Homework 9 Math 116

Due Nov 29, 2012

Remember: No credit will be given for answers without mathematical or logical justification.

Chapter 4

1) In Homework 7 we proved that $\frac{d}{dx}x^n = nx^{n-1}$ when $n \in \mathbb{N}$, and in Homework 8 we used this prove that $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}ln\,x = \frac{1}{x}$ (aside from some possible issues concerning convergence of infinite sums). Using the chain rule along with the fact that $x^n = e^{n\,ln\,x}$, prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{R}$ (and $x > 0$).

Chapter 5

2) Evaluate the following integrals:

   a) $\int_{-\frac{1}{2}}^{\frac{1}{2}} x\sqrt{1-x^2}\,dx$

   b) Evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-x^2}\,dx$ by making the substitution $x = \sin u$. You might have to make use of the identity $2\cos^2 v = 1 + \cos(2v)$.

   c) Evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ by making the same substitution as above.

Chapter 12

3) We have seen a connection between quaternions and 3-dimensional vector space theory (via the cross product). There are connections between quaternions and the 4-dimensional theory as well. For instance if $x = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$, $y = \beta_1 + \beta_2 i + \beta_3 j + \beta_4 k$ are quaternions and $\vec{x} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\vec{y} = (\beta_1, \beta_2, \beta_3, \beta_4)$ are the corresponding 4-vectors, show that

$$\vec{x} \cdot \vec{y} \;=\; Re\,(x\overline{y}) \;=\; \frac{1}{2}\,(x\overline{y} \,+\, y\overline{x})$$

(where the "$\cdot$" symbol indicates the bilinear inner product, and $Re(x)$ denotes the real part of the quaternion x). Note: if a, b are quaternions, then $\overline{ab} = \overline{b}\overline{a}$.

Chapter 13

4) #1 in §13.14

5) #7 in §13.14

6) Determine whether the following sets of vectors are linearly independent or not:

   a) $\{\,(1,\,3,\,5)\,,\,(-1,\,2,\,5)\,,\,(1,\,2,\,-5)\,\}$

   b) $\{\,(-15,\,-1,\,0,\,1)\,,\,(1,\,1,\,-2,\,-1)\,,\,(3,\,0,\,2,\,1)\,,\,(1,\,4,\,-1,\,1)\,\}$

Chapter 14

7) #1 in §14.7

8) #4 in §14.7

9) #5 in §14.7

10) We define the *n-sphere of radius* r (centered at the origin) to be the set of points $\vec{x} \in \mathbb{R}^{n+1}$ such that

$$\|\vec{x}\| \;=\; r. \tag{1}$$

The n-sphere of radius r is denoted $\mathbb{S}^n(r)$ (the n-sphere of radius 1 is often called the *unit n-sphere*).

a) Make a sketch of the 0-sphere of radius 3. Make a sketch of the 1-sphere of radius 4. Make a sketch of the unit 2-sphere.

b) Show that $\vec{f}(t) = \frac{1}{\sqrt{2}}(\cos(t),\, \sin(t),\, \cos(t),\, \sin(t))$ lies on the unit 3-sphere inside $\mathbb{R}^4$.

c) Assume $\vec{f}(t)$ is a path in $\mathbb{R}^{n+1}$ that lies in $\mathbb{S}^n(r)$. Prove that $\vec{f}'(t)$ is perpendicular to $\vec{f}(t)$.

11) Compute

$$\left\| \int_0^\tau (1,\, t)\; dt \right\|$$

and

$$\int_0^\tau \|(1,\, t)\|\; dt$$

where τ is any real number.

12) Let $\vec{f}(t) = (f_1(t), \ldots,\, f_n(t))$ be a vector-valued function. Prove that

$$\left\| \int_a^b \vec{f}(t)\, dt \right\| \;=\; \int_a^b \left\| \vec{f}(t) \right\|\; dt \tag{2}$$

provided that each component $f_i(t)$ is a constant multiple of some fixed function $g(t)$ (that is, constants $a_1, \ldots, a_n \in \mathbb{R}$ exist so that $\vec{f}(t) = (a_1 g(t),\, \ldots,\, a_n g(t))$), under the condition that $g(t) \geq 0$.

13) Determine the unit tangent vector and the principle normal vector for the function in problem (10)b.

14) Consider the helix curve $\vec{f}(t) = (\cos(t),\, \sin(t),\, t)$ in $\mathbb{R}^3$. Compute $\vec{T}$, $\vec{N}$, and $\vec{B}$.