Homework 7 — Math 116

Due Nov 1, 2012

Remember: No credit will be given for answers without mathematical or logical justification.

Chapter 1

1) As in the previous homework set, let $f(x)$ be the function on the interval $[0,1]$ defined by

$$f(x) \;=\; \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Assume $s(x)$ is some step function defined on $[0,1]$. Assume $t(x)$ is some step function defined on $[0,1]$. As before, we know, by definition, that $t(x)$ is subordinate to some partition $P = \{x_0, x_1, \ldots, x_N\}$, and that $t(x)$ has the form

$$t(x) \;=\; t_k \text{ on the } k^{th} \text{ interval}$$

where each t_k is a constant. Finally, assume that $t(x) \le f(x)$ on $[0,1]$.

a) Make a rough sketch of $f(x)$. Explain, on an intuitive level, why it stands to reason that $\underline{\int}_0^1 f(x)\,dx \le 0$.

b) Relative to the partition P, what, specifically, is I_k, and what is $\triangle I_k$? (Here I_k is the k^{th} interval and $\triangle I_k$ is the length of the interval I_k.)

c) Prove that $t_k \le 0$.

d) Prove that $\int_0^1 t(x)\,dx \;\le\; 0$.

e) Prove that $\underline{\int}_0^1 f(x)\,dx \;\le\; 0$.

Chapter 4

2) The *binomial theorem* states that if z, w are any complex (or real) numbers and $n \in \mathbb{N}$, then

$$(z+w)^n \;=\; \sum_{i=0}^{n} \binom{n}{i} z^{n-i}\, w^i$$

where the symbol $\binom{a}{b}$ is defined to be the number

$$\binom{a}{b} \;\triangleq\; \frac{a!}{b!\,(a-b)!} \tag{1}$$

(by convention, we always take $0! = 1$). We will accept the binomial theorem as true; you are not asked to prove it.

a) Using the binomial theorem, determine $(z+w)^5$ as a sum of monomial terms.

b) If $f(x) = x^n$ and $n \in \mathbb{N}$, $n \ge 2$, prove that

$$f(x+h) \;=\; x^n \;+\; n\cdot h\cdot x^{n-1} \;+\; h^2 \sum_{i=0}^{n-2} \binom{n}{i+2} x^{n-i-2}\, h^i$$

c) If $x \in \mathbb{R}$ and $n \in \mathbb{N}$ and $n \geq 2$, prove that

$$\lim_{h \to 0} \sum_{i=0}^{n-2} \binom{n}{i+2} x^{n-i-2}\, h^i$$

exists (and in particular is not "infinite"), and is usually non-zero.

d) Assuming $n \in \mathbb{N}$, prove, directly from the definition of the derivative, that

$$f'(x) \;=\; n\, x^{n-1}$$

Chapter 12

3) In class we proved the *triangle inequality*, namely the statement that, if $\vec{v}, \vec{w} \in \mathbb{R}^n$ then

$$\|\, \vec{v} \,+\, \vec{w} \,\| \;\leq\; \|\vec{v}\| \,+\, \|\vec{w}\|.$$

Using this, prove the following three versions of the triangle inequality:

a) $\|\vec{v}\| \,-\, \|\vec{w}\| \;\leq\; \|\vec{v} \,-\, \vec{w}\|$.

b) $\|\vec{w}\| \,-\, \|\vec{v}\| \;\leq\; \|\vec{v} \,-\, \vec{w}\|$.

c) $\Big|\|\vec{v}\| \,-\, \|\vec{w}\|\Big| \;\leq\; \|\vec{v} \,-\, \vec{w}\|$.

4) In this problem we take an axiomatic approach to the notion of the norm of a vector.

Definition. Let V be any vector space (whose scalars can be either $\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : V \to \mathbb{R}$ is called a *norm* provided the following properties hold:

i) (positivity) $\|\vec{v}\| \;\geq\; 0$, with equality if and only if $\vec{v} = \mathcal{O}$

ii) (homogeneity) $\|c\,\vec{v}\| \;=\; |c|\,\|\vec{v}\|$ where c is any scalar and $\vec{v} \in \mathbb{C}^n$ (or $\mathbb{R}^n$)

iii) (triangle inequality) $\|\vec{v} \,+\, \vec{w}\| \;\leq\; \|\vec{v}\| \,+\, \|\vec{w}\|$ where $\vec{v}, \vec{w} \in \mathbb{C}^n$ (or $\mathbb{R}^n$)

a) Define the function $\|\cdot\|_\infty : \mathbb{C}^n \to \mathbb{R}$ by

$$\|\vec{v}\|_\infty \;=\; \max_{i \in \{1,\ldots,n\}} |v_i|\,, \quad \textit{where}\;\; \vec{v} \;=\; (v_1,\, v_2,\, \ldots,\, v_n)\,.$$

This is called the *max-norm* (or sometimes the *sup-norm*). Give the following vectors in $\mathbb{C}^3$, calculate both $\|\vec{v}\|_\infty$ and the usual norm $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

1) $\vec{v} \;=\; (1-i,\, 0,\, i)$
2) $\vec{v} \;=\; (1+i,\, 0,\, -i)$
3) $\vec{v} \;=\; (1,\, 0,\, 0)$
4) $\vec{v} \;=\; (i,\, i,\, i)$

b) Prove that $\|\cdot\|_\infty$ indeed satisfies (*i*), (*ii*), and (*iii*), and so is a norm.

5) Unfortunately, there are many norms in the sense of problem (3), most of which have nothing to do with dot products. Here we will define what are called the "l^p-norms." We have defined vectors $\vec{v}$ in $\mathbb{C}^n$ (or $\mathbb{R}^n$) to be lists of complex (or real) numbers:

$$\vec{v} \;=\; (\,v_1,\, v_2,\, \ldots,\, v_n)\,.$$

Given any $p \in \mathbb{R}$ and $p > 0$, define the l^p*-norm* on $\mathbb{C}^n$ (or $\mathbb{R}^n$) to be

$$\|\vec{v}\|_p \;=\; \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}\,. \tag{2}$$

a) Prove that axiom (i) holds for $\|\cdot\|_p$

b) Prove that axiom (ii) holds for $\|\cdot\|_p$

(Property (iii), the triangle inequality, only holds when $1 \leq p < \infty$. The proof is a little more difficult and will be left for the extra credit.)

6) The formula for the area of a parallelogram is $A = bh$ where b is base length and h is the altitude. Using this, a little trigonometry, and what you know about dot products, prove that the area of the parallelogram spanned by any two vectors $\vec{v}$ and $\vec{w}$ is $\sqrt{\|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v}\cdot\vec{w})^2}$.

Chapter 13

7) Let $\mathcal{K} = M(\vec{P}; \vec{v}_1, \ldots, \vec{v}_k)$ be a k-plane and let $\mathcal{L} = M(\vec{Q}; \vec{w})$ be a line. In this and the next two problems, we will provide the conditions for $\mathcal{L}$ to lie within the plane $\mathcal{K}$. For this problem, state what it means for $\mathcal{L}$ to lie within $\mathcal{K}$. Put another way, what, precisely, does it mean for a point $\vec{v} \in \mathbb{R}^n$ to be a point of $\mathcal{L}$ and also be a point of $\mathcal{K}$?

8) In the context of problem (7), prove that if

   i) $\vec{Q} - \vec{P} \in span\{v_1, \ldots, v_k\}$

   ii) $\vec{w} \in span\{v_1, \ldots, v_k\}$

   are both met, then $\mathcal{L} \subseteq \mathcal{K}$.

9) In the context of problem (7), prove that if $\mathcal{L} \subseteq \mathcal{K}$, then conditions ($i$) and ($ii$) of problem (8) are both met.

10) Do #12 in §13.8. (Hint: you can use the theorem proved in problems (7)-(9).)