Homework 6 — Math 116

Due October 25, 2012

Remember: No credit will be given for answers without mathematical or logical justification.

Chapter 1

1) Let $f(x)$ be the function on the interval $[0,1]$ defined by

$$f(x) \quad = \quad \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Assume $s(x)$ is some step function defined on $[0,1]$. By the definition of a step function, we know $s(x)$ must be subordinate to some partition $P = \{x_0, x_1, \ldots, x_N\}$, and that $s(x)$ has the form

$$s(x) \;=\; s_k \text{ on the } k^{th} \text{ interval}$$

where the s_k are constants. Finally, suppose that $s(x) \geq f(x)$.

a) Make a rough sketch of $f(x)$. Explain, on an intuitive level, why it stands to reason that $\overline{\int}_0^1 f(x)\,dx \geq \frac{1}{2}$.

b) Relative to the partition P, what, specifically, is I_k, and what is $\triangle I_k$? (Here I_k is the k^{th} interval and $\triangle I_k$ is the length of the interval I_k.)

c) Prove that $s_k \geq x_k$.

d) Prove that $\int_0^1 s(x)\,dx \;\geq\; \sum_{k=1}^N x_k(x_k \,-\, x_{k-1})$.

e) Prove that $\int_0^1 s(x)\,dx \;\geq\; \sum_{k=1}^N \frac{1}{2}(x_k + x_{k-1})(x_k \,-\, x_{k-1})$.

f) Prove that $\int_0^1 s(x)\,dx \;\geq\; \frac{1}{2}$.

g) Prove that $\overline{\int}_0^1 f(x)\,dx \;\geq\; \frac{1}{2}$.

Chapter 12

2) Prove (a) and (b) of Theorem 12.2.

3) Let $\vec{v}, \vec{w} \in \mathbb{R}^n$, and assume that the angle between $\vec{v}$ and $\vec{w}$ is acute. Prove that the angle between $\vec{v}$ and $-\vec{w}$ is obtuse.

4) Let $\vec{v} = (1,2,3,4,5) \in \mathbb{R}^5$. Determine the projection of $\vec{v}$ onto the following vectors:

a) $\vec{w} = (0,0,0,0,1)$

b) $\vec{w} = (0,0,0,0,167)$

c) $\vec{w} = (1,1,1,1,1)$

d) $\vec{w} = (1,2,3,4,5)$

e) $\vec{w} = (1,-1,0,0,0)$

5) Prove that $\text{proj}_{\vec{a}}\vec{b}$ is linear in the second variable: specifically, if c_1, c_2 are any constants and $\vec{a}, \vec{b}_1, \vec{b}_2$ are any vectors in $\mathbb{R}^n$, prove that

$$\text{proj}_{\vec{a}}(c_1\vec{b}_1 \,+\, c_2\vec{b}_2) \;=\; c_1\,\text{proj}_{\vec{a}}(\vec{b}_1) \,+\, c_2\,\text{proj}_{\vec{a}}(\vec{b}_2)$$

6) By contrast, prove that $\text{proj}_{\vec{a}}\vec{b}$ is invariant under constant multiplication in the first variable: specifically, if c is any constant, prove that

$$\text{proj}_{c\vec{a}}\vec{b} \;=\; \text{proj}_{\vec{a}}\vec{b}.$$

7) Below are listed several sets of vectors in $\mathbb{R}^4$. In each case, state why the set is not a basis of $\mathbb{R}^4$:

   a) $S \;=\; \{(1,0,0,0),\,(0,1,0,0),\,(0,0,1,0)\}$
   b) $S \;=\; \{(1,2,1,-1),\,(1,1,1,1),\,(-2,-1,1,-1),\,(0,2,3,-1)\}$
   c) $S \;=\; \{(2,1,2),\,(0,0,1),\,(0,1,-1)\}$

8) We defined a sesquilinear inner product on $\mathbb{C}^n$ by

$$A\cdot B \;=\; \sum_{i=1}^{n} a_i\bar{b}_i$$

(the Hermitian inner product). However it is certainly possible to define a bilinear inner product on $\mathbb{C}^n$ as well, simply by setting

$$A\cdot_b B \;=\; \sum_{i=1}^{n} a_i b_i$$

as in the real case. In the context of $\mathbb{C}^2$, show that $A\cdot_b A$ need not be a positive real number (meaning, for instance, that it would make little sense to define "length" by $\|A\| = \sqrt{A\cdot_b A}$).

9) Let $\mathbb{R}^\infty$ be the vector space whose elements are *ordered lists* of real numbers. To be precise, $\mathbf{v}$ is in $\mathbb{R}^\infty$ if

$$\mathbf{v} \;=\; (v_1,\, v_2,\, \ldots,\, v_i,\, \ldots)$$

and $v_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. As with vectors in $\mathbb{R}^n$, we can add an subtract elements of $\mathbb{R}^\infty$, and multiply by constants. Specifically, if

$$\begin{aligned}\mathbf{v} &\;=\; (v_1,\, v_2,\, \ldots,\, v_i,\, \ldots)\\ \mathbf{w} &\;=\; (w_1,\, w_2,\, \ldots,\, w_i,\, \ldots)\end{aligned}$$

are elements of $\mathbb{R}^\infty$ then we define addition and constant multiplication by

$$\begin{aligned}\mathbf{v}+\mathbf{w} &\;=\; (\,v_1+w_1,\, v_2+w_2,\, \ldots,\, v_i+w_i,\, \ldots)\\ c\mathbf{v} &\;=\; (cv_1,\, cv_2,\, \ldots,\, cv_i,\, \ldots)\end{aligned}$$

where $c \in \mathbb{R}$. We define the *zero element*, or *origin*, to be the ordered list consisting entirely of zeros:

$$\mathcal{O} \;=\; (0,\, 0,\, \ldots,\, 0,\, \ldots).$$

With these definitions, prove that $\mathbb{R}^\infty$ is a vector space.

10) §12.15 #9

11) §12.15 #13