Character sums, automorphic forms, equidistribution, and Ramanujan graphs

Part II. Eigenvalues of Terras graphs[1]

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Abstract

We study two types of character sums related to Terras graphs using the method of ℓ-adic cohomology. These character sums also arise as traces of Frobenii of some two dimensional linear representations of a global function field. Detailed information about these Galois representations at ramified places are obtained from analysis of vanishing cycles. Consequently we give a complete description of the automorphic forms of which these character sums appear as Fourier coefficients. These character sums are shown to be equidistributed with respect to the Sato-Tate measure.

§1. Introduction

Let $\mathbb{F}$ be a finite field with q elements. For convenience we assume its characteristic p is odd although similar results for $p = 2$ also hold. Denote by $\mathbb{F}'$ a quadratic extension of $\mathbb{F}$. Choosing a nonsquare element $\delta \in \mathbb{F}$, we embed the multiplicative group of $\mathbb{F}'$ into $GL_2(\mathbb{F})$ as the subgroup K_δ consisting of matrices $\begin{pmatrix} a & b\delta \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{F}$. The coset space $GL_2(\mathbb{F})/K_\delta$ may be represented by the subgroup

$$H = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : y \in \mathbb{F}^\times, x \in \mathbb{F} \right\},$$

which resembles the classical Poincaré upper-half plane. Let S be a K_δ-double coset of $GL_2(\mathbb{F})$ with cardinality greater than that of K_δ. It can be shown that S has coset representatives $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, where (x, y) runs through all $\mathbb{F}$-rational points of an ellipse $x^2 = \delta(ay + (y-1)^2)$ for some $a \in \mathbb{F}$. Further, there are $q - 2$ such double cosets, parametrized by the elements a in $\mathbb{F}$ other than 0 and 4, which we denote by $S_{a\delta}$.

The Terras graph X_a is the Cayley graph $Cay(GL_2(\mathbb{F})/K_\delta, S_{a\delta}/K_\delta)$. Different choices of δ result in isomorphic graphs. It is a $(q+1)$-regular graph whose eigenvalues can be explicitly expressed in terms of character sums of the following two types. The first type is associated to multiplicative characters χ of $\mathbb{F}$:

$$\lambda_{a,\chi} = \sum_{\substack{x,y \in \mathbb{F} \\ \delta(ax+(x-1)^2)=y^2}} \chi(x),$$

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while the second type is associated to *regular* multiplicative characters ω of $\mathbb{F}'$, that is, characters which are nontrivial on the kernel of Nm:

$$\lambda_{a,\omega} = \sum_{z \in \mathbb{F}',\, \mathrm{Nm}(z)=1} \varepsilon(a - 2 - \mathrm{Tr}(z))\omega(z),$$

where $\mathrm{Tr}(z)$ and $\mathrm{Nm}(z)$ are the trace and norm of z from $\mathbb{F}'$ to $\mathbb{F}$, and $\varepsilon(x)$ is 1, 0, -1 according to $x \in (\mathbb{F}^\times)^2, x = 0$ or $x \in \mathbb{F} \smallsetminus \mathbb{F}^2$. We remark that $\lambda_{a,\chi}$ is independent of the choice of δ. Using the Riemann hypothesis for curves, one can show that the nontrivial eigenvalues have absolute value majorized by $2\sqrt{q}$, hence the Terras graphs X_a are Ramanujan graphs. The reader is referred to [2], [1], and Chap 9 of [9] for more details on Terras graphs.

As in Part I, let H be the quaternion algebra over the rational function field K with the field of constants $\mathbb{F}$ which is ramified only at 0 and ∞, let D be the multiplicative group of H divided by its center, and let $X_{\mathcal{K}}$ be the Ramanujan graph with vertices the double coset space $D(K)\backslash D(A_K)/D(K_\infty)\mathcal{K}$ for some congruence subgroup $\mathcal{K}$ of $\prod_{v \neq \infty} D(\mathcal{O}_v)$ and the edges inherited from the tree structure of $D(K_0)/D(\mathcal{O}_0)$ at place 0. Here $D(\mathcal{O}_v)$ is defined by a fixed maximal order of the quaternion algebra H. The connection between Terras graphs and Ramanujan graphs constructed by Morgenstern [11] based on D was proved in [10] as follows.

(1.1) Proposition *For $b \in \mathbb{F}, b \neq 0, 1$, the Terras graph $X_{4(b-1)/b}$ is a quotient of the Morgenstern graph $X_{\mathcal{K}_b}$ with $\mathcal{K}_b = \kappa_b \prod_{v \neq b,\infty} D(\mathcal{O}_v)$, where κ_b is the compact subgroup of $D(\mathcal{O}_b)$ consisting of all elements in $D(\mathcal{O}_b)$ congruent to the identity element modulo $t - b$.*

Therefore the eigenvalues $\lambda_{4(b-1)/b,\chi}$ and $\lambda_{4(b-1)/b,\omega}$ of Terras graphs are among the eigenvalues of the Hecke operator T_0 at place 0 acting on automorphic forms on the double coset space $X_{\mathcal{K}_b}$. Two questions arise naturally :

(1.2) Questions (i) Find automorphic forms on $X_{\mathcal{K}_b}$ whose Fourier coefficients are eigenvalues of Terras graphs.

(ii) Find the distribution of eigenvalues of Terras graphs.

(1.3) Let K be a (not necessarily rational) function field with $\mathbb{F}$ as its field of constants. The purpose of this paper is to construct automorphic forms of GL_2 over K whose Fourier coefficients are given by the two types of eigenvalues of Terras graphs, analogous to what we did for norm graphs in Part I. These forms are parametrized by nonzero elements in K. We also show that the Sato-Tate conjecture holds for these forms when the parameter is not a constant. In particular, when K is the rational function field $\mathbb{F}(t)$, for special choices of the parameter, we compute the associated L-functions explicitly and show that they are also L-functions attached to automorphic forms on $X_{\mathcal{K}_b}$ via Jacquet-Langlands correspondence. This answers question (i). For question (ii), the numerical data given by Terras in [12] suggests that the normalized eigenvalues of Terras graphs are uniformly distributed with respect to the Sato-Tate measure. We give a theoretic explanation of this phenomenon, as a consequence of the stronger result that the relevant automorphic forms satisfy the Sato-Tate conjecture.

Our approach is geometrical, using the theory of ℓ-adic cohomology. We study character sums of the second type in section 2 and the first type in section 3, by investigating the actions of $\mathrm{Gal}(K^{sep}/K)$ on certain sheaves, computing vanishing cycles and local factors at bad places, and determining the geometric monodromy group. The reformulation of the results in terms of automorphic forms is given in section 4, where applications to Terras graphs are also explored.

For readers primarily interested in the aspects of automorphic forms and graph theory, § 2 and § 3 may look overly technical. Here is an outline of how one uses informations from geometry. Once one knows that a family of character sum comes from a rank-two smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{M}$ *of geometric origin* on an algebraic curve C over a finite field , one deduces from Weil's converse theorem that the L-function attached to this family of character sum comes from an automorphic representation π of $\mathrm{GL}(2)$. If the neutral component of the geometric monodromy group of $\mathcal{M}$ is $\mathrm{SL}(2)$, then the automorphic representation π is cuspidal, and the eigenvalues of Hecke operators are equidistributed according to the Sato-Tate law. The weight information about $\mathcal{M}$ gives estimates of Hecke eigenvalues at unramified places. If one knows that there are at least two ramified places where the local components of π are either special or supercuspidal, then the automorphic representation π can be obtain from from an automorphic representation of a quaternion division algebra over the function field of C via the Jacquet-Langlands correspondence. Our calculation of the vanishing cycles in §§2–3 provides detailed information about the local Galois representations at the ramified places, which are local Langlands parameters of the ramified components of π. Readers who are more geometrically inclined may regard §§2–3 as examples in the theory of ℓ-adic cohomology whose singularities are explicitly computable.

§2. A family of character sums of the second type

The goal of this section is to study the family of exponential sums of the second type, also known as "Soto-Andrade type" in the literature,

$$SA_{b,\varepsilon,\omega} := \sum_{u\in U(\mathbb{F})} \varepsilon(\mathrm{Tr}(u)+b)\,\omega(u)\,,$$

where b is an element of $\mathbb{F}$, $U(\mathbb{F})$ is the norm-one subgroup of the quadratic extension $\mathbb{F}'$ of $\mathbb{F}$, and ω (resp. ε) is a character of $U(\mathbb{F})$ (resp. $\mathbb{F}^\times$).

(2.1) In this section, we follow the notations of [7] closely.

- Let $\mathbb{F}$ be a finite field with q elements, where q is a power of an *odd* prime number p. Let $\mathbb{F}'$ be a quadratic extension field of $\mathbb{F}$.
- Let $\underline{\mathbb{F}'^\times} = \mathrm{Res}_{\mathbb{F}'/\mathbb{F}}\mathbb{G}_m$ (resp. $\underline{\mathbb{F}'} = \mathrm{Res}_{\mathbb{F}'/\mathbb{F}}\mathbb{G}_a$) be the Weil restriction of scalars of $\mathbb{G}_m$ (resp. $\mathbb{G}_a$) for the extension $\mathbb{F}'/\mathbb{F}$. For any $\mathbb{F}$-algebra R, we have $\underline{\mathbb{F}'^\times}(R) = (\mathbb{F}'\otimes_{\mathbb{F}} R)^\times$ and $\underline{\mathbb{F}'}(R) = \mathbb{F}'\otimes_{\mathbb{F}} R$. Let $\mathrm{Nm} : \underline{\mathbb{F}'^\times} \to \mathbb{G}_m$ and $\mathrm{Tr} : \underline{\mathbb{F}'} \to \mathbb{G}_a$ be the $(\mathbb{F}'\otimes_{\mathbb{F}} R/R)$-norm and the $(\mathbb{F}'\otimes_{\mathbb{F}} R/R)$-trace respectively.
- Denote by U the kernel of $\mathrm{Nm} : \underline{\mathbb{F}'^\times} \to \mathbb{G}_m$; it is a one-dimensional torus over $\mathbb{F}$ such that for every $\mathbb{F}$-algebra R, $U(R)$ consists of all elements $u \in (\mathbb{F}'\otimes_{\mathbb{F}} R)^\times$ with $\mathrm{Nm}(u) = 1$. Especially $U(\mathbb{F})$ is the norm-one subgroup of $\mathbb{F}'$.
- Let $\omega : U(\mathbb{F}) \to \overline{\mathbb{Q}_\ell}^\times$ be a character of $U(\mathbb{F})$, and let $\varepsilon : \mathbb{F}^\times \to \overline{\mathbb{Q}_\ell}^\times$ be a character of $\mathbb{F}^\times$.

- Let $\mathcal{L}_\omega$ be the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf over U, given by the push-out of the Lang torsor $\mathrm{Id}\cdot \mathrm{Fr}_q{}^{-1} : U \to U$ by ω. For every closed point x of U, the geometric Frobenius Fr_x acts on the geometric generic fiber $\mathcal{L}_{\omega,\bar\eta}$ via the scalar $\omega(\mathrm{Nm}_{\mathbb{F}(x)/\mathbb{F},U}(x))$.
- Let $\mathcal{L}_\varepsilon$ be the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{G}_m$ obtained from the Lang torsor construction, using the character ε of $\mathbb{F}^\times = \mathbb{G}_m(\mathbb{F})$.
- Let $\tilde\varepsilon$ be the composition of $\mathrm{Nm}_{\mathbb{F}'/\mathbb{F},\mathbb{G}_\mathrm{m}} : \mathbb{F}'^\times \to \mathbb{F}^\times$ with $\varepsilon : \mathbb{F}^\times \to \overline{\mathbb{Q}_\ell}^\times$. Let $\tilde\omega$ be the composition of $\mathrm{Nm}_{\mathbb{F}'/\mathbb{F},U} : U(\mathbb{F}') \to U(\mathbb{F})$ with $\omega : U(\mathbb{F}) \to \overline{\mathbb{Q}_\ell}^\times$.
- Denote by Y the scheme $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{A}^1[\frac{1}{\mathrm{Tr}+t}]$, where t is the regular function on $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{A}^1$ which corresponds to the projection $\mathrm{pr}_2 : U \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{A}^1 \to \mathbb{A}^1$.
- Denote by $\mathcal{L}_{\varepsilon(\mathrm{Tr}+t)}$ the pull-back of $\mathcal{L}_\varepsilon$ by $\mathrm{Tr}+t : Y \to \mathbb{G}_m$.
- Denote by $\mathcal{F} = \mathcal{F}_{\varepsilon,\omega}$ the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_{\varepsilon(\mathrm{Tr}+t)} \otimes \mathrm{pr}_1^*\mathcal{L}_\omega$ on Y. For every closed point x of Y, the geometric Frobenius Fr_x acts on the geometric generic fiber $\mathcal{F}|_{\bar\eta}$ of $\mathcal{F}_{\varepsilon,\omega}$ via the scalar $\varepsilon\left(\mathrm{Nm}_{\mathbb{F}(x)/\mathbb{F},\mathbb{G}_\mathrm{m}}(\mathrm{Tr}(u)+b)\right)\cdot\omega\left(\mathrm{Nm}_{\mathbb{F}(x)/\mathbb{F},U}(u)\right)$, where (u,b) is the tautological point of $Y(\mathbb{F}(x))$ given by x, and $\mathrm{Tr}(u) \in \mathbb{G}_a(\mathbb{F}(x))$ is the image of u under $\mathrm{Tr} : U \to \mathbb{G}_a$.
- Denote by $\pi : Y \to \mathbb{A}^1$ the composition of the inclusion $Y \hookrightarrow U \times \mathbb{A}^1$ and the projection $U \times \mathbb{A}^1 \to \mathbb{A}^1$.

The theory of ℓ-adic cohomology yields the relative cohomology sheaves with compact support $\mathrm{R}^i\pi_!\mathcal{F}$ on $\mathbb{A}^1$; the alternating sum of the trace of the action of the geometric Frobenii of a closed point of $\mathbb{A}^1$ on these cohomology sheaves is a character sum of Soto-Andrade type. The following properties on sheaves are immediate from the definition.

(2.1.1) Lemma (i) *The sheaf $\mathcal{F}_{\varepsilon^{-1},\omega^{-1}}$ is naturally isomorphic to the dual of $\mathcal{F}_{\varepsilon,\omega}$.*

(ii) *The pull-back and the push-forward of the the sheaf $\mathcal{F}_{\varepsilon,\omega}$ under the involution $u \mapsto u^{-1}$ of U are both naturally isomorphic to $\mathcal{F}_{\varepsilon,\omega^{-1}}$.*

In the rest of this section we will analyze the sheaves $\mathrm{R}^i\pi_!\mathcal{F}$ on $\mathbb{A}^1$, and will obtain information on the action of the decomposition group at points where the sheaf $\mathrm{R}^i\pi_!\mathcal{F}$ on $\mathbb{A}^1$ is ramified. These information are sufficient to determine local factors of the L-function attached to $\mathcal{F}$ at the ramified points of $\mathrm{R}^i\pi_!\mathcal{F}$ on $\mathbb{A}^1$.

(2.2) Since the torus U splits over $\mathbb{F}'$, it is convenient to extend the base field from $\mathbb{F}$ to $\mathbb{F}'$. The scheme $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$ is isomorphic to $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{F}'[z,z^{-1}]$. The regular function z on $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$ is the character of U which induces the identity when restricted to $U(\mathbb{F}) \subset \mathbb{F}'^\times$. Moreover we have

$$Y \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}' \cong \mathbb{G}_m \times_{\mathrm{Spec}\,\mathbb{F}'} \mathbb{A}^1\left[\frac{1}{z+z^{-1}+t}\right] = \mathbb{G}_m \times_{\mathrm{Spec}\,\mathbb{F}'} \mathbb{A}^1\left[\frac{1}{z^2+tz+1}\right],$$

since z is an invertible function on $Y \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$.

We identify $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$ with $\mathbb{G}_\mathrm{m}$ over $\mathbb{F}'$ using the regular function z, and consequently we identify $\tilde\omega$ with the character

$$\tilde\omega : (\mathbb{F}')^\times \to \overline{\mathbb{Q}_\ell}^\times, \qquad x \mapsto \tilde\omega(x) = \omega(x/\tau(x))$$

on $(\mathbb{F}')^\times$, where τ denotes the non-trivial element of $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$.

Over $Y_{/\mathbb{F}'} = \operatorname{Spec} \mathbb{F}'[z, z^{-1}, t, (z^2 + tz + 1)^{-1}]$, the sheaf $\mathcal{F}$ becomes

$$\mathcal{L}_{\tilde{\varepsilon}(z+z^{-1}+t)\,\tilde{\omega}(z)} = \mathcal{L}_{\tilde{\varepsilon}(z+z^{-1}+t)} \otimes \mathrm{pr}_1^* \mathcal{L}_{\tilde{\omega}} \,.$$

(2.3) Proposition *Assume either that the character ε^2 of $\mathbb{F}^\times$ is not trivial, or that the character ω^2 of $U(\mathbb{F})$ is not trivial.*

(i) *If $i \neq 1$, then $\mathrm{R}^i\pi_!\mathcal{F}|_{\mathbb{A}^1} = 0$.*

(ii) *The restriction $\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}$ of $\mathrm{R}^1\pi_!\mathcal{F}$ to the open subscheme $\mathbb{A}^1 - \{2, -2\}$ of $\mathbb{A}^1$ is a smooth $\overline{\mathbb{Q}_\ell}$-sheaf of rank two over $\mathbb{A}^1 - \{2, -2\}$.*

(iii) *The stalks of $\mathrm{R}^1\pi_!\mathcal{F}$ at a geometric point $\bar{b} = \pm 2$ of $\mathbb{A}^1$ is a one-dimensional vector space over $\overline{\mathbb{Q}_\ell}$.*

(iv) *Assume moreover that the character ε of $\mathbb{F}^\times$ is not trivial. Then the smooth rank-two $\overline{\mathbb{Q}_\ell}$-sheaf $\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}$ is pure of weight one.*

(v) *The sheaf $\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon,\omega}|_{\mathbb{A}^1-\{2,-2\}}$ is naturally isomorphic to $\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon,\omega^{-1}}|_{\mathbb{A}^1-\{2,-2\}}$.*

(vi) *If the character ε of $\mathbb{F}^\times$ is non-trivial, then the dual of $\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon,\omega}|_{\mathbb{A}^1-\{2,-2\}}$ is naturally isomorphic to $\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon^{-1},\omega^{-1}}|_{\mathbb{A}^1-\{2,-2\}}(1)$. If ε has order two, then $\det\big(\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon,\omega}|_{\mathbb{A}^1-\{2,-2\}}\big) = \overline{\mathbb{Q}_\ell}(-1)$.*

Proof. This proposition is proved in [7, Thm. 1]. We reproduce the proof for the convenience of the reader.

Some remark on the hypothesis is in order. The hypothesis on ε and ω means that the characters $\tilde{\varepsilon}^2$ and $\tilde{\omega}^2$ of $(\mathbb{F}')^\times$ cannot be both trivial, or equivalently that the characters $\tilde{\varepsilon} \cdot \tilde{\omega}$ and $\tilde{\varepsilon} \cdot \tilde{\omega}^{-1}$ of $(\mathbb{F}')^\times$ cannot both be trivial. Since the order of ε divides $q - 1$, and the order of ω divides $q + 1$, $\tilde{\varepsilon} = \tilde{\omega}$ if and only if $\tilde{\varepsilon} = \tilde{\omega}^{-1}$.

We claim that the above hypothesis on the characters ε, ω implies that for any geometric point $\bar{b} \in \mathbb{A}^1$, the sheaf $\mathcal{F}|Y_{\bar{b}}$ on $Y_{\bar{b}} \subset \mathbb{G}_{\mathrm{m}} \subset \mathbb{P}^1$ is ramified at both 0 and ∞. Consider first the point ∞. We have

$$\mathcal{F}|_{Y_{\bar{b}}} = \mathcal{L}_{\tilde{\varepsilon}(1+\frac{\bar{b}}{z}+\frac{1}{z^2})} \otimes \mathcal{L}_{(\tilde{\varepsilon}\cdot\tilde{\omega})(z)} \,.$$

Moreover $\mathcal{L}_{\tilde{\varepsilon}(1+\frac{\bar{b}}{z}+\frac{1}{z^2})}$ is unramified at ∞. Hence $\mathcal{F}|_{Y_{\bar{b}}}$ is unramified at ∞ if and only if $\tilde{\varepsilon} \cdot \tilde{\omega}$ is equal to the trivial character of $(\mathbb{F}')^\times$, which is ruled out by our hypothesis. Similarly we conclude from

$$\mathcal{F}|_{Y_{\bar{b}}} = \mathcal{L}_{\tilde{\varepsilon}(z^2+\bar{b}z+1)} \otimes \mathcal{L}_{(\tilde{\varepsilon}^{-1}\cdot\tilde{\omega})(z)}$$

that $\mathcal{F}|_{Y_{\bar{b}}}$ is unramified at $0 \in \mathbb{P}^1$ if and only if $\tilde{\varepsilon}^{-1} \cdot \tilde{\omega}$ is equal to the trivial character of $(\mathbb{F}')^\times$, which again is ruled out by our hypothesis. The claim is proved.

By the proper base change theorem, the first statement that $\mathrm{R}^i\pi_!\mathcal{F} = 0$ for $i \neq 1$ can be checked fiber by fiber. Over the fiber $Y_{\bar{b}}$ of a geometric point $\bar{b} \in \mathbb{A}^1$, the sheaf $\mathcal{F}|_{Y_{\bar{b}}}$ can be thought of as a sheaf on $U \times_{\operatorname{Spec} \mathbb{F}} \operatorname{Spec} \mathbb{F}^{\mathrm{sep}}$. The sheaf $\mathcal{F}|_{Y_{\bar{b}}}$ is ramified at the zero locus of $\mathrm{Tr} + \bar{b}$ in $U \times_{\operatorname{Spec} \mathbb{F}} \operatorname{Spec} \mathbb{F}^{\mathrm{sep}}$, if ε is not trivial. This zero locus is finite of degree two over $\bar{b}$, and is finite étale of degree two over $\bar{b}$ if $\bar{b} \neq \pm 2$.

The cohomology group $\mathrm{H}^0_c(Y_{\bar b}, \mathcal{F}|_{Y_{\bar b}})$ vanishes because $Y_{\bar b}$ is not complete. The cohomology group

$$\mathrm{H}^2_c(Y_{\bar b}, \mathcal{F}|_{Y_{\bar b}}) \cong \left(\mathcal{F}_{\overline{\eta_{Y_{\bar b}}}}\right)_{\pi_1^{\mathrm{geom}}(Y_{\bar b})}(-1)$$

vanishes because $\mathcal{F}_{Y_{\bar b}}$ is ramified at both $z = 0$ and $z = \infty$.

The statement that the stalks of $\mathrm{R}^1\pi_!\mathcal{F}$ over the open subscheme $\mathbb{A}^1 - \{2, -2\}$ are two-dimensional vector spaces over $\overline{\mathbb{Q}_\ell}$ is a consequence of Grothendieck's Euler-Poincaré characteristic formula. The point is that the sheaf $\mathcal{L}_\varepsilon$ on $\mathbb{G}_m$ is tamely ramified at 0 and ∞, and the sheaf $\mathcal{L}_\omega$ on U is also tamely ramified at the complement of U in the smooth projective completion of U. Hence the sheaf $\mathcal{F}|_{Y_{\bar b}}$ is tamely ramified at the complement of $Y_{\bar b}$ in the smooth projective completion of $Y_{\bar b}$. Suppose that $\bar b$ is a geometric point of $\mathbb{A}^1 - \{2, -2\}$. The sheaf $\mathcal{F}|_{Y_{\bar b}}$ being tamely ramified, we have

$$\chi_c(Y_{\bar b}, \mathcal{F}|_{Y_{\bar b}}) = \chi_c(Y_{\bar b}, \overline{\mathbb{Q}_\ell}) = \chi_c(U, \overline{\mathbb{Q}_\ell}) - 2 = -2\,.$$

Therefore $\dim_{\overline{\mathbb{Q}_\ell}}(\mathrm{H}^1(Y_{\bar b}, \mathcal{F}|_{Y_{\bar b}})) = 2$. This argument actually shows that for any geometric point $\bar b \in \mathbb{A}^1(\mathbb{F}^{\mathrm{sep}})$,

$$-\chi_c\left(Y_{\bar b}, \mathcal{F}|Y_{\bar b}\right) = -\chi_c\left(Y_{\bar b}, \overline{\mathbb{Q}_\ell}\right)$$

is equal to the number of zeros of the polynomial $Z^2 + \bar b Z + 1$ in $(\mathbb{F}^{\mathrm{sep}})^\times$, counted *without* multiplicity. In particular if $\bar b = \pm 2$, then

$$-\chi_c\left(Y_{\bar b}, \mathcal{F}\right) = 1 = \dim_{\overline{\mathbb{Q}_\ell}} \mathrm{H}^1(Y_{\bar b}, \mathcal{F}|_{Y_{\bar b}})\,.$$

We have proved statement (iii).

The smoothness of the $\overline{\mathbb{Q}_\ell}$-sheaf $\mathrm{R}^1\pi_!\mathcal{F}$ on Y follows from [8, Cor. 2.1.2]: It suffices to verify the smoothness after base change to $\mathbb{F}'$. The scheme $Y \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'|_{\mathbb{A}^1-\{2,-2\}}$ is an open subscheme of $\mathbb{P}^1 \times (\mathbb{A}^1 - \{2, -2\})$, whose complement Z is finite étale over $\mathbb{A}^1 - \{2, -2\}$ of degree 4. Moreover the sum of the Swan conductor at the "missing points" is constant (being zero) for every point b in the base $\mathbb{A}^1 - \{2, -2\}$. Therefore the theorem [8, Thm. 2.1.1] of Deligne applies.

In order to prove (iv), we may and do extend the base field from $\mathbb{F}$ to $\mathbb{F}'$, and we identify $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$ with $\mathbb{G}_\mathrm{m}$ as in 2.2. Let $\bar b$ be a geometric point over a closed point b of $\mathbb{A}^1 - \{2, -2\}$. Let j_b be the injection of $\mathbb{G}_m\left[\frac{1}{z^2+bz+1}\right] = Y_b$ into $\mathbb{P}^1$. The assumption that ε is non-trivial implies that $\mathcal{F}|Y_{\bar b}$ is ramified at the zero locus of $z^2 + \bar b z + 1$ in $\mathbb{G}_\mathrm{m}$, and we have seen that $\mathcal{F}|Y_{\bar b}$ is ramified at 0 and ∞. Hence

$$j_{b*}\mathcal{F}|_{Y_b} = j_{b!}\mathcal{F}|_{Y_b}\,.$$

for any closed point b of $\mathbb{A}^1 - \{2, -2\}$. The statement (iv) follows from [3, Thm. 3.2.3].

The statement (v) follows immediately from Lemma 2.1.1 (ii). By [4, dualité, Thm. 1.3, 2.1], we have a perfect pairing

$$\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon,\omega} \times \mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon^{-1},\omega^{-1}} \to \mathrm{R}^2\pi_!\overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}(-1)$$

induced by the cup product. When ε is the character of order two of $\mathbb{F}^\times$, this pairing is isomorphic to the determinant map for $\mathrm{R}^1\pi_!\mathcal{F}_{\varepsilon,\omega}$. This proves (vi). ∎

We would like to compute the Swan conductor of the smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}$ over $\mathbb{A}^1-\{2,-2\}$ at the "missing points" $\{2,-2,\infty\}\subset\mathbb{P}^1$. Our strategy is to compute the cohomology of $\mathcal{F}$ using the "first projection" $f:Y\to U$ of $Y\subset U\times_{\mathrm{Spec}\,\mathbb{F}}\mathbb{A}^1$.

(2.4) Lemma *Let $f:Y\to U$ be the morphism given by the restriction of the projection $U\times_{\mathrm{Spec}\,\mathbb{F}}\mathbb{A}^1\supset Y$ to the first factor U. Then $\mathrm{R}^if_!\mathcal{F}=(0)$ for all $i\geq 0$.*

Proof. For any point $a\in U$, the fiber $f^{-1}(a)\subset\mathbb{A}^1$ is equal to $\mathrm{Spec}\,\mathbb{F}(a)\left[t,\frac{1}{\mathrm{Tr}(a)+t}\right]$. Hence the complement W of Y in $U\times_{\mathrm{Spec}\,\mathbb{F}}\mathbb{P}^1$ is finite étale over U of degree 2. Moreover by the same argument used in the proof of 2.3, the smooth $\overline{\mathbb{Q}_\ell}$ sheaf $\mathcal{F}$ is tamely ramified along the relative Cartier divisor $W\subset U\times_{\mathrm{Spec}\,\mathbb{F}}\mathbb{P}^1$. The Euler-Poincaré characteristic formula gives

$$\chi_c\left((f^{-1}(\bar b),\mathcal{F}|_{f^{-1}(\bar b)}\right)=\chi_c\left(f^{-1}(\bar b),\overline{\mathbb{Q}_\ell}\right)=0$$

for any geometric point b of U. On the other hand, the smooth sheaf $\mathcal{F}|_{f^{-1}(\bar b)}$ is *ramified* at the divisor $W|_{f^{-1}(\bar b)}\subset\mathbb{P}^1$ of degree two, hence neither $\mathcal{F}|_{f^{-1}(\bar b)}$ nor its dual can be geometrically constant. Therefore

$$\mathrm{H}^0_c\left((f^{-1}(\bar b),\mathcal{F}|_{f^{-1}(\bar b)}\right)=\mathrm{H}^2_c\left((f^{-1}(\bar b),\mathcal{F}|_{f^{-1}(\bar b)}\right)=(0)\,.$$

We conclude that

$$\mathrm{H}^1_c\left((f^{-1}(\bar b),\mathcal{F}|_{f^{-1}(\bar b)}\right)=(0)$$

as well since the Euler-Poincaré characteristic is equal to 0. ∎

(2.5) Proposition *The rank-two smooth $\overline{\mathbb{Q}_\ell}$-sheaf $\mathrm{R}^1\pi_!|_{\mathbb{A}^1-\{2,-2\}}\mathcal{F}$ on $\mathbb{A}^1-\{2,-2\}$ is tamely ramified at $\{2,-2,\infty\}\subset\mathbb{P}^1$.*

Proof. Recall from 2.3 (iii) that the stalk of $\mathrm{R}^1\pi_!\mathcal{F}$ at 2 and -2 are both one-dimensional. From the short exact sequence

$$0\to\left(\mathrm{R}^1\pi_!\mathcal{F}\right)|_{\mathbb{A}^1-\{2,-2\}}\to\mathrm{R}^1\pi_!\mathcal{F}\to\left(\mathrm{R}^1\pi_!\mathcal{F}\right)|_{\{2,-2\}}\to 0$$

we get

$$\chi_c\left(\mathbb{A}^1-\{2,-2\}_{/\mathbb{F}^{\mathrm{sep}}},\mathrm{R}^1\pi_!\mathcal{F}\right)=\chi_c\left(\mathbb{A}^1_{/\mathbb{F}^{\mathrm{sep}}},\mathrm{R}^1\pi_!\mathcal{F}\right)-2\,.$$

On the other hand, from the Leray spectral sequence we have

$$\begin{aligned}-\chi_c\left(\mathbb{A}^1_{/\mathbb{F}^{\mathrm{sep}}},\mathrm{R}^1\pi_!\mathcal{F}\right)&=\chi_c\left(Y\times_{\mathrm{Spec}\,\mathbb{F}}\mathrm{Spec}\,\mathbb{F}^{\mathrm{sep}},\mathcal{F}\right)\\&=\textstyle\sum_i(-1)^i\,\chi_c\left(U\times_{\mathrm{Spec}\,\mathbb{F}}\mathrm{Spec}\,\mathbb{F}^{\mathrm{sep}},\mathrm{R}^if_!\mathcal{F}\right)=0\end{aligned}$$

by Lemma 2.4. Therefore $\chi_c\left(\mathbb{A}^1-\{2,-2\}_{/\mathbb{F}^{\mathrm{sep}}},\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}\right)=-2$.

The Euler-Poincaré characteristic $\chi_c\left(\mathbb{A}^1-\{2,-2\}_{/\mathbb{F}^{\mathrm{sep}}},\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}\right)=-2$ can also be computed by Grothendieck's formula, which gives

$$\begin{aligned}-2&=\chi_c\left(\mathbb{A}^1-\{2,-2\}_{/\mathbb{F}^{\mathrm{sep}}},\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}\right)\\&=2\,\chi_c\left(\mathbb{A}^1-\{2,-2\}_{/\mathbb{F}^{\mathrm{sep}}},\overline{\mathbb{Q}_\ell}\right)-\textstyle\sum_{b\in\{2,-2,\infty\}}\mathrm{Sw}_b\left(\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}\right)\end{aligned}$$

Since $\chi_c\left(\mathbb{A}^1-\{2,-2\}_{/\mathrm{Spec}\,\mathbb{F}^{\mathrm{sep}}},\overline{\mathbb{Q}_\ell}\right)=-1$, the sum of the Swan conductor at $2,-2,\infty$ is equal to zero, hence the Swan conductor at these three missing points are all equal to zero. We have proved the tameness of $\mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}$. ∎

(2.6) So far we have seen that $\mathcal{G} = \mathcal{G}_{\varepsilon,\omega} := \mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}$ is a smooth rank-two $\overline{\mathbb{Q}_\ell}$-sheaf which is tamely ramified at $2, -2, \infty$. This sheaf corresponds to a linear representation ρ of the Galois group $\mathrm{Gal}(\mathbb{F}(\mathbb{P}^1)^{\mathrm{sep}}/\mathbb{F}(\mathbb{P}^1))$ on a two dimensional vector space $\mathcal{G}_{\eta_{\overline{\mathbb{A}^1}}}$ over $\overline{\mathbb{Q}_\ell}$, the geometric generic fiber of $\mathcal{G}$. We would like to understand the restriction of the representation ρ to a decomposition group D_b, $b = 2, -2, \infty$, using the theory of vanishing cycles.

We recall some standard notation about vanishing cycles. Let S be a smooth curve over a finite field $\mathbb{F}$, and let Y be a scheme of finite type over S. Let $\mathcal{K} \in \mathrm{D}^b_c(Y, \overline{\mathbb{Q}_\ell})$ be a "constructible complex of $\overline{\mathbb{Q}_\ell}$-sheaves" on Y. Let s be a closed point of the base scheme S. Let $S_{(s)}$ be the henselization of S at s, and let η_s be the generic point of $S_{(s)}$. Let $\overline{\eta_s} = \mathrm{Spec}(\kappa(\eta_s)^{\mathrm{sep}})$ be the geometric point lying over η_s. Let $\overline{S_{(s)}}$ be the spectrum of the normalization of S in $\kappa(\eta_s)^{\mathrm{sep}}$. Let

$$j_{\overline{\eta_s}} : Y_{\overline{\eta_s}} = Y \times_S \overline{\eta_s} \hookrightarrow Y_{\overline{S_{(s)}}} = Y \times_S \overline{S_{(s)}}$$

be the natural open embedding, and let

$$i_{\bar{s}} : Y_{\bar{s}} = Y \times_S \bar{s} \hookrightarrow Y_{\overline{S_{(s)}}} = Y \times_S \overline{S_{(s)}}$$

be the natural closed embedding. The complex of near-by cycles over s for $\mathcal{K}$ is defined to be the object

$$\mathrm{R}\Psi_s(\mathcal{K}) := i_{\bar{s}}^* \, \mathrm{R}j_{\overline{\eta_s}*}\, \mathcal{K}$$

in $\mathrm{D}^b_c(Y_{\bar{s}}, \overline{\mathbb{Q}_\ell})$, endowed with the natural action of the decomposition group

$$D_s := \mathrm{Gal}\left(\kappa(\eta_s)^{\mathrm{sep}}/\kappa(\eta_s)\right) .$$

The complex of vanishing cycles $\mathrm{R}\Phi_s(\mathcal{K})$ over s for $\mathcal{K}$ is defined as the mapping cone of the natural arrow $i_{\bar{s}}^*\mathcal{K} \to \mathrm{R}\Psi_s(\mathcal{K})$, endowed with the natural action of D_s. The cohomology sheaves of $\mathrm{R}\Psi(\mathcal{K})$ and $\mathrm{R}\Phi(\mathcal{K})$ are denoted by $\Psi^i(\mathcal{K})$ and $\Phi^i(\mathcal{K})$ respectively.

(2.7) Denote by $\overline{U}$ the projective completion of U, geometrically isomorphic to $\mathbb{P}^1$. Define $\overline{Y}$ to be $\overline{U} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{P}^1$, which contains $Y = U \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{A}^1$ as a dense open subscheme. Let $j : Y \hookrightarrow \overline{Y}$ be the natural open embedding.

Recall that $\mathcal{F}$ is the rank-one $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_{\varepsilon(\mathrm{Tr}+t)} \otimes \mathcal{L}_\omega$ on Y. Let $\overline{\pi} : \overline{Y} = \overline{U} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{P}^1 \to \mathbb{P}^1$ be the natural projection.

Let $\overline{\mathcal{F}}$ be the sheaf $j_!\mathcal{F}$, the sheaf $\mathcal{F}$ on Y extended by 0 to $\overline{Y}$. We would like to compute the vanishing cycle complex $\mathrm{R}\Phi_b\overline{\mathcal{F}}$ of $\overline{\mathcal{F}}$ for $\overline{\pi}$, where $b \in \{2, -2, \infty\}$ is one of the "bad points" of the base scheme $\mathbb{P}^1$.

(2.8) Theorem *Assume that ε is non-trivial, and that the order of ω is not equal to two if ε has order two. Let b be either 2 or -2.*

(i) *The vanishing cycle complex $\mathrm{R}\Phi_b(\overline{\mathcal{F}})$ is concentrated at one geometric point z_0 of Y_b, where $z_0 = -1$ if $b = 2$, and $z_0 = 1$ if $b = -2$.*

(ii) *The i-th cohomology sheaf $\Phi_b^i(\overline{\mathcal{F}})_{z_0}$ of the stalk at z_0 of $\mathrm{R}\Phi_b(\overline{\mathcal{F}})$ is zero if $i \neq 1$, and $\Phi_b^1(\overline{\mathcal{F}})_{z_0}$ is one-dimensional.*

(iii) *Suppose that ε is non-trivial and has order two, and ω is not the non-trivial character of $U(\mathbb{F})$ of order two. Then the inertia group I_b operates trivially on $\Phi_b^1(\overline{\mathcal{F}})_{z_0}$, and $\Phi_b^1(\overline{\mathcal{F}})_{z_0}$ is pure of weight two as a module of the decomposition group D_b.*

(iv) *Suppose that ε^2 is non-trivial. Then $\Phi_b^1(\overline{\mathcal{F}})_{z_0}$ is pure of weight one as a module of the decomposition group D_b.*

Proof of (i) and (ii). The statement (i) follows from [8, Cor. 2.1.2]. It is easy to see that $\Phi_b^{-1}(\overline{\mathcal{F}})_{z_0} = 0$ and that $\Phi_b^0(\overline{\mathcal{F}})_{z_0} = 0$. The long exact sequence

$$\cdots \to \mathrm{H}^i(\overline{Y}_{\bar{b}}, i_{\bar{b}}^*\overline{\mathcal{F}}) \to \mathrm{H}_c^i(Y_{\overline{\eta_b}}, j_{\overline{\eta_b}}^*\mathcal{F}) \to \Phi_b^i(\overline{\mathcal{F}})_{z_0} \to \mathrm{H}^{i+1}(\overline{Y}_{\bar{b}}, i_{\bar{b}}^*\overline{\mathcal{F}}) \to \cdots$$

gives $\Phi_b^i(\overline{\mathcal{F}})_{z_0} = 0$ for all $i \geq 2$, and it reduces to the following short exact sequence

$$0 \to \mathrm{H}^1(\overline{Y}_{\bar{b}}, i_{\bar{b}}^*\overline{\mathcal{F}}) \to \mathrm{H}_c^1(Y_{\overline{\eta_b}}, j_{\overline{\eta_b}}^*\mathcal{F}) \to \Phi_b^1(\overline{\mathcal{F}})_{z_0} \to 0\,.$$

By the same argument as in the proof of Proposition 2.3, essentially a Euler characteristic calculation, we see that $\mathrm{H}^1(\overline{Y}_{\bar{b}}, i_{\bar{b}}^*\mathcal{F}) = \mathrm{H}_c^1(Y_{\bar{b}}, \mathcal{F}|Y_{\bar{b}})$ is one dimensional. This proves the statement (ii). Also notice that the inertia group I_b operates trivially on the one-dimensional subspace $\mathrm{H}^1(\overline{Y}_{\bar{b}}, i_{\bar{b}}^*\overline{\mathcal{F}})$ of $\mathcal{G}_{\bar{\eta}}$. ∎

Proof of (iv). From the proof of (i), (ii) above, we see that $\mathrm{H}_c^1(Y_{\bar{b}}, \mathcal{F}|Y_{\bar{b}})$ is a one-dimensional D_b-submodule of $\mathcal{G}_{\bar{\eta}}$. Let $j_b : Y_b \hookrightarrow \overline{Y}_b$ be the canonical inclusion. Extending the base to $\mathbb{F}'$, the map j_b becomes the inclusion of $\mathbb{G}_{\mathrm{m}} - \{z_b\}$ in $\mathbb{P}^1$, where $z_b = 1$ (resp. $z_b = -1$) if $b = -2$ (resp. $b = 2$); the sheaf $\mathcal{F}_b$ becomes the sheaf $\mathcal{L}_{\tilde{\varepsilon}^2(z-z_b)} \otimes \mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}}$ on $\mathbb{G}_{\mathrm{m}} - \{z_b\}$.

The assumption that ε^2 is nontrivial implies that $\tilde{\varepsilon} \neq \tilde{\omega}^{\pm 1}$. Therefore $\mathcal{F}_b$ is ramified at the points z_b, 0 and ∞ of $\mathbb{P}^1$. Hence $(j_b)_!\mathcal{F}_b = (j_b)_*\mathcal{F}_b$. By Theorem 3.2.3 in [3], the action of Fr_b on $\mathrm{H}_c^1(Y_{\bar{b}}, \mathcal{F}_{\bar{b}})$ is pure of weight one. By Deligne's theorem on the monodromy weight filtration, Theorem 1.8.4 of [3], we conclude that the D_b-module $\mathcal{G}_{\bar{\eta}}$ is pure of weight one. ∎

To prove (iii), it suffices to show that the D_b-module $\mathrm{H}_c^1(Y_{\bar{b}}, \mathcal{F}|Y_{\bar{b}})$ is unramified and pure of weight zero, by Deligne's theorem on the monodromy weight filtration [3, 1.8.4]. That the inertia group I_b operates trivially on $\mathrm{H}_c^1(Y_{\bar{b}}, \mathcal{F}|Y_{\bar{b}})$ is obvious. As before we extend the base field to $\mathbb{F}'$. Since ε^2 is trivial, the sheaf $\mathcal{F}|Y_{\bar{b}}$ on $\mathbb{G}_{\mathrm{m}} - \{z_b\}$ becomes the restriction to $\mathbb{G}_{\mathrm{m}} - \{z_b\}$ of the smooth rank-one sheaf $\mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}$ on $\mathbb{G}_{\mathrm{m}}$.

We have $\mathrm{H}_c^i(\mathbb{G}_{\mathrm{m}/\mathbb{F}'^{\mathrm{sep}}}, \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}) = (0)$ for $i = 0, 2$ since $\mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}$ is ramified at 0 and ∞. Furthermore $\chi_c(\mathbb{G}_{\mathrm{m}/\mathbb{F}'^{\mathrm{sep}}}, \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}) = 0$ by Grothendieck's Euler-Poincaré characteristic formula. This implies that $\mathrm{H}_c^i(\mathbb{G}_{\mathrm{m}/\mathbb{F}'^{\mathrm{sep}}}, \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}) = (0)$ for all i.

From the short exact sequence

$$0 \to \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}|_{\mathbb{G}_{\mathrm{m}}-\{z_b\}} \to \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}} \to \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}|_b \to 0$$

we see that $\mathrm{H}_c^1(Y_{\bar{b}}, \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}) \cong \mathcal{L}_{\tilde{\varepsilon}\cdot\tilde{\omega}}|_{\bar{b}}$ as D_b-modules. This finishes the proof of Theorem 2.8. ∎

(2.8.1) Remark In the case (iv) of Theorem 2.8, the inertia group I_b acts via a non-trivial finite cyclic quotient μ_m on $\Phi_b^1(\overline{\mathcal{F}})_{z_0}$ with $(m, p) = 1$. The order, m, of this cyclic group depends only on the order of the character ε and is given as follows. Let n be the order of ε^2; write $n = 2^a n_1$, $a \geq 0$, and $(2, n_1) = 1$. Then $m = 2n$ if $a = 0$, $m = n_1 = \frac{n}{2}$ if $a = 1$, and $m = n$ if $a \geq 2$.

(2.9) Theorem *Assume that p is an odd prime number. Suppose that ε is the non-trivial character of $\mathbb{F}^\times$ of order two, and ω is not the non-trivial character of $U(\mathbb{F})$ of order two.*

(i) *Suppose that $b = -2$. Then the decomposition group D_b operates via the unramified character of order two $\lambda_2 : D_b \twoheadrightarrow \mu_2$ of D_b on the one-dimensional $\overline{\mathbb{Q}_\ell}$-space $(\mathcal{G}|_{\overline{\eta}})^{I_b}$ consisting of all inertia invariants in the geometric generic fiber of $\mathcal{G}$. In other words every geometric Frobenius element Fr_q operates as -1.*

(ii) *Suppose that $b = 2$. Then the one-dimensional D_b-module $(\mathcal{G}|_{\overline{\eta}})^{I_b}$ is unramified, on which every geometric Frobenius element $\mathrm{Fr}_q \in D_b$ operates as $-\varepsilon(-1)\omega(-1)$.*

(iii) *The geometric monodromy group for $\mathcal{G}$ is equal to $\mathrm{SL}(2)$.*

Proof of (i). Suppose that $b = -2$, and denote by z_1 a geometric point lying above $(1, -2) \in U \times \mathbb{A}^1(\mathbb{F}) = U(\mathbb{F}) \times \mathbb{F}$ This is the only point in the fiber Y_b where the sheaf $\mathcal{F}$ is ramified. Denote by $\bar{b}$ a geometric point lying above b.

In the proof of Theorem 2.8 (iii) we saw that there is a short exact sequence

$$0 \to \mathrm{H}^1(\overline{Y}_{\bar{b}}, \overline{\mathcal{F}}|_{\overline{Y}_{\bar{b}}}) \to \mathcal{G}|_{\bar{b}} \to \Phi^1_b(\overline{\mathcal{F}})_{z_1} \to 0\,.$$

The one-dimensional $\overline{\mathbb{Q}_\ell}$ space $\mathrm{H}^1(\overline{Y}_b, \overline{\mathcal{F}}|_{\overline{Y}_b})$ is equal to $(\mathcal{G}|_b)^{I_b}$, the inertia invariants in the two-dimensional D_b-module $\mathcal{G}|_b$. The sheaf $\overline{\mathcal{F}}|_{\overline{Y}_b}$ on $\overline{Y}_b \cong \overline{U}$ is canonically isomorphic to the extension by zero to $\overline{U}$ of the sheaf $\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega$ on $U - \{1\}$. Let $j_{U-\{1\}} : U - \{1\} \hookrightarrow \overline{U}$ and $j_U : U \hookrightarrow \overline{U}$ be the natural inclusions. Over $\mathbb{F}'$, $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$ is isomorphic to $\mathbb{G}_\mathrm{m}$, and the sheaf $\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega$is isomorphic to

$$\mathcal{L}_{\tilde{\varepsilon}^2(z-1)} \otimes \mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}} = \mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}}|_{U-\{1\}}\,,$$

since ε has order two. Clearly $(U - \{1\} \hookrightarrow U)_* \mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}}$ is equal to the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}}$ on $U_{/\mathbb{F}'}$, which is tamely ramified at 0 and ∞. By Grothendieck's Euler-Poincaré characteristic formula, $\chi_c(U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}^{\mathrm{sep}}, \mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}}) = 0$, hence $\mathrm{H}^i_c(U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}^{\mathrm{sep}}, \mathcal{L}_{\tilde{\varepsilon}^{-1}\cdot\tilde{\omega}}) = (0)$ for all i.

From the above we know that $(U - \{1\} \hookrightarrow U)_* \mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega$ is a smooth rank-one sheaf on U, and we have a short exact sequence

$$\begin{aligned} 0 \to (U - \{1\} \hookrightarrow U)_!(\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega) &\to (U - \{1\} \hookrightarrow U)_*(\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega) \\ &\to (U - \{1\} \hookrightarrow U)_*(\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega)|_{u=1} \to 0\,. \end{aligned}$$

From the associated long exact sequence of cohomologies we get an isomorphism

$$\mathrm{H}^1(\overline{Y}_b, \overline{\mathcal{F}}|_{\overline{Y}_b}) \cong (U - \{1\} \hookrightarrow U)_*(\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega)|_{u=1}$$

of D_b-modules.

To finish (i) we have to compute the action of D_b on $(U - \{1\} \hookrightarrow U)_*(\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)} \otimes \mathcal{L}_\omega)|_{u=1}$. By definition, the geometric Frobenius element Fr_1 as $\omega(1) = 1$ on the restriction of the smooth rank-one sheaf $\mathcal{L}_\omega$ to the point $1 \in U(\mathbb{F})$, so we only need to compute the Galois action on the restriction of $(U - \{1\} \hookrightarrow U)_* \mathcal{L}_{\varepsilon(\mathrm{Tr}-2)}$ to $1 \in U(\mathbb{F})$. Since ε is the non-trivial character of $\mathbb{G}_\mathrm{m}(\mathbb{F})$ of order two, the sheaf $\mathcal{L}_\varepsilon$ on $\mathbb{G}_\mathrm{m}$ is the push-forward of the short exact sequence

$$1 \to \mu_2 \to \mathbb{G}_\mathrm{m} \xrightarrow{[2]} \mathbb{G}_\mathrm{m} \to 1$$

by the non-trivial character of μ_2. Hence the action of $D_b/I_b \cong \mathrm{Fr}_q^{\mathbb{Z}}$ on the one-dimensional $\overline{\mathbb{Q}_\ell}$-vector space $(U - \{1\} \hookrightarrow U)_*\mathcal{L}_{\varepsilon(\mathrm{Tr}-2)}|_{u=1}$ can be computed as follows. Pick a generator π_1 of the maximal ideal of the local ring $\mathcal{O}_{U,1}$ and consider the leading term $a_m \cdot \pi_1^m$ in expansion of the element $\mathrm{Tr}-2 \in \mathcal{O}_{U,1}$ as a power series in π_1 with coefficients in $\mathbb{F}$. Then every Frobenius element Fr_q operates as $\varepsilon(a_m) \cdot (-1)^m$ on the one-dimensional space above.

We still need to perform the power series expansion of $\mathrm{Tr}-2 \in \mathcal{O}_{U,1}$. Pick an element $a \in \mathbb{F}'$, $a \notin \mathbb{F}$. Let z be the standard coordinate function of $\mathbb{G}_{\mathrm{m}/\mathbb{F}'} \cong U \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$. Then the function $\pi_1 = a\,z + a^q\,z^{-1} - a - a^q \in \Gamma(\mathcal{O}_U) \otimes_{\mathbb{F}} \mathbb{F}' = \Gamma(\mathbb{G}_{\mathrm{m}/\mathbb{F}'}, \mathcal{O}_{\mathbb{G}_{\mathrm{m}/\mathbb{F}'}})$ is a regular function on U and gives a uniformizing element of $\mathcal{O}_{U,1}$. Expanded as a series in the monomials of $(z-1)$, the leading term of $a\,z + a^q\,z^{-1} - a - a^q$ is $(a - a^q)\,(z-1)$. On the other hand, the leading term of the expansion of $\mathrm{Tr}-2$ in powers of $(z-1)$ is $(z-1)^2$. Hence the leading term of the expansion of $\mathrm{Tr}-2$ in the powers of π_1 is $(a-a^q)^{-2}\,\pi_1^2$. Since $\varepsilon((a-a^q)^2) = -1$, we have proved the assertion in (i). ∎

Proof of (ii). The case $b = 2$ is similar to the case $b = -2$ before. The only difference is the power series expansion of $\mathrm{Tr}+2$. Pick an element $a \in \mathbb{F}'$, $a \notin \mathbb{F}$ as before. Then $\pi_{-1} = a\,z + a^q\,z^{-1} + a + a^q$ is a uniformizing element for $\mathcal{O}_{U,-1}$, and the leading term of the expansion of $\mathrm{Tr}+2$ in the monomials of π_{-1} is $-(a-a^q)^2\,\pi_{-1}^2$. ∎

Proof of (iii). The geometric monodromy group is contained in $SL(2)$ by Prop. 2.3 (vi). On the other hand, we know by [3, Cor. 1.3.9] that the neutral component of the geometric monodromy group is semisimple, hence it is either trivial or is equal to SL(2) in the present case. According to the theory of monodromy weight filtration in [3, §1.8], the geometric monodromy group contain the exponential of the logarithm of N of the inertia action at $b = 2$ and $b = -2$. By (i) and (ii), the monodromy weight filtration is non-trivial both for $b = 2$ and $b = -2$, hence the logarithm N of the inertia action is nontrivial at $b = \pm 2$ according to Deligne's theory in [3, §1.8]. In particular the geometric monodromy group cannot be finite. This finishes the proof of Theorem 2.9. ∎

(2.9.1) Remarks (1) Suppose that ε has order two. From (i) and (ii) one sees that $\det(\mathcal{G})$ is unramified at ± 2. We already know that it is tamely ramified at ∞, therefore $\det(\mathcal{G})$ is unramified since the tame fundamental group of $\mathbb{A}^1$ is trivial. This means that $\det(\mathcal{G})$ is the pull-back from $\mathrm{Spec}\,\mathbb{F}$. Then we can determine $\det(\mathcal{G})$ from its stalk at 2 or -2 using (i), (ii); or one use the fact that the characteristic polynomials of Frobenii for $\mathcal{G}$ are real to deduce that $\det(\mathcal{G}) = \overline{\mathbb{Q}_\ell}(-1)$. This gives an alternative proof of the fact that $\det(\mathcal{G})$ is $\overline{\mathbb{Q}_\ell}(-1)$.

(2) If ε^2 is not trivial, then $\det(\mathcal{G})$ is ramified, hence is not equal to $\overline{\mathbb{Q}_\ell}(-1)$.

(2.10) Theorem *Assume that ε is not the trivial character of $\mathbb{F}^\times$, that ω is not the trivial character of $U(\mathbb{F})$, and that ε^2 and ω^2 are not both trivial. Consider the geometric fiber $\mathcal{G}|_{\overline{\eta}}$ of $\mathcal{G} := \mathrm{R}^1\pi_!\mathcal{F}|_{\mathbb{A}^1-\{2,-2\}}$ as a module of the decomposition group D_∞ at ∞. Then the wild inertia group P_∞ acts trivially, and I_∞ operates via a finite quotient with no non-zero fixed element in $\mathcal{G}|_{\overline{\eta}}$. As a D_∞-module, $\mathcal{G}|_{\overline{\eta}}$ is a direct sum of two one-dimensional D_∞-submodules corresponding to two characters of D_∞; each character is ramified (tamely) and pure of weight one.*

(2.11) Over $\mathbb{F}'$, the complement of $Y \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$ in

$$\overline{Y} \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}' = \overline{U} \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{P}^1 \times_{\mathrm{Spec}\,\mathbb{F}} \mathrm{Spec}\,\mathbb{F}'$$

is the union of four divisors:

- $S_0 = \{0\} \times \mathbb{P}^1$,
- $S_\infty = \{\infty\} \times \mathbb{P}^1$,
- $\overline{Y}_v = \overline{U} \times \{\infty\}$,
- D_h, the Zariski closure in $\overline{Y}$ of the zero locus of $\mathrm{Tr} + t$ in Y.

The Frobenius element Fr_q interchanges S_0 and S_∞. The divisors $\overline{Y}_v$ and D_h are defined over $\mathbb{F}$. The three divisors $S_0, \mathrm{D}_h, \overline{Y}_v$ intersect at the point $y_0 = (0, \infty)$; the three divisors $S_\infty, \mathrm{D}_h, \overline{Y}_v$ intersect at the point $y_\infty = (\infty, \infty)$.

Let $X \to \overline{Y}$ be the blowing-up of $\overline{Y}$ at the two points $\{y_0, y_\infty\}$. The inverse image in Z of the union of the four divisors in $\overline{Y}$ above is the union of the following six divisors in Z.

- E_0, the exceptional divisor above y_0,
- E_∞, the exceptional divisor above y_∞,
- $\widetilde{\mathrm{D}}_0$, the strict transform of S_0,
- $\widetilde{\mathrm{D}}_\infty$, the strict transform of S_∞,
- $\widetilde{\mathrm{D}}_h$, the strict transform of S_h,
- Z_v, the strict transform of $\overline{Y}_v$.

Let Z be the union of six divisors above; it is a reduced divisor with normal crossings. This divisor has six singularities:

- $x_{0,0}$, the intersection of $\widetilde{\mathrm{D}}_0$ and E_0,
- $x_{0,h}$, the intersection of $\widetilde{\mathrm{D}}_h$ and E_0,
- $x_{0,v}$, the intersection of Z_v and E_0,
- $x_{\infty,\infty}$, the intersection of $\widetilde{\mathrm{D}}_\infty$ and E_∞,
- $x_{\infty,h}$, the intersection of $\widetilde{\mathrm{D}}_h$ and E_∞,
- $x_{\infty,v}$, the intersection of Z_v and E_∞

Denote by $\tilde{j} = \tilde{j}_{Y,X} : Y \hookrightarrow X$ the open immersion of Y in X, and let $\tilde{\mathcal{F}} := \tilde{j}_!\mathcal{F}$. Let $\tilde{\pi} : X \to \mathbb{P}^1$ be the projection from X to the base scheme $\mathbb{P}^1$, and let X_∞ be the divisor $E_0 \cup Z_v \cup E_\infty$, the inverse image of ∞ under $\tilde{\pi}$. We would like to understand the vanishing cycle complex $\mathrm{R}\Phi_\infty(\tilde{\mathcal{F}})$ with respect to $\tilde{\pi}$.

(2.12) Theorem *Assume as in the statement of 2.10 that ε is not the trivial character of $\mathbb{F}^\times$, that ω is not the trivial character of $U(\mathbb{F})$, and that ε^2 and ω^2 are not both trivial.*

(i) *The i-th cohomology sheaf $\Phi^i_\infty(\tilde{\mathcal{F}})$ of $\mathrm{R}\Phi_\infty(\tilde{\mathcal{F}})$ is equal to zero for all $i \geq 1$.*

(ii) *The stalk of* $\Phi^0_\infty(\tilde{\mathcal{F}})_x$ *is equal to zero if* $x \in \{x_{0,0}, x_{0,h}, x_{0,v}, x_{\infty,\infty}, x_{\infty,h}, x_{\infty,v}\}$.

(iii) *The sheaf* $\Phi^0_\infty(\tilde{\mathcal{F}})|_{Z_v - \{x_{0,v}, x_{\infty,v}\}}$ *is smooth of rank one, tamely ramified at the two points* $\{x_{0,v}, x_{\infty,v}\}$ *with respect to* $Z_v - \{x_{0,v}, x_{\infty,v}\} \hookrightarrow Z_v$. *Moreover*

$$
\begin{aligned}
&(Z_v - \{x_{0,v}, x_{\infty,v}\} \hookrightarrow Z_v)_! \left(\Phi^0_\infty(\tilde{\mathcal{F}})|_{Z_v - \{x_{0,v}, x_{\infty,v}\}}\right) \\
&= (Z_v - \{x_{0,v}, x_{\infty,v}\} \hookrightarrow Z_v)_* \left(\Phi^0_\infty(\tilde{\mathcal{F}})|_{Z_v - \{x_{0,v}, x_{\infty,v}\}}\right)
\end{aligned}
$$

The inertia group I_∞ *acts on this sheaf via a one-dimensional tame character whose order is equal to the order of* ε.

(iv) *The sheaf* $\Phi^0_\infty(\tilde{\mathcal{F}})|_{E_0 - \{x_{0,v}, x_{0,h}, x_{0,0}\}}$ *is smooth of rank one, tamely ramified at the three point* $\{x_{0,v}, x_{0,h}, x_{0,0}\}$ *with respect to* $E_0 - \{x_{0,v}, x_{0,h}, x_{0,0}\}$. *Moreover*

$$
\begin{aligned}
&(E_0 - \{x_{0,v}, x_{0,h}, x_{0,0}\})_! \left(\Phi^0_\infty(\tilde{\mathcal{F}})|_{E_0 - \{x_{0,v}, x_{0,h}, x_{0,0}\}}\right) \\
&= (E_0 - \{x_{0,v}, x_{0,h}, x_{0,0}\})_* \left(\Phi^0_\infty(\tilde{\mathcal{F}})|_{E_0 - \{x_{0,v}, x_{0,h}, x_{0,0}\}}\right)
\end{aligned}
$$

The inertia group I_∞ *acts on this sheaf via a tame character whose order is equal to the order of* $\tilde{\varepsilon}^{-1} \cdot \tilde{\omega}$.

(v) *The sheaf* $\Phi^0_\infty(\tilde{\mathcal{F}})|_{E_\infty - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\}}$ *is smooth of rank one, tamely ramified at the three points* $\{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\}$ *with respect to* $E_\infty - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\}$. *Moreover*

$$
\begin{aligned}
&(E_\infty - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\})_! \left(\Phi^0_\infty(\tilde{\mathcal{F}})|_{E_\infty - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\}}\right) \\
&= (E_\infty - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\})_* \left(\Phi^0_\infty(\tilde{\mathcal{F}})|_{E_\infty - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\}}\right)
\end{aligned}
$$

The inertia group I_∞ *acts on this sheaf via a tame character whose order is equal to the order of* $\tilde{\varepsilon} \cdot \tilde{\omega}$.

Deduction of Theorem 2.10 from Theorem 2.12. Since the restriction of $\tilde{\mathcal{F}}$ to X_∞ is zero, $\mathcal{G}_{\bar{\eta}}$ is isomorphic to $\mathrm{H}^1(X_{\overline{\infty}}, \mathrm{R}\Phi_\infty(\tilde{\mathcal{F}}))$ as representations of the decomposition group D_∞, where $X_{\overline{\infty}}$ is the geometric fiber of X over the geometric point $\overline{\infty}$ above $\infty \in \mathbb{P}^1$. From the statements (i), (ii) of Theorem 2.12, we see that $\mathrm{H}^1(X_{\overline{\infty}}, \mathrm{R}\Phi_\infty(\tilde{\mathcal{F}}))$ is isomorphic to the direct sum of

$\mathrm{H}^1_c(Z_v \times \operatorname{Spec} \mathbb{F}^{\mathrm{sep}} - \{x_{0,v}, x_{\infty,v}\}, \Phi^0_\infty(\tilde{\mathcal{F}}))$,

$\mathrm{H}^1_c(E_0 \times \operatorname{Spec} \mathbb{F}^{\mathrm{sep}} - \{x_{0,v}, x_{0,h}, x_{0,0}\}, \Phi^0_\infty(\tilde{\mathcal{F}}))$, and

$\mathrm{H}^1_c(E_\infty \times \operatorname{Spec} \mathbb{F}^{\mathrm{sep}} - \{x_{\infty,v}, x_{\infty,h}, x_{\infty,\infty}\}, \Phi^0_\infty(\tilde{\mathcal{F}}))$.

From the Euler-Poincaré characteristic formula we see that first of the three cohomology groups is zero according to 2.10 (iii). Again by the Euler-Poincaré formula, the latter two cohomology groups are both one dimensional, and are pure of weight one as representations of the inertia group D_∞ by [3, 3.2.3]. ∎

(2.13) Lemma *Let* $j : \mathbb{G}_m \times_{\operatorname{Spec}\mathbb{F}} \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \times_{\operatorname{Spec}\mathbb{F}} \mathbb{A}^1$ *be the natural open immersion. Let* $\chi_1, \chi_2 : \mathbb{G}_m(\mathbb{F}) \to \overline{\mathbb{Q}_\ell}^\times$ *be characters of* $\mathbb{G}_m(\mathbb{F}) = \mathbb{F}^\times$*. We assume that* χ_1 *is non-trivial. Let* (x,y) *be the standard coordinates on* $\mathbb{A}^1 \times \mathbb{A}^1$ *and on* $\mathbb{G}_m \times \mathbb{G}_m$*. Denote by* $\mathcal{L}_{\chi_1(x)\chi_2(y)}$ *the smooth rank-one* $\overline{\mathbb{Q}_\ell}$*-sheaf* $\mathrm{pr}_1^*\mathcal{L}_{\chi_1} \otimes \mathrm{pr}_2^*\mathcal{L}_{\chi_2}$ *on* $\mathbb{G}_m \times_{\operatorname{Spec}\mathbb{F}} \mathbb{G}_m$*. Let* $\mathrm{R}\Phi\left(j_!\mathcal{L}_{\chi_1(x)\chi_2(y)}\right)$ *be the complex of vanishing cycle for* $j_!\left(\mathcal{L}_{\chi_1(x)\chi_2(y)}\right)$ *over* $\mathbb{A}^1 \times \{0\}$ *with respect to the second projection* $\mathrm{pr}_2 : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$*. Then the stalk* $\mathrm{R}\Phi\left(j_!\mathcal{L}_{\chi_1(x)\chi_2(y)}\right)_{(0,0)}$ *of the complex of vanishing cycles at* $(0,0)$ *is acyclic.*

Proof. According to the definition of vanishing cycles we have a natural isomorphism

$$\mathrm{R}\Phi\left(j_!\mathcal{L}_{\chi_1(x)\chi_2(y)}\right) = \left(\mathcal{L}_{\chi_2}|_{\overline{\eta}}\right) \otimes \mathrm{R}\Phi\left(j_!\mathcal{L}_{\chi_1(x)}\right),$$

where $\mathcal{L}_{\chi_2}|_{\overline{\eta}}$ is the stalk of the sheaf $\mathcal{L}_{\chi_2}$ over a geometric generic point $\overline{\eta}$ of $\mathbb{G}_m$, regarded as a representation of the decomposition group D_0 over the point 0 of the base $\mathbb{A}^1$. Therefore we may and do assume that χ_2 is trivial.

Let $k : \mathbb{G}_m \times_{\operatorname{Spec}\mathbb{F}} \mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \times_{\operatorname{Spec}\mathbb{F}} \mathbb{A}^1$ be the standard natural inclusion. Denote by $\mathrm{pr}_1^*\mathcal{L}_{\chi_1}$ the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{G}_m \times_{\operatorname{Spec}\mathbb{F}} \mathbb{A}^1$ obtained by pulling back the sheaf $\mathcal{L}_{\chi_1}$ on $\mathbb{G}_{m/\mathbb{F}}$ via the first projection. We have a short exact sequence

$$0 \to j_!\left(\mathrm{pr}_1^*\mathcal{L}_{\chi_1}\right) \to k_!\left(\mathrm{pr}_1^*\mathcal{L}_{\chi_1}\right) \to k_!\left(\mathrm{pr}_1^*\mathcal{L}_{\chi_1}\right)|_{\mathbb{A}^1\times\{0\}} \to 0\,.$$

By SGA 7 II, exposé XIII, Lemma 2.1.11,

$$\mathrm{R}\Phi\left(k_!\left(\mathrm{pr}_1^*\mathcal{L}_{\chi_1}\right)\right) = 0\,.$$

It follows that

$$\mathrm{R}\Phi\left(j_!\mathcal{L}_{\chi_1(x)\chi_2(y)}\right)_{(0,0)} = k_!\left(\mathrm{pr}_1^*\mathcal{L}_{\chi_1}\right)|_{(0,0)}[-1] = 0\,.$$

∎

(2.14) Proposition *Let* $j : \mathbb{G}_m \times_{\operatorname{Spec}\mathbb{F}} \mathbb{G}_m \hookrightarrow \mathbb{A}^2_{/\mathbb{F}}$ *be the natural inclusion. Let* $f : \mathbb{A}^2 = \operatorname{Spec}\mathbb{F}[x,y] \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{F}[t]$ *be the morphism over* $\mathbb{F}$ *given by the* $\mathbb{F}$*-algebra homomorphism which sends* t *to* xy*. Let*

$$\chi_1, \chi_2 : \mathbb{G}_m(\mathbb{F}) = \mathbb{F}^\times \to \overline{\mathbb{Q}_\ell}^\times$$

be two characters of $\mathbb{F}^\times$*,* $\chi_1 \neq \chi_2$*. Let* $\mathcal{L}_{\chi_1(x)\cdot\chi_2(y)} = \mathrm{pr}_1^*\mathcal{L}_{\chi_1} \otimes \mathrm{pr}_2^*\mathcal{L}_{\chi_2}$*, where* $\mathcal{L}_{\chi_i}$ *is the smooth rank-one* $\overline{\mathbb{Q}_\ell}$*-sheaf on* $\mathbb{G}_m$ *given by pushing out the Lang torsor by* χ_i*,* $i = 1,2$*. Then the stalk* $\mathrm{R}\Phi(j_!\mathcal{L}_{\chi_1(x)\cdot\chi_2(y)})_{(0,0)}$ *at* $(0,0)$ *of the complex* $\mathrm{R}\Phi(j_!\mathcal{L}_{\chi_1(x)\cdot\chi_2(y)})$ *of vanishing cycles with respect to* f *is an acyclic complex.*

Proof. After tensoring with $f^*\mathcal{L}_{\chi_2^{-1}}$, we may and do assume that χ_2 is trivial, therefore χ_1 is non-trivial.

We partially compactify the morphism f involved as follows. Let $U = \mathbb{G}_m \times_{\operatorname{Spec}\mathbb{F}} \mathbb{G}_m$, contained in $\mathbb{P}^1 \times_{\operatorname{Spec}\mathbb{F}} \mathbb{A}^1$. Let (x,s) be the standard coordinate functions for $\mathbb{G}_m \times \mathbb{G}_m$, and let $y = \frac{s}{x}$. Let X be the blowing-up of $\mathbb{P}^1 \times_{\operatorname{Spec}\mathbb{F}} \mathbb{A}^1$ at $(0,0)$. Let $\overline{f} : X \to \mathbb{A}^1$ be the composition of the blowing-up morphism $X \to \mathbb{P}^1 \times \mathbb{A}^1$ and the second projection $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. This is compatible with the geometric set-up in the statement of Theorem 2.14, since X contains $\mathrm{Spec}\mathbb{F}[x,y]$ as an open subscheme. The fiber $\overline{f}^{-1}(0)$ is the union of the strict transform $\widetilde{D}$ of $\mathbb{P}^1 \times \{0\} \subset \mathbb{P}^1 \times \mathbb{A}^1$ and the exceptional divisor E of $\overline{f}$. Let $\tilde{x}_0$ be the intersection of $\widetilde{D}$ with E. Let $x_\infty \in \widetilde{D}$ be the intersection of $\widetilde{D}$ with the strict transform of $\{\infty\} \times \mathbb{A}^1$. Let $x'_\infty \in E$ be the point on E which lies outside the affine open subscheme $\mathrm{Spec}\,\mathbb{F}[x,y]$ of X. Let $k : \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \hookrightarrow X$ be the inclusion map. Over $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} = \mathrm{Spec}\,[x, x^{-1}, s, s^{-1}] = \mathrm{Spec}\,[x, x^{-1}, y, y^{-1}]$ we have the smooth rank-one $\overline{\mathbb{Q}_\ell}$ sheaf $\mathcal{L}_{\chi_1(x)}$. Let $\mathcal{G} = k_! \mathcal{L}_{\chi_1(x)}$. Let $D' = D - \{\tilde{x}_0, x_\infty\}$, and let $E' = E - \{\tilde{x}_0, x'_\infty\}$. We compute the restriction to $D' \sqcup E'$ of the complex of vanishing cycles for $\mathcal{G}$.

The open subscheme $\mathbb{G}_\mathrm{m} \times \mathbb{A}^1$ of $\mathbb{P}^1 \times \mathbb{A}^1$ contains D'. Over this open subscheme the sheaf $\mathcal{L}_{\chi_1(x)}$ on $\mathrm{Spec}\,\mathbb{F}[x, x^{-1}, y, y^{-1}]$ extends to a smooth rank-one smooth sheaf, the pull-back of the sheaf $\mathcal{L}_{\chi_1}$ on $\mathbb{G}_\mathrm{m}$ to $\mathbb{G}_\mathrm{m} \times \mathbb{A}^1$ via the second projection $\mathbb{G}_\mathrm{m} \times \mathbb{A}^1 \to \mathbb{G}_\mathrm{m}$. From this we deduce that the restriction of $\mathrm{R}\Phi(\mathcal{G})$ to $D' = \mathrm{Spec}[x, x^{-1}] \cong \mathbb{G}_\mathrm{m}$ is isomorphic to $\mathcal{L}_{\chi_1}[1]$, with trivial action by the decomposition group.

On the other hand, E' is contained in the open subscheme $\mathrm{Spec}\,\mathbb{F}[s, y, y^{-1}]$. Over the open subscheme $\mathrm{Spec}\,\mathbb{F}[s, s^{-1}, y, y^{-1}] \subset \mathrm{Spec}\,\mathbb{F}[s, y, y^{-1}]$, we have $\mathcal{L}_{\chi 1(x)} = \mathcal{L}_{\chi_1(s)\chi_1^{-1}(y)}$. From the definition of vanishing cycles we get

$$\mathrm{R}\Phi(\mathcal{G})|_{E'=\mathrm{Spec}\,\mathbb{F}[y,y^{-1}]} = \left(\mathcal{L}_{\chi_1}\right)_{\overline{\eta}} \otimes \mathcal{L}_{\chi_1^{-1}(y)}[1]\,,$$

where the second factor $\mathcal{L}_{\chi_1^{-1}}$ is the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf on $E' = \mathrm{Spec}\,\mathbb{F}[y, y^{-1}] = \mathbb{G}_\mathrm{m}$ attached to the character χ_1^{-1}, while the first factor $\left(\mathcal{L}_{\chi_1}\right)_{\overline{\eta}}$ is the geometric generic fiber of the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_{\chi_1}$ regarded as a linear representation of the decomposition group D_0 at 0.

From Grothendieck's Euler-Poincaré characteristic formula and the computation above, we deduce that

$$\mathbb{H}^i_c(D', \mathrm{R}\Phi(\mathcal{G})|_{D'}) = 0 \qquad \forall i \in \mathbb{Z}$$

and

$$\mathbb{H}^i_c(E', \mathrm{R}\Phi(\mathcal{G})|_{E'}) = 0 \qquad \forall i \in \mathbb{Z}\,.$$

Similarly from the Euler-Poincaré formula we have

$$\mathrm{H}^i_c(\mathbb{G}_\mathrm{m}, \mathcal{L}_{\chi_1}) = 0 \qquad \forall i \in \mathbb{Z}\,.$$

We have a short exact sequence

$$\begin{gathered} 0 \to (D' \sqcup E' \hookrightarrow D \cup E)_!\,(\mathrm{R}\Phi(\mathcal{G})|_{D' \sqcup E'}) \to \mathrm{R}\Phi(\mathcal{G}) \\ \to \mathrm{R}\Phi(\mathcal{G})|_{\tilde{x}_0} \oplus \mathrm{R}\Phi(\mathcal{G})|_{\tilde{x}_\infty} \oplus \mathrm{R}\Phi(\mathcal{G})|_{x'_\infty} \to 0 \end{gathered}$$

Let $\overline{\eta}$ be a geometric generic fiber of the base scheme $\mathbb{A}^1$. The theory of vanishing cycles gives

$$0 = \mathrm{H}^i_c(\mathbb{G}_{\mathrm{m}\overline{\eta}}, \mathcal{L}_{\chi_1}) = \mathrm{H}^i(\mathbb{P}^1_{\overline{\eta}}, \mathcal{G}_{\overline{\eta}}) = \mathbb{H}^i_c(D \cup E, \mathrm{R}\Phi(\mathcal{G})) \quad \forall i\,.$$

From the long exact sequence we deduce that the stalk at $\tilde{x}_0$ $\mathrm{R}\Phi(\mathcal{G})|_{\tilde{x}_0}$ of the complex of vanishing cycles $\mathrm{R}\Phi(\mathcal{G})$ is acyclic. This proves Proposition 2.14 since $\mathrm{R}\Phi(\mathcal{G})|_{\tilde{x}_0}$ is isomorphic to $\mathrm{R}\Phi(j_! \mathcal{L}_{\chi_1(x)})|_{(0,0)}$. ∎

Proof of Theorem 2.12. The proof is a little tedious, consisting of computation of the restriction of the complex of vanishing cycles $\mathrm{R}\Phi_\infty(\tilde{F})$ to the disjoint open subsets $E_0' := E_0 - \{x_{00}, x_{0,h}, x_{0,v}\}$, $Z_v' = Z_v - \{x_{0,v}, x_{\infty,v}\}$, $E_\infty' = E_v - \{x_{\infty,\infty}, x_{\infty,h}, x_{\infty,v}\}$ of X_∞, and the stalks of the complex of vanishing cycles at the six points $x_{00}, x_{0,h}, x_{0,v}, x_{\infty,\infty}, x_{\infty,h}, x_{\infty,v}$.

Notice that the automorphism $(u,t) \mapsto (u^{-1}, t)$ of $U \times \mathbb{G}_\mathrm{m}$ sends $\mathcal{L}_{\varepsilon(\mathrm{Tr}+t)} \otimes \mathcal{L}_{\omega(t)}$ to $\mathcal{L}_{\varepsilon(\mathrm{Tr}+t)} \otimes \mathcal{L}_{\omega^{-1}(t)}$, and interchanges y_0 and y_∞. Therefore it suffices to do the calculation for E_0', Z_v', and $x_{00}, x_{0,h}, x_{0,v}$.

(a) $\mathrm{R}\Phi_\infty(\tilde{F})|_{E_0'}$.

Let $s = \frac{1}{t}$, a coordinate for $\mathbb{P}^1$ at $\infty \in \mathbb{P}^1$. Let $u = \frac{z}{s}$. The affine open subscheme $\operatorname{Spec}\mathbb{F}'[u, u^{-1}, \frac{1}{s^2u^2+1+u}, s] \subset X$ contains E_0'. On $\operatorname{Spec}\mathbb{F}'[u, u^{-1}, \frac{1}{s^2u^2+1+u}, s] \subset X$, the sheaf $\tilde{\mathcal{F}}$ is equal to

$$\mathcal{L}_{\tilde{\varepsilon}(s^2u^2+1+u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(s)}\,.$$

Since $\mathcal{L}_{\tilde{\varepsilon}(s^2u^2+1+u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)}$ is smooth on $\operatorname{Spec}\mathbb{F}'[u, \frac{1}{s^2u^2+1+u}, s]$, we get

$$\mathrm{R}\Phi_\infty(\tilde{F})|_{E_0'} = \mathcal{L}_{\tilde{\varepsilon}(1+u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)} \otimes \mathrm{R}\Phi_{t=\infty}(\tilde{j}_!\mathrm{pr}_2^*\mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}-1)(s)})|_{E_0'}\,.$$

By the definition of the vanishing cycles,

$$\mathrm{R}\Phi_{t=\infty}(\tilde{j}_!\mathrm{pr}_2^*\mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(s)})|_{E_0'} \cong \mathrm{R}\Psi_{t=\infty}(\tilde{j}_!\mathrm{pr}_2^*\mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(s)})|_{E_0'}$$

is equal to $\mathrm{pr}_2^*\mathrm{R}\Psi_{t=\infty}\big((\mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{P}^1)_!(\mathcal{L}_{(\tilde{\omega}^{-1}\tilde{\varepsilon})(t)})\big)|_{E_0'}$. By definition, $\mathrm{R}\Phi_{t=\infty}(\tilde{j}_!\mathcal{L}_{(\tilde{\omega}^{-1}\tilde{\varepsilon})(t)})$ is represented by the D_∞-module $\mathcal{L}_{(\tilde{\omega}^{-1}\tilde{\varepsilon})(t)}|_{\overline{\eta_\infty}}$, "concentrated at degree zero". In other words, $\mathrm{R}\Phi_\infty(\tilde{\mathcal{F}})|_{E_0'}$ "is" the smooth sheaf $\mathcal{L}_{\tilde{\varepsilon}(1+u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)}$, with D_∞ action via the character for the D_∞-module $\mathcal{L}_{(\tilde{\omega}^{-1}\tilde{\varepsilon})(t)}|_{\overline{\eta_\infty}}$.

(b) $\mathrm{R}\Phi_\infty(\tilde{F})|_{Z_v'}$.

Near Z_v', we have coordinates z, s, with $s = \frac{1}{t}$: $Z_v' \subset \operatorname{Spec}\mathbb{F}'[z, z^{-1}, s] \subset X$. Over this open subscheme $\operatorname{Spec}\mathbb{F}'[z, z^{-1}, s, s^{-1}, (sz^2+s+z)^{-1}]$, $\tilde{\mathcal{F}}$ is equal to

$$\mathcal{L}_{\tilde{\varepsilon}(sz^2+s+z)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(z)} \otimes \mathcal{L}_{\tilde{\varepsilon}^{-1}(z)}\,.$$

Hence

$$\mathrm{R}\Phi_\infty(\tilde{F})|_{Z_v'} = \mathcal{L}_{\tilde{\omega}(z)} \otimes \mathrm{R}\Phi_\infty(\tilde{j}_!\mathrm{pr}_2^*\mathcal{L}_{\varepsilon^{-1}(s)})|_{Z_v'}\,.$$

In other words, $\mathrm{R}\Phi_\infty(\tilde{F})|_{Z_v'}$ "is" equal to the smooth sheaf $\mathcal{L}_{\tilde{\omega}(z)}$ on Z_v', with the decomposition group D_∞ operating via the character for the D_∞-module $\mathcal{L}_{\varepsilon(t)}|_{\overline{\eta}_\infty}$.

(c) $\mathrm{R}\Phi_\infty(\tilde{F})_{x_{0,0}}$.

For $u = \frac{z}{s}$, the affine scheme $V_{0,0} = \operatorname{Spec}\mathbb{F}'[u, s, \frac{1}{s^2u^2+1+u}]$ is an open neighborhood of $x_{0,0}$. Let $U_{0,0} = \operatorname{Spec}\mathbb{F}'[u, s, \frac{1}{(s^2u^2+1+u)su}]$, and let $j_{0,0} : U_{0,0} \hookrightarrow V_{0,0}$ be the natural inclusion. Over $V_{0,0}$ we have

$$\tilde{\mathcal{F}} = j_{0,0!}\left(\mathcal{L}_{\tilde{\varepsilon}(s^2u^2+1+u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(s)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon})^{-1}(u)}\right)\,.$$

Hence

$$\mathrm{R}\Phi_\infty(\tilde{F})_{x_{0,0}} = \mathcal{L}_{\tilde{\varepsilon}(1+u)}|_{u=0} \otimes \mathcal{L}_{(\tilde{\omega}^{-1}\cdot\tilde{\varepsilon})(t)}|_{\overline{\eta_\infty}} \otimes \mathrm{R}\Phi_\infty\left(j_{0,0!}(\mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)})\right)_{x_{0,0}}$$

which is acyclic by Lemma. 2.13, since $\tilde{\omega}\cdot\tilde{\varepsilon}^{-1}$ is non-trivial.

(d) $\mathrm{R}\Phi_\infty(\tilde{F})_{x_{0,h}}$.

Let $V_{0,h} = \operatorname{Spec}\mathbb{F}'[u,s,\frac{1}{u}]$, an affine open subscheme which contains $x_{0,h}$, and let $U_{0,h} = \operatorname{Spec}\mathbb{F}'[u,s,\frac{1}{su(s^2u^2+1+u)}]$. Let $j_{0,h} : U_{0,h} \hookrightarrow V_{0,h}$ be the open immersion. The elements (s, s^2u^2+1+u) form a regular system of parameters at $x_{0,h}$. Over $V_{0,h}$ we have

$$\widetilde{\mathcal{F}} = j_{0,h!}(\mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)} \otimes \mathcal{L}_{\tilde{\varepsilon}(s^2u^2+1+u)} \otimes \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(s)}) \,.$$

So

$$\mathrm{R}\Phi_\infty(\tilde{F})_{x_{0,h}} = \mathcal{L}_{(\tilde{\omega}\cdot\tilde{\varepsilon}^{-1})(u)}|_{u=-1} \otimes \mathcal{L}_{(\tilde{\omega}^{-1}\cdot\tilde{\varepsilon})(t)}|_{\overline{\eta_\infty}} \otimes \mathrm{R}\Phi_\infty\left(j_{0,h!}(\mathcal{L}_{\tilde{\varepsilon}(s^2u^2+1+u)})\right)_{x_{0,h}} \,,$$

which is acyclic by Lemma 2.13, since $\tilde{\omega}^{-1}\cdot\tilde{\varepsilon}$ is non-trivial.

(e) $\mathrm{R}\Phi_\infty(\tilde{F})_{x_{0,v}}$.

Let $v = \frac{s}{z}$. The scheme $\operatorname{Spec}\mathbb{F}'[s,v,(z^2v+v+1)^{-1}]$ is an open neighborhood of the point $x_{0,v}$, given by $s=v=0$. Over $\operatorname{Spec}\mathbb{F}'[s,v,(z^2v+v+1)^{-1}]$ we have

$$\mathcal{L}_{\tilde{\varepsilon}(z+z^{-1}+s^{-1})} \otimes \mathcal{L}_{\tilde{\omega}(z)} = \mathcal{L}_{\tilde{\varepsilon}(z^2v+v+1)} \otimes \mathcal{L}_{(\tilde{\varepsilon}^{-1}\cdot\tilde{\omega})(z)} \otimes \mathcal{L}_{\tilde{\varepsilon}^{-1}(v)}$$

By Prop. 2.14 we conclude that $\mathrm{R}\Phi(\tilde{\mathcal{F}})_{x_{0,v}}$ is acyclic, because $\tilde{\omega}$ is non-trivial.

∎

§3. A family of character sums of the first type

In this section we study the family of character sums denoted by $\lambda_{a,\chi}$ in §1.

(3.1) The geometric set-up

Let $\mathbb{F}$ be a finite field with q elements, where q is a power of an *odd* prime number p. Let B be the base curve $\mathbb{P}^1$ with rational function field $\mathbb{F}(t)$. Let $\mathfrak{X}$ be the surface contained in $B\times\mathbb{P}^2$ defined by the equation

$$SY^2 = S(X-Z)^2 + 4TXZ \,.$$

Here S,T are the homogeneous coordinates of B, while X,Y,Z are the homogeneous coordinates of $\mathbb{P}^2$; let t,x,y denote the rational functions $\frac{T}{S},\frac{X}{Z},\frac{Y}{Z}$ respectively. Let $f:\mathfrak{X}\to B$ be the natural projection. Let η be the generic point of B.

It is easy to see that $\mathfrak{X}$ is a proper smooth surface over $\mathbb{F}$. Moreover, $f : \mathfrak{X} \to B$ is smooth over $V = B - \{b_0, b_1, b_\infty\}$, where b_0, b_1, b_∞ are the points of B where the value of t is equal to $0, 1, \infty$ respectively. The three degenerate fibers will be denoted by $\mathfrak{X}_0$, $\mathfrak{X}_1$ and $\mathfrak{X}_\infty$ respectively; one checks easily that they are all reduced with normal crossings. Moreover, each of the three singular fibers is a union of two copies of $\mathbb{P}^1$ intersecting transversally at exactly one $\mathbb{F}$-rational point. The singular point s_0 (resp. s_1, resp. s_∞) of f is given by equations $t = 0, x = 1, y = 0$ (resp. $t = 1, x = -1, y = 0$, resp. $\frac{S}{T} = 0, \frac{X}{Y} = \frac{Z}{Y} = 0$).

Let D_0 (resp. D_∞) be the divisor of $\mathfrak{X}$ given by $X = 0$ (resp. $Z = 0$). Let U be the complement in $\mathfrak{X}$ of the union of D_0 and D_∞. One checks that D_0 is the union of a divisor D_0^{h} finite étale over B with degree two and the component $E_{X,S}$ of $\mathfrak{X}_\infty$ where X vanishes. Similarly D_∞ is the union of a divisor D_∞^{h} finite étale over B with degree two and the component $E_{Z,S}$ of $\mathfrak{X}_\infty$ where Z vanishes. Below we give an explicit description of the divisors involved.

- D_0^{h} is defined by the equations $\{X = 0, Y^2 - Z^2 = 0\}$. It is the disjoint union of two divisors $D_{Y+Z,X}^{\mathrm{h}}$ and $D_{Y-Z,X}^{\mathrm{h}}$, where the subscripts are the defining equations for the two divisors, each isomorphic to the base curve B under f.
- $E_{X,S}$ is defined by the equations $\{X = 0, S = 0\}$.
- D_∞^{h} is defined by the equations $\{Z = 0, Y^2 - X^2 = 0\}$. It is the disjoint union of two divisors $D_{X-Y,Z}^{\mathrm{h}}$ and $D_{X+Y,Z}^{\mathrm{h}}$, indexed by their defining equations, and each is isomorphic to B.
- $E_{Z,S}$ is defined by the equations $\{Z = 0, S = 0\}$.
- $\mathfrak{X}_0$ is defined by the equations $\{T = 0, Y^2 - (X - Z)^2 = 0\}$. It is the union of two divisors $D_{Y-X+Z,T}^{\mathrm{v}}$ and $D_{Y+X-Z,T}^{\mathrm{v}}$, indexed by their defining equations.
- $\mathfrak{X}_1$ is defined by the equations $\{S - T = 0, Y^2 - (X + Z)^2 = 0\}$. It is the union of two divisors $D_{Y+X+Z,S-T}^{\mathrm{v}}$ and $D_{Y-X-Z,S-T}^{\mathrm{v}}$, indexed by their defining equations.
- $\mathfrak{X}_\infty$ is defined by the equations $\{S = 0, XZ = 0\}$; it is the union of $E_{X,S}$ and $E_{Z,S}$.

The union of the horizontal divisors meet each of the three singular fibers $\mathfrak{X}_0$, $\mathfrak{X}_1$, $\mathfrak{X}_\infty$ transversally at four $\mathbb{F}$-rational points. For instance the horizontal divisor D_∞^{h} meets $E_{Z,S}$ transversally at two points $x_{\infty,\infty}$, $x'_{\infty,\infty}$, with projective coordinates $([X : Y : Z :] = [1 : 1 : 0], [S : T] = [0 : 1])$ and $([X : Y : Z :] = [1 : -1 : 0], [S : T] = [0 : 1])$ respectively; the horizontal divisor D_0^{h} meets $E_{X,S}$ transversally at two points $x_{0,0}$, $x'_{0,0}$, with projective coordinates $([X : Y : Z] = [0 : 1 : 1], [S : T] = [0 : 1])$ and $([X : Y : Z] = [0 : 1 : -1], [S : T] = [0 : 1])$ respectively. Denote these four points on $\mathfrak{X}_\infty$ by $s_{\infty,i},\ i = 1, 2, 3, 4$. Similarly we denote by $s_{0,i},\ i = 1, \ldots, 4$ (resp. $s_{1,i},\ i = 1, 2, 3, 4$) the four intersection points of $\mathfrak{X}_0$ (resp. $\mathfrak{X}_0$) with the horizontal divisors; the actual numbering will be unimportant for us.

The rational function $x = \frac{X}{S}$ on $\mathfrak{X}$ defines a morphism from U to $\mathbb{G}_m$; denote by $\mathcal{F}(\chi)$ the pull-back $x^*\mathcal{L}_\chi$ of the rank-one smooth sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_m$ defined by the character χ of $\mathbb{F}^\times$. So $\mathcal{F}(\chi)$ is a smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf on U. We assume that χ is non-trivial. Let $j : U \to \mathfrak{X}$ be the inclusion map. Let $\overline{\mathcal{F}(\chi)} = j_!\mathcal{F}(\chi)$. The complement of U in $\mathfrak{X}$ can be regarded as the ramification divisor of the $\overline{\mathbb{Q}_\ell}$-sheaf $\overline{\mathcal{F}(\chi)}$; it is the union of the horizontal divisors and $\mathfrak{X}_\infty$.

(3.2) Proposition *Assume that χ is a non-trivial character of $\mathbb{F}^\times$. Then*

$$\mathrm{R}^i(f|_U)_!\,\overline{\mathcal{F}(\chi)} = 0 \qquad \text{if } i \neq 1\,.$$

The $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}(\chi) := \mathrm{R}^1(f|_U)_!\,\overline{\mathcal{F}(\chi)}$ on $B - \{b_0, b_1, b_\infty\}$ is smooth of rank two on $V = B - \{b_0, b_1, b_\infty\}$, and is pure of weight one.

Proof. For every geometric point $\bar{b} \in V$, the restriction of the sheaf $\overline{\mathcal{F}(\chi)}$ to the geometric fiber $\mathfrak{X}_b$ is smooth outside the four intersection points with the horizontal divisors, and is tamely ramified at these four intersection points since χ is not trivial. This implies the vanishing statement, and the statement about the rank follows from the Euler-Poincaré characteristic formula. The purity of the sheaf $\mathcal{G}(\chi) := \mathrm{R}^1(f|_U)_!\,\overline{\mathcal{F}(\chi)}$ follows from [3, Thm. 3.2.3]. The smoothness of $\mathcal{G}(\chi) := \mathrm{R}^1(f|_U)_!\,\overline{\mathcal{F}(\chi)}$ follows from Lemma 2.13; it is also a consequence of the local acyclicity of the vanishing cycle complex for smooth morphisms, since $\mathcal{F}(\chi)$ is tamely ramified along the horizontal divisors, which are smooth over B. ∎

(3.3) Lemma *The sheaf $\mathcal{G}(\chi) := \mathrm{R}^1(f|_U)_!\,\overline{\mathcal{F}(\chi)}$ is canonically isomorphic to $\mathcal{G}(\chi^{-1})$, and the coefficients of the characteristic polynomial of the Frobenii on $\mathcal{G}(\chi)$ are totally real algebraic numbers. If χ is not trivial, then* $\det(\mathcal{G}(\chi))$ *is equal to* $\overline{\mathbb{Q}_\ell}(-1)$.

Proof. We have a B-involution ι of $\mathfrak{X}$, which interchanges the projective coordinates X, Z and leaves Y, S, T fixed. Moreover $\iota_*(\mathcal{F}(\chi)) = \mathcal{F}(\chi^{-1})$. When χ is non-trivial, the cup product gives a perfect pairing

$$\mathrm{R}^1(f|_U)_!\,\overline{\mathcal{F}(\chi)} \times \mathrm{R}^1(f|_U)_!\,\overline{\mathcal{F}(\chi^{-1})} \to \mathrm{R}^2(f|_U)_!\,\overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}(-1)$$

by [4, dualité, Thm. 1.3, 2.1], and this pairing factors through $\det(\mathcal{G})$. ∎

(3.4) Proposition *Assume that χ is a non-trivial character of $\mathbb{F}^\times$. Denote by $\mathrm{R}\Phi_{b_0}(\overline{\mathcal{F}(\chi)})$ (resp. $\mathrm{R}\Phi_{b_1}(\overline{\mathcal{F}(\chi)})$ the complex of vanishing cycles for the sheaf $\overline{\mathcal{F}(\chi)}$ with respect to the map $f : \mathfrak{X} \to B$ over the point $b_0 \in B$ (resp. $b_1 \in B$.)*

(i) *The stalks of the vanishing cycle complex at the intersection points with the horizontal divisors are acyclic:*

$$\Phi^j_{b_0}(\overline{\mathcal{F}(\chi)})_{s_{0,i}} = 0 \quad \forall j \geq 0,\ i = 1, 2, 3, 4,$$
$$\Phi^j_{b_1}(\overline{\mathcal{F}(\chi)})_{s_{1,i}} = 0 \quad \forall j \geq 0,\ i = 1, 2, 3, 4.$$

(ii) *At the singular points s_0, s_1 of $\mathfrak{X}_0$, $\mathfrak{X}_1$, we have for $i = 0, 1$*

$$\dim\bigl(\Phi^j_{b_i}(\overline{\mathcal{F}(\chi)})_{s_i}\bigr) = \begin{cases} 0 & \text{if } \ j \neq 1 \\ 1 & \text{if } \ j = 1 \end{cases}$$

(iii) *The action of the decomposition group D_{b_i} on the one-dimensional space $\Phi^1_{b_i}(\overline{\mathcal{F}(\chi)})_{s_i}$ is unramified. The geometric Frobenius Fr_{b_0} operates as q on $\Phi^1_{b_0}(\overline{\mathcal{F}(\chi)})_{s_0}$, while the geometric Frobenius Fr_{b_1} operates as $\chi(-1)q$ on $\Phi^1_{b_1}(\overline{\mathcal{F}(\chi)})_{s_1}$.*

Proof. The statement (i) follows from Lemma 2.13. The statements (ii) and (iii) are consequences of [3], (3.1.3) case (a), since the restriction of $\mathcal{F}(\chi)$ to the $\mathbb{F}$-rational point b_0 (resp. b_1) is a rank-one $\overline{\mathbb{Q}_\ell}$-sheaf over $\operatorname{Spec}\mathbb{F}$ on which Fr_q operates as $\chi(1) = 1$ (resp. $\chi(-1)$.) ∎

(3.5) Corollary *Assume that χ is non-trivial. Then the representation of the decomposition group D_b on the geometric generic fiber $\mathcal{G}_{\bar\eta}$ of $\mathcal{G}$ is tamely ramified for $b = b_0$ and $b = b_1$. Moreover*

(1) *At b_0, the local Galois representation corresponds to the representation $sp(2) \otimes \overline{\mathbb{Q}_\ell}(-1)$ of the Deligne-Weil group; i.e. it corresponds to the special representation $\sigma(\|\cdot\|^{-1}, \mathbf{1})$ of $GL(2)$ under the local Langlands correspondence. The local Artin conductor is equal to 1, and the local L-factor is $(1 - q^{-s})^{-1}$.*

(2) *At b_1, the local Galois representation corresponds to the representation $sp(2) \otimes \chi_{-1}(-1)$, where $\chi_{-1}(-1)$ denotes the unramified character whose value at the geometric Frobenius is equal to $\chi(-1)q$. Its local L-factor is equal to $(1 - \chi(-1)q^{-s})^{-1}$.*

(3.6) Theorem *Assume that χ is not the trivial character of $\mathbb{F}^\times$. Let $\mathcal{L}_{\chi(t)}$ (resp. $\mathcal{L}_{\chi^{-1}(t)}$) be the rank-one smooth sheaf on $\operatorname{Spec}\mathbb{F}[t, t^{-1}] = B - \{b_0, b_\infty\}$ attached to χ (resp. χ^{-1}), where t is the rational function $\frac{T}{S}$ on B. Let $s_{\infty,1}, s_{\infty,2}$ be the two intersection points of $E_{Z,S}$ with the horizontal divisors, and let $E'_{Z,S} = E_{Z,S} - \{s_\infty, s_{\infty,1}, s_{\infty,2}\}$. Similarly let $s_{\infty,3}, s_{\infty,4}$ be the two intersection points of $E_{X,S}$ with the horizontal divisors, and let $E'_{Z,X} = E_{Z,X} - \{s_\infty, s_{\infty,3}, s_{\infty,4}\}$.*

(i) *Write $E'_{Z,S} = \operatorname{Spec}\mathbb{F}[u, \frac{1}{u^2-1}]$, where u is the restriction to $E'_{Z,S}$ of the rational function $\frac{Y}{X}$ on $\mathfrak{X}$. The restriction of the vanishing cycle complex $\mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})$ to $E'_{Z,S}$ is represented by the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_{\chi^{-1}(4^{-1}(u^2-1))}$ in degree zero, and the decomposition group D_{b_∞} acts on this rank-one sheaf via its natural action on $\left(\mathcal{L}_{\chi(t)}\right)_{\bar\eta}$.*

(ii) *Write $E'_{X,S} = \operatorname{Spec}\mathbb{F}[y, \frac{1}{y^2-1}]$, where u is the restriction to $E'_{X,S}$ of the rational function $\frac{Y}{T}$ on $\mathfrak{X}$. The restriction of the vanishing cycle complex $\mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})$ to $E'_{X,S}$ is represented by the smooth rank-one $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{L}_{\chi(4^{-1}(y^2-1))}$ in degree zero, and the decomposition group D_{b_∞} acts on this rank-one sheaf via its natural action on $\left(\mathcal{L}_{\chi^{-1}(t)}\right)_{\bar\eta}$.*

(iii) *The stalk of the vanishing cycle complex $\mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})$ at $s_{\infty,i}$ is acyclic for $i = 1, 2, 3, 4$.*

(iv) *If χ^2 is non trivial, then the stalk of the vanishing cycle complex $\mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})$ at s_∞ is acyclic.*

(v) *If χ^2 is trivial, then*

$$\dim\left(\Phi^j_{b_\infty}(\overline{\mathcal{F}(\chi)})_{s_\infty}\right) = \begin{cases} 0 & \text{if} \quad j \neq 0, 1 \\ 1 & \text{if} \quad j = 0, 1\,. \end{cases}$$

The decomposition group D_{b_∞} operates on the one-dimensional $\overline{\mathbb{Q}_\ell}$-vector space

$$\Phi^0_{b_\infty}(\overline{\mathcal{F}(\chi)})_{s_\infty} \qquad (\textit{resp. } \Phi^1_{b_\infty}(\overline{\mathcal{F}(\chi)})_{s_\infty}\,)$$

via its natural action on $\left(\mathcal{L}_{\chi(t)}(-1)\right)_{\bar\eta}$ (resp. $\left(\mathcal{L}_{\chi(t)}\right)_{\bar\eta}$.)

Proof. Since the proof of (i) and (ii) are essentially the same, we only give the proof of (i). Let $u = \frac{Y}{X}$, $v = \frac{Z}{X}$ and $s = \frac{S}{T}$. Then near $E'_{Z,S}$ the surface $\mathfrak{X}$ is defined by the equation

$$s\,(u^2 - (1 - v)^2) = 4v\,.$$

We have

$$\mathcal{L}_{\chi(x)} = \mathcal{L}_{\chi(v^{-1})} = \mathcal{L}_{\chi(s^{-1})} \otimes \mathcal{L}_{\chi^{-1}\left(\frac{u^2-(1-v)^2}{4}\right)} ,$$

and the sheaf $\mathcal{L}_{\chi^{-1}\left(\frac{u^2-(1-v)^2}{4}\right)}$ is smooth at points of $E'_{Z,S}$. This proves (i).

The statement (iii) follows from Lemma 2.13. For (iv) and (v), let $u_1 = \frac{X}{Y}$, $v_1 = \frac{Z}{Y}$, $s = \frac{S}{T}$; u_1, v_1 is a regular system of parameters for $\mathcal{O}_{\mathfrak{X},s_\infty}$. The equation for $\mathfrak{X}$ near s_∞ is

$$s\,(1-(u_1-v_1)^2) = 4u_1v_1\,.$$

We have

$$\mathcal{L}_{\chi(x)} = \mathcal{L}_{\chi(u_1)\chi^{-1}(v_1)} = \mathcal{L}_{\chi(s)} \otimes \mathcal{L}_{\chi^{-2}(v_1)} \otimes \mathcal{L}_{\chi\left(\frac{1-(u_1-v_1)^2}{4}\right)} ,$$

and $\mathcal{L}_{\chi\left(\frac{1-(u_1-v_1)^2}{4}\right)}$ is smooth at s_∞. The statement (v) follows directly from the above by the definition of vanishing cycles, while we deduce (iv) from Proposition 2.14. ∎

(3.6.1) Remarks It may be instructive to explain the calculation of vanishing cycles at s_∞ in an analogous complex analytic situation.

(1) When the bad point c is the double point s_∞, the variety of vanishing cycles W is

$$W = \left\{(\tau,u,v) \in \mathbb{H}\times\Delta^*\times\Delta^* \,|\, \exp(2\pi\sqrt{-1}\tau) = uv\right\},$$

where $\mathbb{H}$ is the upper half-plane and Δ^* is the punctured unit disk. We have a map $f : W \to \Delta^*$ which sends (τ,u,v) to $\frac{u}{v}$. Let $\mathbb{L}$ be a non-trivial rank-one local system on Δ^*. Then all cohomologies $\mathrm{H}^i(W, f^*\mathbb{L})$ vanish, for all $i \geq 0$ if $\mathbb{L}^{\otimes 2}$ is non-trivial. When $\mathbb{L}^{\otimes 2}$ is trivial, both $\mathrm{H}^0(W, f^*\mathbb{L})$ and $\mathrm{H}^1(W, f^*\mathbb{L})$ are one-dimensional; all other cohomologies vanish. By homotopy invariance, the above statements quickly reduce to the fact that all cohomologies of a non-trivial rank-one local system on Δ^* vanish, while the zeroth and the first cohomology of the trivial rank-one local system on Δ^* are one-dimensional.

(2) When the bad point c is one of the four intersection points of the horizontal ramification divisors with $\mathfrak{X}_\infty$, the situation is even easier. The variety of vanishing cycles is $W = \mathbb{H}\times\Delta$, where Δ is the unit disk. We have a map $g : \mathbb{H}\times\Delta^* \to \Delta^*$, which sends (τ, u) to $\exp(2\pi\sqrt{-1}\tau)u$. Let $\mathbb{L}$ again be a non-trivial rank-one local system on Δ^*. Let $j : \mathbb{H}\times\Delta^* \to W$ be the inclusion. Then $\mathrm{H}^i(W, j_!g^*\mathbb{L}) = 0$ for all $i \geq 0$. Homotopy invariance quickly reduces this to the fact that all cohomologies $\mathrm{H}^i(\Delta, (\Delta^* \hookrightarrow \Delta)_!\mathbb{L})$ vanish.

(3.7) Proposition *Let χ be a nontrivial character of $\mathbb{F}^\times$.*

(1) *The smooth rank-two sheaf $\mathcal{G} = \mathrm{R}^1 f_*\overline{\mathcal{F}(\chi)}$ over V is tamely ramified at b_∞ with local conductor equal to 2, and the local L-factor is trivial.*

(2) *The geometric monodromy group for $\mathcal{G}$ is equal to $\mathrm{SL}(2)$.*

(3) *When $\chi^2 \neq \mathbf{1}$, the local Galois representation at b_∞ is a direct sum of two one-dimensional characters λ_1, λ_2 of weight one, and $\lambda_1\lambda_2 = \|\cdot\|^{-1}$. Both λ_1 and λ_2 are tamely ramified, and the restriction of each λ_i to the tame ramification group is equal to $\chi^{\pm 1}$ if we identify $\mathbb{F}^\times$ as the canonical quotient of the tame ramification group of $\mathbb{G}_m$ at 0 or ∞ via the Lang torsor.*

(4) *When $\chi^2 = \mathbf{1}$, the local Galois representation at b_∞ corresponds to the special representation $\sigma(\lambda\|\cdot\|^{-1}, \lambda)$ of $GL(2)$ under the local Langlands correspondence, where λ is a tamely ramified character with $\lambda^2 = \mathbf{1}$ whose restriction to the tame ramification group is the unique character of order two.*

Proof. One deduces statements (1), (3), (4) from Theorem 3.6, using the natural isomorphism

$$\mathcal{G}_{\bar{\eta}} = \mathrm{H}^1\big(\mathfrak{X}_\infty \times_{\operatorname{Spec}\mathbb{F}} \operatorname{Spec}\overline{\mathbb{F}}, \mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})\big)$$

of D_{b_∞}-modules. For instance if χ^2 is not trivial, then $\mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)}$ is represented by the smooth rank-one sheaf on $E'_{Z,S} \sqcup E'_{X,S}$ described in (i), (ii) of Theorem 3.6, extended by zero to $\mathfrak{X}_\infty$. According to Thm. 3.6 (i), the restriction of $\mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)}$ to $E'_{Z,S}$ is isomorphic to $\mathcal{L}_{\chi^{-1}(\frac{u^2-1}{4})}$, which is ramified at $u = 1, -1, \infty$; similarly for the restriction to $E'_{X,S}$. Thus both $\mathrm{H}^1_c\big(E'_{Z,S} \times_{\operatorname{Spec}\mathbb{F}} \operatorname{Spec}\overline{\mathbb{F}}, \mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})\big)$ and $\mathrm{H}^1_c\big(E'_{X,S} \times_{\operatorname{Spec}\mathbb{F}} \operatorname{Spec}\overline{\mathbb{F}}, \mathrm{R}\Phi_{b_\infty}(\overline{\mathcal{F}(\chi)})\big)$ are one-dimensional, pure of weight one, with tame action by the decomposition group D_∞ as described in (i), (ii) of Thm. 3.6. This proves (3) and the statement (1) when χ^2 is not trivial; the proof of (4) and the rest of (1) is similar.

Since the determinant of $\mathcal{G}$ comes from the sheaf $\overline{\mathbb{Q}_\ell}(-1)$ on $\operatorname{Spec}\mathbb{F}$, its geometric monodromy group is contained in SL(2). On the other hand we see from Prop. 3.4 that the geometric monodromy group of $\mathcal{G}$ is not finite, so this semisimple group must be equal to SL(2). This proves (2). ∎

§4. Automorphic forms and applications to Terras graphs

We shall first reformulate the results in the previous two sections in terms of automorphic forms for $GL(2)$, and then show their connections to automorphic forms on quaternion group D and eigenvalues of Terras graphs. Let K be a function field with the field of constants $\mathbb{F}$. For convenience, the residue field of K at a place v will be denoted $\mathbb{F}_v$, which has cardinality Nv.

(4.1) Let C be the underlying curve of K. Given a rational function a in K, it defines a morphism from C to $\mathbb{P}^1$. Denote by $\mathfrak{X}_a$ the pull-back surface in $C \times \mathbb{P}^2$ of the surface $\mathfrak{X}$ in §3 via a and by $\mathcal{F}_a(\chi)$ the pullback sheaf on $\mathfrak{X}_a$ of the sheaf $\mathcal{F}(\chi)$ on $\mathfrak{X}$. The sheaf $\mathrm{R}^1(\mathrm{pr}_1)_!\mathcal{F}_a(\chi)$ is smooth of rank two when restricted to a sufficiently small open subscheme $U_{\chi,a} \subset C$; it coincides with the pull-back by the morphism $U_{\chi,a} \hookrightarrow C \xrightarrow{a} \mathbb{P}^1$ of the sheaf $\mathcal{G}(\chi)$ in the notation of Lemma 3.3. Let $\rho_{a,\chi}$ be the degree two representation of $\mathrm{Gal}(K^{sep}/K)$ attached to $\mathrm{R}^1(\mathrm{pr}_1)_!\mathcal{F}_a(\chi)|_{U_{\chi,a}}$. By the results of Grothendieck, Deligne, and the converse theorem of $GL(2)$, we know that the L-function attached to $\rho_{a,\chi}$ is an automorphic L-function for $GL(2)$ over K. We examine its local factors.

Let $g(x) = (x-1)^2 + 4ax$. Denote by $g_v(x)$ the polynomial $g(x) \pmod v$. Our computation in the previous section shows that at a place v which is not a pole of a and where $a \not\equiv 0, 1 \pmod v$, the local factor of $L(s, \rho_{a,\chi})$ at v is a polynomial of degree 2 in Nv^{-s}, with the coefficient of Nv^{-s} being

$$-\lambda_\chi(\mathbb{F}_v; a) \;=\; \sum_{\substack{x,y\in\mathbb{F}_v \\ y^2=g_v(x)}} \chi \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(x),$$

and the coefficient of Nv^{-2s} being Nv by Lemma 3.3. Hence the local factor at v is

$$1 - \lambda_\chi(\mathbb{F}_v; a)Nv^{-s} + Nv^{1-2s}.$$

We have shown

(4.2) Theorem *Let K be a function field with the field of constants $\mathbb{F}$. Given a nonzero element a in K and a nontrivial character χ of $\mathbb{F}^\times$, there is an automorphic form $f_{a,\chi}$ of GL_2 over K which is an eigenfunction of the Hecke operator T_v at all places v of K, which is not a pole of a and where $a \not\equiv 0, 1 \pmod v$, with eigenvalue $\lambda_\chi(\mathbb{F}_v; a)$ as defined above. In other words,*

$$L(s, f_{a,\chi}) \sim \prod_{v \text{ good}} \frac{1}{1 - \lambda_\chi(\mathbb{F}_v; a)Nv^{-s} + Nv^{1-2s}}.$$

Here and later we use $L_1 \sim L_2$ to mean that two Euler products L_1 and L_2 agree at all but finitely many factors.

At each place v choose a nonsquare δ in $\mathbb{F}_v$. Observe (cf. [5]) that

$$-\lambda_\chi(\mathbb{F}_v; a) = -\sum_{\substack{x,y\in\mathbb{F}_v \\ y^2=\delta g_v(x)}} \chi \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(x)$$

since

$$\sum_{\substack{x,y\in\mathbb{F}_v \\ y^2=g_v(x)}} \chi \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(x) + \sum_{\substack{x,y\in\mathbb{F}_v \\ y^2=\delta g_v(x)}} \chi \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(x) = 2\sum_{x\in\mathbb{F}_v} \chi \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(x) = 0.$$

Note that the character sum is independent of the choice of δ. Therefore $\lambda_\chi(\mathbb{F}_v; a)$ is nothing but the eigenvalue $\lambda_{4a,\chi\circ\mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}}$ of a Terras graph with base field $\mathbb{F}_v$. This theorem shows that the eigenvalues of $f_{a,\chi}$ with respect to the Hecke operators are eigenvalues of Terras graphs of first type, parallel to Theorem 3.6.1 in Part I. Its existence for the case $K = \mathbb{F}(t)$ was conjectured in [10].

(4.2.1) As a consequence of Proposition 3.7 (2), the geometric monodromy group of the ℓ-adic representation $\rho_{a,\chi}$ is $SL(2)$ provided that a is not a constant. In other words, the Sato-Tate conjecture holds for $f_{a,\chi}$.

(4.2.2) Corollary *With the same notation as in Theorem 4.2, suppose that a is not a constant. Then the normalized eigenvalues $\lambda_\chi(\mathbb{F}_v; a)/\sqrt{|\mathbb{F}_v|}$ are uniformly distributed with respect to the Sato-Tate measure*

$$\mu_{ST}(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx$$

when Nv tends to infinity.

(4.3) Let D be the algebraic group over K attached to the multiplicative group of the quaternion algebra H over K ramified exactly at 0 and ∞ as in §1. Given an element $b \in \mathbb{F}$ not equal to 0 or 1, we specialize Theorem 4.2 to the case $K = \mathbb{F}(t)$ and $a = (t-1)/(t-b)$. The bad places for the automorphic form $f_{a,\chi}$ are 1, b and ∞, and the local components of the corresponding representation of $GL(2)$ over K are special representations at the places 1 and ∞ with nontrivial local L-factors as shown in Corollary 3.5, and is either a principal series or a special representation at the place b with trivial local L-factor as shown in Proposition 3.7. Moreover, the local component of $f_{a,\chi}$ at the place b is a new vector for the congruence subgroup $\Gamma_0((t-b)^2)$ with central character $\| \cdot \|^{-1}$. Conjugation of $f_{a,\chi}$ by $\begin{pmatrix} t-b & 0 \\ 0 & 1 \end{pmatrix}$ yields another automorphic form which has the same properties as $f_{a,\chi}$ at all places except b, and at b it is invariant by the principal congruence subgroup $\Gamma(t-b)$. By the global correspondence between automorphic representations of the quaternion group D over K and those of $GL(2)$ over K proved in [6], there is an automorphic form $f_{b,\chi}$ on $D(A_K)$ right invariant by $\mathcal{K}_b$ (as defined in Introduction) which has the same L-function as $f_{a,\chi}$. In particular, $f_{b,\chi}$ is an eigenfunction of the Hecke operator T_0 at the place 0 with eigenvalue $\lambda_\chi(\mathbb{F}; 1/b)$. We compare the eigenvalue $\lambda_\chi(\mathbb{F}; 1/b)$ with $\lambda_{4(b-1)/b,\chi}$.

Let $z = -x$ and note that

$$\delta(x-1)^2 + 4\delta(b-1)x/b = \delta(x+1)^2 - 4\delta x/b = \delta(z-1)^2 + 4\delta z/b,$$

so we have

$$\lambda_\chi(\mathbb{F}; 1/b) = \chi(-1)\lambda_{4(b-1)/b,\chi}.$$

Thus if χ is an even character, then $f_{b,\chi}$ is an eigenfunction of T_0 with eigenvalue $\lambda_{4(b-1)/b,\chi}$, while if χ is an odd character, then the eigenvalue of T_0 is opposite to what we are looking for. Therefore, the unramified twist $g_{b,\chi}$ of $f_{b,\chi}$ defined by

$$g_{b,\chi}(x) = \chi(-1)^{\deg \mathrm{Nm}_{\mathrm{rd}}(x)} f_{b,\chi}(x)$$

for $x \in D(A_K)$ has eigenvalue $\lambda_{4(b-1)/b,\chi}$ at place 0 in both cases. This proves

(4.4) Theorem *Let K be the rational function field $\mathbb{F}(t)$. Given a nonzero element $b \in \mathbb{F}$ not equal to 1 and a nontrivial character χ of $\mathbb{F}^\times$, there is an automorphic form $g_{b,\chi}$ on $D(A_K)$, right invariant by $\mathcal{K}_b$ as in Introduction, which is an eigenfunction of Hecke operators T_v for all places v other than $1, b, \infty$, and whose associated L-function is*

$$L(s, g_{b,\chi}) = \frac{1}{1-\chi(-1)q^{-s}} \cdot \frac{1}{1-q^{-s}} \prod_{v \neq 1, b, \infty} \frac{1}{1 - \chi(-1)^{\deg v}\lambda_\chi(\mathbb{F}_v; \frac{t-1}{t-b})Nv^{-s} + Nv^{1-2s}}.$$

In particular, $g_{b,\chi}$ is an eigenfunction of T_0 with eigenvalue $\lambda_{4(b-1)/b,\chi}$.

In the above expression of L-function, the first factor is at place 1, and the second is at place ∞, by Corollary 3.5.

(4.5) Next we reformulate the results from section 2. Recall that $\mathbb{F}$ has cardinality q. As in the beginning of this section, a rational function a in K defines a morphism from the curve C to the projective line $\mathbb{P}^1$, which in turn yields a map from $U \times_{\mathrm{Spec}\,\mathbb{F}} C$ to $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{P}^1$. This allows us to pull back the open subset Y of $U \times_{\mathrm{Spec}\,\mathbb{F}} \mathbb{A}^1$ to an open subset Y_a of $U \times_{\mathrm{Spec}\,\mathbb{F}} C$. Given a nontrivial character ε of $\mathbb{F}^\times$ and a regular character ω of the quadratic extension $\mathbb{F}'$ (which induces a nontrivial character of $U(\mathbb{F})$), denote by $\mathcal{F}_{a,\varepsilon,\omega}$ the sheaf on Y_a which is the pull-back of the sheaf $\mathcal{F}_{\varepsilon,\omega}$ on Y. The restriction of the sheaf $\mathrm{R}^1(\mathrm{pr}_1)_!\mathcal{F}_{a,\varepsilon,\omega}$ to a sufficiently small open subset $U_{a,\varepsilon,\omega} \subset \mathbb{P}^1$ is smooth of rank two; it is equal to the pull-back of the sheaf $\mathcal{G}_{\varepsilon,\omega}$ in 2.6 by the composition $U_{a,\varepsilon,\omega} \hookrightarrow C \xrightarrow{a} \mathbb{P}^1$. Let $\rho_{a,\varepsilon,\omega}$ be the representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ associated to the smooth rank-two sheaf $\mathrm{R}^1(\mathrm{pr}_1)_!\mathcal{F}_{a,\varepsilon,\omega}|_{U_{a,\varepsilon,\omega}}$ on $U_{a,\varepsilon,\omega}$, and write $\eta_{a,\varepsilon,\omega}$ for the determinant of $\rho_{a,\varepsilon,\omega}$. As before, the global L-function attached to the representation $\rho_{a,\varepsilon,\omega}$ is an automorphic L-function for $GL(2)$ over K with central character $\eta_{a,\varepsilon,\omega}$. We examine the local factors of the L-function.

At each place v away from the poles of a and where $a \not\equiv \pm 2 \pmod v$, let

$$-\lambda_{\varepsilon,\omega}(\mathbb{F}_v; a) \;=\; \sum_{u \in U(\mathbb{F}_v)} \varepsilon \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(\mathrm{Tr}_{\mathbb{F}'_v/\mathbb{F}_v}(u) + a_v)\omega \circ \mathrm{Nm}_{\mathbb{F}'_v/\mathbb{F}'}(u),$$

where a_v denotes $a \pmod v$ and $\mathbb{F}'_v = \mathbb{F}_v \otimes_{\mathbb{F}} \mathbb{F}'$. When $\deg v$ is odd, $\mathbb{F}'_v$ is a quadratic extension of $\mathbb{F}_v$ and

$$-\lambda_{\varepsilon,\omega}(\mathbb{F}_v; a) = \sum_{u \in \mathbb{F}'_v,\, \mathrm{Nm}_{\mathbb{F}'_v/\mathbb{F}_v}(u)=1} \varepsilon \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(\mathrm{Tr}_{\mathbb{F}'_v/\mathbb{F}_v}(u) + a_v)\omega \circ \mathrm{Nm}_{\mathbb{F}'_v/\mathbb{F}'}(u).$$

When $\deg v$ is even, $\mathbb{F}_v$ contains $\mathbb{F}'$ so that $\mathbb{F}'_v = \mathbb{F}_v \times \mathbb{F}_v$ and elements in $U(\mathbb{F}_v)$ are the pairs $(x, 1/x)$ as x runs through all nonzero elements in $\mathbb{F}_v$. In this case

$$\begin{aligned} -\lambda_{\varepsilon,\omega}(\mathbb{F}_v; a) &= \textstyle\sum_{u \in \mathbb{F}_v^\times} \varepsilon \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}(u + \frac{1}{u} + a_v)\omega \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}'}(\frac{u}{u^q}) \\ &= \textstyle\sum_{u \in \mathbb{F}_v} \varepsilon \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}((u-1)^2 + (a_v+2)u)(\varepsilon \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}} \cdot \omega^{1-q} \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}'})(u). \end{aligned}$$

It follows from the computations in §2 that the L-factor at v is

$$1 - \lambda_{\varepsilon,\omega}(\mathbb{F}_v; a)Nv^{-s} + \eta_{a,\varepsilon,\omega}(\pi_v)Nv^{-2s}.$$

Here π_v is a uniformizer at the place v. We summarize the above discussion in

(4.6) Theorem *Let K be a function field with the field of constants $\mathbb{F}$. Let a be a nonzero element in K, ε be a nontrivial character of $\mathbb{F}^\times$, and ω be a regular character of $\mathbb{F}'^\times$, such that either ε^2 is a nontrivial character of $\mathbb{F}^\times$, or ω^2 is a regular character of $\mathbb{F}'^\times$. Then there is an automorphic form $f_{a,\varepsilon,\omega}$ of GL_2 over K, with central character $\eta_{a,\varepsilon,\omega}$, which is an eigenfunction of the Hecke operator T_v at all places v of K, which is not a pole of a and where $a \not\equiv \pm 2 \pmod v$, with eigenvalue $\lambda_{\varepsilon,\omega}(\mathbb{F}_v; a)$ as defined above. In other words,*

$$L(s, f_{a,\varepsilon,\omega}) \sim \prod_{v \text{ good}} \frac{1}{1 - \lambda_{\varepsilon,\omega}(\mathbb{F}_v; a)Nv^{-s} + \eta_{a,\varepsilon,\omega}(\pi_v)Nv^{-2s}}.$$

Moreover, if ε has order two, then $\eta_{a,\varepsilon,\omega}$ is the unramified idele class character $\|\cdot\|^{-1}$.

The last statement follows from Proposition 2.3, (vi). For the remainder of this section, assume ε has order two. We remark that, up to sign, the eigenvalue $\lambda_{\varepsilon,\omega}(\mathbb{F}_v; a)$ of $f_{a,\varepsilon,\omega}$ at a good place v is an eigenvalue of a Terras graph with base field $\mathbb{F}_v$. Specifically, the computation above shows that if the degree of v is odd, then it is equal to $-\varepsilon(-1)\lambda_{2-a_v,\omega\circ \mathrm{Nm}_{\mathbb{F}'_v/\mathbb{F}'}}$; while if the degree of v is even, it is equal to $\lambda_{a_v+2,\chi}$ with the character χ of $\mathbb{F}_v^\times$ being $\varepsilon \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}}\, \omega^{1-q} \circ \mathrm{Nm}_{\mathbb{F}_v/\mathbb{F}'}$. Let $g_{a,\varepsilon,\omega}$ be the twist of $f_{a,\varepsilon,\omega}$ by the unramified character $-\varepsilon(-1)$, that is,

$$g_{a,\varepsilon,\omega}(x) = (-\varepsilon(-1))^{\deg \det(x)} f_{a,\varepsilon,\omega}(x)$$

for all $x \in GL_2(A_K)$. Then the eigenvalues of $g_{a,\varepsilon,\omega}$ at places of odd degree are eigenvalues of Terras graphs of the second type, while those at places of even degree are eigenvalues of Terras graphs of the first type.

(4.7) When ε has order two and a is not a constant, we know from Theorem 2.9 that the geometric monodromy of the representation $\rho_{a,\varepsilon,\omega}$ is $SL(2)$. Therefore, the Sato-Tate conjecture holds for $f_{a,\varepsilon,\omega}$.

(4.7.1) Theorem *With the same notation as in Theorem 4.6, assume further that ε is a character of order two and that a is not a constant. Then the normalized Fourier coefficients $\lambda_{\varepsilon,\omega}(\mathbb{F}_v; a)/\sqrt{|\mathbb{F}_v|}$ of $f_{a,\varepsilon,\omega}$ are uniformly distributed with respect to the Sato-Tate measure μ_{ST} as Nv tends to infinity.*

(4.8) We specify Theorem 4.6 to the case $K = \mathbb{F}(t)$ and $a = \frac{2t-4+2b}{t-b}$ for an element $b \in \mathbb{F}$ and $b \neq 0, 1$. Assume that ε has order two and ω has order greater than two. The bad places for the automorphic form $g_{a,\varepsilon,\omega}$ are at $1, \infty$ and b. The local components of the corresponding representation of $GL(2)$ over K are special representations at 1 and ∞ with nontrivial local L-factors by Theorem 2.9, and it is a principal series at place b with trivial local L-factor by Theorem 2.12. Moreover, the local component of $g_{a,\varepsilon,\omega}$ at the place b is a new vector for the congruence subgroup $\Gamma_0((t-b)^2)$ with central character $\|\cdot\|^{-1}$. By the same argument as in the previous case, we obtain an automorphic form $g_{b,\varepsilon,\omega}$ of $D(A_K)$ right invariant by $\mathcal{K}_b$ (as defined in Introduction) which has the same L-function as $g_{a,\varepsilon,\omega}$. In particular, as remarked right after Theorem 4.6, $g_{b,\varepsilon,\omega}$ is an eigenfunction of the Hecke operator T_0 at place 0 with eigenvalue $-\varepsilon(-1)\lambda_{\varepsilon,\omega}(\mathbb{F}; -2+4/b)$, which is equal to the eigenvalue $\lambda_{4(b-1)/b,\omega}$ of the Terras graph $X_{4(b-1)/b}$ as defined in Introduction.

Combining the above discussion with Theorems 2.9 and 2.12, we obtain

(4.8.1) Theorem *Let K be the rational function field $\mathbb{F}(t)$. Given a nonzero element $b \in \mathbb{F}$ not equal to 1, a character ε of $\mathbb{F}^\times$ of order two, and a regular character ω of $\mathbb{F}'$ of order greater than two, there is an automorphic form $g_{b,\varepsilon,\omega}$ on $D(A_K)$, right invariant by $\mathcal{K}_b$ as in Introduction, which is an eigenfunction of the Hecke operators T_v for all places v outside $1, b, \infty$, and whose associated L-function is*

$$L(s, g_{b,\varepsilon,\omega}) = \frac{1}{1-\omega(-1)q^{-s}} \cdot \frac{1}{1-\varepsilon(-1)q^{-s}} \prod_{v\neq 1,b,\infty} \frac{1}{1-(-\varepsilon(-1))^{\deg v}\lambda_{\varepsilon,\omega}(\mathbb{F}_v; \frac{2t-4+2b}{t-b})Nv^{-s} + Nv^{1-2s}}.$$

In particular, $g_{b,\varepsilon,\omega}$ is an eigenfunction of T_0 with eigenvalue $\lambda_{4(b-1)/b,\omega}$.

(4.8.2) Remarks (1) The eigenvalues of $g_{b,\varepsilon,\omega}$ are eigenvalues of Terras graphs, as discussed in detail after Theorem 4.6.

(2) In writing $L(s, g_{b,\varepsilon,\omega})$ as an Euler product above, the first factor is the local factor at the place ∞ and the second factor is at the place 1.

(4.9) Recall from Proposition 1.1 that the Terras graph $X_{4(b-1)/b}$ is a quotient graph of the Morgenstern graph $X_{\mathcal{K}_b}$. For each nontrivial character χ of $\mathbb{F}^\times$, we have found an automorphic form $g_{b,\chi}$ on $X_{\mathcal{K}_b}$ which realizes the eigenvalue $\lambda_{4(b-1)/b,\chi}$ of the Terras graph $X_{4(b-1)/b}$, and for each regular character ω of $\mathbb{F}'$ of degree greater than two, we have found an automorphic form $g_{b,\varepsilon,\omega}$ on $X_{\mathcal{K}_b}$ which realizes the eigenvalue $\lambda_{4(b-1)/b,\omega}$ of the Terras graph $X_{4(b-1)/b}$. This answers question (i) in Introduction. As for question (ii), we conclude from Theorems 4.2.2 and 4.7.1 that eigenvalues of type $\lambda_{a,\chi}$ (resp. of type $\lambda_{a,\omega}$) are uniformly distributed with respect to the Sato-Tate measure, as the cardinality of the underlying finite field of Terras graph tends to infinity.

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