Homework 10 Math 116

Due Dec 6, 2012

Remember: No credit will be given for answers without mathematical or logical justification.

1) Assume $\vec{f}(t)$ is a path in $\mathbb{R}^n$ with zero curvature. Formally prove that the graph of $\vec{f}(t)$ is a straight line. (Hint: take a look at the $\vec{T}$ and $\vec{N}$ components of acceleration, and integrate.)

2) In this problem, we give a concrete illustration of the (intuitively obvious) fact that arc length does not change when parametrization changes. Set

$$\vec{f}(v) \;=\; (\cos(v),\, \sin(v),\, v)\,. \tag{1}$$

Set $v_0 = 0$ and $v_1 = \pi$.

a) Compute the arclength between $v = v_0$ and $v = v_1$ by evaluating the arclength integral

$$\int_{v=v_0}^{v_1} \left\| \frac{d\vec{f}}{dv} \right\| \, dv \tag{2}$$

b) Now suppose $v = v(t)$ is itself a function of t—for this problem we'll choose the reparametrization $v(t) = t^2 - 1$. Express $\vec{f}(v) = \vec{f}(v(t))$ as a function of t. Show that the reparametrization is positive. Also, as v varies from v_0 to v_1, t varies from t_0 to t_1. What are the numbers t_0 and t_1?

c) Compute the arclength between $t = t_0$ and $t = t_1$ by evaluating the arclength integral

$$\int_{t=t_0}^{t} \left\| \frac{d\vec{f}}{dt} \right\| \, dt. \tag{3}$$

Explain, on an intuitive level, why you got the same number as you did in (*a*).

3) Given any function $\vec{f}(v) = (f_1(v), \ldots, f_n(v))$ of v, and picking any numbers $v_0 < v_1$, we know that the length of arc between $v = v_0$ and $v = v_1$ is

$$\int_{v_0}^{v_1} \left\| \frac{d\vec{f}}{dv} \right\| \, dv. \tag{4}$$

Now assume f has been reparametrized: $v = v(t)$ is a function of t, so that $\vec{f}(v) = \vec{f}(v(t))$ is now a function of t. Further assume that the parametrization is positive, meaning $\frac{dv}{dt} > 0$. Set $t_0 = v(t_0)$ and $t_1 = v(t_1)$. Formally prove that the arclegth in invariant under reparametrization. That is, prove that

$$\int_{v_0}^{v_1} \left\| \frac{d\vec{f}}{dv} \right\| \, dv \;=\; \int_{t_0}^{t_1} \left\| \frac{d\vec{f}}{dt} \right\| \, dt \tag{5}$$

4) In this problem, we will reparametrize a function according to arclength. Set

$$\vec{f}(t) \;=\; \left(\frac{1}{2}\cos(t^2),\, \frac{1}{2}\sin(t^2),\, \frac{1}{3}t^3\right). \tag{6}$$

Let s be the length of arc between time 0 and time t, meaning

$$s \;=\; \int_0^t \left\|\frac{d\vec{f}}{d\tau}\right\| d\tau \tag{7}$$

a) Compute $s = s(t)$ as a function of t.

b) Compute $t = t(s)$ as a function of s, and compute the reparametrization of $\vec{f} = \vec{f}(t(s))$ as a function of arclength s.

c) With your new parametrization of $\vec{f}$ in terms of s, show by computation that $\frac{d\vec{f}}{ds}$ is a unit vector.

5) From Problem (4) we now have two parametrizations of the path $\vec{f}$:

$$\begin{aligned} \vec{f} \;&=\; \left(\frac{1}{2}\cos(t^2),\, \frac{1}{2}\sin(t^2),\, \frac{1}{3}t^3\right) \\ \vec{f} &= \left(\frac{1}{2}\cos\left((3s+1)^{\frac{2}{3}}-1\right),\, \frac{1}{2}\sin\left((3s+1)^{\frac{2}{3}}-1\right),\, \frac{1}{3}\left((3s+1)^{\frac{2}{3}}-1\right)^{\frac{3}{2}}\right) \end{aligned} \tag{8}$$

Each of the following computations may either be horrific or merely quite bad, depending on how you choose to proceed. Before beginning each step, stop and ask yourself what method would be most efficient.

a) Compute $\vec{T}$ as a function of s and as a function of t.

b) Compute $\vec{N}$ as a function of s and as a function of t.

c) Compute $\vec{B}$ as a function of s and as a function of t.

d) Compute κ as a function of s and as a function of t.

e) Compute τ as a function of s and as a function of t.

6) With the $\vec{f} = \vec{f}(t)$ from Problems (4) and (5), determine both the tangential and normal acceleration.

7) Consider the path $\vec{f}(t) = (\cos(t), \sin(t), 1)$.

a) Describe, in words, the path that $\vec{f}$ traces out as t goes from 0 to 2π.

b) Compute $\vec{T}$, $\vec{N}$, $\vec{B}$ as functions of time.

c) Without doing a single calculation, determine τ. What is the osculating plane in this case, and why does it make sense that τ is what it is?

8) The path $\vec{f}(t) = (r\cos(t), r\sin(t))$ traces out a circle in $\mathbb{R}^2$ of radius r. What is the curvature of the circle of radius r?

9) Compute the length of the path $\vec{\gamma}(t) = (t+\ln t,\, t-\ln t)$ between time 0 and 1 (evaluate if you can, otherwise leave the solution in implicit form).

10) If $\vec{f}(t)$ is the position vector and $\vec{v}(t) = \frac{d}{dt}\vec{f}$ is velocity and $\vec{a}(t) = \frac{d^2}{dt^2}\vec{f}$ is acceleration, then prove the following:

a) If the triple product $\vec{T} \cdot (\vec{a} \times \vec{B})$ vanishes, then the motion is in a straight line or there is no motion.

b) If the triple product $\vec{N} \cdot (\vec{a} \times \vec{B})$ vanishes, then the motion has constant (possibly zero) speed.

11) Do #19 in §14.13

12) Consider the path $\vec{f}(t) = (\cos(t/5), \sin(t/5), \cos(t/3), \sin(t/3))$.

a) Show that the position and velocity at time $t = 2 \cdot 5 \cdot 3\pi$ coincide with the position and velocity at time 0. Explain why $\vec{f}(t)$, $0 \le t < 30\pi$ forms a smooth, closed loop in $\mathbb{R}^4$.

b) Show that the parametrization has constant speed. What is the length of the loop it describes?

c) At each time t, determine the osculating 2-plane.

d) The gyrations of the osculating 2-plane not withstanding, prove that the path's curvature is constant.