Homework 8 Math 116

Due Nov 8, 2012

Remember: No credit will be given for answers without mathematical or logical justification.

Chapter 1

1) As in the previous homework set, let $f(x)$ be the function on the interval $[0, 1]$ defined by

$$f(x) \quad = \quad \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Using the results from your previous two homeworks, formally prove that $f(x)$ is not integrable.

Chapter 4

2) From the previous homework, we proved that if $n \in \mathbb{N}$, then $\frac{d}{dx}x^n = nx^{n-1}$. It is customary to define

$$e^x \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

Although you are not asked to do so, one can easily prove that this sum always converges. Using this definition along with the result from the previous homework, prove that $\frac{d}{dx}e^x = e^x$. You may assume that $\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{1}{n!}x^n\right)$, even though strictly speaking we have not proven this.

3) This definition of e^x is a little difficult to use; in particular, if x is negative, how can you tell whether e^x is positive or negative? In a later homework, we will prove formally that e^x is both positive and monotonically increasing; for now we just assume this is true. But this implies e^x has an *inverse* function, called the natural logarithm, denoted $ln\, x$. Using the fact that $e^{ln\, x} = x$ along with the chain rule, prove that $\frac{d}{dx}ln\, x = \frac{1}{x}$.

Chapter 12

4) Let $\{\vec{v}_1, \vec{v}_2\}$ be a set of two independent vectors in $\mathbb{R}^n$. Prove that $\vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2$ is orthogonal to $\vec{v}_1$.

5) (Orthogonalization) As before let $\{\vec{v}_1, \vec{v}_2\}$ be a set of two independent vectors in $\mathbb{R}^n$. Setting $\vec{w}_1 = \vec{v}_1$ and $\vec{w}_2 = \vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2$, prove that $\vec{w}_1$ and $\vec{w}_2$ are orthogonal, and that $span\{\vec{w}_1, \vec{w}_2\}$ is the same as $span\{\vec{v}_1, \vec{v}_2\}$.

6) (Orthonormalization) As before let $\{\vec{v}_1, \vec{v}_2\}$ be a set of two independent vectors in $\mathbb{R}^n$. Setting

$$\begin{aligned} \vec{w}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ \vec{w}_2 &= \frac{\vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2}{\|\vec{v}_2 - proj_{\vec{v}_1}\vec{v}_2\|} \end{aligned} \tag{1}$$

prove that $\{\vec{w}_1, \vec{w}_2\}$ is an *orthonormal set*, and has the same span as the original set $\{\vec{v}_1, \vec{v}_2\}$.

Chapter 13

7) Given $\vec{A}, \vec{B} \in \mathbb{R}^3$, where $\vec{A} = (a_1, a_2, a_3)$ and $\vec{B} = (b_1, b_2, b_3)$, we defined

$$\vec{A} \times \vec{B} \;=\; (a_2 b_3 - a_3 b_2,\; a_3 b_1 - b_3 a_1,\; a_1 b_2 - a_2 b_1)\,. \tag{2}$$

Computing $\|\vec{A} \times \vec{B}\|^2$, we get

$$\begin{aligned} \left\|\vec{A} \times \vec{B}\right\|^2 &= (a_2 b_3 - a_3 b_2)^2 \;+\; (a_3 b_1 - b_3 a_1)^2 \;+\; (a_1 b_2 - a_2 b_1)^2 \\ &= (a_2 b_3)^2 \;-\; 2 a_2 a_3 b_2 b_3 \;+\; (a_3 b_2)^2 \;+ \\ &\quad (a_3 b_1)^2 \;-\; 2 a_1 a_3 b_1 b_3 \;+\; (a_1 b_3)^2 \;+ \\ &\quad (a_1 b_2)^2 \;-\; 2 a_1 a_2 b_1 b_2 \;+\; (a_2 b_1)^2 \end{aligned} \tag{3}$$

by carrying out similar computations for $\|\vec{A}\|^2$, $\|\vec{B}\|^2$, and $(\vec{A} \cdot \vec{B})^2$, prove that

$$\left\|\vec{A} \times \vec{B}\right\|^2 \;=\; \|\vec{A}\|^2 \|\vec{B}\|^2 \;-\; \left(\vec{A} \cdot \vec{B}\right)^2. \tag{4}$$

Using this and your result from Problem 6 of Homework 7, what is the geometric interpretation of $\|\vec{A} \times \vec{B}\|$?

8) Let $\vec{v}_1$, $\vec{v}_2$ be two independent vectors in $\mathbb{R}^3$. Given any point $\vec{P} = (p_1, p_2, p_3) \in \mathbb{R}^3$, consider the 2-plane $\mathcal{M} = M(\vec{P};\, \vec{v}_1, \vec{v}_2)$, and the hyperplane given by the solutions to $\vec{x}$ that satisfy $\vec{N} \cdot (\vec{x} - \vec{P}) = 0$ where $\vec{N} = \vec{v}_1 \times \vec{v}_2$.

   i) If $\vec{x}$ is any point of $\mathcal{M}$, then prove $\vec{x}$ satisfies the given hyperplane equation.

   ii) If $\vec{x}$ satisfies the hyperplane equation, then prove $\vec{x} \in \mathcal{M}$. (Hint: We know that $\vec{v}_1, \vec{v}_2, \vec{v}_1 \times \vec{v}_2$ are independent, so therefore form a basis of $\mathbb{R}^3$. Since $\vec{x} - \vec{P} \in \mathbb{R}^n$, is has a unique expression as a linear combination of these basis elements. In terms of this linear combination, what information is provided by the fact that $\vec{x}$ satisfies the hyperplane equation?)

9) This exercise is meant to help you get used to quaternion arithmetic. A quaternion is any number of the form

$$\alpha_1 \;+\; \alpha_2\, i \;+\; \alpha_3\, j \;+\; \alpha_4\, k, \tag{5}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and the imaginary units i, j, k obey the following relations:

$$i^2 \;=\; -1, \quad j^2 \;=\; -1, \quad k^2 \;=\; -1, \quad ijk \;=\; -1. \tag{6}$$

The set of all quaternions is denoted $\mathbb{H}$.

   i) Compute xy where

      a) $x = i$, $y = i + j$

      b) $x = 1 + j$, $y = 1 + k$

      c) $x = 1 + i + j$, $y = 1 - i - j$

   ii) Prove the following:

$$ij \;=\; -ji, \quad ik \;=\; -ki, \quad jk \;=\; -kj. \tag{7}$$

   In particular, the quaternionic algebra is not commutative.

iii) If

$$x = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k \tag{8}$$

is any quaternion, its *quaternionic conjugate* is

$$\bar{x} = \alpha_1 - \alpha_2 i - \alpha_3 j - \alpha_4 k. \tag{9}$$

Prove that $x\bar{x} = (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 + (\alpha_4)^2$.

10) Just as $\mathbb{R}^2$ can be identified with the complex numbers $\mathbb{C}$, we can identify with $\mathbb{R}^4$ with $\mathbb{H}$, thus giving $\mathbb{R}^4$ a "product" operation. Similarly $\mathbb{R}^3$ can be identified with the purely imaginary quaternions simply by identifying $i, j, k \in \mathbb{H}$ with the basis vectors $\hat{\imath} = (1,0,0), \hat{\jmath} = (0,1,0), \hat{k} = (0,0,1)$. Obviously the usual product on quaternions does *not* give $\mathbb{R}^3$ a product operation. However the *commutator product* on $\mathbb{H}$, defined by

$$[x,\, y] \;=\; \frac{1}{2}\,(xy \;-\; yx) \tag{10}$$

does indeed give $\mathbb{R}^3$ a product operation.

i) Prove that the commutator product on $\mathbb{H}$ is bilinear and antisymmetric.

ii) We already know that

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \quad \hat{\imath} \times \hat{k} = -\hat{\jmath}, \quad \hat{\jmath} \times \hat{k} = i. \tag{11}$$

Prove that

$$[i,\, j] \;=\; k, \quad [i,\, k] \;=\; -j, \quad [j,\, k] \;=\; i \tag{12}$$

iii) Using (*i*) and (*ii*), prove that the cross product on $\mathbb{R}^3$ and the commutator product on the purely imaginary quaternions are the same thing. Specifically, if

$$\begin{aligned} \vec{x} \;&=\; x_1\hat{\imath} \,+\, x_2\hat{\jmath} \,+\, x_3\hat{k} \\ \vec{y} \;&=\; y_1\hat{\imath} \,+\, y_2\hat{\jmath} \,+\, y_3\hat{k} \end{aligned} \tag{13}$$

are vectors and

$$\begin{aligned} x \;&=\; x_1 i \,+\, x_2 j \,+\, x_3 k \\ y \;&=\; y_1 i \,+\, y_2 j \,+\, y_3 k \end{aligned} \tag{14}$$

are the corresponding purely imaginary quaternions, show that $\vec{x} \times \vec{y}$ and $[x,\, y]$ have the same formal expressions.

iv) Because the cross product and the quaternionic commutator are the same thing, proving properties of the cross products can always be done with quaternions. Prove the Jacobi identity for the cross product: if $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$, then

$$\vec{x} \times (\vec{y} \times \vec{z}) \;+\; \vec{z} \times (\vec{x} \times \vec{y}) \;+\; \vec{y} \times (\vec{z} \times \vec{x}) \;=\; 0 \tag{15}$$

(Hint: do NOT attempt to write out $x = x_1 i + x_2 j + x_3 k$, etc. Just compute commutators:

$$[x, [y, z]] = \frac{1}{2}\,(x[y, z] - [y, z]x) = \frac{1}{4}(xyz - xzy - yzx + zyx) \tag{16}$$

and so on.)