THE HECKE ORBIT CONJECTURE: A SURVEY AND OUTLOOK

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1. INTRODUCTION

Entry 15 of [41], *Hecke orbits*, contains the following prediction.

> *Given any $\overline{\mathbb{F}}_p$-point $x = [(A_x, \lambda_x)]$ in the moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties in characteristic $p > 0$, the Hecke orbit of x, which consists of all points $[(B, \nu)] \in \mathscr{A}_g$ related to $[(A_x, \lambda_x)]$ by symplectic isogenies, is Zariski dense in the Newton polygon stratum of $\mathscr{A}_g$ which contains x.*

The obvious generalization of this prediction, for (good) modular varieties of PEL-type in positive characteristic p or more generally for reduction modulo p of Shimura varieties, has since become known as (part of) the Hecke orbit conjecture. In this chapter we will survey results as well as new structures and methods which have been developed since 1995, in response to the above conjecture. We will also formulate several rigidity-type questions related to Hecke symmetries on reductions of Shimura varieties. These questions arose naturally in the pursuit of, and are similar in spirit to, the Hecke orbit conjecture.

1.1. What is the Hecke orbit conjecture?

We will consider modular varieties of PEL type over $\overline{\mathbb{F}}_p$ in this chapter. A salient feature of such a modular variety $\mathcal{M}$ is that it has a large collection of symmetries, called prime-to-p Hecke correspondences; see §2. These symmetries lift to symmetries on the universal family of abelian scheme with PEL structure on $\mathcal{M}$, preserving all p-adic invariants coming from the p-divisible groups attached to the universal abelian scheme.

Central leaves in a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$. Given a point $x_0 \in \mathcal{M}$, the locus of all points of $\mathcal{M}$ having "the same p-adic invariants as x_0" is a smooth locally closed algebraic subvariety $\mathcal{C}(x_0)$ of $\mathcal{M}$, call the *central leaf* in $\mathcal{M}$ passing through x_0. More precisely, $\mathcal{C}(x_0)$ is characterized by the following property that for every algebraically closed extension field Ω of $\overline{\mathbb{F}}_p$ and every geometric point $y \in \mathcal{M}(\Omega)$, the p-divisible groups with imposed polarization and endomorphism structure attached to the fibers at y and x_0 of the universal abelian scheme are isomorphic over Ω. See 3.3 for the definition of $\mathcal{C}(x_0)$.

The following prediction is known as the *Hecke orbit conjecture* for the PEL modular variety $\mathcal{M}$.

Hecke orbit conjecture for a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$.
The prime-to-p Hecke orbit $\mathcal{H}^{(p)} \cdot x_0$ of a point $x_0 \in \mathcal{M}(\overline{\mathbb{F}}_p)$ is Zariski dense in the central leaf $\mathcal{C}_\mathcal{M}(x_0) \subseteq \mathcal{M}$ containing x_0.

Note that this conjecture *is a rigidity statement*. It asserts that there are not "too many" subvarieties of a modular variety of PEL-type over $\overline{\mathbb{F}}_p$ which are stable under all prime-to-p Hecke correspondences.

The Hecke orbit conjecture was first proved in the case when the central leaf is the ordinary locus of the Siegel moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}}_p$ in [1]. Subsequently it was shown for all central leaves in $\mathscr{A}_g$; see [3] for a sketch of a proof and also §8. The method used for $\mathscr{A}_g$ uses a special property of Siegel modular varieties and extends to some modular varieties of PEL type which also has this property; those modular varieties are of PEL type C. The Hecke orbit conjecture for general PEL-type modular varieties remains open.

1.2. New methods, structures and phenomena inspired by the Hecke orbit problem.

A challenging question in mathemtics often stimulates the development of new methods, triggers discoveries of new structures and phenomena, and begets further questions. The Hecke orbit conjecture is an example this paradigm.

(1) Foliation on moduli spaces underlying Hecke symmetries.
Every modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ of PEL type is a disjoint union of central leaves. Each central leaf $\mathcal{C}$ is a smooth locally closed subvariety $\mathcal{M}$ stable under all prime-to-p Hecke correspondences such that every p-adic invariant of the universal family of crystals of PEL type over $\mathcal{M}$ has "the same value" at all points of $\mathcal{C}$. See §3.3 for the precise definition.

Central leaves were first defined in [45] for $\mathscr{A}_g$, motivated by the problem of determining the Zariski closure of the prime-to-p Hecke orbit of an arbitrary point in $\mathscr{A}_g(\overline{\mathbb{F}}_p)$. The general Hecke orbit conjecture asserts that central leaves are the minimal elements in the collection of all reduced locally closed subvarieties of $\mathcal{M}$ which are stable under all prime-to-p Hecke correspondences.

(2) Sustained p-divisible groups.
Central leaves was first defined in [45] using the concept of *geometrically fiberwise constant p-divisible groups*. It is a "point-wise" definition and useful only for reduced base schemes. The notion of *sustained p-divisible groups* generalizes the concept of geometrically fiberwise constant p-divisible groups, and helps to elucidate structural properties of central leaves, including (a) the existence of slope filtration on the restriction to central leaves of the universal p-divisible group over $\mathcal{M}$, and (b) generalization of Serre–Tate coordinates for the local structure of central leaves. See 3.2 for a precise definition and [11], [13] for more information.

(3) Generalized Serre–Tate local coordinates on central leaves.

In several aspects central leaves in a modular variety $\mathcal{M}$ of PEL type over $\overline{\mathbb{F}_p}$ deserve to be thought of as analogs of Shimura varieties in positive characteristic p. A prominent property of a central leaf $\mathcal{C}$ is that the formal completion $\mathcal{C}^{/x_0}$ of $\mathcal{C}$ at a point $x_0 \in \mathcal{C}(\overline{\mathbb{F}_p})$ is "built up" from a system of fibrations whose fibers are p-divisible formal groups.

We would like to think of such generalized Serre–Tate coordinates on the formal completions $\mathcal{C}^{/x_0}$ as a sort of *Tate-linear structure* of $\mathcal{C}$—at the infinitesimal neighborhood of each closed point of a central leaf $\mathcal{C}$. Here "being Tate-linear" means "resembles a p-divisible groups". We emphasize that the resulting "coordinates" live entirely inside a central leaf $\mathcal{C}$, in characteristic p.

Let's illustrate the Tate-linear structure on $\mathcal{C}^{/x_0}$ with two easy examples.
(i) The first example is the Serre–Tate coordinates for the ordinary locus of $\mathscr{A}_g^{\mathrm{ord}}$, where the whole ordinary locus $\mathscr{A}_g^{\mathrm{ord}}$ is a single leaf, and the formal completion at a closed point $\mathscr{A}_g^{\mathrm{ord}}$ is a formal torus of dimension $g(g+1)/2$.
(ii) The next example is the central leaf in $\mathscr{A}_g$ attached to a moduli point $x_0 = [(A_{x_0}, \mu_{x_0})] \in \mathscr{A}_g(\overline{\mathbb{F}_p})$, such that the p-divisible group $A_{x_0}[p^\infty]$ is a direct product of isoclinic p-divisible groups Y, Z of height g and slopes s and $1 - s$ respectively, with $s < \frac{1}{2}$. Then $\mathcal{C}(x_0)^{/x_0}$ is an isoclinic p-divisible group over $\overline{\mathbb{F}_p}$ of height $g(g+1)/2$ and slope $1 - 2s$.

See §4.2.3 for an example where biextension of p-divisible formal groups appears, and also 4, [11, §5], [10, §8] and [13] for more information.

(4) Monodromy and irreducibility of Hecke-invariant subvarieties.

This is a general property of for Hecke-invariant subvarieties of a modular variety $\mathcal{M}$ of PEL-type over $\overline{\mathbb{F}_p}$. We assume for simplicity that the monodromy group G attached to the PEL input data is a connected reductive linear algebraic group over $\mathbb{Q}$ whose derived group is absolutely simple and simply connected.

Let V be a smooth locally closed subvariety V of $\mathcal{M}$ which is stable under all prime-to-p Hecke correspondences. Assume moreover that every prime-to-p Hecke orbit contained in V is infinite. Then for every prime number $\ell \neq p$, the Zariski closure of the ℓ-adic monodromy for the restriction to V of the universal smooth PE-type $\mathbb{Q}_\ell$-local system over $\mathcal{M}$ contains the derived group of G. Moreover V is irreducible if and only if the prime-to-p Hecke correspondences operate transitively on the set of irreducible components of V.

See [2] for a proof in the case when $\mathcal{M}$ is $\mathscr{A}_g$. That proof, based on group theory, also works for PEL-type modular varieties.

The above irreducibility criterion is an effective tool for showing irreducibility of subvarieties of moduli spaces of PEL type over $\overline{\mathbb{F}_p}$ defined by p-adic invariants. See [9] for applications to Siegel modular varieties and [5] for how to combine the above method with ℓ-adic monodromy with a geometric version of the product formula to show that the p-adic monodromy of a Hecke invariant subvariety of $\mathcal{M}$ is also large.

(5) Action of the local stabilizer subgroup.

For any $\overline{\mathbb{F}_p}$-point x_0 in a PEL-type modular variety $\mathcal{M}$ over $\overline{\mathbb{F}_p}$, there are many prime-to-p Hecke correspondences which have x_0 as a fixed-point. Through the

product formula, the complementry p-adic components of these Hecke correspondences form a dense subset of a positive dimensional p-adic Lie group Γ_{x_0}. There is a natural action of this p-adic Lie group Γ_{x_0} on the formal completion $\mathcal{M}^{/x_0}$ of $\mathcal{M}$ at x_0. The following fact, called the *local stabilizer principle*, is an important *local property* of subvarieties of $\mathcal{M}$ stable under all prime-to-p Hecke correspondences.

Local stabilizer principle. *If V is a locally closed subvariety of $\mathcal{M}$ containing x_0 and is stable under the action of all prime-to-p Hecke correspondences, then there exists an open subgroup Γ' of Γ_{x_0} such that the formal completion $V^{/x_0}$ of V at x_0 is a closed formal subscheme of $\mathcal{M}^{/x_0}$ stable under the action of every element of Γ'.*

See §5 for the precise definition of the local stabilizer subgroup Γ_{x_0} and more information, including another version of the local stabilizer subgroup, denoted by $\tilde{\Gamma}_{x_0}$ in §5.

Four types of applications of the local stabilizer principle are explained in 5.4. The primary one, 5.4 (a), is for the case when V is a closed subset of a central leaf and produces a serious constraint on Hecke-invariant closed subvarieties of a central leaf $\mathcal{C}(x_0)$. This result is obtained using the Tate-linear structure of $\mathcal{C}^{/x_0}$ in (3) above and the local rigidity results 6.3 for p-divisible groups and biextensions. The combined output of (5), (3) and local rigidity results similar to 6.3 for formal schemes with Tate-linear structure similar to formal completion of central leaves is expected to have the following geneneral shape.

(6) General shape of the local rigidity for central leaves. *Let Z be a reduced closed subvariety of a central leaf $\mathcal{C}(x_0)$ in a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ passing through an $\overline{\mathbb{F}}_p$-point x_0 of $\mathcal{M}$. If Z is stable under all prime-to-p Hecke correspondences on $\mathcal{M}$ and the formal completion $Z^{/x_0}$ is a Tate-linear formal subscheme of $\mathcal{C}(x_0)^{/x_0}$ with respect to the Tate-linear structure of $\mathcal{C}(x_0)^{/x_0}$.*

This "general shape statement" is a theorem when $\mathcal{C}(x_0)^{/x_0}$ is a p-divisible formal group, or a biextension of p-divisible formal group. In the former case a Tate-linear formal subscheme is a p-divisible formal subgroup. In the latter case a Tate-linear formal subscheme W of a biextension E of (Y_1, Y_2) by Z, where Y_1, Y_2, Z are p-divisible formal groups over $\overline{\mathbb{F}}_p$, is a smooth formal subscheme of E such that (a) the intersection of W with Z is a p-divisible subgroup Z_1 of Z, (b) W is stable under the translation action by Z_1, and (c) the quotient W/Z_1 is isomorphic to a p-divisible formal subgroup of $Y_1 \times Y_2$.

We expect that the method for proving local rigidity for biextensions will be extended to the Tate-linear structure for central leaves in the near future. Then the "general shape statement" above will become a full-fledged theorem.

One may think of the local rigidity for central leaves, which is a consequence of the local stabilizer principle and rigidity results for formal schemes with Tate-linear structure, as a sort of *linerization method* for closed subvarieties of a central leaf $\mathcal{C}$ which are stable under all prime-to-p Hecke correspondences. It puts a stringent constraint on such a Hecke-invariant subvariety W of central leaves: at every point x of W, the formal completion $W^{/x}$ is compatible with the Tate-linear structure

on the central leaf $\mathcal{C}$. We will use the phrase "W is a Tate-linear subvariety of $\mathcal{C}$" to describe this property.

1.3. Further questions and outlook.

The Hecke orbit conjecture has spawned new tools and generated new insights. Despite the progress made in the last twenty years, the general Hecke orbit conjecture remains open. Or to put it more cheerfully, the story is not yet finished, and the conjecture may lead to further advances in mathematics.

In §9 is a list of open questions motivated by the Hecke orbit conjecture. Here we just mention a strong local rigidity question, which is conjecture 9.3.

A strong local rigidity conjecture at supersingular point.
Let $\mathscr{A}_g$ be the moduli space of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}}_p$. Let $x \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ be a supersingular principally polarized abelian variety over $\overline{\mathbb{F}}_p$. Let Z b a reduced irreducible closed formal subscheme of the formal completion $\mathscr{A}_g^{/s}$ of $\mathscr{A}_g$ at s. Suppose that Z is stable under the natural action of an open subgroup of the p-adic $\mathrm{Aut}((A_s, \lambda_s)[p^\infty])$ on $(\mathscr{A}_{g,1} \otimes \overline{\mathbb{F}}_p)^{/s}$, and the fiber over the generic point of Z of the universal p-divisible group is ordinary. Show that $Z = (\mathscr{A}_{g,1} \otimes \overline{\mathbb{F}}_p)^{/s}$.

The conjecture 9.2 for the Lubin–Tate local moduli space is related to but simpler than the above conjecture.

The local rigidity question at supersingular points, represented by 9.2 and 9.3, has a flavor quite different from the local rigidity for central leaves. The question is harder for two reasons. Firstly, the structure of the formal completion of the moduli space $\mathscr{A}_g$ a supersingular point s is not as simple as the formal completion of a central leaf—there is no Tate-linear structure to piggy back on. Secondly the action the local stabilizer subgroup of Γ_s on $\mathscr{A}_g^{/s}$ is poorly understood.

On the other hand, the local stabilizer subgroup Γ_s is really big and the action of Γ_s carries information about "almost all" prime-to-p Hecke correspondences. There are good reasons to believe that being stable under the action of an open subgroup of the local stabilizer subgroup at a supersingular point s translates into a very strict constraint on formal subvarieties of $\mathscr{A}_g^{/s}$ with this property. We just need to persuade the local stabilizer subgroups to reveal their secrets.

2. HECKE SYMMETRY ON MODULAR VARIETIES OF PEL TYPE

2.1. PEL input data.

Recall from [26] that a *PEL input data unramified at p* is an 8-tuple $\left(B, \mathscr{O}_B, *, V, (\cdot, \cdot), h, \Lambda_p, K_f^{(p)}\right)$, where

- B is a finite dimensional simple algebra over $\mathbb{Q}$, whose center L is an algebraic number field which is unramified above p,
- $\mathscr{O}_B$ is a $\mathbb{Z}_{(p)}$-order of B whose p-adic completion is $(\mathscr{O}_L \otimes_\mathbb{Z} \mathbb{Z}_p)$-linearly isomorphic to a matrix ring $\mathrm{M}_m(\mathscr{O}_L \otimes_\mathbb{Z} \mathbb{Z}_p)$, where $m = \sqrt{\dim_L(B)}$,
- $*$ is a positive involution on B,
- V is a finite dimensional left B-module,

- $(\cdot,\cdot)$ is a non-degenerate $\mathbb{Q}$-valued alternating form on V such that
$$(bv_1, v_2) = (v_1, b^*v_2)$$
for all $v_1, v_2 \in V$ and all $b \in B$,
- $h : \mathbb{C} \to \mathrm{End}_B(V) \otimes_{\mathbb{Q}} \mathbb{R}$ is a $*$-homomorphism such that the map
$$V_{\mathbb{R}} \times V_{\mathbb{R}} \longrightarrow \mathbb{R} \qquad (v_1, v_2) \mapsto (v_1, h(\sqrt{-1})v_2)$$
is a positive definite symmetric bilinear form on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$,
- Λ_p is a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ which is stable under the action of $\mathscr{O}_B$ such that the alternating pairing $(\cdot,\cdot)$ induces a $\mathbb{Z}_p$-valued alternating self-dual pairing
$$(\cdot,\cdot)_p : \Lambda_p \times \Lambda_p \longrightarrow \mathbb{Z}_p$$
on Λ_p, and
- $K_f^{(p)}$ is a compact open subgroup of $G(\mathbb{A}_f^{(p)})$, where
$$\mathbb{A}_f^{(p)} = \prod_{\ell \neq p}{}' \mathbb{Q}_\ell$$
is the ring of all finite prime-to-p adeles over $\mathbb{Q}$, and
$$G = \mathrm{U}(\mathrm{End}_B(V), *')$$
is the unitary group attached to the simple algebra $\mathrm{End}_B(V)$ with involution $*'$ induced by the pairing $(\cdot,\cdot)$.

2.1.1. The left $B_{\mathbb{C}}$-module $V_{\mathbb{C}}$ decomposes into a direct sum $V_{\mathbb{C}} = V_1 \oplus V_2$ of $B_{\mathbb{C}}$-submodules, where
$$V_1 := \{v \in V_{\mathbb{C}} \mid h(z) \cdot v = z \cdot v\}, \quad V_2 := \{v \in V_{\mathbb{C}} \mid h(z) \cdot v = \bar{z} \cdot v\}.$$
The *reflex field* E of a PEL input data as above is the field of definition of the isomorphism class of the $\mathbb{C}$-linear representation of B afforded by the $B_{\mathbb{C}}$-module V_1; it is a subfield of $\mathbb{C}$ of finite degree over $\mathbb{Q}$.

The meaning of "field of definition" in algebraic geometry can be unclear at times, but here we have an alternative explicit definition of the reflex field E: it is the field $\mathbb{Q}(\mathrm{tr}(b|V_1) : b \in B)$ generated by the traces of the action of elements of B on V_1. Denote by $\mathscr{O}_E$ the ring of integers of the reflex field E tensored with $\mathbb{Z}_{(p)}$. As we will see soon, the ring $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$ will serve as the base ring of PEL-type modular varieties associated to the given PEL input data.

2.2. Modular varieties of PEL type

We will use an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ as the base field for modular varieties of PEL type. More precisely, the base ring of our modular variety $\mathcal{M}$ will be of the form $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$, where $\mathscr{O}_E$ is the ring of integers of the reflex field of the given PEL data. Since the PEL data is assumed to be unramified at p, the ring $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$ is isomorphic to a product of a finite number of copies of $\overline{\mathbb{F}}_p$.

Given a PEL input data $(B, \mathscr{O}_B, *, V, (\cdot,\cdot), h, \Lambda_p, K_f^{(p)})$ unramified at p such that $K_f^{(p)}$ is *sufficiently small*, there is an associated moduli scheme $\mathcal{M} = \mathcal{M}_{K_f^{(p)}}$ over $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$, which is a quasi-projective scheme over $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$, such that

for every commutative ring R over $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$, the set $\mathcal{M}(R)$ of R-valued points of $\mathcal{M}$ is the set of isomorphism classes

$$(A, \iota : \mathscr{O}_B \to \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \lambda : A \to A^t, \bar{\eta})$$

where

- A is an abelian scheme over R up to prime-to-p isogeny,
- ι is a ring homomorphism,
- λ is a polarization of A, and
- $\bar{\eta}$ is a level structure of type $K_f^{(p)}$,

such that the following properties are satisfied.

(i) The homomorphism ι is compatible with the involution $*$ on $\mathscr{O}_B$ and the Rosati involution induced by λ.

(ii) The "determinant condition" below is satisfied; c.f. [26, §5 p. 390]. This condition ensures that the first homology of the polarized $\mathscr{O}_B$-linear abelian variety (A, ι, λ) is modeled on the given PEL input data.

Choose a $\mathbb{Z}_{(p)}$-basis $b_1, \ldots, b_m$ of $\mathscr{O}_B$. Let $t_1, \ldots, t_m$ be variables. The determinant of the action of the element

$$b_t := t_1 b_1 + \cdots + t_m b_m \in \mathscr{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}[t_1, \ldots, t_n]$$

on V_1 is an element f_{b_t} of the polynomial ring $\mathscr{O}_E[t_1, \ldots, t_m]$ over $\mathscr{O}_E$. The determinant condition is the requirement that, after the base change $R \to R[t_1, \ldots, t_m]$, the determinant of action of the element b_t on the Lie algebra

$$\operatorname{Lie}\left(A \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R[t_1, \ldots, t_m])\right)$$

of the base change of A to $R[t_1, \ldots, t_m]$ is equal to the image of f_{b_t} in the polynomial ring $R[t_1, \ldots, t_m]$ over R.

2.2.1. A sufficient condition for $K_f^{(p)}$ to be "sufficiently small" is the following: there exists a $\mathbb{Z}$-lattice $\Lambda_{\mathbb{Z}}$ in V and a positive integer $n \geq 3$ which is prime to p such that

$$K_f^{(p)} \subseteq \left\{ \gamma_f^{(p)} \in G(\mathbb{A}_f^{(p)}) \,\middle|\, (\gamma_f^{(p)} - 1) \cdot (\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \subseteq n \cdot (\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \quad \forall \ell | n \right\}.$$

2.2.2. When one varies the compact open subgroup $K_f^{(p)} \subseteq G(\mathbb{A}_f^{(p)})$ while the other ingredients of the PEL input data remain fixed, one obtains a filtered[1] projective system

$$\widetilde{\mathcal{M}} := \left(\mathcal{M}_{K_f^{(p)}}\right)_{K_f^{(p)}}$$

of modular varieties over $\mathscr{O}_E \otimes \overline{\mathbb{F}}_p$.

Remark. Since we are considering PEL-type modular varieties over $\overline{\mathbb{F}}_p$, the base ring of our modular varieties $\mathcal{M}_{K_f^{(p)}}$ is $\mathscr{O}_E \otimes_{\mathbb{Z}_{(p)}} \overline{\mathbb{F}}_p$. The latter ring is isomorphic to the product of a finite number of copies of $\overline{\mathbb{F}}_p$ because the PEL input data is assumed to be unramified at p.

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[1] Here being filtered means that for any two compact open subgroups K_1, K_2 of $G(\mathbb{A}_f^{(p)})$, there is a third compact open subgroup contained in both K_1 and K_2, e.g. $K_1 \cap K_2$.

2.2.3. A first example of modular varieties of PEL type is the Siegel modular variety $\mathscr{A}_g$, which classifies g-dimensional principally polarized abelian varieties (with prime-to-p level structures) in characteristic p. In this case the central simple algebra B is $\mathbb{Q}$, the B-module V is a $2g$-dimensional vector space over $\mathbb{Q}$ with a perfect alternating pairing, and G is the split symplectic group Sp_{2g}.

A remark about notation: unless otherwise specified, we consider PEL modular varieties over $\overline{\mathbb{F}}_p$ in this chapter. The polarization of the PEL type abelian varieties up to prime-to-p isogeny being classified is prime to p. The Siegel modular variety $\mathscr{A}_g$ over $\overline{\mathbb{F}}_p$ considered here is often denoted by $\mathscr{A}_{g,1,n}$ in the literature, with principal polarization and symplectic level-n structure, where n is prime to p.

2.3. Hecke symmetries on the prime-to-p tower of modular varieties

We keep the notation in 2.2. Let $\widetilde{\mathcal{M}}$ be the filtered projective system of moduli schemes over $\mathscr{O}_E \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$. There is a natural action of the locally compact group $G(\mathbb{A}_f^{(p)})$ on the projective system $\widetilde{\mathcal{M}}$ over $\mathrm{Spec}\big(\mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\big)$. For every sufficiently small open compact subgroup $K_f^{(p)} \subseteq G(\mathbb{A}_f^{(p)})$, the modular variety $\mathcal{M}_{K_f^{(p)}}$ is the quotient of $\widetilde{\mathcal{M}}$ by $K_f^{(p)}$, so that $K_f^{(p)}$ is naturally isomorphic to the Galois group of the pro-finite etale Galois cover $\widetilde{\mathcal{M}} \to \mathcal{M}_{K_f}$.

When one fixes a (sufficiently small) level subgroup $K_f^{(p)} \subseteq G(\mathbb{A}_f^{(p)})$, we no longer have a group acting on the modular variety $\mathcal{M}_{K_f^{(p)}}$. What's left of the action of the group $G(\mathbb{A}_f^{(p)})$ on the tower $\widetilde{\mathcal{M}}$ is a family of finite etale algebraic correspondences

$$\mathcal{M}_{K_f^{(p)}} \xleftarrow{\mathrm{pr}} \mathcal{M}_{K_f^{(p)} \cap \gamma^{-1} \cdot K_f^{(p)} \cdot \gamma} \xrightarrow{\gamma \,\cdot} \mathcal{M}_{K_f^{(p)}}, \qquad \gamma \in K_f^{(p)} \backslash G(\mathbb{A}_f^{(p)}) / K_f^{(p)}$$

on $\mathcal{M}_{K_f^{(p)}}$, indexed by the double coset $K_f^{(p)} \backslash G(\mathbb{A}_f^{(p)}) / K_f^{(p)}$. These finite etale algebraic correspondences are the prime-to-p Hecke symmetries on $\mathcal{M}_{K_f^{(p)}}$. For any prime number $\ell \neq p$, those etale algebraic correspondences defined by elements $\gamma \in G(\mathbb{Q}_\ell)$ will be called ℓ-adic Hecke correspondences or ℓ-adic Hecke symmetries.

2.4. Remark. (a) Hecke correspondences were introduced by Hecke in [16], [17], [18]. They induce linear operators on elliptic modular forms.

(b) A PEL input data actually defines PEL moduli schemes $\mathbf{M}_{K_f^{(p)}}$ flat over $\mathscr{O}_E$ such that $\mathbf{M}_{K_f^{(p)}} \otimes_{\mathscr{O}_E} (\mathscr{O}_E \otimes \overline{\mathbb{F}}_p)$ is $\mathcal{M}_{K_f^{(p)}}$. For every ring homomorphism $\tau : \mathscr{O}_E \to \mathbb{C}$, every ℓ-adic Hecke orbit is dense in $(\mathcal{M}_{K_f} \otimes_{\mathscr{O}_E} \mathbb{C})(\mathbb{C})$ for every prime number ℓ. In contrast, in characteristic $p > 0$ there are p-adic invariants which are preserved by all prime-to-p Hecke correspondences. In the case of $\mathcal{A}_g$, Newton polygons of abelian varieties and kernels of "multiplication by p" on abelian varieties are two familiar p-adic invariants.

Every such p-adic invariant imposes upper bounds on the Zariski closure of Hecke orbits. For instance the prime-to-p Hecke orbit of any non-ordinary point of $\mathscr{A}_g$ is contained in the zero locus of the Hasse invariant, and the prime-to-p Hecke orbit of any supersingular point of $\mathscr{A}_g$ is finite.

3. Sustained p-divisible groups and central leaves

The notion of a *sustained p-divisible group* extends the concept of a *geometrically fiberwise constant p-divisible group* in [45]. In this section we define *central leaves* in a PEL modular variety over $\overline{\mathbb{F}}_p$ in terms of sustained p-divisible groups with imposed PL structure (polarization and endomorphism). We refer to [11] and [13] for more information about sustained p-divisible groups. The structure with imposed endomorphisms is not considered in [11] and [13], but the generalization to the PL situation is straight forward.

3.1. Definition. Let B be finite dimensional simple algebra over $\mathbb{Q}_p$, let $\mathscr{O}_B$ be a maximal order of B, and let $*$ be an involution of B. A polarized $\mathscr{O}_B$-linear p-divisible group over a scheme S is a triple

$$(X \to S, \iota : \mathscr{O}_B \to \mathrm{End}_S(X), \lambda : X \to X^t),$$

where $X \to S$ is a p-divisible group over S, ι is a ring homomorphism, and λ is an isogeny of p-divisible groups compatible with the involution $*$, in the sense that

$$\iota(b)^t \circ \lambda = \lambda \circ \iota(b^*) \quad \forall b \in \mathscr{O}_B.$$

3.2. Definition. Let B be finite dimensional simple algebra over $\mathbb{Q}_p$, let $\mathscr{O}_B$ be a maximal order of B, and let $*$ be an involution of B. Let κ be a field of characteristic $p > 0$, and let S be a scheme over κ. Let

$$(X \to S, \iota : \mathscr{O}_B \to \mathrm{End}_S(X), \lambda : X \to X^t)$$

be a polarized $\mathscr{O}_B$-linear p-divisible group over S.

(a) A polarized $\mathscr{O}_B$-linear p-divisible group

$$(X \to S, \iota : \mathscr{O}_B \to \mathrm{End}_S(X), \lambda : X \to X^t)$$

over S is *strongly κ-sustained modeled on an $\mathscr{O}_B$-linear polarized p-divisible group* $(X_0, \iota_0, \lambda_0)$ over κ if the S-scheme

$$\mathscr{ISOM}_S \big((X_0[p^n]_S, \iota_0, \lambda_0[p^n]_S), (X[p^n], \iota[p^n], \lambda[p^n]) \big) \to S$$

is faithfully flat over S for every $n \in \mathbb{N}$.

(b) An $\mathscr{O}_B$-linear polarized p-divisible group $(X \to S, \iota, \lambda)$ is *κ-sustained* if

$$\mathscr{ISOM}_{S \times_{\mathrm{Spec}\,\kappa} S} \big(\mathrm{pr}_1^*(X[p^n], \iota[p^n], \lambda[p^n]), \mathrm{pr}_2^*(X[p^n], \iota[p^n], \lambda[p^n]) \big) \to S \times_{\mathrm{Spec}\,\kappa} S$$

is faithfully flat for every $n \in \mathbb{N}$. A κ-sustained $\mathscr{O}_B$-linear polarized p-divisible group $(X_2 \to K, \iota_2, \lambda_2)$ over an extension field K of κ is a *K/κ-model* of a κ-sustained $\mathscr{O}_B$-linear polarized p-divisible group $(X \to S, \iota, \lambda)$ if the morphism

$$\mathscr{ISOM}_{S_K} \big((X_2[p^n], \iota_2[p^n], \lambda_2[p^n]) \times_{\mathrm{Spec}\,K} S_K, (X[p^n], \iota[p^n], \lambda[p^n]) \times_S S_K \big) \to S_K$$

is faithfully flat for every $n \in \mathbb{N}$.

3.3. Definition. Let $\mathcal{M}_{K_f^{(p)}}$ be the modular variety over $\mathscr{O}_E \otimes \overline{\mathbb{F}}_p$ associated to a PEL input data $\big(B, \mathscr{O}_B, *, V, (\cdot, \cdot), h, \Lambda_p, K_f^{(p)} \big)$ unramified at p, where E is the reflex field. Let $x_0 = [(A_0, \iota_0, \lambda_0)] \in \mathcal{M} = \mathcal{M}_{K_f^{(p)}}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$-point of $\mathcal{M}$.

(i) Define $\mathcal{CL}(x_0) = \mathcal{CL}_{\mathcal{M}}(x_0)$ to be the maximal element among all locally closed subscheme $S \subseteq \mathcal{M}$ such that the restriction to S of the universal polarized $\mathscr{O}_B$-linear p-divisible group is strongly $\overline{\mathbb{F}}_p$-sustained modeled on $(A_0[p^\infty], \iota_0[p^\infty], \lambda_0[p^\infty])$.

(ii) Let $\mathcal{C}(x_0) = \mathcal{CL}(x_0)_{\mathcal{M}, \mathrm{red}}$ be the reduced scheme underlying $\mathcal{CL}(x_0)$; it has the same underlying topological space as $\mathcal{CL}(x_0)$, and its structure sheaf is the quotient of $\mathscr{O}_{\mathcal{CL}(x_0)}$ by the radical of $\mathscr{O}_{\mathcal{CL}(x_0)}$.

The two subscheme $\mathcal{CL}(x_0)$ and $\mathcal{C}(x_0)$ will be called the *schematic central leaf* and the *reduced central leaf* in $\mathcal{M}_{K_f^{(p)}}$ passing through x_0 respectively. In practice there is not much real difference in view of 3.4 (iii).

No detailed proof of the statement 3.4 (iii) below, other than the case of moduli spaces of polarized abelian varieties, has been written down. Therefore 3.4 (iii) should be taken with a grain of salt, and the word "proposition" in 3.4 is set in lower case. For the Hecke orbit conjecture 3.4 (iii) is unimportant. The statements 3.4 (i) is obvious, and the statement 3.4 (ii) follows from the fact that deformation functors at closed points of $\mathcal{CL}(x_0)$ are isomorphic.

3.4. proposition. *Notation as in 3.3. Let x_0 be an $\overline{\mathbb{F}}_p$-point of $\mathcal{M}_{K_f^{(p)}}$.*

(i) *Both $\mathcal{CL}(x_0)$ and $\mathcal{C}(x_0)$ are stable under all prime-to-p Hecke correspondences on $\mathcal{M}_{K_f^{(p)}}$.*

(ii) *The reduced scheme $\mathcal{C}(x_0)$ is a smooth locally closed subscheme of $\mathcal{M}_{K_f^{(p)}}$.*

(iii) *The scheme $\mathcal{CL}(x_0)$ is smooth, therefore coincides with $\mathcal{C}(x_0)$.*

The general Hecke orbit conjecture below for PEL-type modular varieties over $\overline{\mathbb{F}}_p$ has been stated in 1.1. We repeat it here after the definition of central leaves in PEL-type modular varieties.

3.5. The Hecke orbit conjecture for PEL modular varieties. *The prime-to-p Hecke orbit of x_0 is Zariski dense in the central leaf $\mathcal{C}(x_0)$, for every $\overline{\mathbb{F}}_p$-point x_0 of $\mathcal{M}_{K_f^{(p)}}$. Equivalently, the only non-empty closed subscheme of the central leaf $\mathcal{C}(x_0)$ which is stable under all prime-to-p Hecke correspondences is $\mathcal{C}(x_0)$ itself.*

4. LOCAL STRUCTURE OF LEAVES

4.1. The phenomenon: Serre–Tate theory on central leaves

Let $\mathcal{M} = \mathcal{M}_{K_f^{(p)}}$ be a modular variety over $\overline{\mathbb{F}}_p$ attached to a PEL input data unramified at p. Let x_0 be an $\overline{\mathbb{F}}_p$-point of $\mathcal{M}$, and let $\mathcal{C}(x_0)$ be the central leaf in $\mathcal{M}$ through x_0. It turns out that the formal completion of the central leaf $\mathcal{C}(x_0)$ admits a structural description similar to the Serre–Tate theory for deformations of ordinary abelian varieties and p-divisible groups.

> *The formal completion $\mathcal{C}(x_0)^{/x_0}$ of a central leaf $\mathcal{C}(x_0)$ at an $\overline{\mathbb{F}}_p$-point x_0 of $\mathcal{C}(x_0)$ is "built up" from a web of fibrations in a cascade-like fashion in the sense of [32], where each fibration is a torsor for a p-divisible formal group.*

The exact formulation in the general case of central leaves in modular varieties of PEL type is a bit complicated. We refer to [11] and [13] for more information in the case of Siegel modular variety $\mathscr{A}_g$. The case with imposed endomorphism structure is similar.

An important property of central leaves is that the restriction to a central leaf of the universal p-divisible group over a PEL type modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ admits a *slope filtration* in the sense of [55] and [47]; see [11] and [13]. In addition, sustained deformations of a p-divisible group all carry slope filtration. That means that sustained deformations of a p-divisible group X_0 are contained in deformations of the slope filtration of X_0. So the slope filtration provides a convenient instrument for analyzing the local structure of central leaves. We refer to [13] for details.

4.2. Examples. We would like to think of the above cascade-like structure of fibering $\mathcal{C}_{x_0}^{/x_0}$ by p-divisible formal groups as a "*Tate-linear structure*" on $\mathcal{C}_{x_0}^{/x_0}$ of some sort, with "Tate-linear" interpreted as "similar to p-divisible formal groups".

We offer three examples, all for central leaves in $\mathscr{A}_g$; see [11] and [13] for more information.

4.2.1. The prototype of such "Tate-linear structure" is the ordinary locus $\mathscr{A}_g^{\mathrm{ord}}$ in $\mathscr{A}_g$, a dense opens subscheme of $\mathscr{A}_g$ which forms a single central leaf in $\mathscr{A}_g$. The formal completion $\mathscr{A}_g^{/x_0}$ of $\mathscr{A}_g^{\mathrm{ord}}$ at an $\overline{\mathbb{F}}_p$-point has a natural structure as a $\frac{g(g+1)}{2}$-dimensional formal torus over $\overline{\mathbb{F}}_p$; see [29] and [25].

4.2.2. In this example $x_0 = [(A_{x_0}, \lambda_{x_0})]$ is an $\overline{\mathbb{F}}_p$-point of $\mathscr{A}_g$ such that the p-divisible group $A_{x_0}[p^\infty]$ is isomorphic to the product of an isoclinic p-divisible group Y over $\overline{\mathbb{F}}_p$ of height g and slope s, with an isoclinic p-divisible group Z over $\overline{\mathbb{F}}_p$ of height g and slope $1-s$, $s < \frac{1}{2}$, and the principal polarization μ_{x_0} induces an isomorphism $Y \xrightarrow{\sim} Z^t$, where Z^t is the Serre-dual of Y. In this case the formal completion $\mathcal{C}(x_0)^{/x_0}$ of the central leaf $\mathcal{C}(x_0)$ in $\mathcal{A}_g$ has a natural structure as an isoclinic p-divisible formal group over $\overline{\mathbb{F}}_p$ of height $\frac{g(g+1)}{2}$ and slope $1-2s$.

4.2.3. In this example $x_0 = [(A_{x_0}, \lambda_{x_0})]$ is an $\overline{\mathbb{F}}_p$-point of $\mathscr{A}_g$ such that the p-divisible group $A_{x_0}[p^\infty]$ is isomorphic to the product of three isoclinic p-divisible groups Y_1, Y_2, Y_3 over $\overline{\mathbb{F}}_p$; their heights are h_1, h_2, h_1 and their slopes are $s, \frac{1}{2}, 1-s$ respectively, with $2h_1 + h_2 = 2g$ and $s < \frac{1}{2}$. induces an isomorphism $Y_1 \xrightarrow{\sim} (Y_3)^t$ and also a principal polarization on Y_2. We assume for simplicity that $p > 2$.

The Tate-linear structure of the formal completion $\mathcal{C}(x_0)^{/x_0}$ is better understood as a formal subscheme of another formal scheme $\mathcal{D}ef(A_{x_0}[p^\infty])_{\mathrm{sus}}$; the latter is the largest closed formal subscheme of the characteristic-p deformation space $\mathcal{D}ef(A_{x_0}[p^\infty])$ of the p-divisible group $A_{x_0}[p^\infty]$ over which the universal p-divisible group is $\overline{\mathbb{F}}_p$-sustained. The formal scheme $\mathcal{D}ef(A_{x_0}[p^\infty])$ has a natural structure as a biextension

$$\left(\pi : \mathcal{D}ef(A_{x_0}[p^\infty]) \longrightarrow (U_1, U_2), \, Z, \, +_1, \, +_2 \right)$$

of p-divisible formal groups in the sense of [33, (iii) on p. 310], where U_1, U_2, Z are p-divisible formal groups, $+_1$ and $+_2$ are two relative group laws making

$\mathcal{D}ef(A_{x_0}[p^\infty])$ an extension of (the base change to U_2 of) U_1 by (the base change to U_2 of) Z over U_2, and also an extension of (the base change to U_1 of) U_2 by (the base change to U_1 of) Z over U_1, such that the two group laws are compatible as spelled out in [33].

The three p-divisible formal groups U_1, U_2, Z are $\mathcal{D}ef(Y_1 \times Y_2)_{\mathrm{sus}}$, $\mathcal{D}ef(Y_2 \times Y_3)_{\mathrm{sus}}$ and $\mathcal{D}ef(Y_1 \times Y_3)_{\mathrm{sus}}$ respectively. Here $\mathcal{D}ef(Y_1 \times Y_2)_{\mathrm{sus}}$ is the largest closed formal subscheme of the characteristic-p deformation space $\mathcal{D}ef(Y_1 \times Y_2)$ over which the universal deformation of $Y_1 \times Y_2$ is $\overline{\mathbb{F}}_p$-sustained; similarly for $\mathcal{D}ef(Y_2 \times Y_3)_{\mathrm{sus}}$ and $\mathcal{D}ef(Y_1 \times Y_3)_{\mathrm{sus}}$. These three p-divisible formal groups are isoclinic, with slopes $\frac{1}{2} - s$, $\frac{1}{2} - s$, $1 - 2s$ and heights $h_1 h_2$, $h_1 h_2$, $h_1 h_3$ respectively.

The rest of 4.2.3 depends on the assumption that $p > 2$. The principal polarization μ_{x_0} defines an automorphism ι on $\mathcal{D}ef(A_{x_0}[p^\infty])$ and also a group automorphism $\bar{\iota}$ on $U_1 \times U_2$ satisfying the following conditions:

- $\iota^2 = \mathrm{id}_{\mathcal{D}ef(A_{x_0}[p^\infty])}$ and $\bar{\iota}^2 = \mathrm{id}_{U_1 \times U_2}$,
- $\bar{\iota}(U_1) = U_2$ and $\bar{\iota}(U_2) = U_1$, and
- $\bar{\iota} \circ \pi = \pi \circ \iota$.

Let $\mathcal{D}ef(A_{x_0}[p^\infty])_{\mathrm{sus}}^{\iota=\mathrm{id}}$ be the largest closed formal subscheme of $\mathcal{D}ef(A_{x_0}[p^\infty])_{\mathrm{sus}}^{\iota=\mathrm{id}}$ over which the involution ι is equal to the identity map. Similarly we have fixed-point formal subschemes $(U_1 \times U_2)^{\iota=\mathrm{id}} \subsetneqq U_1 \times U_2$, $Z^{\iota=\mathrm{id}}$. Then $\mathcal{C}(x_0)^{/x_0}$ is naturally isomorphic to $\mathcal{D}ef(A_{x_0}[p^\infty])_{\mathrm{sus}}^{\iota=\mathrm{id}}$, and the morphism

$$\pi|_{\mathcal{D}ef(A_{x_0}[p^\infty])_{\mathrm{sus}}^{\iota=\mathrm{id}}} : \mathcal{C}(x_0)^{/x_0} = \mathcal{D}ef(A_{x_0}[p^\infty])_{\mathrm{sus}}^{\iota=\mathrm{id}} \longrightarrow (U_1 \times U_2))^{\iota=\mathrm{id}}$$

gives $\mathcal{C}(x_0)^{/x_0}$ a natural structure as a torsor for $Z^{\iota=\mathrm{id}}$ over $(U_1 \times U_2))^{\iota=\mathrm{id}}$.

5. Action of the local stabilizer subgroup

5.1. Given an $\overline{\mathbb{F}}_p$-point x_0 of a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$, the prime-to-p Hecke orbit $\mathcal{H}^{(p)} \cdot x_0$ of x_1 is a countable set of $\overline{\mathbb{F}}_p$-points of $\mathcal{M}$. There is no general tool in algebraic geometry for studying a countable set of closed points.

The *local stabilizer principle* 5.3 provides a path to obtain local information of a subvariety Z of $\mathcal{M}$ which is stable under all prime-to-p Hecke correspondences, such as the Zariski closure of a prime-to-p Hecke orbit. It has the effect of (partly) transforming a seemingly intractable discrete problem into a continuous problem, which is more tractable—at least in principle.

The desired local information of a Hecke-invariant subvariety Z of a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ is wrapped inside the "change of marking" action of a p-adic Lie group $\tilde{\Gamma}_z$ on the formal completion $\mathcal{M}^{/z}$ at z of $\mathcal{M}$. Note that $\mathcal{M}^{/z}$ is naturally identified with the deformation space $\mathcal{D}ef((A_z, \iota_z, \lambda_z)[p^\infty])$ of the $(\mathscr{O}_B \otimes \mathbb{Z}_p)$-linear polarized p-divisible group $(A_z, \iota_z, \lambda_z)[p^\infty]$.

5.2. Local stabilizer subgroups.

5.2.1. Let $\left(B, \mathscr{O}_B, *, V, (\cdot, \cdot), h, K_f^{(p)}\right)$ be the PEL input data which defines the modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ as in [26]; see §2 for a quick review. Let $(A_z, \beta_z : B \to \mathrm{End}^0(A_z), \mu_z)$ be the abelian variety with endomorphism by the simple algebra B up to isogeny plus a B-linear polarization parametrized by the point z of the modular variety $\mathcal{M}$. Let $*_z$ be the Rosati involution on $\mathrm{End}_B^0(A_z)$.

(1) Let H_z be the unitary group over $\mathbb{Q}_p$ attached to the semisimple algebra with involution $\big(\operatorname{End}_B^0(A_z)[p^\infty], *_z \big)$ over $\mathbb{Q}_p$. It is the affine algebraic group over $\mathbb{Q}_p$ such that for every commutative $\mathbb{Q}_p$-algebra R we have

$$H_z(R) = \big\{ u \in (\operatorname{End}_B^0(A_z)[p^\infty] \otimes_{\mathbb{Q}_p} R)^\times \mid uu^* = 1 \big\}.$$

(2) Let U_z be the unitary group attached to the semisimple algebra with involution $\big(\operatorname{End}_B^0(A_z) \otimes_{\mathbb{Q}} \mathbb{Q}_p, *_z \big)$ over $\mathbb{Q}_p$. In other words U_z is the affine algebraic group over $\mathbb{Q}_p$ such that

$$U_z(R) = \big\{ u \in (\operatorname{End}_B^0(A_z) \otimes_{\mathbb{Q}} R)^\times \mid uu^* = 1 \big\}.$$

for every commutative $\mathbb{Q}_p$-algebra R. In particular

$$U_z(\mathbb{Q}_p) = \big\{ u \in (\operatorname{End}_B^0(A_z) \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \mid uu^* = 1 \big\}.$$

(3) The p-adic local stabilizer subgroup $\tilde{\Gamma}_z$ is the compact open subgroup of $U_z(\mathbb{Q}_p)$ defined by

$$\tilde{\Gamma}_z := H_z(\mathbb{Q}_p) \cap \operatorname{Aut}(A_z[p^\infty]).$$

(4) The *local stabilizer subgroup* Γ_z at z is the compact open subgroup of $U_z(\mathbb{Q}_p)$ defined by

$$\Gamma_z := U_z(\mathbb{Q}_p) \cap \operatorname{Aut}(A_z[p^\infty]).$$

5.2.2. In the situation of 5.2.1, we have a natural action of the p-adic local stabilizer subgroup $\tilde{\Gamma}_z$ operates on the formal completing $\mathcal{M}^{/z}$ of $\mathcal{M}$ at z, by functoriality of deformation theory, because $\mathcal{M}^{/z}$ is canonically isomorphic to the characteristic-p deformation space $\mathcal{D}ef\big((A_z, \iota_z, \lambda_z)[p^\infty]\big)$. The local stabilizer subgroup Γ_z is a subgroup of $\tilde{\Gamma}_z$, and inherits from $\tilde{\Gamma}_z$ an action on $\mathcal{M}^{/z}$. The local stabilizer principle is the almost obvious statement that, because the p-adic closure in $U_z(\mathbb{Q}_p)$ of the subset of all Hecke symmetries having z as a fixed point contains an open neighborhood of $U_z(\mathbb{Q}_p)$, the formal completion $\mathcal{Z}^{/z}$ must be stable under the action of an open subgroup of Γ_z.

5.2.3. Remark. (a) The $\mathbb{Q}_p$-group U_z contains the *Frobenius torus* T_z at the $\overline{\mathbb{F}}_p$-rational point z of $\mathcal{M}$. In particular the local stabilizer subgroup Γ_z is "not too small". This is the reason why we formulated the local stabilizer principle for $\overline{\mathbb{F}}_p$-points.
(b) The local stabilizer principle holds for closed subvarieties Z of the minimal compactification $\overline{\mathcal{M}}$ as well. The precise formulation of the local stabilizer subgroup is omitted here.

5.3. Local stabilizer principle. *Let $z \in Z(\overline{\mathbb{F}}_p)$ be a point of a closed subvariety $Z \subseteq \mathcal{M}$ stable under all prime-to-p Hecke correspondences. The formal completion $\mathcal{Z}^{/z}$ of Z at z is stable under the action of an open subgroup of the local stabilizer subgroup Γ_z.*

Remark. The local stabilizer principle is an analog of the following fact in the context of group actions: suppose that an algebraic group G operates on an algebraic variety X, Y is a subvariety of X stable under G, and y is an $\overline{\mathbb{F}}_p$-point of Y, then $Y^{/y} \subseteq X^{/y}$ is stable under the action of the stabilizer subgroup $G_y = \operatorname{Stab}_G(y)$ at y.

5.4. Exploring the action of the local stabilizer subgroups. A wealth of information is encoded in the action of the local stabilizer subgroups. The challenge is to figure out how to effectively mine this source and make the hidden information accessible. The difficulty one faces in exploring the action of the local stabilizer subgroup and the success achieved so far vary greatly on the structure of $(A_z[p^\infty], \beta_z[p^\infty], \iota_z[p^\infty], U_z)$. We comment on four situations below. In situations (b) and (c) below, the local stabilizer subgroup Γ_z is quite big and the action of Γ_z is understood, so the local stabilizer principle provides substantial information. The general Hecke orbit conjecture would follow if one can show that the Zariski closure of a Heck-invariant subvariety Z of $\mathcal{C}$ contains points as those in (b) and (c); proving this statement remains a challenge.

(a) At points of a central leaf $\mathcal{C}$: Local rigidity for subvarieties of leaves

The formal completion at a closed point z of a central leaf $\mathcal{C}$ has a "linear structure", in the sense that $\mathcal{C}^{/z}$ is built up from p-divisible formal groups; see 4. The action of the local stabilizer subgroup Γ_z on $\mathcal{C}^{/z}$ is understood. It is expected that the method used in [12] will allow one to show that the formal completion at z of a Hecke-invariant closed subvariety $Z \subseteq \mathcal{C}$ is a "linear subvariety", in the sense that it is assembled from p-divisible formal subgroups of the building blocks of $\mathcal{C}^{/z}$.

(b) At points the boundary of the modular variety $\mathcal{M}$

If the Zariski closure $\overline{Z}$ of Z in the minimal compactification $\mathcal{M}^*$ of the moduli space $\mathcal{M}$ contains a point z of the boundary $\mathcal{M}^* \smallsetminus \mathcal{M}$, the local stabilizer principle at z yields substantial information. An example is the calculation in [1, §1], where it is shown that if the Zariski closure of a Hecke orbit of an ordinary point of $\mathscr{A}_g$ contains a zero-dimensional cusp, then that Hecke orbit is dense in $\mathscr{A}_g$.

(c) At hypersymmetric points

See 7.3 for the definition of hypersymmetric points for abelian varieties with imposed endomorphisms. The notion of hypersymmetric points is defined in [7] and applied in [5], [20], [21] and [9], in combination with the method in [2] to prove irreducibility results in characteristic p.

For an application to the Hecke orbit problem, the difficulty lies in showing that a given Hecke invariant subvariety Z of a central leaf $\mathcal{C}$ contains a point z such that the local stabilizer subgroup is substantially bigger than the local stabilizer subgroups of typical $\overline{\mathbb{F}}_p$-points of $\mathcal{C}$. In the case of the Siegel modular variety $\mathscr{A}_g$ one can show that Z contains hypersymmetric points with the help of Hilbert modular subvarieties. Note that the property that every $\mathbb{F}_p$-point is contained in a positive-dimensional Shimura subvariety is special to PEL type C: there exist PEL modular varieties of type A and modular varieties of type D such that the above property does *not* hold.

(d) At supersingular points

It is shown in [1] (as a consequence of [1, §1, Prop. 1]) that the Zariski closure of every Hecke-invariant subvariety of $\mathscr{A}_g$ contains a supersingular point. The argument therein shows that the same statement holds for PEL-type modular varieties if "supersingular" is replaced by "basic" (in the sense of [27]).

In some sense the local stabilizer subgroup at a supersingular point $z_1 \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ contains a subset "of finite index" in the set of all prime-to-p Hecke symmetries on $\mathscr{A}_g$. Stated in a different way, the phenomenon is that the local stabilizer subgroup at a supersingular point of $\mathscr{A}_g$ "knows about" at least a positive fraction of Hecke symmetries. So it is not too far-fetched to expect strong rigidity statements for formal subvarieties $\hat{W}$ of $\mathscr{A}_g^{/z_1}$ stable under the action of an open subgroup of the local stabilizer subgroup Γ_z. For instance if such a formal subvariety $\hat{W}$ is generically ordinary, one expects that $\hat{W} = \mathscr{A}_g^{/z_1}$. A big obstacle here is that the action of the local stabilizer subgroup Γ_{z_1} on $\mathscr{A}_g^{/z_1}$ at a supersingular point z_1 is not understood when compared with the action of a local stabilizer group Γ_{z_2} on the formal completion $\mathcal{C}(z_2)^{/z_2}$ at a point z_2 of a positive dimensional central leaf $\mathcal{C}(z_2)$. The formal completion $\mathcal{C}(z_2)^{/z_2}$ has a linear structure assembled from p-divisible formal groups, and the action of Γ_{z_2} on $\mathcal{C}(z_2)$ preserves the linear structure. In contrast we only know that the formal completion $\mathscr{A}^{/z_1}$ is the spectrum of a formal power series ring over $\overline{\mathbb{F}}_p$, and we have little idea about the general pattern of the action of elements of Γ_{z_1} on $\mathscr{A}_g^{/z_1}$, nor do we possess a serviceable asymptotic expansion for the action of Γ_{z_1} on the formal completion $\mathscr{A}_g^{/z_1}$ at a supersingular point z_1.

6. Local rigidity for subvarieties of leaves

6.1. As outlined in 5.4 (a), it is expected that every irreducible formal subvariety of the formal completion $\mathcal{C}^{/x_0}$ of a central leaf $\mathcal{C}(x_0)$ in a PEL modular variety $\mathcal{M}$ which is stable under the action of the local stabilizer subgroup Γ_{x_0} is a "linear subvariety" of $\mathcal{C}^{/x_0}$, for any $\overline{\mathbb{F}}_p$-point of $\mathcal{M}$.

We recall that $\mathcal{C}^{/x_0}$ is built up from a finite family of torsors of p-divisible formal groups. A "linear subvariety" $\mathcal{W}$ in $\mathcal{C}^{/x_0}$ is an irreducible formal subscheme $\mathcal{C}^{/x_0}$ which is also built up from a family of torsors of p-divisible formal groups, such that the "building blocks" of $\mathcal{W}$ are p-divisible subgroups of the "building blocks" of $\mathcal{C}^{/x_0}$, and the inclusion map $\mathcal{W} \hookrightarrow \mathcal{C}^{/x_0}$ is compatible with the torsor structures on $\mathcal{W}$ and $\mathcal{C}^{/x_0}$.

In this section we explain two local rigidity results, one for p-divisible formal groups and one for biextensions of p-divisible formal groups. They imply the above linearity statement in the case when $\mathcal{M} = \mathscr{A}_g$, $x_0 = (A_0, \lambda_0)$, and $A_0[p^\infty]$ has at most three slopes. The proofs are in [6] and [12] respectively.

6.2. Definition. Let G be a compact p-adic Lie group and let $\mathrm{Lie}(G)$ be the Lie algebra of G. Let k be a perfect field $k \supset \mathbb{F}_p$.

  (a) Let X be a p-divisible group over k. let $\mathrm{M}(X)$ be the covariant Dieudonné module of X, and let $\mathrm{M}(X)_\mathbb{Q} := \mathrm{M}(X) \otimes_\mathbb{Z} \mathbb{Q}$. Let $\rho : G \to \mathrm{Aut}(X)$ be an action of G on X. We say that the action of G on X is *strongly non-trivial* if the trivial representation of $\mathrm{Lie}(G)$ does not appear in the Jordan-Hölder series of the representation of the Lie algebra $\mathrm{Lie}(G)$ on $\mathrm{M}(X)_\mathbb{Q}$. In other words for every open subgroup G' of G and any two p-divisible subgroups $Y \subsetneqq Z$ of X stable under G', there exists an element $\gamma \in G'$ such that $(\rho(\gamma) - 1)(Z) \not\subseteq Y$.

(b) Let X, Y, Z be p-divisible groups over k. Let $\pi : E \to X \times Y$ be a bi-extension of (X, Y) by Z. An action $\rho : G \to \mathrm{Aut}(E \to X \times Y)$ of G on the bi-extension $\pi : E \to X \times Y$ is *strongly non-trivial* if the induced actions of G on X, Y and Z are all strongly non-trivial as in (a) above.

6.3. Proposition. *Let G be a compact p-adic Lie group and let $\mathrm{Lie}(G)$ be the Lie algebra of G. Let k be an algebraically closed field $k \supset \mathbb{F}_p$.*

(a) *Let X be a p-divisible formal group over k, and let $\rho : G \to \mathrm{Aut}(X)$ be a strongly non-trivial action of G on X. Suppose that W is an irreducible closed formal subscheme of X stable under the action of G. Then W is a p-divisible formal subgroup of X.*

(b) *Let X, Y, Z be p-divisible formal groups over k. Let $\pi : E \to X \times Y$ be a bi-extension of (X, Y) by Z. Let $\rho : G \to \mathrm{Aut}(E \to X \times Y)$ be a strongly non-trivial action of G on E compatible with the bi-extension structure. Let W be an irreducible closed formal subscheme of X stable under the action of G. Suppose that every slope of Z is strictly bigger than every slope of X and every slope of Y.*

  (i) *There exists a p-divisible subgroup Z_1 of Z and a p-divisible subgroup U of $X \times Y$ such that W is stable under the action of Z_1 and the projection $\pi : E \to X \times Y$ induces an isomorphism $W/Z_1 \xrightarrow{\sim} U$. In other words W is a Z_1-torsor over U and the inclusion map $W \hookrightarrow E$ is equivariant with respect to $Z_1 \hookrightarrow Z$. Note that the p-divisible subgroups $Z_1 \subseteq Z$ and $U \subseteq X \times Y$ in (i) are uniquely determined by W.*

  (ii) *Suppose that X and Y do not have any slope in common. Then there exists p-divisible subgroups $X_1 \subseteq X$ and $Y_1 \subseteq Y$, uniquely determined by W, such that $U = X_1 \times Y_1$ and W is a sub-bi-extension of E. In other words W is stable under both relative group laws of the bi-extension E.*

6.4. Remark. The key ingredient of the proof of 6.3 (a) is the following "identity principle" for formal power series over k, proved in [6, §3], which produces identities of the form $f(\underline{x}, \underline{y}) = 0$ in two sets of variables $\underline{x}, \underline{y}$, from a sequence of congruence relations of the form

$$f(\underline{x}, \underline{x}^{p^n}) \equiv 0 \bmod (\underline{x})^{d_n}, \quad \text{with} \quad \lim_{n \to \infty} \frac{p^n}{d_n} = 0.$$

Identity principle. *Let $k \supset \mathbb{F}_p$ be a field. Let $\mathbf{u} = (u_1, \ldots, u_a)$, $\mathbf{v} = (v_1, \ldots, v_b)$ be two tuples of variables. Let $f(\mathbf{u}, \mathbf{v}) \in k[[\mathbf{u}, \mathbf{v}]]$ be a formal power series in the variables $u_1, \ldots, u_a, v_1, \ldots, v_b$ with coefficients in k. Let $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ be two new sets of variables. Let $(g_1(\mathbf{x}), \ldots, g_a(\mathbf{x}))$ be an a-tuple of power series such that $g_i(\mathbf{x}) \in (\mathbf{x})k[[\mathbf{x}]]$ for $i = 1, \ldots, a$. Let $(h_1(\mathbf{y}), \ldots, h_b(\mathbf{y}))$ be a b-tuple of power series with $h_j(\mathbf{y}) \in (\mathbf{y})k[[\mathbf{y}]]$ for $j = 1, \ldots, b$. Let $q = p^r$ be a power of p for some positive integer r. Let $n_0 \in \mathbb{N}$ be a natural number. Let $(d_n)_{n \in \mathbb{N}, \, n \geq n_0}$ be a sequence of natural numbers such that $\lim_{n \to \infty} \frac{q^n}{d_n} = 0$. Suppose we are given power series $\phi_{j,n}(\mathbf{v}) \in k[[\mathbf{v}]]$ for all $j = 1, \ldots, b$ and all $n \geq n_0$ such*

that

$$R_{j,n}(\mathbf{v}) := \phi_{j,n} - v_j^{q^n} \equiv 0 \pmod{(\mathbf{v})^{d_n}} \quad \forall j = 1, \ldots, b, \ \forall n \geq n_0.$$

and

$$f\left(g_1(\mathbf{x}), \ldots, g_a(\mathbf{x}), \phi_{1,n}(h(\mathbf{x})), \ldots, \phi_{b,n}(h(\mathbf{x}))\right) \equiv 0 \pmod{(\mathbf{x})^{d_n}}$$

in $k[[\mathbf{x}]]$, *for all* $n \geq n_0$. *Then*

$$f(g_1(\mathbf{x}), \ldots, g_a(\mathbf{x}), h_1(\mathbf{y}), \ldots, h_b(\mathbf{y})) = 0 \ \text{in} \ \ k[[\mathbf{x}, \mathbf{y}]].$$

6.5. Remark. The proof of 6.3 (b) uses the notion of *complete restricted perfection* of a complete Noetherian local ring in equi-characteristic p. An example of such a ring is the ring

$$k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}$$

consisting of all formal series over k whose underlying abelian group is the set of all formal series $\sum_I b_I\, \underline{t}^I$ with $b_I \in \kappa$ for all I, where I runs through all elements in $\mathbb{N}[1/p]^m$ such that

($\flat$) $$|I|_p \leq \mathrm{Max}\left(C \cdot (|I|_\sigma + d)^E, 1\right).$$

Here the parameters E, C, d are positive real numbers; $\mathbb{N}[1/p]$ is the additive semi-group of all non-negative rational numbers whose denominators divide a power of p. The p-adic absolute value $|I|_p$ and the archimedean absolute value $|I|_\sigma$ of an element $I = (i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m$ are defined by

$$|I|_p := \mathrm{Max}(|i_1|_p, \ldots, |i_m|_p) \quad \text{and} \quad |I|_\sigma := i_1 + \cdots + i_m$$

respectively, where $|\cdot|_p$ is the p-adic absolute value on $\mathbb{Q}$, normalized by $|p|_p = p^{-1}$. The subset $S_{E,C,d}$ of $\mathbb{N}[1/p]^m$ consisting of all elements of $\mathbb{N}[1/p]^m$ satisfying the condition ($\flat$) satisfies the following property: for every real number M, the set of all elements of $S_{E,C,d}$ such that $|I|_\sigma \leq M$ is *finite*. This finiteness property implies that the standard formula for multiplication of two such formal series give a ring structure on $k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}$.

6.6. Remark. We indicate how the complete restricted perfections enters the proof of 6.3 (b), in the case when Z is isoclinic. Furthermore we assume for simplicity that the intersection $Z \cap W$ is a p-divisible subgroup Z_1 of Z. From local rigidity for p-divisible group, we know that the image of W under the projection map $\pi : E \to X \times Y$ is a p-divisible subgroup U of $X \times Y$. We have to show that W is stable under the action of Z_1.

(i) If the bi-extension $E \to X \times Y$ of (X, Y) by Z is trivial, we have a natural retraction morphism $\mathbf{r} : E \to Z$. Then we use the identity principle 6.4 to show that $W \times W$ maps to W under the composition

$$E \times E \xrightarrow{\mathbf{r} \times 1_E} Z \times E \to E,$$

where the second maps $Z \times E \to E$ comes from the Z-torsor structure of E. This is the strategy for the proof of 6.3 (a). However when the given bi-extension $E \to X \times Y$ is non-split, we won't be able to produce a natural "retraction map" $\mathbf{r} : W \to Z_1$, even after modifying the bi-extension E by an isogeny. The complete restricted perfection comes to the rescue: there exists a natural "generalized retraction map" $W \dashrightarrow Z_1$ after passing from W to (the formal spectrum of) a

suitable complete restricted perfections of the coordinate rings of W. If we view elements of a restricted perfection as a sort of "generalized functions" on W, then such a generalized retraction $W \dashrightarrow Z_1$ can be viewed as a map "with coefficients in a ring of generalized functions".

(ii) The identity principle 6.4 holds when the formal series in the statement belong to a ring of the form $k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle\rangle_{C;d}^{E,\flat}$. This generalization of 6.4 depends crucially on the finiteness property of the support sets $S_{E,C,d}$. Using this generalized identity principle, one shows that W is stable under the translation action by the image of the "retraction map" $W \dashrightarrow Z_1$. The last statement implies that W is stable under translation by Z_1.

7. Monodromy of Hecke invariant subvarieties

In this section we discuss a general property for the prime-to-p monodromy of a subvariety Z of a modular variety $\mathcal{M}$ of PEL type over $\overline{\mathbb{F}_p}$, with the property that Z is stable under all prime-to-p Hecke correspondences on $\mathcal{M}$. The argument is based largely on group theory. In the case when the Hecke-invariant subvariety in question contains a point with very large local stabilizer subgroups, called *hypersymmetric points*, the same group-theoretic argument leads to information on p-adic monodromy.

We will formulate the results in the case of Siegel modular varieties, then comment on generalization to general PEL modular varieties.

7.1. Proposition. *Let $n \geq 3$ be a positive integer prime to p. Let $\mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$ be the moduli space of g-dimensional principally polarized abelian varieties with symplectic level-n structure over $\overline{\mathbb{F}_p}$. Let Z be a smooth locally closed subscheme of $\mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$ which is stable under the action of all prime-to-p Hecke correspondences. Let ℓ be a prime number such that $\ell \nmid pn$. Let $A \to \mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$ be the universal abelian scheme over $\mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$. Assume that Z is not contained in the supersingular locus of $\mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$.*

- (i) *The subscheme Z is irreducible if and only if the prime-to-p Hecke correspondences operates transitively on the set $\pi_0(Z)$ of connected components of Z.*
- (ii) *For any connected component Z_0 of Z, the image of the Galois representation attached to ℓ-power torsion points $(A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}_p}}} Z_0)[\ell^\infty] \to Z_0$ over Z_0 is isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.*

7.2. Remark. (1) Proposition 7.1 was proved in [2]. The proof is essentially group-theoretic.

(2) The "if" part of the statement 7.1 (i) implies that if a subvariety W of $A \to \mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$ is defined by some "p-adic invariant" fixed under all prime-to-p Hecke correspondences, to show that W is irreducible, it suffices to show that $\pi_0(W)$ consists of a single prime-to-p Hecke orbit. Using this method, the authors proved in [9] that the every non-supersingular Newton polygon stratum of $\mathscr{A}_{g,n,\overline{\mathbb{F}_p}}$, as well as every non-supersingular central leaf, is irreducible.

(3) The proof of 7.1 depends crucially on the fact that the algebraic group Sp_{2g} is simply connected. Therefore for general PEL type Shimura varieties one needs

to pass to the $G_{\mathrm{der}}^{\mathrm{sc}}(\mathbb{A}_f^{(p)})$-tower first and use the Hecke correspondences defined by this tower, before applying the group-theoretic argument. Here G is the reductive group over $\mathbb{Q}$ attached to the given PEL input data, and $G_{\mathrm{der}}^{\mathrm{sc}}$ is the simply connected cover of the derived group of G.

The statement of 7.1 also depends on the fact that Sp_{2g} is almost simple over $\mathbb{Q}$. The way we defined PEL input data has the consequence that the associated reductive group G has the property that G_{der} is almost simple over $\mathbb{Q}$. If the definition of modular varieties of PEL type is generalized to allow the possibility that G_{der} is not necessarily almost $\mathbb{Q}$-simple, the statement of 7.1 needs to be adjusted, to the effect that the $H(\mathbb{A}_f^{(p)})$-Hecke symmetries applied to any $\overline{\mathbb{F}}_p$-point of Z is infinite, for every $\mathbb{Q}$-factor H of $G_{\mathrm{der}}^{\mathrm{sc}}$.

7.3. Definition. Let B be a finite dimensional simple algebra over $\mathbb{Q}$. An abelian variety A over a field $K \supset \mathbb{F}_p$ with an action by B up to isogeny is said to be *hypersymmetric* if the natural map

$$\mathrm{End}_{\overline{K},B}^0\big(A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow \mathrm{End}_{\overline{K},B}^0\big(A[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

7.4. Remark. (1) The notion of hypersymmetric abelian varieties was introduced in [7].

(2) Clearly an abelian variety which is isogenous to a hypersymmetric abelian variety is itself hypersymmetric. Therefore we formulated 7.3 (b) for abelian varieties up to isogeny.

(3) For a point $z_0 = [(A_0, \lambda_0)] \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ such that A_0 is hypersymmetric, the local stabilizer subgroup Γ_{z_0} is $\mathrm{U}(\mathrm{End}(A_0[p^\infty]), *_0) \cap \mathrm{Aut}(A_0[p^\infty])$, where $*_0$ is the Rosati involution on $\mathrm{End}(A_0[p^\infty])$ attached to the principal polarization λ_0.

It is easy to show that the prime-to-p Hecke orbit of a hypersymmetric point $z_0 = [(A_0, \lambda_0)] \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ is dense in the central leaf $\mathcal{C}(z_0)$. This is an easy consequence of the local stabilizer principle, the Tate-linear structure on the formal completion at a point of a central leaf, and basic representation theory. This example leads to the following observation: if we can show that the Zariski closure in the central leaf $\mathcal{C}(x_0)$ of the prime-to-p Hecke orbit of a point $x_0 \in \mathcal{A}_g(\mathbb{F}_p)$ contains a point y_0 with a large stabilizer subgroup Γ_{y_0}, we have shown that the Zariski closure of this Hecke orbit is not too small.

(4) The notion of hypersymmetric points combined with the argument for the irreducibility result 7.1 (i) leads to an effective way to show that the p-adic monodromy of a Hecke invariant subvariety is large; see [5]. This method is used in [9] to show that the p-adic monodromy of every non-supersingular central leaf in $\mathcal{A}_g$ is maximal.

(5) For modular varieties of PEL type, it can happen that there are central leaves which do not contain any hypersymmetric point in the sense of 7.3. A simple example is a central leaf with Newton polygon $\mathrm{NP}\big((3,1) + (2,1) + (1,2) + (1,3)\big)$ in a Hilbert modular variety associated to a totally real number field F with $[F : \mathbb{Q}] = 14$. We refer to [56] for a complete solution of the existence problem of hypersymmetric points on modular varieties of PEL type.

8. The Hecke orbit conjecture for $\mathscr{A}_g$

8.1. In this section we outline a proof of the Hecke orbit conjecture for the moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}}_p$. A more detailed sketch of the proof of the Hecke orbit conjecture for $\mathscr{A}_g$ can be found in [3]; see also [4].

The proof uses a special property of $\mathscr{A}_g$, that every $\overline{\mathbb{F}}_p$-point of $\mathscr{A}_g$ is contained in a Hilbert modular subvariety of $\mathscr{A}_g$; see 8.3. It is a consequence of the fact that the endomorphism algebra (A, λ) of every polarized abelian variety over $\overline{\mathbb{F}}_p$ contains a product of totally real fields $F_1 \times \cdots F_r$ fixed by the Rosati involution, with $[F_1 : \mathbb{Q}] + \cdots + [F_r : \mathbb{Q}] = \dim(A)$. The same train of thought, together with the consideration of the action of local stabilizer subgroups, leads to the trick of "splitting at supersingular point"; see 8.2.

8.2. Proposition. *Let $n \geq 3$ be a positive integer prime to p. Let $x_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$ point of $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$. Let $\mathcal{C}(x_0)$ be the central leaf in $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ containing x_0. There exist*

(i) *an $\overline{\mathbb{F}}_p$-point $x_1 = [(A_1, \lambda_1)]$ in the Zariski closure in $\mathcal{C}(x_0)$ of the prime-to-p Hecke orbit of x_0,*

(ii) *totally real number fields $F_1, \ldots, F_r$ with $[F_1 : \mathbb{Q}] + \cdots + [F_r : \mathbb{Q}] = g$ such that $F_i \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field for $i = 1, \ldots, r$, and*

(iii) *a subring of the endomorphism algebra $\mathrm{End}^0(A_1) := \mathrm{End}(A_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ of fixed by the Rosati involution attached to λ_1, which is isomorphic to $F_1 \times \cdots \times F_r$.*

Clearly if we can show that the prime-to-p Hecke orbit of x_1 is dense in $\mathcal{C}(x_1)$, then the prime-to-p Hecke orbit of x_0 is dense in $\mathcal{C}(x_0)$.

8.3. Proposition. *We keep the notation in 8.2. There exist*

(i) *a positive integer $m \geq 3$ prime to p,*

(ii) *a Hecke-equivariant finite morphism*

$$f : \mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p} \times \cdots \mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p} \longrightarrow \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$$

with respect to the embedding $\mathrm{SL}_2(F_1) \times \cdots \mathrm{SL}_2(F_r) \to \mathrm{Sp}_{2g}$ of algebraic groups, where $\mathcal{M}_{F_i,m}$ is the Hilbert modular variety with level-m structure attached to F_i,

(iii) *Hecke-equivariant finite morphisms*

$$h_i : \mathcal{M}_{F_i,m,\overline{\mathbb{F}}_p} \longrightarrow \mathscr{A}_{[F_i:\mathbb{Q}],n,\overline{\mathbb{F}}_p}$$

with respect to the embedding $\mathrm{SL}_2(F_i) \to \mathrm{Sp}_{2[F_i:\mathbb{Q}]}$ for $i = 1, \ldots, r$,

(iii) *an $\overline{\mathbb{F}}_p$-point $(z_1, \ldots, z_r)$ of $\mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p} \times \cdots \mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p}$ with*

$$f_1(z_1, \ldots, z_r) = x_1,$$

such that for any r-tuple of points $y_1, \ldots, y_r$ of $\mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p}, \ldots, \mathcal{M}_{F_r,m,\overline{\mathbb{F}}_p}$ corresponding to $\mathscr{O}_{F_i}$-linear $[F_i : \mathbb{Q}]$-dimensional abelian varieties $B_1, \ldots, B_r$, the abelian variety corresponding to $f_1(y_1, \ldots, y_r)$ is $(F_1 \times \cdots \times F_r)$-linearly isogenous to $B_1 \times \cdots B_r$, and the abelian variety corresponding to $h_i(y_i)$ is isogenous to B_i for $i = 1, \ldots, r$.

The general linearization method implies quickly that the Zariski closure of the prime-to-p Hecke orbit of z_i on the Hilbert modular variety $\mathcal{M}_{F_i,m,\overline{\mathbb{F}}_p}$ contains an open subset of the central leaf $\mathcal{C}_F(z_i)$ in $\mathcal{M}_{F_i,m,\overline{\mathbb{F}}_p}$ passing through z_i for $i = 1,\dots,r$, because $F_i \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field. The irreducibility result [53] of C.-F. Yu says that $\mathcal{C}_F(z_i)$ is irreducible unless z_i is supersingular, in which case $\mathcal{C}_F(z_i)$ is a finite set and the prime-to-p Hecke correspondences operates transitively on $\mathcal{C}_F(z_i)$. Note that there exist hypersymmetric points in $\mathcal{C}_F(z_i)$ for every i. The statement 8.4 follows.

8.4. Corollary. *Notation as in* 8.3. *There exists an* $\overline{\mathbb{F}}_p$-*point* $x_2 = [(A_2, \lambda_2)] \in \mathcal{C}(x_1)$ *which lies in the Zariski closure the prime-to-p Hecke orbit of* x_1 *such that* A_2 *is isogenous to a product of hypersymmetric abelian varieties.*

8.5. Theorem. *Notation as in* 8.2 *and* 8.3. *The prime-to-p Hecke orbit* $\mathcal{H}^{(p)} \cdot x_1$ *of every* $\overline{\mathbb{F}}_p$-*point* x_1 *of* $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ *is Zariski dense in the central leaf* $\mathcal{C}(x_1)$ *in* $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$.

As we have mentioned earlier, the combination of the local stabilizer principle and the generalized Serre–Tate theory for central leaves shows that the Zariski closure of the prime-to-p Hecke orbit of the hypersymmetric point $h_i(z_i)$ in the Siegel modular variety $\mathscr{A}_{[F_i:\mathbb{Q}],m,\overline{\mathbb{F}}_p}$ is dense in $\mathcal{C}(h_i(z_i))$. Another application of this method plus easy representation theory allows us conclude that the Zariski closure of the prime-to-p Hecke orbit of x_1 contains an open subset of $\mathcal{C}(x_1)$. However we know that the central leaf $\mathcal{C}(x_1)$ is irreducible unless x_1 is supersingular, so we are done.

9. Open questions

9.1. In this section we list several open questions related to the Hecke orbit conjecture. We have not attempted to put these questions in the most general setting possible. Instead we have formulated them in relatively simple cases where essential aspects of the difficulties are preserved.

- Conjectures 9.2 and 9.3 are samples of strong local rigidity predictions in the direction of 5.4 (d). To make progress on them, the first step would be developing a theory of "asymptotic expansion" for the action of elements of the local stabilizer subgroup which are close to 1, on the characteristic p deformation space (or the formal subvariety W in question). Note that in the situation of 4.2.2 and 4.2.3 the linear structure of $\mathcal{C}^{/z_0}$ provides such asymptotic expansions, which allows us to apply the identity principle.
- Problem 9.4 is a global rigidity statement with a flavor somewhat different from the Hecke orbit conjecture. Among other things, it asserts that a local geometric condition on a subvariety Z of a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$, that it is "linear" at one point, implies the existence of many Tate cycles on Z, to the extent that these Tate cycles "cut out" Z in $\mathcal{M}$. The assertion is tempting, but we have no evidence other than our inability to come up with a counter-example.
- Problem 9.5 is a possible approach to 9.4 via p-adic monodromy. Of course both 9.4 and 9.5 can be formulated in the context of "linear subvarieties"

of a central leaf $\mathcal{C}$ of a PEL modular variety. But we feel that the charm of relative simplicity has its virtue.

- Problem 9.6 is an important and natural question, but we don't have a good idea about the shape of the answer at this point. It bears some connections with 9.3 and also some connection with 9.4.

9.2. Conjecture. *Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let X_0 be a one-dimensional p-divisible formal group of height $h > 1$. Suppose that $\mathcal{W}$ is a reduced irreducible closed formal subscheme of the characteristic-p deformation space $\mathcal{D}ef(X_0)$ which is stable under the action of an open subgroup of $\mathrm{Aut}(X_0)$. Then $\mathcal{W}$ is formally smooth over k, and $\mathcal{W}$ is the locus in $\mathcal{D}ef(X_0)$ over which the universal p-divisible group has p-rank at most $\dim(\mathcal{W})$.*

9.3. Conjecture. *Let (A_0, λ_0) be a supersingular principally polarized abelian variety over an algebraically closed field $k \supset \mathbb{F}_p$. Suppose that $\mathcal{W}$ is an irreducible formal subscheme of the characteristic-p deformation space $\mathcal{D}ef(A_0, \lambda_0) = \mathcal{D}ef(A_0[p^\infty], \lambda_0[p^\infty])$. Assume that $\mathcal{W}$ is stable under the action of an open subgroup of $\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty])$. If the restriction to the generic point of the universal p-divisible group is ordinary, then $\mathcal{W} = \mathcal{D}ef(A_0, \lambda_0)$.*

9.4. Problem. Let $n \geq 3$ be a positive integer prime to p, and let $\mathscr{A}_{g,1,n}/W(\overline{\mathbb{F}}_p)$ be the moduli space of g-dimensional principally polarized abelian varieties with symplectic level-n structure over $W(\overline{\mathbb{F}}_p)$. Let Z be an irreducible closed subscheme of $\mathscr{A}_{g,1,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{F}}_p)$. Let $z_0 = [(A_0, \lambda_0)] \in Z(\overline{\mathbb{F}}_p)$ be a closed point of Z. Suppose that A_0 is an ordinary abelian variety, and the formal completion $Z^{/z_0}$ of Z is a formal subtorus of the Serre–Tate formal torus

$$\left(\mathscr{A}_{g,1,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{F}}_p) \right)^{/z_0}.$$

Show that there exists a finite extension field L of $W(\overline{\mathbb{F}}_p)[1/p]$ and a reduced closed subscheme $V \subseteq \mathscr{A}_{g,1,n} \times_{\mathrm{Spec}(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\mathscr{O}_L)$ such that the following statements hold.

- (i) There exists a $\mathbb{Q}_p$-linear embedding $L \hookrightarrow \overline{\mathbb{Q}}_p$ such that geometric generic fiber $V \times_{\mathrm{Spec}(\mathscr{O}_L)} \mathrm{Spec}(\overline{\mathbb{Q}}_p)$ of V is a Shimura subvariety of the moduli space $\mathscr{A}_{g,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{Q}}_p)$.
- (ii) The scheme Z is an irreducible component of the closed fiber $V \times_{\mathrm{Spec}(\mathscr{O}_L)} \mathrm{Spec}(\overline{\mathbb{F}}_p)$ of V.

9.5. Problem. Let Z be an irreducible closed subscheme of the Siegel moduli scheme $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p} := \mathscr{A}_{g,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{F}}_p)$ over $\overline{\mathbb{F}}_p$ which is linear at an ordinary point as in 9.4; i.e. there exists a closed point $z_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,n}(\overline{\mathbb{F}}_p)$ with A_0 ordinary such that $Z^{/z_0}$ is a formal subtorus of $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}^{/z_0}$. Let $A \to \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ be the universal abelian scheme over $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$, and let Z_{ord} be the largest open subset of Z such that the abelian scheme $A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}} \to Z_{\mathrm{ord}}$ over Z is ordinary.

- (i) Show that the Zariski closure of the image of the Galois representation attached to the maximal etale quotient $\left(A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}}\right)[p^\infty]_{\mathrm{et}}$ of the p-divisible group $\left(A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}}\right)[p^\infty]$ over Z_{ord} is a reductive subgroup

G_{Z,naive,z_0} of $\mathrm{GL}_g(\mathrm{V}_p(A_0[p^\infty]_{\mathrm{et}}))$. Here $A_0[p^\infty]_{\mathrm{et}}$ is the maximal etale quotient of the p-divisible group $A_0[p^\infty]$, and $\mathrm{V}_p(A_0[p^\infty]_{\mathrm{et}})$ is the p-adic Tate module of $A_0[p^\infty]_{\mathrm{et}}$, non-canonically isomorphic to $\mathbb{Q}_p^g$.

(ii) Let $\mathcal{T}_{Z_{\mathrm{ord}}}$ be the $\mathbb{Q}_p$-linear Tannakian subcategory generated by the isocrystal attached the p-divisible group $A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}}[p^\infty]$ in the Tannakian category of all overconvergent isocrystals over Z_{ord}.

  (a) Show that the Galois group $\mathrm{Gal}(\mathcal{T}_{Z_{\mathrm{ord}},z_0})$ attached to the Tannakian category $\mathcal{T}_{Z_{\mathrm{ord}}}$ and the fiber functor at z_0 is reductive.

  (b) Show that there exists a parabolic subgroup P of $\mathrm{Gal}(\mathcal{T}_{Z_{\mathrm{ord}},z_0})$ such that the unipotent U radical of P is naturally isomorphic to the cocharacter group of the formal torus $Z^{/z_0}$, and the reductive quotient of P/U is naturally isomorphic to G_{Z,naive,z_0}.

9.6. Problem. Let $\mathcal{C}$ be a central leaf in $\mathscr{A}_g$ over $\overline{\mathbb{F}}_p$, let $\overline{\mathcal{C}}$ be the Zariski closure of $\mathcal{C}$ in $\mathscr{A}_g$, and let s_0 be an $\overline{\mathbb{F}}_p$-point of $\overline{\mathcal{C}} \smallsetminus \mathcal{C}$. Analyze the structure of the formal completion $\overline{\mathcal{C}}^{/s_0}$ of $\overline{\mathcal{C}}$ at s_0, and relate the structure of $\overline{\mathcal{C}}^{/s_0}$ to the family of linear structures of $\mathcal{C}^{/z}$ as z varies over points of $\mathcal{C}$.

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