the complex conjugation on L.

Suppose (A_1, λ_1) is an ordinary polarized abelian variety with the above properties. Let F be the totally real subfield in L, namely the subfield of L consisting of all totally real elements in L. Let $\mathcal{O}_F$ be the ring of algebraic integers in F, $g = [F : \mathbb{Q}]$. Denote by $\mathscr{M}_{\mathcal{O}_F}$, or for short $\mathscr{M}_F$, the moduli stack of abelian schemes of dimension g with multiplication by $\mathcal{O}_F$ as in [7] 2.1. In other words $\mathcal{M}_F$ is the stack of groupoids whose objects are pairs $(A \to S, \iota)$, where $A \to S$ is an abelian scheme of relative dimension g and $\iota : \mathcal{O}_F \to \mathrm{End}(A/S)$ gives A the structure of an $\mathcal{O}_F$-module. Given $(A \to S, \iota)$ as above, let

$$\mathscr{P} = \mathscr{P}(A, \iota) = \underline{Hom}_{\mathcal{O}_F}(A, A^t)^{sym}$$

be the étale sheaf over S of $\mathcal{O}_F$-linear quasi-polarizations of (A, ι). Let $\mathscr{P}_+ \subset \mathscr{P}$ be the étale sheaf over S of $\mathcal{O}_F$-linear polarizations of (A, ι). We know that $\mathscr{P}$ is an étale sheaf over S of projective $\mathcal{O}_F$-modules of rank 1, and $\mathscr{P}_+$ defines a notion of positivity on $\mathscr{P}$. In other words, for every real embedding σ of F we have an orientation of $\mathscr{P} \otimes_{\mathcal{O}_F, \sigma} \mathbb{R}$, and $\mathscr{P}_+$ is the subsheaf of totally positive elements. Denote by $\mathcal{O}_{F,+}^{\times}$ the group of totally positive units in $\mathcal{O}_F$. Then $(\mathscr{P}, \mathscr{P}_+)$ is an $\mathcal{O}_{F,+}^{\times}$-torsor, which has to be constant if S is normal and connected, c.f. [31] 1.18. By [7] Théorème 2.2 $\mathcal{M}_F$ is geometrically normal over $\mathrm{Spec}(\mathbb{Z})$, hence $(\mathscr{P}, \mathscr{P}_+)$ is constant on each connected component of $\mathcal{M}_F$.

We can also specify the sheaf of polarizations and arrive at a variant of the above definition. Assume that Λ is a projective $\mathcal{O}_F$-module of rank 1 with a notion of positivity; that is for every real embedding σ of F we are given an orientation of $\Lambda \otimes_{\mathcal{O}_F, \sigma} \mathbb{R}$. We can define a moduli stack $\mathcal{M}_{\mathcal{O}_F}^{(\Lambda, \Lambda_+)} = \mathcal{M}_F^{\Lambda}$ whose objects consists of triples $(A \to S, \iota, h)$ where $(A \to S, \iota)$ is an abelian schemes of relative dimension g with multiplication by $\mathcal{O}_F$ as before, and

$$h : A \otimes_{\mathcal{O}_F} \Lambda \xrightarrow{\sim} A^t$$

is an isomorphism, which sends $\lambda \in \Lambda$ to $\phi_\lambda \in \mathscr{P}(A, \iota)$ and $\mathscr{P}_+(A, \iota)$ to Λ_+. The algebraic stack $\mathcal{M}_F$ is isomorphic to a disjoint union $\coprod_i \mathcal{M}_F^{\Lambda_i}$, where Λ_i runs through a system of representatives of the strict ideal class group of F. Notice that we have used the definition of the Hibert-Blumenthal moduli spaces given in [7] 2.1, which modifies the definition in [31] 1.1.

There are natural Hecke correspondences on $\mathcal{M}_F$ coming from $\mathrm{GL}_2(\mathbb{A}_{f,F})$, the group of finite adelic points of $GL(2, F)$. To distinguish the Hecke orbits on the Hilbert-Blumenthal moduli spaces from the Hecke orbits on the Siegel moduli spaces, we shall use the term '*F-Hecke orbits*' for Hecke orbits on $\mathcal{M}_F$, $\mathcal{M}_F^{\Lambda}$ or their variants. For instance the prime-to-p F-Hecke orbits are the orbits of prime-to-p Hecke correspondences which come from $\mathrm{GL}_2(\mathbb{A}_{f,F}^{(p)})$; the ℓ-power F-Hecke orbits are the orbits of ℓ-power Hecke correspondences which come from $\mathrm{GL}_2(\mathbb{Q}_\ell)$. Here

$$\mathbb{A}_{f,F}^{(p)} = \mathbb{A}_{\mathbb{Q},F}^{(p)} \otimes_{\mathbb{Q}} F = \left(\textstyle\prod'_{\ell \neq p} \mathbb{Q}_\ell\right) \otimes_{\mathbb{Q}} F$$

is the ring of finite prime-to-p adeles of F. If $[(A, \iota_A)]$ is a point of $\mathcal{M}_F$ over an algebraically closed field of characteristic $p > 0$, the prime-to-p F-Hecke orbit can be described geometrically as follows. It consists of all points of $\mathcal{M}_F(k)$ of the form $[(B, \iota_B)]$ such that (B, ι_B) is isogenous to (A, ι_A) via an $\mathcal{O}_F$-linear isogeny whose kernel is killed by some prime-to-p integer. For $\mathcal{M}_F^{\Lambda}$ the F-Hecke operators coming from $\mathrm{GL}_2(\mathbb{A}_{f,F})$ do not preserve $\mathcal{M}_F^{\Lambda}$ in general. Consider a point $[(A, \iota_A, h_A)]$ in $\mathcal{M}_F^\Lambda(k)$, where k is an algebraically closed field of characteristic $p > 0$, and h_A is thought of as an $\mathcal{O}_F$-linear isomorphism from (Λ, Λ_+) to $(\mathcal{P}(A, \iota_A), \mathcal{P}(A, \iota_A)_+)$. The intersection of $\mathcal{M}_F^\Lambda(k)$ with the $\mathrm{GL}_2(\mathbb{A}_{f,F})$-Hecke orbit of $[(A, \iota_A, h_A)]$ can be described as follows. It consists of all points $[(B, \iota_B, h_B)]$ in $\mathcal{M}_F^\Lambda(k)$ such that there exists an $\mathcal{O}_F$-linear isogeny $\phi : B \to A$ and a totally positive element $c \in \mathcal{O}_F$ such that $Ker(\phi)$ is killed by some prime-to-p integer, and

$$ {}^t\phi \circ h_A(\lambda) \circ \phi = c \cdot h_B(\lambda) \quad \text{for every } \lambda \in \Lambda. $$

In terms of group theory, an element γ of $\mathrm{GL}_2(\mathbb{A}_{f,F}^{(p)})$ changes the isomorphism class of the polarization sheaf via the translation by an element of the strict ideal class group; this element in the strict ideal class group comes from the determinant of γ.

From now on we shall restrict ourselves to schemes over $\mathbb{F}_p$, therefore $\mathcal{M}_F$ and $\mathcal{M}_F^\Lambda$ stand for the Hilbert moduli stacks over $\mathbb{F}_p$. Let $\mathcal{M}_F^{or}$ and $\mathcal{M}_F^{\Lambda,or}$ be the ordinary locus of $\mathcal{M}_F$ and $\mathcal{M}_F^\Lambda$ respectively. It is known that $\mathcal{M}_F^{or}$ (resp. $\mathcal{M}_F^{\Lambda,or}$) is open and dense in $\mathcal{M}_F$ (resp. $\mathcal{M}_F^\Lambda$).

Now we return to the previous situation: A_1 is an ordinary abelian variety over $\overline{\mathbb{F}}_p$, λ_1 is a polarization of A_1, L is a CM field with $[L : \mathbb{Q}] = 2\dim(A)$, $\mathcal{O}_L$ operates on A_1 via an embedding $\tilde{\iota}_1 : \mathcal{O}_F \to \mathrm{End}(A_1)$, and the Rosati involution $*_{\lambda_1}$ sends L into itself and induces the complex conjugation on L. Let $\iota_1 : \mathcal{O}_F \to \mathrm{End}(A_1)$ be the restriction of $\tilde{\iota}_1$ to $\mathcal{O}_F$. Then (A_1, ι_1) is a point of $\mathcal{M}_F(\overline{\mathbb{F}}_p)$. The polarization λ_1 gives a section of $\mathcal{P}(A_1, \iota_1)_+$ over $\mathrm{Spec}(\overline{\mathbb{F}}_p)$. The polarization λ_1 on A_1 has a type δ, characterized by $K(\delta) \cong Ker(\lambda_1)$. If we write $\delta = (d_1, \ldots, d_g)$, $d_1 | \cdots | d_g$, then $\mathcal{P}(A_1, \iota_1)/\mathcal{O}_F \cdot \lambda_1$ is isomorphic to $\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z}$ as abelian groups. Let $(\Lambda, \Lambda_+) = (\mathcal{P}(A_1, \iota_1), \mathcal{P}(A_1, \iota_1)_+)$, so that λ_1 is an element of $\Lambda_+ \subset \Lambda$.

We define a morphism

$$ j : \mathcal{M}_F^{\Lambda,or} \longrightarrow \mathcal{A}_{g,\delta}^{or} $$

which sends a triple $(A \to S, \iota, h)$ in $\mathcal{M}_F^{\Lambda,or}(S)$ to the polarized ordinary abelian variety $(A \to S, h(\lambda_1))$ in $\mathcal{A}_{g,\delta}^{or}(S)$, where $h(\lambda_1)$ is the $\mathcal{O}_F$-linear polarization given by the element λ_1 in Λ. On $\mathcal{A}_{g,\delta}^{or}$ there are prime-to-p Hecke correspondences coming from the group $\mathrm{GSp}(\mathbb{A}_{f,\mathbb{Q}}^{(p)})$. We shall explain the compatibility of the morphism j with Hecke operators. First we need some notation. The first homology group of A_1 with coefficients in $\mathbb{A}_{f,\mathbb{Q}}^{(p)}$,

$$ V_f^{(p)}(A_1) = \mathrm{H}^1(A_1, \mathbb{A}_{f,\mathbb{Q}}^{(p)}) := \textstyle\prod'_{\ell \neq p} \mathrm{H}^1(A_1, \mathbb{Q}_\ell) , $$

is a free $\mathbb{A}_{f,\mathbb{Q}}^{(p)}$-module of rank $2\dim(A_1) = [L : \mathbb{Q}]$. The polarization $h(\lambda_1)$ of A_1 gives a nondegenerate symplectic pairing

$$ \langle\ ,\ \rangle_{\lambda_1} \ : \ V_f^{(p)}(A_1) \times V_f^{(p)}(A_1) \longrightarrow \mathbb{A}_{f,\mathbb{Q}}^{(p)}(1). $$

This defines a group of symplectic similitudes, which we denote by $\mathrm{GSp}(V_f^{(p)}(A_1), \lambda_1)$, or $\mathrm{GSp}(V_f^{(p)}(A_1))$ for short. It is isomorphic to $\mathrm{GSp}_{2g}(\mathbb{A}_{f,\mathbb{Q}}^{(p)})$. Now we turn to the other side for $\mathcal{M}_F^\Lambda$. The group $\mathrm{GL}_{\mathbb{A}_{f,\mathbb{Q}}^{(p)}}(V_f^{(p)}(A_1))$

is isomorphic to $\mathrm{GL}_2(\mathbb{A}^{(p)}_{f,\mathbb{Q}})$. But $\mathrm{GL}_{\mathbb{A}^{(p)}_{f,\mathbb{Q}}}(V_f^{(p)}(A_1))$ does not embed into $\mathrm{GSp}(V_f^{(p)}(A_1))$ in a natural way. We have to cut down the size of the center of $\mathrm{GL}_{\mathbb{A}^{(p)}_{f,\mathbb{Q}}}(V_f^{(p)}(A_1))$ to make that happen. The nondegenerate symplectic pairing $\langle\ ,\ \rangle_{\lambda_1}$ factorizes through an $\mathbb{A}^{(p)}_{f,F}$-linear nondegenerate alternating pairing

$$\langle\ ,\ \rangle_F\ :\ V_f^{(p)}(A_1)\times V_f^{(p)}(A_1)\longrightarrow \mathbb{A}^{(p)}_{f,\mathbb{F}}(1)$$

such that $\langle\ ,\ \rangle_{\lambda_1}=\mathrm{tr}_{F/\mathbb{Q}}\circ\langle\ ,\ \rangle_F$. Since $\bigwedge^2_{\mathbb{A}^{(p)}_{f,\mathbb{F}}}(V_f^{(p)}(A_1))$ is a free $\mathbb{A}^{(p)}_{f,\mathbb{F}}$-module of rank 1, the involution $*=*_F$ on $\mathrm{End}_{\mathbb{A}^{(p)}_{f,\mathbb{F}}}(V_f^{(p)}(A_1))$ induced by $\langle\ ,\ \rangle_F$ is the standard involution. Especially

$$\alpha\cdot\alpha^*=\alpha^*\cdot\alpha=\det(\alpha)\cdot\mathrm{Id},\quad \forall\alpha\in\mathrm{End}_{\mathbb{A}^{(p)}_{f,\mathbb{F}}}(V_f^{(p)}(A_1)).$$

Let $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ be the subgroup of $\mathrm{GL}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ consisting of all elements $\alpha\in\mathrm{GL}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ such that $\det(\alpha)\in\mathbb{A}^{(p)\times}_{f,\mathbb{Q}}$. Clearly the center of the group $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ is $\mathbb{A}^{(p)\times}_{f,\mathbb{Q}}$, and the derived group of $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ is $\mathrm{SL}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$. The action of $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ on $V_f^{(p)}(A_1)$ identifies the group $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ as a subgroup of $\mathrm{GSp}(V_f^{(p)}(A_1))$.

We go back to the morphism

$$j:\mathscr{M}_F^{\Lambda,or}\longrightarrow\mathscr{A}^{or}_{g,\delta}.$$

Clearly j extends naturally to a morphism $j^{(p)}$ from the prime-to-p tower of $\mathscr{M}_F^{\Lambda,or}$ to the prime-to-p tower of $\mathscr{A}^{(or)}_{g,\delta}$. The group $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ operates on the prime-to-p tower of $\mathscr{M}_F^{\Lambda,or}$, while $\mathrm{GSp}(V_f^{(p)}(A_1))$ operates on the prime-to-p tower of $\mathscr{A}^{or}_{g,\delta}$. Moreover $j^{(p)}$ is equivariant with respect to the embedding of $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$ in $\mathrm{GSp}(V_f^{(p)}(A_1))$. These statements are not hard to check, and we omit their verification since we do not need it. For us the important consequence is that the image of the $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$-Hecke orbit of a point $[(A_1,\iota_1,h_1)]$ in $\mathscr{M}_F^{\Lambda,or}(\overline{\mathbb{F}_p})$ is contained in the prime-to-p Hecke orbit of the point $[(A_1,\lambda_1)]$ in $\mathscr{A}^{(or)}_{g,\delta}(\overline{\mathbb{F}_p})$. For an algebraically closed field k of characteristic $p>0$, the $\mathrm{GL}^{\mathrm{red}}_{\mathbb{A}_{f,F^{(p)}}}(V_f^{(p)}(A_1))$-Hecke orbit of the point $[(A_1,\iota_1,h_1)]$ in $\mathscr{M}_F^{\Lambda,or}(\overline{\mathbb{F}_p})(k)$ is explicitly given below, and will be referred to as the *reduced prime-to-p F-Hecke orbit* of the point $[(A_1,\iota_1,h_1)]$ in $\mathscr{M}_F^{\Lambda,or}(\overline{\mathbb{F}_p})(k)$. It consists of all points $[B,\iota_B,h_B)]$ in $\mathscr{M}_F^{\Lambda}(\overline{\mathbb{F}_p})$ such that there exists an $\mathcal{O}_F$-linear isogeny $\phi:B\to A$, a totally positive unit $u\in\mathcal{O}_F^\times$ and a positive integer $n\in\mathbb{Z}_{(p)}$ such that

$${}^t\phi\circ h_A(\lambda)\circ\phi=n\cdot u\cdot h_B(\lambda),\quad \forall\lambda\in\Lambda.$$

It is clear from this description that j sends the reduced prime-to-p F-Hecke orbit of the point $[(A_1,\iota_1,h_1)]$ in $\mathscr{M}_F^{\Lambda,or}(\overline{\mathbb{F}_p})$ into the prime-to-p Hecke orbit of the point $[(A_1,\lambda_1)]$ in $\mathscr{A}^{or}_{g,\delta}(\overline{\mathbb{F}_p})$. Of course for a positive integer D with $(D,p)=1$, there is a similar notion of '*reduced D-power F-Hecke orbit*' in $\mathscr{M}_F^{\Lambda,or}(k)$, and j sends a reduced D-power F-Hecke orbit in $\mathscr{M}_F^{\Lambda,or}(k)$ into a D-power Hecke orbit of $\mathscr{A}^{or}_{g,\delta}(k)$.

Proposition 3. *Let k be an algebraically closed field of characteristic $p > 0$. Let D be a positive integer with $(D, p) = 1$. Assume that the reduced prime-to-p (resp. reduced D-power) F-Hecke orbit of every ordinary point of $\mathcal{M}_F^\Lambda(\overline{\mathbb{F}_p})$ is Zariski dense in $\mathcal{M}_F^\Lambda$, for any totally real number field F and any projective rank-one $\mathcal{O}_F$-module Λ with a notion of positivity Λ_+. Then the prime-to-p (resp. D-power) Hecke orbit of every ordinary point in $\mathcal{A}_{g,\delta}(k)$ is dense in $\mathcal{A}_{g,\delta}$, for any $g > 0$ and any type δ.*

Proof. Recall that at the boundary of $\mathcal{M}_F^\Lambda$, the universal abelian scheme with multiplication by $\mathcal{O}_F$ degenerates to a torus. Therefore if the reduced prime-to-p (resp. D-power) F-Hecke orbit of (A_1, ι_1) is Zariski dense in $\mathcal{M}_F^\Lambda$, then the Zariski closure Z of the prime-to-p (resp. D-power) Hecke orbit of (A_1, λ_1) contains a one-parameter family which degenerates to a torus. Proposition 2″ concludes the proof. □

In the rest of this section we collect a few facts about Hilbert-Blumenthal abelian varieties. They are all well known. Proofs are supplied for the convenience of the reader.

Lemma 5. *Let F be a totally real number field. Let (A_1, ι_1) be an ordinary abelian variety of dimension $[F : \mathbb{Q}]$ over a perfect field k of characteristic $p > 0$ with multiplication $\iota_1 : \mathcal{O}_F \to \mathrm{End}_k(A_1)$ by $\mathcal{O}_F$. Then its Lie algebra* $\mathrm{Lie}(A_1)$ *is a free module over $\mathcal{O}_F \otimes_{\mathbb{Z}} k$ of rank 1.*

Proof. The p-divisible group $A_1[p^\infty]$ associated to A_1 is the direct sum of its toric part $A_1[p^\infty]_{\text{toric}}$ and its étale part $A_1[p^\infty]_{\acute{e}t}$. Therefore its contravariant Dieudonné module $\mathbb{D}(A_1[p^\infty])$ is the direct sum $\mathbb{D}(A_1[p^\infty]_{\text{toric}}) \oplus \mathbb{D}(A_1[p^\infty]_{\acute{e}t})$ of the Diedonné modules of its toric and étale part. By [31] 1.3, $\mathbb{D}(A_1[p^\infty])$ is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} W(k)$-module of rank 2. Notice that [31] 1.3 is formulated generally and does not depend on the freeness condition (*) in Def. 1.1, *loc.cit.* Hence $\mathbb{D}(A_1[p^\infty]_{\text{toric}})$ is a projective module over $\mathcal{O}_F \otimes_{\mathbb{Z}} W(k)$. Since $\mathbb{D}(A_1[p^\infty]_{\text{toric}}) \otimes_{W(k)} k \cong \mathrm{Lie}(A_1)^\vee$, $\mathrm{Lie}(A_1)^\vee$ is a projective module over $\mathcal{O}_F \otimes_{\mathbb{Z}} k$. This implies that $\mathrm{Lie}(A_1)$ is a projective module over $\mathcal{O}_F \otimes_{\mathbb{Z}} k$. On the other hand by [7] 2.7, we know that $\mathrm{Lie}(A_1)$ and $\mathcal{O}_F \otimes_{\mathbb{Z}} k$ have the same class in the Grothendieck group of $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-modules of finite type. Taking into account that $\mathcal{O}_F \otimes_{\mathbb{Z}} k$ is an artinian k-algebra, this implies that $\mathrm{Lie}(A_1)$ is isomorphic to $\mathcal{O}_F \otimes_{\mathbb{Z}} k$ as an $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-module. □

Lemma 6. *Let F be a totally real number field. Let (A, ι) be an abelian variety of dimension $g = [F : \mathbb{Q}]$ with multiplication by $\mathcal{O}_F$ over an algebraically closed field k. Then A is isogenous to B^n for some simple abelian variety B over k. Let $D = \mathrm{End}_k(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, so $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_{n\times n}(D)$. There are four possibilities*

i. (*Type I*) *The algebra D is a totally real number field K_0, and $[K_0 : \mathbb{Q}] = \dim(B)$. F contains K_0, and $[F : K_0] = n$. The centralizer of F in* $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is equal to F itself.*

ii. (*Type II*) *The algebra D is a totally indefinite quaternion division algebra over a totally real number field K_0, and* $\dim(B) = 2[K_0 : \mathbb{Q}]$. *The field F contains K_0, and $[F : K_0] = 2n$. The centralizer of F in* $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is equal to F itself.*

iii. (*Type III*) *The algebra D is a totally definite quaternion division algebra over a totally real number field K_0. There are 2 cases:*

  (a) *The field k has characteristic p, $K_0 = \mathbb{Q}$ and* $\dim(B) = 1$. *The algebra D is the quaternion division algebra over $\mathbb{Q}$ ramified only at p and ∞. The algebra B is a supersingular elliptic curve over k. The centralizer of F in* $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is the quaternion division algebra over F which is ramified at all infinite places of F and all places v of F above p such that $[F_v : \mathbb{Q}_p]$ is odd, and is unramified at all other finite places.*

  (b) *The simple abelian variety B is not an elliptic curve;* $\dim(B) = 2[K_0 : \mathbb{Q}]$. *The field F contains K_0, and $[F : K_0] = 2n$. The centralizer of F in* $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is equal to F itself.*

iv. (*Type IV*) *The algebra D is a central division algebra over a totally imaginary quadratic extension K of a totally real number field K_0. Write* $\dim_K(D) = d^2$. *Then $F \supseteq K_0$, and $[F : K_0] = nd$. The centralizer of F in* $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is equal to $F \cdot K$, the subalgebra in* $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *generated by F and K. It is isomorphic to $F \otimes_{K_0} K$, a totally imaginary quadratic extension of F. If $\mathrm{char}(k) = 0$, then $d = 1$ and $D = K$.*

The abelian variety A does not have sufficiently many complex multiplication (smCM) in the following cases: (Type I), (Type II), (Type III (b)), and has smCM in the other cases. If $k = \overline{\mathbb{F}}_p$ and A is ordinary, then D is a CM-field, an example of the (Type IV) case. Here to say that A has smCM means that $\mathrm{End}^0(A)$ *contains a commutative semisimple subalgebra of dimension* $2\dim(A)$ *over $\mathbb{Q}$.*

Proof. The abelian variety A is isogenous to a product $\prod_{i=1}^m B_i^{n_i}$, where B_i is not isogenous to B_j if $i \neq j$. Let $B = B_1$, $n = n_1$, $D = \mathrm{End}_k(B_1) \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural map of $F \to \mathrm{M}_{n\times n}(D)$ is an embedding. The division algebra D has one of the four possibilities as in [26], p. 202. See also [32] §2 for the characteristic 0 case.

i. (Type I) The algebra D is a totally real algebraic number field. The table in [26] p. 202 gives $[K_0 : \mathbb{Q}] | \dim(B)$, hence $n[K_0 : \mathbb{Q}] \leqq n\dim(B) \leqq [F : \mathbb{Q}]$. Since F is a commutative semisimple subalgebra of $M_{n\times n}(D)$, $[F : \mathbb{Q}] \leqq n[K_0 : \mathbb{Q}]$. Therefore $n[K_0 : \mathbb{Q}] = n\dim(B) = \dim(A) = [F : \mathbb{Q}] = \dim(A)$ and $m = 1$. This implies that F is a maximal commutative semisimple subalgebra of $\mathrm{M}_{n\times n}(K_0)$, therefore F contains K_0.

ii. (Type II) The algebra D is a totally indefinite quaternion division algebra over a totally real field K_0. In this case [26] p. 202 gives $2[K_0 : \mathbb{Q}] | \dim(B)$. Hence $2n[K_0 : \mathbb{Q}] \leqq n\dim(B) \leqq \dim(A) = [F : \mathbb{Q}]$. On the other hand $[F : Q] \leqq 2n[K_0 : \mathbb{Q}]$ since F is a commutative semisimple subalgebra of $M_{n\times n}(D)$. So $[F : Q] = 2n[K_0 : \mathbb{Q}]$, $2[K_0 : \mathbb{Q}] = \dim(B)$, and $m = 1$. This implies that F is a maximal commutative semisimple subalgebra of $\mathrm{M}_{n\times n}(D)$, therefore F contains K_0.

iii. (Type III) The algebra D is a totally definite quaternion division algebra over a totally real number field K_0. The table in [26], p. 202 gives $[K_0 : \mathbb{Q}] | \dim(B)$, and $2[K_0 : \mathbb{Q}] | \dim(B)$ if char.$(k) = 0$. First assume that $[K_0 : \mathbb{Q}] = \dim(B)$. In this case the field k has to have positive characteristic. The abelian variety B has smCM. Grothendieck has shown that in positive characteristic, an abelian variety with smCM is isogenous to an abelian variety defined over a finite field, see [30]. Therefore we may assume that B is defined over a finite field $\mathbb{F}_q$. Moreover we may assume that all elements of $\mathrm{End}_{\overline{\mathbb{F}}_q}(B)$ are defined over $\mathbb{F}_q$. The Frobenius element $\pi_{B\,\mathbb{F}_q}$ for $A_1/\mathbb{F}_q$ belongs to K_0 because it commutes with all elements of $\mathrm{End}_{\overline{\mathbb{F}}_q}(B)$. Since K_0 has real places, we conclude that B is a supersingular elliptic curve; see [33], §1 example a). The rest of the statement (a) follow from standard results on central simple algebras.

Next assume that $\dim(B) \geqq 2[K_0 : \mathbb{Q}]$. Then as in the (Type II) case we have $2n[K_0 : \mathbb{Q}] \leqq n \dim(B) \leqq \dim(A) = [F : \mathbb{Q}]$ and $[F : Q] \leqq 2n[K_0 : \mathbb{Q}]$ because F is a commutative semisimple subalgebra of $M_{n\times n}(D)$. Therefore $[F : Q] = 2n[K_0 : \mathbb{Q}]$, $2[K_0 : \mathbb{Q}] = \dim(B)$, $m = 1$ and F contains K_0.

iv. (Type IV) The algebra D is a central division algebra over a totally imaginary quadratic extension K of a totally real number field K_0. Write $\dim_K(D) = d^2$. The table in [26], p. 202 says that $[K_0 : \mathbb{Q}] \mp | \dim(\mathbb{B})$, and $[K_0 : \mathbb{Q}]d^2 | \dim(B)$ if char.$(k) = 0$. So $nd[K_0 : \mathbb{Q}] \leqq n \dim(B) \leqq \dim(A) = [F : \mathbb{Q}]$. On the other hand $[F : \mathbb{Q}] \leqq nd[K_0 : \mathbb{Q}]$ since $F \cdot K$ is a commutative semisimple subalgebra of $\mathrm{M}_{n\times n}(D)$. Therefore $[F : \mathbb{Q}] = nd[K_0 : \mathbb{Q}]$, $m = 1$, $d[K_0 : \mathbb{Q}] = \dim(B)$, $F \supseteq K_0$ and $[F : K_0] = nd$. If char.$(k) = 0$, we have $[K_0 : \mathbb{Q}]d^2 | \dim(B)$, therefore $d = 1$ and $D = K$. To compute the centralizer of F in $\mathrm{M}_{n\times n}(D)$, we use descent from K to K_0: $\mathrm{M}_{n\times n}(D) \otimes_{K_0} K$ is isomorphic to $\left(\mathrm{M}_{n\times n}(D) \otimes_{K_0} K\right) \times \left(\mathrm{M}_{n\times n}(D) \otimes_{K_0} K\right)$. Via this isomorphism $F \otimes_{K_0} K$ becomes a subalgebra of $\left(\mathrm{M}_{n\times n}(D) \otimes_{K_0} K\right) \times \left(\mathrm{M}_{n\times n}(D) \otimes_{K_0} K\right)$. Its image in either factor is equal to $F \cdot K$, a maximal commutative semisimple subalgebra of $\mathrm{M}_{n\times n}(D)$. Therefore the centralizer of $F \otimes_{K_0} K$ is contained in $(F \cdot K) \times (F \cdot K) = (F \times K) \otimes_{K_0} K$. Consequently the centralizer of F in $\mathrm{M}_{n\times n}(D)$ is contained in $F \cdot K$. The other inclusion is obvious.

When $k = \overline{\mathbb{F}}_p$ and A is ordinary, we know that D is a CM-field by Lemma 3, therefore belongs to (Type IV). □

The following lemma is an analogue of Proposition 1 for Hilbert-Blumenthal abelian varieties.

Lemma 7. *Let F be a totally real algebraic number field field, $\mathcal{O}_F$ be the ring of algebraic integers in F. Let k be an algebraically closed field of characteristic $p > 0$. Assume that A_1 is an abelian variety over k, with a structure of $\mathcal{O}_F$-module $\iota_1 : \mathcal{O} \to \mathrm{End}(A)$. If the reduced ℓ-power F-Hecke orbit of (A_1, ι_1) in $\mathcal{M}_F$ is finite for a prime number ℓ different from p, then A_1 is supersingular. Conversely, if A_1 is supersingular, then the prime-to-p F-Hecke orbit of (A_1, ι_1) in $\mathcal{M}_F$ is finite.*

Proof. Since the proof of this lemma is very similar to the proof of Proposition 1, we shall only give a sketch and leave the complete proof to the reader as an exercise. First if A_1 is supersingular, then $\mathrm{End}(A_1, \iota_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the centralizer of F in $\mathrm{End}(A_1) \otimes_{\mathbb{Z}} \mathbb{Q}$, hence is a quaternion algebra over F. That is it is a central simple algebra over F of relative dimension 4. The expression of the prime-to-p F-Hecke orbit of (A_1, ι_1) in $\mathscr{M}_F$ as a double coset shows that it is finite. Conversely, assume that the reduced ℓ-power F-Hecke orbit of (A_1, ι_1) in $\mathscr{M}_F$ is finite for a prime number $\ell \neq p$. Then we conclude as in the proof of Proposition 1 that the $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-linear span of

$$(\mathrm{End}(A_1, \iota_1) \otimes_{\mathbb{Z}} \mathbb{Q})^\times \cap \mathrm{GL}^{\mathrm{red}}_{F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell}(\mathrm{H}_1(A_1), \mathbb{Q}_\ell)$$

is $\mathrm{End}_{F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell}(\mathrm{H}_1(A_1), \mathbb{Q}_\ell)$. Hence $\mathrm{End}(A_1, \iota_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a quaternion algebra over F. An inspection of the cases in Lemma 6 finishes the proof. Alternatively we can finish the proof directly as follows: The abelian variety A_1 has smCM, therefore is isogenous to an abelian variety A_2 defined over a finite field $\mathbb{F}_q$, and we may assume that all elements of $\mathrm{End}_{\overline{\mathbb{F}}_q}(A_2)$ are defined over $\mathbb{F}_q$. The Frobenius element $\pi_{A_2, \mathbb{F}_q}$ for $A_1/\mathbb{F}_q$ belongs to F because it commutes with all elements of $\mathrm{End}_{\overline{\mathbb{F}}_q}(A_2)$. Since F has real places, we conclude that A_2 is supersingular, so is A_1. See [33], §1 Example a). □

Lemma 8. *Suppose that F be a totally real number field, (A, ι) is an ordinary abelian variety of dimension $g = [F : \mathbb{Q}]$ over a perfect field k of characteristic $p > 0$ with multiplication $\iota : \mathcal{O}_F \to \mathrm{End}_k(A)$ by $\mathcal{O}_F$. Let $T_p(A[p^\infty]_{\acute{e}t})$ be the p-adic Tate module for the étale quotient of $A[p^\infty]$, $V_p(A[p^\infty]_{\acute{e}t}) = T_p(A[p^\infty]_{\acute{e}t}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Similarly for the dual A^t of A. Then*

(1) *The p-adic Tate modules $T_p(A[p^\infty]_{\acute{e}t})$ and $T_p(A^t[p^\infty]_{\acute{e}t})$ are both free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-modules of rank 1.*
(2) *If (A, ι) is defined over a finite field, then $\left(T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p(A^t[p^\infty]_{\acute{e}t})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a free module over $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ of rank 1, where $L = \mathrm{End}_k(A, \iota) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the endomorphism algebra of the abelian variety (A, ι) up to isogeny with multiplication by F. The field L is a totally imaginary quadratic extension of F which is split at every place $\mathfrak{p}$ of F above p. If moreover $\mathcal{O}_L = \mathrm{End}_k(A, \iota)$, then $T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p(A^t[p^\infty]_{\acute{e}t})$ is a free $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module of rank 1.*

Proof. (1) Since $T_p(A[p^\infty]_{\acute{e}t})$ is torsion-free, it is a projective module over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We know that $\mathrm{Lie}(A^t)^\vee \cong T_p(A[p^\infty]_{\acute{e}t}) \otimes_{\mathbb{Z}_p} k$ as $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-modules, and *Lie*$(A^t)$ is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-module of rank 1 by Lemma 5. Hence $T_p(A[p^\infty]_{\acute{e}t})$ is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module of rank 1. Similarly for A^t.

(2) If $\mathcal{O}_L = \mathrm{End}_k(A, \iota)$, then $T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p(A^t[p^\infty]_{\acute{e}t})$ is a projective module over $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ because it is torsion-free. So the second statement in (2) follows from the first one. We know from Lemma 6 that we are in the Type IV case of Lemma 6. Therefore L is a totally imaginary quadratic extension of F. In the notation of Lemma 6 iv, we have $L = F \cdot K$, and the quadratic extension K/K_0 is split at every place of K_0 above p by

Lemma 3, therefore the extension L/F is split at every place of F above p. Clearly $\left(T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p\left(A^t[p^\infty]^\vee_{\acute{e}t}\right)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a projective module over $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Moreover we know that the trace of the natural linear action of any element of L on $\left(T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p\left(A^t[p^\infty]^\vee_{\acute{e}t}\right)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is in $\mathbb{Q}$, because as an $L \otimes W(k)$-module $\left(T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p\left(A^t[p^\infty]^\vee_{\acute{e}t}\right)\right) \otimes_{\mathbb{Z}_p} \mathbb{W}(k)$ is naturally isomorphic to the $W(k)$-dual of the first crystalline cohomology group of $A/W(k)$. So $\left(T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p\left(A^t[p^\infty]^\vee_{\acute{e}t}\right)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a free module of rank 1 over $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Since over a discrete valuation ring every free module of finite rank is non-canonically isomorphic to its dual, it follows that $\left(T_p(A[p^\infty]_{\acute{e}t}) \oplus T_p\left(A^t[p^\infty]_{\acute{e}t}\right)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a free module over $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ of rank 1. □

4. Calculation at smooth ordinary points over finite fields

We keep the notation in the previous section: F is a totally real algebraic number field, $\mathcal{O}_F$ denotes the ring of algebraic integers in F. Let k be an algebraically closed field of characteristic $p > 0$. We have the Hilbert moduli space $\mathcal{M}_F$ and its variants $\mathcal{M}_F^A$.

Suppose that $\tilde{Z}$ is the Zariski closure of the reduced prime-to-p Hecke orbit of an ordinary point in $\mathcal{M}_F^A(k)$. Then the intersection of the open subset of smooth points of Z with $\mathcal{M}_F^{A,or}$ is an open dense subscheme $\tilde{Z}_{sm}^{or}$ of $\tilde{Z}$. Moreover $\tilde{Z}_{sm}^{or}$ is stable under the reduced prime-to-p F-Hecke correspondences, and contains a dense subset of points defined over finite fields. Therefore for any point $\tilde{x} \in \tilde{Z}_{sm}^{or}$ defined over $\overline{\mathbb{F}}_p$, the formal completion $\tilde{Z}_{\tilde{x}}^\wedge$ of $\tilde{Z}$ at $\tilde{x}$ is stable under the action of the stabilizer subgroup of $\tilde{x}$ in the reduced prime-to-p F-Hecke correspondences. In this section we shall decipher this information by an explicit calculation using the Serre-Tate coordinates.

We first recall the Serre-Tate coordinates. A good reference is [18]. Let A_1 be an ordinary abelian variety over k. Denote by $T_p(A_1)(k)$ the 'physical' p-adic Tate module $T_p\left(A_1[p^\infty]_{\acute{e}t}\right)(k)$. Let Def_{A_1} be the deformation functor which sends a pair (R,ε) consisting of an artinian local ring R and an isomorphism $\varepsilon : \kappa_R = R/\mathfrak{m}_R \longrightarrow k$ to

$$\mathrm{Def}_{A_1}(R,\varepsilon) = \left\{ (A,\alpha) \,\middle|\, \begin{array}{l} A \text{ is an abelian scheme over } R\text{; and} \\ \alpha : A \otimes_{R,\varepsilon} k \xrightarrow{\sim} A_1 \text{ is an isomorphism} \end{array} \right\} \Big/ (\text{isomorphisms}).$$

The Serre-Tate theory says that the deformation functor Def_{A_1} is represented by the formal scheme

$$\underline{Hom}_{\mathbb{Z}_p}\left(T_p(A_1)(k) \otimes_{\mathbb{Z}_p} T_p(A_1^t)(k),\ \hat{\mathbb{G}}_m\right),$$

or equivalently the formal scheme

$$\left(T_p(A_1)(k) \otimes_{\mathbb{Z}_p} T_p(A_1^t)(k)\right)^\vee \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m.$$

Now suppose that (A_1, ι_1) is an ordinary abelian variety of dimension $[F : \mathbb{Q}]$ with multiplication by $\mathcal{O}_F$. Then both $T_p(A_1)(k)$ and $T_p(A_1^t)(k)$ are free modules over $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of rank 1 by Lemma 8. Serre-Tate theory tells us that the formal completion of $\mathcal{M}_F$ at the point $[(A_1, \iota_1)] \in \mathcal{M}_F(k)$ is naturally isomorphic to

$$\underline{Hom}_{\mathbb{Z}_p}\left(T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k),\ \hat{\mathbb{G}}_m\right)$$
$$\cong T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} \left(\mathcal{D}_F^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m.$$

The same statement is true for $\mathcal{M}_F^A$, where $(A, A_+) = (\mathcal{P}(A_1, \iota_1), \mathcal{P}(A_1, \iota_1)_+)$, since $\mathcal{M}_F^A$ is étale over $\mathcal{M}_F$. Assume in addition that (A_1, ι_1) is defined over $\overline{\mathbb{F}}_p$. Then the ring $\mathcal{O}_{(A_1, \iota_1)} := \mathrm{End}_{\mathcal{O}_F}(A_1, \iota_1)$ is an order in a totally imaginary quadratic extension L of F, which is split at every place of F above p by Lemma 8. Its group of p-adic units $U_p := \left(\mathcal{O}_{(A_1, \iota_1)} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)^\times$ operates on the deformation space as follows: an element $u \in U_p$ sends an element

$$q((A_1, \iota_1); -, -) \in \underline{Hom}_{\mathbb{Z}_p}\left(T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k),\ \hat{\mathbb{G}}_m\right)$$

to the element

$$T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k) \ni \quad a \otimes a^t \longmapsto q\left((A_1, \iota_1); u^{-1}(a), u^t(a^t)\right),$$
$$\forall a \in T_p(A_1)(k), \quad \forall a^t \in T_p(A_1^t)(k).$$

It is clear that the subgroup $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ of U_p operates trivially on the deformation space.

Now we resume the situation in the beginning of this section. In the rest of this section, we only consider equi-characteristic deformations, and all universal deformation functors and the formal tori representing them are understood to be over k. Let $\mathfrak{L}^\times$ (resp. $\mathfrak{F}^\times$) be the algebraic torus over $\mathbb{Q}$ such that $\mathfrak{L}^\times(\mathbb{Q}) = L^\times$ (resp. $\mathfrak{F}^\times(\mathbb{Q}) = F^\times$). The relative norm defines a homomorphism

$$\mathrm{N}_{L/F} : \mathfrak{L}^\times \longrightarrow \mathfrak{F}^\times.$$

Let $\mathfrak{L}_0^\times = \mathrm{N}_{L/F}^{-1}(\mathbb{G}_m)$, an algebraic torus over $\mathbb{Q}$. Clearly

$$\mathfrak{L}_0^\times(\mathbb{Q}) = \mathrm{N}_{L/F}^{-1}(\mathbb{Q}^\times) \subseteq L^\times,$$

and

$$\mathfrak{L}_0^\times(\mathbb{Q}_p) = \mathrm{N}_{L/F}^{-1}(\mathbb{Q}^\times) \subseteq (L \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times.$$

The group $U_p^0 = \mathfrak{L}_0^\times(\mathbb{Q}_p) \cap U_p$ is a compact open subgroup of $\mathfrak{L}_0^\times(\mathbb{Q}_p)$. For any point $\tilde{x}_1 = [(A_1, \iota_1)] \in \tilde{Z}_{sm}^{or}$ defined over $\overline{\mathbb{F}}_p$, the stabilizer subgroup of $\tilde{x}_1$ in the reduced prime-to-p F- Hecke correspondences contains those which come from $\mathfrak{L}_0^\times(\mathbb{Q}) \cap U_p$. By the weak approximation theorem, $\mathfrak{L}_0^\times(\mathbb{Q}) \cap U_p$ is dense in U_p^0. Therefore the formal completion $\tilde{Z}_{\tilde{x}}^{\wedge}$ of $\tilde{Z}$ at $\tilde{x}$ is stable under the action of U_p^0. (Had we used the reduced ℓ-power F-Hecke correspondences, we would have to replace $\mathfrak{L}_0^{\times}(\mathbb{Q}) \cap U_p$ by $\mathfrak{L}_0^{\times}(\mathbb{Q}) \cap \left(\mathcal{O}_{(A_1,\imath_1)} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\ell}\right]\right)^{\times}$ in the above, and run into the difficulty that the closure of $\mathfrak{L}_0^{\times}(\mathbb{Q}) \cap \left(\mathcal{O}_{(A_1,\imath_1)} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{\ell}\right]\right)^{\times}$ may not be an open subgroup of U_p^0.) Consider the homomorphism

$$\mathfrak{L}_0^{\times} \longrightarrow \mathfrak{L}^{\times}/\mathfrak{F}^{\times}$$

induced by the inclusion $\mathfrak{L}_0^{\times} \to \mathfrak{L}^{\times}$. One checks easily that this homomorphism is faithfully flat. Its kernel is 1-dimensional but is not connected. Clearly the image of U_p^0 in $(\mathfrak{L}^{\times}/\mathfrak{F}^{\times})(\mathbb{Q}_p)$ is an open subgroup of $(\mathfrak{L}^{\times}/\mathfrak{F}^{\times})(\mathbb{Q}_p)$. Since L/F is split at every place of L above p,

$$(\mathfrak{L}^{\times}/\mathfrak{F}^{\times})(\mathbb{Q}_p) = (L \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}/(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}.$$

Also there is a unique isomorphism

$$(L \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}/(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times} \xrightarrow{\sim} (F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}$$

such that the composition

$$\begin{aligned} U_p \hookrightarrow \mathfrak{L}_0^{\times}(\mathbb{Q}_p) \to (\mathfrak{L}^{\times}/\mathfrak{F}^{\times})(\mathbb{Q}_p) &= (L \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}/(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times} \\ &\xrightarrow{\sim} (F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times} \end{aligned}$$

factorizes as

$$U_p \xrightarrow{\chi} (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times} \hookrightarrow (F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times},$$

and such that χ coincides with the character of the following $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-linear action of the group U_p on the free rank-one $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module $T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k)$:

$$\begin{aligned} U_p \ni u \longmapsto \big(a \otimes a^t &\mapsto u(a) \otimes (u^t)^{-1}(a^t)\big) \\ &\forall a \otimes a^t \in T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k)\,. \end{aligned}$$

From the above discussions and Lemma 8, we see that the formal subscheme $\tilde{Z}_{\tilde{x}}^{\wedge}$ of

$$T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} \left(\mathscr{D}_F^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m$$

is stable under the action of an open subgroup of $\left(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)^{\times}$. Here the action of $\left(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)^{\times}$ on

$$T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} \left(\mathscr{D}_F^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m$$

comes from the natural $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module structure of

$$T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} \left(\mathscr{D}_F^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right).$$

Let

$$Y := T_p(A_1)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} T_p(A_1^t)(k) \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)} \left(\mathcal{D}_F^{-1} \otimes_{\mathbb{Z}} \mathbb{Z}_p\right),$$

a free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module of rank 1. We have seen that

$$\tilde{Z}_{\tilde{x}}^{\wedge} \subseteq Y \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m$$

is stable under the action of an open subgroup of $\left(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)^{\times}$. Write

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_{\mathfrak{p}_1} \times \cdots \times \mathcal{O}_{\mathfrak{p}_r} ,$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the prime ideals of $\mathcal{O}_F$ above p and $\mathcal{O}_{\mathfrak{p}_i}$ is the maximal order in $F_{\mathfrak{p}_i}$. Correspondingly, there is a decomposition

$$Y = Y_1 \oplus \cdots \oplus Y_r ,$$

where each Y_i is a free $\mathcal{O}_{\mathfrak{p}_i}$-module of rank 1: $Y_i = \mathcal{O}_{\mathfrak{p}_i} \cdot Y$. We have seen that there exist open subgroups $U_i \subseteq \mathcal{O}_{\mathfrak{p}_i}^{\times}$ such that

$$\tilde{Z}_{\tilde{x}}^{\wedge} \subseteq \left(Y_1 \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m\right) \times \cdots \times \left(Y_r \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m\right)$$

is stable under the action of $U_1 \times \cdots \times U_r$. We shall see that there are only a finite number of possibilities for $\tilde{Z}_{\tilde{x}}^{\wedge}$:

Proposition 4. *Let p be a prime number. Suppose that for each $i = 1, \ldots, r$, $\mathcal{O}_{\mathfrak{p}_i}$ is a complete discrete valuation ring of characteristic 0 with finite residue field $\kappa_i \cong \mathbb{F}_{p^{n_i}}$. Let Y_i be a free $\mathcal{O}_{\mathfrak{p}_i}$-module of rank 1 for $i = 1, \ldots, r$. Suppose that*

$$W \subseteq \left(Y_1 \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m\right) \times \cdots \times \left(Y_r \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m\right)$$

is a smooth formal subscheme of $\left(Y_1 \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m\right) \times \cdots \times \left(Y_r \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m\right)$ over an algebraically closed field k of characteristic p, which is invariant under the natural action of $U_1 \times \cdots \times U_r$, where each U_i is an open subgroup of $\mathcal{O}_{\mathfrak{p}_i}^{\times}$, $i = 1, \ldots, r$. Then there exists a subset $\omega \subseteq \{1, \ldots, r\}$ such that

$$W = \prod\nolimits_{i \in \omega} Y_i \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m,$$

a product of formal subtori.

Proof. Let I be the ideal of W. Let R_i be the ring of regular formal functions of $Y_1 \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m$, $i = 1, \ldots, r$. Let $\mathrm{gr}^{\bullet}(R_i)$ be the graded ring of R_i. Then $\mathrm{gr}^{\bullet}(R_i)$ is naturally isomorphic to the symmetric k-algebra $\mathrm{S}^{\bullet}\left(X_i \otimes_{\mathbb{Z}_p} k\right)$, where $X_i = \mathrm{Hom}_{\mathbb{Z}_p}(Y_i, \mathbb{Z}_p)$, the dual of Y_i. The graded ring $\mathrm{gr}^{\bullet}(R)$ of $R = R_1 \hat{\otimes}_k \cdots \hat{\otimes}_k R_r$ is naturally isomorphic to $\mathrm{gr}^{\bullet}(R_1) \otimes_k \cdots \otimes_k \mathrm{gr}^{\bullet}(R_r)$. The tangent cone of W is defined by the prime ideal $\mathrm{gr}^{\bullet}(I)$ of $\mathrm{gr}^{\bullet}(R)$. It is generated by a k-vector subspace $\bar{I}_W$ of $\bigoplus_{i=1}^{r} X_i \otimes_{\mathbb{Z}_p} k \subset \mathrm{gr}^{\bullet}(R)$ because W is smooth.

There exists a natural number $N \in \mathbb{N}$ such that $1 + p^N \mathcal{O}_{\mathfrak{p}_i} \subseteq U_i$ for each $i = 1, \ldots, r$. For each i, let

$$\Phi_{X_i, p^n} : X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p \longrightarrow \mathrm{S}^{p^n}\left(X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p\right)$$

be the map sending each element of $X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ to its p^n-th power. Given any non-zero element $\alpha = \sum_{i=1}^{r} \alpha_i$ in $\bar{I}_W$ with $\alpha_i \in X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ for $i = 1,\dots,r$. Suppose that $\alpha_{i_0} \neq 0$, and pick an element $f \in I$ with initial form α. Let $\varepsilon_i : U_i \to U_1 \times \cdots \times U_r$ denote the injection of the i-th factor U_j into $U_1 \times \cdots \times U_r$. Then a straightforward calculation shows that the initial form of $\varepsilon_{i_0}(1 + p^n) \cdot f - f$ is the image of α_0 under

$$\Phi_{X_{i_0}, p^n} \otimes_{\mathbb{F}_p} \mathrm{id}_k : X_{i_0} \otimes_{\mathbb{Z}_p} k = X_{i_0} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \otimes_{\mathbb{F}_p} k \longrightarrow \mathrm{S}^{p^n}\left(X_{i_0} \otimes_{\mathbb{Z}_p} \mathbb{F}_p\right) \otimes_{\mathbb{F}_p} k = \mathrm{S}^{p^n}\left(X_{i_0} \otimes_{\mathbb{Z}_p} k\right).$$

Therefore $\Phi_{X_{i_0}, p^n} \otimes \mathrm{id}_k(\alpha_0) \in \mathrm{gr}^\bullet(I)$ if $n \geqq N$. Since $\mathrm{gr}^\bullet(I)$ is a prime ideal, it follows that the image of α_{i_0} under

$$\mathrm{id}_{X_{i_0}} \otimes \sigma_p^{-n} : X_{i_0} \otimes_{\mathbb{Z}_p} k \longrightarrow X_{i_0} \otimes_{\mathbb{Z}_p} k$$

belongs to $\mathrm{gr}^\bullet(I)$ for all $n \geqq N$, where $\sigma_p^{-n} : k \to k$ denotes the inverse of the n-th power of the Frobenius on k. Since this holds for every element $\alpha \in \bar{I}_W$, it follows that $\bar{I}_W$ is a product of subspaces defined over $\mathbb{F}_p$: There exists $\mathbb{F}_p$-subspaces $\bar{I}_i \subseteq X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, $i = 1,\dots,r$, such that $\bar{I}_W = \left(\bar{I}_1 \oplus \cdots \oplus \bar{I}_r\right) \otimes_{\mathbb{F}_p} k$.

Clearly each subspace $\bar{I}_i \subseteq X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is a module under $\mathcal{O}_{\mathfrak{p}_i} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. We want to show that each $\bar{I}_i \subseteq X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is either equal to the trivial subspace (0) or $X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ itself. Let π_i be a uniformiser of $\mathcal{O}_{\mathfrak{p}_i}$. Let e_i be the absolute ramification index for $\mathcal{O}_{\mathfrak{p}_i}$ and write $p = u_i \cdot \pi^{e_i}$ with $u_i \in \mathcal{O}_{\mathfrak{p}_i}^\times$. If $\bar{I}_i \neq (0)$, then it has to contain the subspace $\pi_i^{e_i - 1} \cdot X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Therefore there exists an element $f_i \in I$ with initial form $\pi_i^{e_i - 1} \cdot \beta_i$, where β_i is the image in $X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ of an $\mathcal{O}_{\mathfrak{p}_i}$-generator of X_i. Of course the initial form of the element $\varepsilon_i(1 + u_i\pi) \cdot f_i - f_i$ is in $\mathrm{gr}^\bullet(I)$. A simple calculation shows that this initial form is equal to $\Phi_{X_i,p}(\beta_i)$, the p-th power of the degree-one form β_i. Since $\mathrm{gr}^\bullet(I)$ is a prime ideal, we get $\beta_i \in \mathrm{gr}^\bullet(I)$ and hence $\beta \in \bar{I}_i$. Thus if $\bar{I}_i \neq (0)$, then it contains a generator of $X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$, hence is equal to $X_i \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. We have shown that there exists a subset $\omega \subseteq \{1,\dots,r\}$ such that $\bar{I}_W = \bigoplus_{j \notin \omega} X_j \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

We still have to show that W itself is a product of formal subtori. Pick a $\mathbb{Z}_p$-basis for each X_i, we arrive at a $\mathbb{Z}_p$-basis $\chi_1,\dots,\chi_a$ for $\oplus_{i \in \omega} X_i$ and a $\mathbb{Z}_p$-basis $\chi_{a+1},\dots,\chi_{a+b}$ for $\bigoplus_{j \notin \omega} X_j$. Then we have

$$R \cong k[[x_1,\dots,x_a, y_1,\dots,y_b]] = k[[\underline{x}, \underline{y}]],$$

where $x_1 = \chi_1 - 1,\dots,x_a = \chi_a - 1$, $y_1 = \chi_{a+1} - 1,\dots,y_b = \chi_{a+b} - 1$. Our result above says that there exist elements in I of the form

$$g_j(\underline{x}, \underline{y}) = y_j + \Delta_j(\underline{x}, \underline{y}) \qquad j = 1,\dots,b,$$

with $\Delta_j(\underline{x}, \underline{y}) \equiv 0 \mod \left(\underline{x}, \underline{y}\right)^2$ for each $j = 1,\dots,b$. They form a set of generators of the ideal I. An easy manipulation of power series shows that we can modify the $g_j(\underline{x}, \underline{y})$'s to get elements in I of the form

$$h_j(\underline{x}, \underline{y}) = y_j + \rho_j(\underline{x}) \qquad j = 1, \dots, b,$$

such that $\rho_j(\underline{x}) \in k[[\underline{x}]]$ and $\rho_j(\underline{x}) \equiv 0 \mod (\underline{x})^2$ for each $j = 1, \dots, b$. Again the $h_j(\underline{x}, \underline{y})$'s generate the ideal I. To conclude the proof of proposition 4, we only need to show that all the $\rho_j(\underline{x})$'s are equal to 0. To see this, consider the elements

$$\begin{aligned} \eta_j(\underline{x}) &= \left(\prod_{i \in \omega} \varepsilon_i (1 + p^N)\right) \cdot h_j(\underline{x}, \underline{y}) - h_j(\underline{x}, \underline{y}) \\ &= \left(\prod_{i \in \omega} \varepsilon_i (1 + p^N)\right) \cdot \rho_j(\underline{x}) - \rho_j(\underline{x}) \qquad j = 1, \dots, b. \end{aligned}$$

The initial form of $\eta_j(\underline{x})$ will be a non-zero form in $k[\underline{x}]$ unless $\rho_j(\underline{x}) = 0$. However we know that $\mathrm{gr}^\bullet(I) \subseteq k[\underline{x}, \underline{y}]$ is the ideal generated by $y_1, \dots, y_b$, which does not contain any non-zero form in $k[\underline{x}]$. Therefore all the $\rho_j(\underline{x})$'s are equal to 0. Hence the ideal I is generated by $y_1, \dots, y_b$, i.e. $W = \prod_{i \in \omega} Y_i \otimes_{\mathbb{Z}_p} \hat{\mathbb{G}}_m$. Proposition 4 is proved. $\square$

The statement in Proposition 4 can be globalized at the level of tangent spaces at the smooth points of $\tilde{Z}$. Recall that the tangent sheaf of $\mathcal{M}_F$ has a natural $\mathcal{O}_F$-module structure. If at a point $[(A_x, \iota_x)]$ of $\mathcal{M}_F(k)$ the Lie algebra $\mathrm{Lie}(A_x)$ is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-module of rank 1, then the stalk $T_{\mathcal{M}_F,x}$ of the tangent sheaf $T_{\mathcal{M}_F}$ of $\mathcal{M}_F$ at x is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_F,x}$-module of rank 1. So if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ are the primes of F above p as before, $T_{\mathcal{M}_F,x}$ of $\mathcal{M}_F$ decomposes as a direct sum:

$$T_{\mathcal{M}_F,x} = \bigoplus_{i=1}^{r} T_{\mathcal{M}_F,x}(\mathfrak{p}_i) \quad ,$$

and each $T_{\mathcal{M}_F,x}(\mathfrak{p}_i)$ is a free $\mathcal{O}_{\mathfrak{p}_i} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{M}_F,x}$-module of rank 1. Because of Lemma 5, $\mathcal{M}_F$ is smooth at ordinary points.

The following is an immediate consequence of Proposition 4 by faithfully flat descent.

Proposition 5. *For each irreducible component W of the smooth locus $\tilde{Z}_{sm}$ of $\tilde{Z}$ there exists a subset $\omega \subseteq \{1, \dots, r\}$ such that the tangent sheaf $T_W \subseteq T_{\mathcal{M}_F} \otimes_{\mathcal{O}_{\mathcal{M}_F}} \mathcal{O}_W$ of W is equal to*

$$\bigoplus_{i \in \omega} T_{\mathcal{M}_F}(\mathfrak{p}_i) \otimes_{\mathcal{O}_{\mathcal{M}_F}} \mathcal{O}_W \ .$$

Remark. In fact one can deduce a stronger consequence of Proposition 4 in the situation of Proposition 5: For each irreducible component W of $\tilde{Z}^{or}_{sm}$ there exists a subset $\omega \subseteq \{1, \dots, r\}$ such that in the direct sum decomposition of the p-divisible group $A_W[p^\infty]$ of the universal abelian scheme A_W with multiplication by $\mathcal{O}_F$

$$A_W[p^\infty] = \bigoplus_{i \in \omega} A_W[\mathfrak{p}_i^\infty] \ \oplus \ \bigoplus_{j \notin \omega} A_W[\mathfrak{p}_j^\infty] \ ,$$

the first factor $\bigoplus_{i \in \omega} A_W[\mathfrak{p}_i^\infty]$ is universal at every point of $W(k)$, while the second factor $\bigoplus_{j \notin \omega} A_W[\mathfrak{p}_j^\infty]$ is the direct sum of a multiplicative p-divisible group with an étale p-divisible group. Hence the statement in Proposition 4 holds for every point of $\tilde{Z}^{\mathrm{or}}_{\mathrm{sm}}(k)$. Since we will not need this stronger statement, the proof is omitted.

5. Inspection at the supersingular points

Proposition 6. *Let F be a totally real number field, Λ be a projective $\mathcal{O}_F$-module of rank 1 with a notion of positivity. Let $\tilde{Z}$ be a Zariski closed subscheme of $\mathcal{M}_F^\Lambda$ over an algebraically closed field k of characteristic $p > 0$, which is stable under all reduced ℓ-power F-Hecke correspondences for a prime number $\ell \neq p$. Then $\tilde{Z}$ contains supersingular points, and is non-proper only if $\tilde{Z}$ is equal to $\mathcal{M}_F^\Lambda$.*

Proof. The idea of the proof is already sketched in §1. Let $\tilde{Z}^*$ be the closure of $\tilde{Z}$ in the minimal compactification $\mathcal{M}_F^{\Lambda *}$ of $\mathcal{M}_F^\Lambda$. The minimal compactification of Hilbert-Blumenthal moduli spaces can be constructed using the methods of [12] chap. 5; it is discussed in [4]. The boundary of $\mathcal{M}_F^{\Lambda *}$ consists of isolated points usually referred to as cusps. Clearly $\tilde{Z}$ is stable under all reduced prime-to-p F-Hecke correspondences. Because $\mathcal{M}_F^{\Lambda *}$ is proper over Spec, the subscheme $\tilde{Z}$ of $\mathcal{M}_F^\Lambda$ is proper over Spec if and only if its closure $\tilde{Z}^*$ in $\mathcal{M}_F^{\Lambda *}$ is equal to $\tilde{Z}$ itself. In other words, $\tilde{Z}$ is non-proper over Spec if and only if $\tilde{Z}^*$ contains a cusp. If $\tilde{Z}^*$ contains a cusp, then by a calculation similar to and simpler than that of Proposition 2, $\tilde{Z}$ is equal to $\mathcal{M}_F^\Lambda$, since $\mathcal{M}_F^\Lambda$ is irreducible and geometrically normal. Of course then $\tilde{Z}$ contains a supersingular point.

Assume now that $\tilde{Z}$ is proper over k. We can apply the main result of [10], since Λ contains separable polarizations, and all results in [10] remain true for separable polarizations and not just principal polarizations: the same proof works. The main result of [10] implies that there is a natural stratification of $\mathcal{M}_F^\Lambda$ which comes from the p-torsion points $A[p]$ of the universal abelian scheme A over $\mathcal{M}_F^\Lambda$, and each stratum is quasi-affine. Therefore unless $\tilde{Z}$ is 0-dimensional, it cannot be contained in the generic stratum. So $\tilde{Z}$ meets some stratum $\mathcal{S}_\alpha$ of lower type than the generic stratum if $\dim(\tilde{Z}) > 0$. Again $\tilde{Z} \cap \mathcal{S}_\alpha$ is stable under all reduced ℓ-power F-Hecke correspondences. If this intersection is not 0-dimensional, its closure meets some stratum $\mathcal{S}_\beta$ of lower type than $\mathcal{S}_\alpha$. So the intersection $\tilde{Z} \cap \mathcal{S}_\beta$ is non-empty. This argument can be continued, eventually we reach some stratum $\mathcal{S}_\gamma$ such that $\tilde{Z} \cap \mathcal{S}_\gamma$ is non-empty and 0-dimensional. Therefore $\tilde{Z}$ contains a point with finite reduced ℓ-power F-Hecke orbit. By Lemma 7, this points has to be supersingular. This proves Proposition 6. □

Proposition 7. *Let k be an algebraically closed field of characteristic $p > 0$. Let F be a totally real number field and let Λ be a projective $\mathcal{O}_F$-module of rank 1, with a notion of positivity. If $\tilde{Z}$ is the Zariski closure of the* *reduced prime-to-p F-Hecke orbit of an ordinary point of* $\mathcal{M}_F^A(k)$, *then* $\tilde{Z} = \mathcal{M}_F^A$.

Proof. Clearly $\tilde{Z}$ is reduced, and the smooth ordinary locus $\tilde{Z}_{sm}^{or}$ is open and dense in $\tilde{Z}$. Assume that $\tilde{Z}$ is not equal to $\mathcal{M}_F^A$. We want to get a contradiction.

By Proposition 6, $\tilde{Z}$ contains a supersingular point $x_s = [(A_s, \iota_s, h_s)] \in \mathcal{M}_F^A(k)$. As before let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be the prime ideals of $\mathcal{O}_F$ above p. The p-divisible group $A_s[p^\infty]$ of A_s decomposes into a direct sum

$$A_s[p^\infty] = A_s[\mathfrak{p}_1^\infty] \oplus \cdots \oplus A_s[\mathfrak{p}_r^\infty] \ ,$$

corresponding to the direct sum decomposition $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_{\mathfrak{p}_1} \oplus \cdots \oplus \mathcal{O}_{\mathfrak{p}_r}$. Each $A_s[\mathfrak{p}_i^\infty]$ is a p-divisible group with an action by $\mathcal{O}_{\mathfrak{p}_i}$, and $\dim(A_1[\mathfrak{p}_i^\infty]) = 2[F_{\mathfrak{p}_i} : \mathbb{Q}_p]$. By the Serre-Tate theorem, the formal completion of $\mathcal{M}_F^A$ at the point x_s is isomorphic to the universal deformation space of $(A_s[p^\infty], \iota_s)$ over k. In turn, this universal deformation space decomposes into the product of the universal deformation spaces of $(A_s[\mathfrak{p}_i^\infty], \iota_i)$ over k, where i ranges from 1 to r, and ι_i denotes the action of $\mathcal{O}_{\mathfrak{p}_i}$ on $A_s[\mathfrak{p}_i^\infty]$. Let $\mathfrak{D}_i$ denote the universal deformation spaces of $(A_s[\mathfrak{p}_i^\infty], \iota_i)$ over k. The formal completion $\tilde{Z}_{x_s}^\wedge$ of $\tilde{Z}$ at x_s is a closed formal subscheme of $\mathfrak{D} = \mathfrak{D} \times \cdots \times \mathfrak{D}_r$.

Consider $\mathrm{End}(A_s, \iota_s)$ and $\mathrm{End}^0(A_s, \iota_s) = \mathrm{End}_{\mathcal{O}_F}(A_s, \iota_s) \otimes_{\mathcal{O}_F} F$. By Lemma 6 iii, $B = \mathrm{End}^0(A_s, \iota_s)$ is a totally definite quaternion algebra over F which is unramified at all places of F which are prime to p, and for places $\mathfrak{p}$ of F above p, B is ramified at $\mathfrak{p}$ if $[F_\mathfrak{p} : \mathbb{Q}_p]$ is odd, unramified if $[F_\mathfrak{p} : \mathbb{Q}_p]$ is even. When tensored with $\mathbb{Q}_p$ the quaternion algebra B decomposes into a direct sum

$$B \otimes_{\mathbb{Q}} \mathbb{Q}_p = (B \otimes_F F_{\mathfrak{p}_1}) \oplus \cdots \oplus (B \otimes_F F_{\mathfrak{p}_r}) \ ,$$

each $B \otimes_F F_{\mathfrak{p}_i}$ is a quaternion algebra over $F_{\mathfrak{p}_i}$. Correspondingly we have a decomposition of the order $\mathrm{End}(A_s, \iota_s) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$:

$$\begin{aligned} \mathrm{End}(A_s, \iota_s) \otimes_{\mathbb{Z}} \mathbb{Z}_p &= (\mathrm{End}(A_s, \iota_s) \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}_1}) \oplus \cdots \oplus (\mathrm{End}(A_s, \iota_s) \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}_r}) \\ &= (\mathrm{End}(A_s[\mathfrak{p}_1^\infty], \iota_1) \oplus \cdots \oplus (\mathrm{End}(A_s[\mathfrak{p}_r^\infty], \iota_r) \end{aligned}$$

Let $B_1^\times$ be the group of elements of B with reduced norm 1. Similarly let $(B \otimes_{\mathbb{Q}} \mathbb{Q}_p)_1^\times$ (resp. $(B \otimes_F F_{\mathfrak{p}_i})_1^\times$) be the group of elements of $(B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ (resp. $(B \otimes_F F_{\mathfrak{p}_i})^\times$) with reduced norm 1. Let $B_{\mathrm{red}}^\times$ be the group of elements of $B^\times$ whose reduced norms are in $\mathbb{Q}^\times$. Let $U_p = (\mathrm{End}(A_s, \iota_s) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$, $U_{\mathfrak{p}_i} = (\mathrm{End}(A_s, \iota_s) \otimes_{\mathcal{O}_F} \mathcal{O}_{\mathfrak{p}_i})^\times = (\mathrm{End}(A_s[\mathfrak{p}_i^\infty], \iota_i)^\times$, $i = 1, \ldots, r$. Then $U_p = U_{\mathfrak{p}_1} \times \cdots \times U_{\mathfrak{p}_r}$. Each $U_{\mathfrak{p}_i}$ operates on $\mathfrak{D}_i$ 'by changing the marking of the closed fiber'. Similarly the group U_p acts on $\mathfrak{D}$, and is just the product of the actions of $U_{\mathfrak{p}_i}$ on $\mathfrak{D}_i$, $i = 1, \ldots, r$. Let $U_p^1 = U_p \cap (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)_1^\times$ and let $U_{\mathfrak{p}_i}^1 = U_{\mathfrak{p}_k} \cap (B \otimes_F f_{\mathfrak{p}_i})_1^\times$. Clearly

$$U_p^1 = U_{\mathfrak{p}_1}^1 \times \cdots \times U_{\mathfrak{p}_r}^1 .$$

The stabilizer subgroup for (A_s, ι_s) in the reduced prime-to-p F-Hecke correspondences contains those which come from $B^{\times}_{\mathrm{red}} \cap U_p$. Since $B^{\times}_{1} \cap U^1_p$ is dense in U^1_p by the weak approximation theorem, the closed subscheme $\tilde{Z}^{\wedge}_{x_s} \subseteq \mathfrak{D}$ is stable under the action of U^1_p.

The supersingular point x_s belongs to the closure of an irreducible component W of $\tilde{Z}^{or}_{sm}$. By Proposition 5, there exists a subset $\omega \subseteq \{1,\dots,r\}$ such that the tangent sheaf $T_W \subseteq T_{\mathcal{M}_F} \otimes_{\mathcal{O}_{\mathcal{M}_F}} \mathcal{O}_W$ of W is equal to

$$\bigoplus_{i \in \omega} T_{\mathcal{M}_F}(\mathfrak{p}_i) \otimes_{\mathcal{O}_{\mathcal{M}_F}} \mathcal{O}_W \ .$$

Since we assumed that $\tilde{Z}$ is not equal to $\mathcal{M}^{\Lambda}_F$, $\omega \neq \{1,\dots,r\}$.

Write $\mathfrak{D}_i = \mathrm{Spf}(R_i)$, $i = 1,\dots,r$, $\mathfrak{D} = \mathrm{Spf}(R_1 \hat{\otimes}_k \cdots \hat{\otimes}_k R_r)$, and $\tilde{Z}^{\wedge}_{x_s} = \mathrm{Spf}(R/I) \subset \mathrm{Spf}(R) = \mathfrak{D}$. There exists a formal curve $\xi : \mathrm{Spf}(k[[t]]) \to \tilde{Z}^{\wedge}_{x_s}$ which is generically ordinary. In other words the universal abelian variety over R, when base changed to $k((t))$ via $R \to k[[t]] \hookrightarrow k((t))$, gives an ordinary abelian variety over $k((t))$. Equivalently, if we write $\xi(t) = (\xi_1(t),\dots,\xi_r(t))$, where $\xi_i : \mathrm{Spf}(k[[t]]) \to \mathfrak{D}_i$ is the i-th component of ξ, then all $\xi_1(t),\dots,\xi_r(t)$ are generically ordinary. Moreover, we may assume that ξ gives a map from $\mathrm{Spec}((t))$ to $\tilde{Z}^{or}_{sm}$. In other words the jacobian criterion for (R,I) is satisfied for the homomorphism $R \to k((t))$ and ξ is generically ordinary.

Pick any element $j \in \{1,\dots,r\}$, $j \notin \omega$. For any element $u_j \in U^1_{\mathfrak{p}_j} \subseteq U_p$, $u_j \cdot \xi$ is again a formal curve in Z. For any $i \neq j$, the i-th component of $u_j \cdot \xi$ is equal to ξ_i. If u_j is close to the identity, then the j-th component of $u_j \cdot \xi$ is close to ξ_j in the sense that these two homomorphisms from R_j to $k[[t]]$ are congruent modulo high powers of t. On the other hand for u_j outside of a subalgebra of $B \otimes_F F_{\mathfrak{p}_j}$ of dimension at most 2 over $F_{\mathfrak{p}_j}$, the j-th component $(u_j \cdot \xi)_j$ is not equal to ξ_j: Since ξ is generically ordinary, the endomorphism algebra of the p-divisible group with $\mathcal{O}_{\mathfrak{p}_j}$-multiplication $(A_s[\mathfrak{p}_j^{\infty}], \iota_j) \otimes_{R_j,\xi_j} k((t))$ has rank at most 2 over $\mathcal{O}_{\mathfrak{p}_j}$. So there exist elements $u_j \in U^1_{\mathfrak{p}_j}$ such that $(u_j \cdot \xi)_j$ is not equal to ξ_j, while congruent to ξ_j to arbitrarily high power of t. This implies that at the $k((t))$-valued point $R \to k((t))$ of W the tangent space of W contains some element with non-zero $\mathfrak{p}_j$-component. This can be checked using the Taylor expansion for instance. By Proposition 5, we must have $j \in \omega$. This is a contradiction. Proposition 7 is proved. □

Remark. In the proof of Proposition 7 we have used very little information on the action of the automorphism group U_p of the closed fiber on the deformation space $\mathfrak{D}$ at a supersingular point x_s. Ideally we should be able to get much more results on this action, and prove an analogue of question (Q 2) in this setting.

Theorem 1. *Let k be an algebraically closed field of characteristic $p > 0$. Let F be a totally real number field and let Λ be a projective $\mathcal{O}_F$-module of rank 1, with a notion of positivity. If $\tilde{Z}_\ell$ is the Zariski closure of the reduced ℓ-power F-Hecke orbit of an ordinary point of $\mathcal{M}^{\Lambda}_F(k)$, then $\tilde{Z}_\ell = \mathcal{M}^{\Lambda}_F$.*

Proof. First we show that $\tilde{Z}_\ell \subseteq \mathcal{M}_F^A$ is stable under all reduced prime-to-p Hecke correspondences. By Proposition 6, $\tilde{Z}_\ell$ contains a supersingular point $x_s = [(A_s, \iota_s, h_s)]$ in $\mathcal{M}_F^A(k)$. We shall use the notation in the proof of Proposition 7. Especially $B = \mathrm{End}^0(A_s, \iota_s)$ is a totally definite quaternion algebra over F. Let $\mathcal{O}_B = \mathrm{End}(A_s, \iota_s)$, an order in B. Let G_{red} be the algebraic group over $\mathbb{Q}$ whose $\mathbb{Q}$-rational points consists of elements in $B^\times$ with reduced norm in $\mathbb{Q}^\times$. For any prime number ℓ', $G_{\mathrm{red}}(\mathbb{Q}_{\ell'})$ consist of elements of $(B \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell'})^\times$ with reduced norm in $\mathbb{Q}_{\ell'}^\times$. Let $K_{\ell'} = G_{\mathrm{red}}(\mathbb{Q}_{\ell'}) \cap (\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell'})^\times$, a compact open subgroup of $G_{\mathrm{red}}(\mathbb{Q}_{\ell'})$. The derived group $G_{\mathrm{red}}^{\mathrm{der}}$ of G_{red} is simply connected and $\mathbb{Q}$-simple. Its group of real points $G_{\mathrm{red}}^{\mathrm{der}}(\mathbb{R})$ is compact, while for any prime number $\ell' \neq p$ the group $G_{\mathrm{red}}^{\mathrm{der}}(\mathbb{Q}_{\ell'})$ is noncompact. The quotient $G_{\mathrm{red}}/G_{\mathrm{red}}^{\mathrm{der}}$ is isomorphic to $\mathbb{G}_m$ via the reduced norm map. By the strong approximation theorem, $G_{\mathrm{red}}(\mathbb{Q}) \cdot G_{\mathrm{red}}(\mathbb{Q}_\ell)$ is dense in $G_{\mathrm{red}}(\mathbb{A}_{\mathbb{Q}})$. Especially $G_{\mathrm{red}}(\mathbb{Q}) \cap \prod_{\ell' \neq \ell} K_{\ell'}$ is dense in K_p. Since $G_{\mathrm{red}}(\mathbb{Q}) \cap \prod_{\ell \neq \ell'} K_{\ell'}$ is contained in the stabilizer subgroup of x_s, the formal completion $Z_{\ell,x_s}^\wedge$ of Z_ℓ at x_s is stable under the action of K_p. The strong approximation theorem also implies that the reduced prime-to-p F-Hecke orbit of x_s is equal to the reduced ℓ-power F-Hecke orbit of x_s, since $G_{\mathrm{red}}(\mathbb{Q}) \cdot G_{\mathrm{red}}(\mathbb{Q}_\ell)$ is dense in $G_{\mathrm{red}}(\mathbb{A}_{f,\mathbb{Q}})$. Therefore for any reduced prime-to-p F-Hecke correspondence γ, the union $\tilde{Z}_\ell \cup \gamma(\tilde{Z}_\ell)$ of $\tilde{Z}_\ell$ with the image of $\tilde{Z}_\ell$ under γ coincides with $\tilde{Z}_\ell$ at the formal completion of the supersingular points of $\tilde{Z}_\ell \cup \gamma(\tilde{Z}_\ell)$. Clearly $\tilde{Z}_\ell \cup \gamma(\tilde{Z}_\ell)$ is stable under all reduced prime-to-ℓ F-Hecke correspondences of $\mathcal{M}_F^A$. So $\tilde{Z}_\ell \cup \gamma(\tilde{Z}_\ell)$ is the union of $\tilde{Z}_\ell$ with a closed subscheme W_γ of $\mathcal{M}_F^A$, such that W_γ does not contain supersingular points and is stable under all reduced ℓ-power F-Hecke correspondences on $\mathcal{M}_F^A$. Hence W_ℓ is empty by Proposition 6. This shows that $\tilde{Z}_\ell$ is stable under all reduced prime-to-p F-Hecke correspondences on $\mathcal{M}_F^A$. Therefore $\tilde{Z}_\ell = \mathcal{M}_F^A$ by Proposition 7. Theorem 1 is proved. □

Clearly Proposition 3 and Theorem 1 imply

Theorem 2. *Let $d_1, \ldots, d_g$ be positive integers such that $d_1 | \cdots | d_g$, and let $\delta = (d_1, \ldots, d_g)$. Let k be an algebraically closed field of characteristic $p > 0$. Then for any prime $\ell \neq p$, the ℓ-power Hecke orbit of any ordinary point of $\mathcal{A}_{g,\delta}$ is Zariski dense in $\mathcal{A}_{g,\delta}$. In particular, the ℓ-power Hecke orbit of any ordinary point of $\mathcal{A}_g$ is Zariski dense in $\mathcal{A}_g$.*

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