Differential Equations Practice Problems

Math 116

Part I

First order DE's:

1) §8.5 #2
2) §8.5 #3
3) §8.5 #5
4) §8.5 #14

Second order homogeneous DE's:

5) §8.14 #1
6) §8.14 #2
7) §8.14 #5
8) §8.14 #7
9) §8.14 #10
10) §8.14 #10
11) §8.14 #11
12) §8.14 #14

Second order non-homogeneous DE's:

13) §8.17 # 1
14) §8.17 # 12
15) §8.17 # 15

Part II

Recall Newton's Second Law:

$$\vec{F}_{net} \;=\; m\,\frac{d^2\vec{x}}{dt^2} \tag{1}$$

where $\vec{x}$ is a particle's position vector, and $\vec{F}_{net}$ is the sum of forces on the particle. In this section, we consider 1-dimensional motion, so the position and force vector-value functions can be regarded as just ordinary funcitons:

$$F_{net} \;=\; m\,\frac{d^2x}{dt^2} \tag{2}$$

We consider the case where the force is given by a combination of Hooke's law:

$$F_{spring} \;=\; -kx \tag{3}$$

where k is the spring constant, a damping force (aka friction):

$$F_{damp} \;=\; -c\frac{dx}{dt} \tag{4}$$

where c is the coefficient of dynamic friction, and some other external force $F_{ext}(t)$ that depends on time.

16) Set $p = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$, and $F(t) = \frac{1}{m}F_{ext}(t)$. Show that the particle's position x obeys the second order linear constant coefficient nonhomogeneous differential equation

$$\ddot{x} \,+\, 2\,p\,\dot{x} \,+\, \omega_0^2\,x \;=\; F(t). \tag{5}$$

Equation (5) is the standard form for a damped harmonic oscillator.

The function $F(t)$ is called the *forcing function.* In the following problems we consider the unforced case: $F(t) = 0$.

17) Given $p = 5$, $\omega_0 = 1$ find the solution for $x(0) = 0$, $\dot{x}(0) = 1$.

18) Given $p = 1$, $\omega_0 = 5$ find the solution for $x(0) = 0$, $\dot{x}(0) = 1$.

19) (Undamped Case) Given $p = 0$ and $F(t) = 0$ (unknown ω_0), find the general solution to (5). Explain why ω_0 is called the *undamped natural frequency.*

20) (Underdampled Case) Assume $p < \omega_0$. Set $\omega_1 = \sqrt{\omega_0^2 - p^2}$, and prove that the general solution is

$$x(t) \;=\; c_1\,e^{-pt}\cos(\omega_1 t) \,+\, c_2\,e^{-pt}\sin(\omega_1 t) \tag{6}$$

Explain why ω_1 is called the *damped natural frequency.* What happens as the damping constant decreases to zero?

21) (Overdamped Case) If $p > \omega_0$, prove that the general solution is a linear combination of two decaying exponentials.

22) (Critically Damped Case) Assume $p = \omega_0$, and show the general solution is a linear combination of e^{-pt} and te^{-pt}.

Now we consider the case of a *forced harmonic oscillator*, where the forcing function is sinusoidal:

$$F(t) \;=\; \cos(\omega t). \tag{7}$$

The constant ω is called the *forcing frequency*.

23) Given $F(t) = \cos(\omega t)$, what is your "guess" for the particular solution $x_p(t)$?

24) From the mathematics, we know the general solution $x(t) = x_c(t) + x_p(t)$ is the sum of the complementary and particular solutions. In a physical context, the complimentary solution is called the *transient solution* and the particular solution is called the *steady-state solution*. Explain, in a precise way, why this terminology makes sense.

25) From Problem (24) the correct "guess" is $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Given $p = 1$, $\omega_0 = \sqrt{10}$, and forcing frequency 2, determine the general solution. Then determine the solution when the oscillator is starting from rest: $x(0) = \dot{x}(0) = 0$.

26) Referring to Problem (25), make a plot of the transient solution, and make a separate plot of the steady-state solution.

Part III (optional; won't be on final)

A general first order equation can be written in the form

$$y' = F(y, t). \tag{8}$$

Most differential equations cannot be solved in closed form; for instance

$$y' = y^2 + t^2 \tag{9}$$

has a solution, but it is impossible to write it down in terms of elementary functions (although you could solve it using Taylor series methods). Given a DE that you can't solve by hand, how do you know if a solution even exists?

Theorem 3.1 (Existence) *The differential equation*

$$y' = F(y, t) \tag{10}$$

has at least one solution with initial conditions $y(t_0) = y_0$ provided the function $F(y,t)$ is continuous with respect to y at (y_0, t_0). Further, the solution is continuous, and continues so long as $F(y,t)$ remains to be continuous with respect to y, and $y(t)$ does not approach $\pm\infty$.

Theorem 3.2 (Uniqueness) *The differential equation*

$$y' = F(y, t) \tag{11}$$

has one and only one solution with initial conditions $y(t_0) = y_0$ provided both $F(y,t)$ and $\frac{\partial F}{\partial y}$ are continuous with respect to y at (y_0, t_0). Further, the solution is continuous, and continues uniquely so long as $F(y,t)$ and $\frac{\partial F}{\partial y}(y,t)$ remain continuous with respect to y, and $y(t)$ does not approach $\pm\infty$.

The proofs go beyond the scope of this class, but are proved in Math 361.

27) Consider the following equation:

$$y' = 3\,y^{\frac{2}{3}}. \tag{12}$$

Show that $y(t) = 0$ and $y(t) = t^3$ both solve (12) with initial condition $y(0) = 0$.

28) Prove directly that the differential equation $y' = \frac{y}{t}$ has no solution with initial condition $y(0) = 1$.

29) Prove that $y' = y^2$, with initial condition $y(0) = 1$ cannot be continued indefinitely. If the initial condition is $y(0) = -1$, show that the solution exists for all time $t \geq 0$.

30) Determine for which initial conditions the following DE's have unique solutions:

a) $y' = ty^{-1}$

b) $y' = sgn(t-1)$

c) $y' = sgn(y-1)$

d) $y' = y^{\frac{1}{3}}$

e) $y' = y^{-\frac{1}{3}}$

f) $y' = sgn(\sin(\pi y))$

g) $y' = \csc(y)$

31) Consider the two DE's $y' = y$ and $y' = -y$, with initial conditions $y(0) = 1$ (obviously the solutions are $y = e^t$ and $y = e^{-t}$, respectively, but for now pretend we don't know this). In either case, show that if some t_1 exists where $y(t_1) = 0$, then in fact $y = 0$ for all t, contradicting the fact that the initial condition was $y(0) = 1$.

32) Use your result from Problem (31) to show that e^t is positive for all t.

33) Show that e^t is continuously differentiable for all t. Prove that e^t is infinitely continuously differentiable for all t. Prove that e^t is convex.